PROPAGATION OF AN ELECTROMAGNETIC WAVE ALONG

A HELIX SURROUNDED BY A RESISTANCE SHEATH

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Abstract

The problem of wave-propagation along a helix when losses are present arises in connection with the design of broadband travelling-wave tubes. Due to imperfect matching, a part of the wave will be reflected at the output end of the tube. To reduce the amplitude of the reflected wave and thus avoid oscillations it is convenient to surround the helix by a sheath of lossy material. In this report the effect of this resistance sheath on the phase velocity of the wave and the resulting attenuation is calculated. Only the case when no electron beam is present has been treated. It is shown that the introduction of losses in this way reduces the phase velocity, and that the amount of this reduction approaches zero for very high and very low frequencies.

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1. Solution of the Boundary-Value Problem

To simplify the problem, certain idealizations are made. The helix is represented by a helical sheath, that is, a cylindrical surface of zero thickness and nonisotropic conductivity (see Fig. 1). If ψ is the pitch-angle of the helix, the conductivity of the sheath is zero in a direction forming the angle ψ with the axis. Normal to this direction the conductivity is infinite.

The current flowing in the resistance sheath is supposed to be a surface current, with its magnitude proportional to the electric field. Finally the assumption is made that only axially symmetrical modes are excited.

For a point of zero current and zero-charge density, Maxwell's equations are

$$\left.\begin{aligned} \nabla \times \vec{H} - i\omega\epsilon\vec{E} = 0 \\[2mm] \nabla \times \vec{E} + i\omega\mu\vec{H} = 0 \end{aligned}\right\} \tag{1.1}$$

when harmonic time dependence of the fields is assumed. The circularly symmetrical solution in the cylindrical coordinate system (r, θ, z) is found to be [1]

$$\left.\begin{aligned} E_z &= \left[A\, I_0(pr) + B\, K_0(pr)\right] e^{-\gamma z} \\[3mm] E_r &= \frac{\gamma}{p}\left[A\, I_1(pr) - B\, K_1(pr)\right] e^{-\gamma z} \\[3mm] H_\theta &= \frac{i\omega\epsilon}{p}\left[A\, I_1(pr) - B\, K_1(pr)\right] e^{-\gamma z} \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} H_z &= \left[C\, I_0(pr) + D\, K_0(pr)\right] e^{-\gamma z} \\[3mm] H_r &= \frac{\gamma}{p}\left[C\, I_1(pr) - D\, K_1(pr)\right] e^{-\gamma z} \\[3mm] E_\theta &= -\frac{i\omega\mu}{p}\left[C\, I_1(pr) - D\, K_1(pr)\right] e^{-\gamma z} \end{aligned}\right\} \tag{1.3}$$

where $I_n(z)$ and $K_n(z)$ are the modified Bessel functions of first and second kind, respectively. The parameter p is defined by

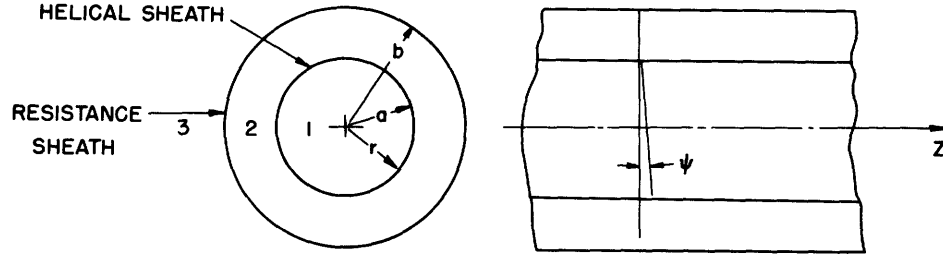$$p^2 = -k^2 - \gamma^2 = -\epsilon\mu\omega^2 - \gamma^2 \tag{1.4}$$

Fig. 1

Helix with the surrounding resistance sheath.

A, B, C, and D are constants whose value in the various regions of the field is determined by the boundary conditions. Let the subscript 1 on any of these constants refer to the region inside the helical sheath; the region between the sheaths and the region outside are denoted by subscripts 2 and 3, respectively.

Since $K_n(z)$ has a singularity for $z = 0$ and $I_n(z)$ has one for $z = \infty$, it is immediately clear that we must have

$$B_1 = D_1 = A_3 = C_3 = 0$$

In the determination of the remaining eight constants the boundary conditions give eight linear homogeneous equations. A solution exists only if the determinant of this system of equations is equal to zero. This condition determines p and hence the propagation constant γ. In the present case, the method of substitution is more convenient.

If the surface conductivity of the resistance sheath is σ, the surface current is

$$\vec{K} = \sigma\left(\vec{E}_{t,3}\right)_{r=b}$$

and we get a discontinuity in the tangential components of the magnetic field across the sheath.

$$\vec{n} \times \left(\vec{H}_2 - \vec{H}_3\right) = \vec{K}$$

When this is expressed by field components, we get

$$\left.\begin{aligned}
B_3 &= -\frac{A_2\, I_1(pb) - B_2\, K_1(pb)}{K_1(pb) - i\sigma Z_o \dfrac{p}{k} K_0(pb)} \\[2em]
D_3 &= \frac{C_2\, I_0(pb) + D_2\, K_0(pb)}{K_0(pb) + i\sigma Z_o \dfrac{p}{k} K_1(pb)}
\end{aligned}\right\} \tag{1.5}$$

where Z_o is the intrinsic impedance of the dielectric

$$Z_o = (\mu/\epsilon)^{1/2}$$

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On the other hand, the tangential components of the electric field must be continuous across the sheath.

$$A_2 \, I_0(pb) + B_2 \, K_0(pb) = B_3 \, K_0(pb)$$

$$C_2 \, I_1(pb) - D_2 \, K_1(pb) = - D_3 \, K_1(pb)$$

Together with Eq. 1.5 this determines the ratios A_2/B_2 and C_2/D_2.

$$\left.
\begin{aligned}
d &= \frac{A_2}{B_2} = \frac{i\sigma Z_0 p^2 b^2 K_0^2(pb)}{kb - i\sigma Z_0 p^2 b^2 I_0(pb) K_0(pb)} \\[2em]
s &= \frac{C_2}{D_2} = \frac{i\sigma Z_0 K_1^2(pb)}{\frac{1}{kb} + i\sigma Z_0 I_1(pb) K_1(pb)}
\end{aligned}
\right\} \tag{1.6}$$

Turning our attention to the helical sheath, the tangential electric field must be zero on both sides in the direction of infinite conductivity.

$$A_1 \, I_0(pa) \sin \psi - \frac{i\omega\mu}{p} C_1 \, I_1(pa) \cos \psi = 0$$

$$\left[A_2 \, I_0(pa) + B_2 \, K_0(pa)\right] \sin \psi - \frac{i\omega\mu}{p} \left[C_2 \, I_1(pa) - D_2 \, K_1(pa)\right] \cos \psi = 0$$

Further, the tangential electric field must be continuous.

$$A_1 \, I_0(pa) = A_2 \, I_0(pa) + B_2 \, K_0(pa)$$

Together with Eq. 1.6, these relations give us all of the constants in terms of A_1.

$$\left.
\begin{aligned}
B_2 &= \frac{I_0(pa)}{d \, I_0(pa) + K_0(pa)} A_1 \quad ; \quad A_2 = d \, B_2 \\[2em]
D_2 &= \frac{p \tan \psi}{i\omega\mu} \frac{I_0(pa)}{s \, I_1(pa) - K_1(pa)} A_1 \quad ; \quad C_2 = s \, D_2 \\[2em]
C_1 &= \frac{p}{i\omega\mu} \frac{I_0(pa)}{I_1(pa)} \tan \psi \, A_1
\end{aligned}
\right\} \tag{1.7}$$

Finally, the tangential magnetic field in the direction of current flow must be continuous across the sheath.

$$\frac{i\omega\epsilon}{p} A_1 \, I_1(pa) \cos \psi + C_1 \, I_0(pa) \sin \psi = \frac{i\omega\epsilon}{p} \left[A_2 \, I_1(pa) - B_2 \, K_1(pa)\right] \cos \psi$$

$$\tag{1.8}$$

$$+ \left[C_2 \, I_0(pa) + D_2 \, K_0(pa)\right] \sin \psi$$

When the values given in Eq. 1.7 of the constants are substituted in Eq. 1.8, we arrive

at the desired equation for p. At this point it is convenient to introduce the notation

$$z = ap; \quad ka = \frac{2\pi a}{\lambda} = u; \quad \frac{b}{a} = v$$

Our determinantal equation then takes the form

$$z^2 \frac{I_0(z) \, K_0(z)}{I_1(z) \, K_1(z)} f(z) = u^2 \cot^2 \psi \tag{1.9}$$

where

$$f(z) = \frac{1 + d(z) \dfrac{I_0(z)}{K_0(z)}}{1 - s(z) \dfrac{I_1(z)}{K_1(z)}}$$

and

$$d(z) = \frac{i\sigma Z_0 v^2 z^2 \, K_0^2(vz)}{vu - i\sigma Z_0 v^2 z^2 \, I_0(vz) \, K_0(vz)}$$

$$s(z) = \frac{i\sigma Z_0 \, K_1^2(vz)}{\dfrac{1}{vu} + i\sigma Z_0 \, I_1(vz) \, K_1(vz)}$$

2. Approximate Solution of Equation 1.9

For extreme values of σ, Eq. 1.9 is considerably simplified. For $\sigma = 0$ we get $d(z) = s(z) = 0$ and $f(z) = 1$. The equation then reduces to the one solved by Chu and Jackson (2). For $\sigma = \infty$, which corresponds, of course, to the case when the helix is surrounded by a perfectly conducting cylinder, the equation is also much simplified. This case has been discussed by Johnsen and Dahl (3) and others.

In the general case, when σ is finite, the solution z of Eq. 1.9 is a complex quantity. Let us write

$$z = x + iy$$

and

$$\gamma = \alpha + i\beta$$

By means of the relation (1.4) α and β may be expressed by x and y. Since the phase velocity is

$$v_p = \frac{\omega}{\beta}$$

we find

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$$\left(\frac{v_p}{c}\right)^2 = \frac{u^2}{\left[\frac{1}{4}\left(x^2 - y^2 + u^2\right)^2 + x^2 y^2\right]^{1/2} + \frac{1}{2}\left(u^2 + x^2 - u^2\right)}$$

$$a^2 \alpha^2 = \left[\frac{1}{4}\left(x^2 - y^2 + u^2\right)^2 + x^2 y^2\right]^{1/2} + \frac{1}{2}\left(y^2 - x^2 - u^2\right)$$

(2.1)

When $|x| \gg |y|$ these formulas simplify to

$$\frac{v_p}{c} = \frac{u}{\left(x^2 + u^2\right)^{1/2}} \qquad a\alpha = \frac{-xy}{\left(x^2 + u^2\right)^{1/2}}$$

(2.2)

For further discussion, we shall restrict ourselves to the case

$$\sigma Z_o = 1$$

The surface impedance of the resistance sheath is equal to the intrinsic impedance of the medium.

(a) $|z| \gg 1$

When $|z|$ is large, the Bessel functions may be replaced by the first term of their asymptotic expansions. We have (4)

$$I_0(z) \approx I_1(z) \approx \frac{e^z}{(2\pi z)^{1/2}}$$

$$K_0(z) \approx K_1(z) \approx \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}$$

(2.3)

With these approximations our equation assumes the form

$$z^2 \frac{1 + i \dfrac{\dfrac{2z}{u}}{1 - i \dfrac{z}{2u}} e^{2z(1-\nu)}}{1 - i \dfrac{1}{1 - i \dfrac{2z}{u}} e^{2z(1-\nu)}} = u^2 \cot^2 \psi$$

(2.4)

From this, it is apparent that when $z \to \infty$

$$\frac{z}{u} \to \cot \psi$$

In most practical cases ψ is a small angle and therefore z/u is a large number. Hence Eq. 2.4 is approximately equivalent to

$$z \left[1 - e^{-2z(1-\nu)}\right]^{1/2} = u \cot \psi$$

(2.5)

This equation is easily solved numerically. If a solution z_1 of this equation is

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found for given u, ψ, and ν, a better approximation is found from Eq. 2.4, namely

$$z_2 = \frac{u \cot \psi}{\left[f(z_1)\right]^{1/2}} \tag{2.6}$$

where

$$f(z_1) = \frac{1 + i \dfrac{\dfrac{z_1}{2u}}{1 - i\dfrac{z_1}{2u}} e^{2z_1(1-\nu)}}{1 - i \dfrac{\dfrac{1}{2z_1}}{1 - i\dfrac{1}{u}} e^{2z_1(1-\nu)}} \tag{2.7}$$

The degree of accuracy that may be obtained from this method of successive approximations is, of course, limited by the errors of the asymptotic representation of the Bessel functions. Fortunately the various errors tend to compensate each other so that the total error of the result is considerably less than in any of the asymptotic expressions.

Solutions of Eq. 2.5 give a good approximation to the real part of z for $z \geq 3$ and $\psi \leq 10°$, and from Eq. 2.2 we find

$$\left.\begin{array}{l} \left(\dfrac{v_p}{c}\right)^2 = \dfrac{\left[1 - e^{2z(1-\nu)}\right] \tan^2 \psi}{1 + \left[1 - e^{2z(1-\nu)}\right] \tan^2 \psi} \\[4ex] u = z \left[1 - e^{2z(1-\nu)}\right]^{1/2} \tan \psi \end{array}\right\} \tag{2.8}$$

for $z \gtrless 3, \psi \lessgtr 10°$.

This set of parametric equations gives directly the phase velocity as a function of $u = 2\pi a/\lambda$. For still smaller values of the pitch angle ψ, Eqs. 2.8 simplify to

$$\left.\begin{array}{l} \dfrac{v_p}{c} = \left[1 - e^{2z(1-\nu)}\right]^{1/2} \tan \psi \\[4ex] u = z \left[1 - e^{2z(1-\nu)}\right]^{1/2} \tan \psi \end{array}\right\} \tag{2.9}$$

A first-order approximation for the attenuation factor α may be found in a similar manner. When we assume $y \ll x$ and $u \ll x$, formula 2.2 for α becomes

$$a\alpha = -y$$

We find y by taking the imaginary part of z_2 as given by Eq. 2.6. The result is

$$a\alpha = \frac{e^{2z(1-\nu)}}{1 - e^{2z(1-\nu)}} \tag{2.10}$$

$$z \left[1 - e^{2z(1-\nu)}\right]^{1/2} = u \cot \psi$$

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This gives α in nepers per unit length as a function of u. We observe that the attenuation factor approaches zero when u increases towards infinity.

$$(b) \quad |z| \ll 1$$

When $|z|$ is very small, the modified Bessel functions may be replaced by the first term of their Taylor expansion.

$$I_0(z) \approx 1; \quad I_1(z) \approx \frac{z}{2}; \quad K_0(z) \approx -\ell n \frac{\gamma z}{2}; \quad K_1(z) = \frac{1}{z}$$

where

$$\ell n \gamma = 0.577.$$

When this is introduced we find

$$f(z) = \frac{u}{u + i \nu z^2 \, \ell n \frac{\gamma \nu z}{2}} \tag{2.11}$$

and Eq. 1.9 gives

$$u = (2)^{1/2} z \left(-\ell n \frac{\gamma z}{2} \tan \psi \right)^{1/2} \tag{2.12}$$

From this it is seen that when u approaches zero we must also have z approach zero, and, further, that the imaginary part of z is a small quantity of higher order than the real part. Hence the approximations of Eq. 2.2 are still valid. It is further seen that

$$\frac{u}{z} \to \infty \text{ as } u \to 0$$

or

$$\frac{v_p}{c} = \frac{u}{\left(x^2 + u^2 \right)^{1/2}} \to 1, \text{ as } u \to 0$$

The phase velocity tends towards the velocity of light as u approaches zero. From the fact that the imaginary part of z decreases faster than the real part we see that

$$\alpha a \to 0, \text{ as } u \to 0$$

The attenuation factor approaches zero for low frequencies.

3. Conclusion

Formulas have been derived for the phase velocity and attentuation of a circularly symmetrical wave propagating along a helix when the surface resistance of the surrounding sheath is equal to the intrinsic impedance of the medium (377 ohms for vacuum).

The phase velocity has been plotted in Figs. 2, 3, and 4 as a function of $u = 2\pi a/\lambda$ for three specific ratios of the diameters of the two sheaths, namely, $\nu = 1.05$, $\nu = 1.10$, and $\nu = 1.20$. In Figs. 5 and 6 the attentuation is plotted for $\nu = 1.05$ and $\nu = 1.20$. For a certain range of values of u the approximations made are not valid. This region is,
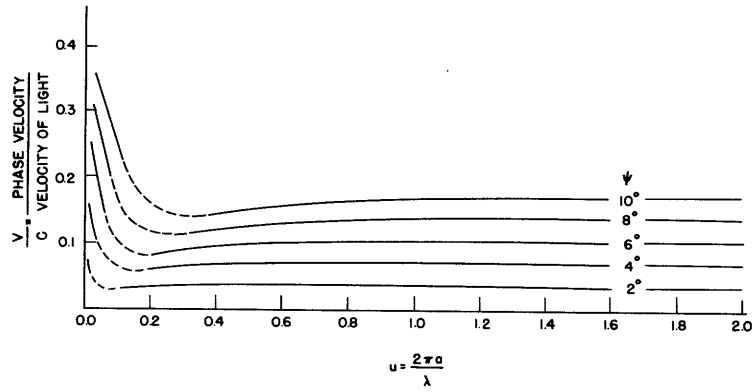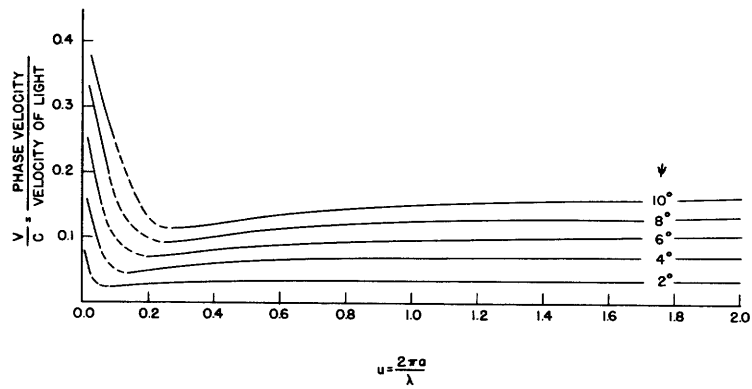
Fig. 2

Phase velocity plotted as a function of $u = 2\pi a/\lambda$
$$\nu = \frac{b}{a} = \frac{radius\ of\ resistance\ sheath}{radius\ of\ helical\ sheath} = 1.2$$
Surface resistance = 377 ohms per square.

Fig. 3

Phase velocity plotted as a function of $u = 2\pi a/\lambda$
$$\nu = \frac{b}{a} = \frac{radius\ of\ resistance\ sheath}{radius\ of\ helical\ sheath} = 1.1$$
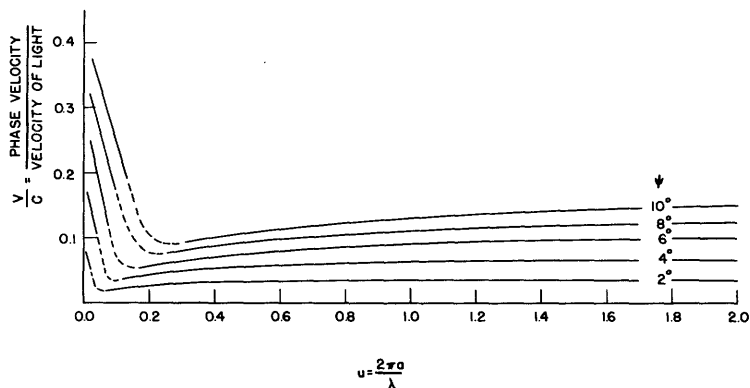Surface resistance = 377 ohms per square.

Fig. 4

Phase velocity plotted as a function of $u = 2\pi a/\lambda$

$$\nu = \frac{b}{a} = \frac{\text{radius of resistance sheath}}{\text{radius of helical sheath}} = 1.05$$
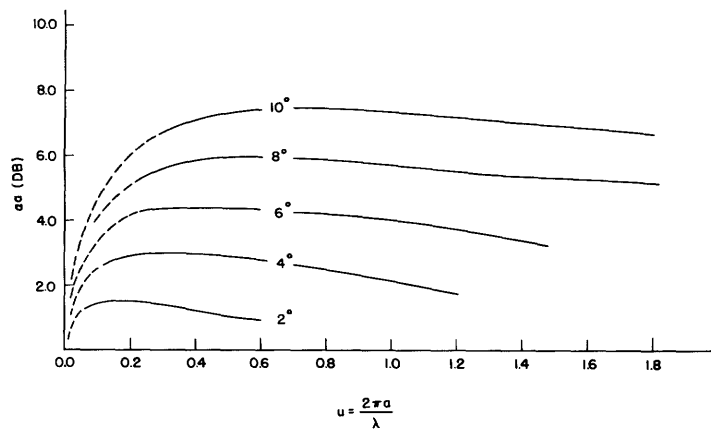
Surface resistance = 377 ohms per square.

Fig. 5

Attenuation plotted as a function of $u = 2\pi a/\lambda$

$$\nu = \frac{b}{a} = \frac{\text{radius of resistance sheath}}{\text{radius of helical sheath}} = 1.05$$

Surface resistance = 377 ohms per square
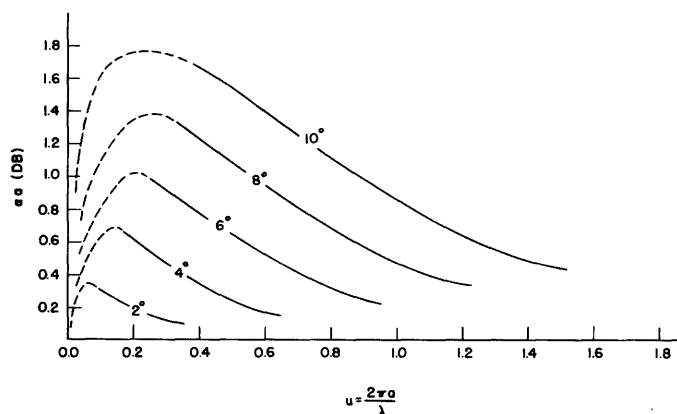a = attenuation in decibels per unit length.

Fig. 6

Attenuation plotted as a function of $u = 2\pi a/\lambda$
$$\nu = \frac{b}{a} = \frac{\text{radius of resistance sheath}}{\text{radius of helical sheath}} = 1.20$$
Surface resistance = 377 ohms per square
a = attenuation in decibels per unit length.

however, quite narrow and it is not difficult to cover it by extrapolation from both sides. This part of the curves is shown dotted. It is seen that the resistance sheath causes a decrease in the phase velocity; and further, that over the major part of the curve, the slope is positive. The group velocity

$$v_g = v_p + u \frac{dv_p}{du}$$

is thus larger than the phase velocity. We have an anomalous dispersion. For $\nu \geq 1.2$, the effect of the resistance sheath in the phase velocity is quite small, but it becomes more pronounced as the distance between the two sheaths decreases. When ν approaches 1 as a limit, the phase velocity approaches zero and the attentuation increases indefinitely.

The qualitative picture is now quite clear: For very high frequencies, the field is concentrated close to the helical sheath; the phase velocity is v_p = c tan ψ; and there are no losses, because the electric field is vanishingly small on the resistance sheath. This is not the case when the frequency becomes smaller. The losses then increase and we get, as a result of this, a decrease in the phase velocity. The losses reach a maximum and thereafter decrease, because the field energy spreads out over such a wide cross section that the intensity in the neighborhood of the resistance sheath again becomes small. As might be expected, the frequency of maximum loss is greater the closer the two sheaths are to each other. When, in the limit, the frequency approaches zero, the sheaths no longer influence the wave propagation. The losses are zero, and the phase velocity approaches the velocity of light.

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