# Make-To-Order vs. Make-To-Stock:
## The Role of Inventory in Delivery-Time Competition

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## Abstract

The paper formulates a set of stochastic models to study the production and inventory policies of a firm that considers explicitly the behavior of customers and competing firms. We start with a single firm production control problem. The optimal policy is solved explicitly, and the optimal mix between make-to-order and make-to-stock operations is determined with customers characterized by patience levels in a simple newsboy-like formula. The model is then extended to a n-firm market game in which firms compete for orders in the aspect of early delivery. One could think of this setting as an oligopoly racing market. The analysis shows that competition can breed a demand for produce-to-stock, just as other economic phenomena such as economies of scale, uncertainty, or seasonality can induce production to inventory, and that competition of this kind increases the buyer's welfare while decreases the producer's welfare. The paper also suggests the analytical and numerical methods which can compute an equilibrium of the stocking game.

## 1. Introduction

Inventories held in the United States exceed one-half trillion dollars in value. Inventory holding is a prominent economic phenomenon that concerns economists as well as operations management academics and practitioners.

The make-to-order versus make-to-inventory question is a fundamental one in organizing and controlling production, but the one that has been subject to relatively little formal modeling in either economics or operations management literatures. One exception is the recent paper by Milgrom and Roberts (1987) that uses a newsboy-type model to formalize the idea that inventories and information about demand are substitutes for one another in production. Their results suggest that firms should employ only one of the two strategies: firms either produce for inventory or produce to order, but not the both.

Many examples support this conclusion, but many others do not. Some firms often shift from one alternative to the other; other firms constantly employ a mix of the two alternatives. For example, in a Burger King Restaurant, much of the sandwich and drink production is triggered by the counter hostess calling in the order to the kitchen during any very slow period of demand. However, in high-demand time intervals, a finished goods inventory of principal burger products is maintained, and the broiler loading operation responds to the finished burger inventory situation rather than to the call-in of orders. (See Schmenner (1981)).

In the Burger King case, the production activity is not a newsboy-like, one-shot decision problem as in the Milgrom and Roberts paper. Therefore, the dynamics of the environment, for example, variations in demand, may provide incentives for holding inventory even when the uncertainty has been resolved at the design stage. Uncertainty can arise from many sources. On the demand side, both the quantity demanded and the demand timing may be uncertain. Uncertainties also arise from stochastic variability in the different stages of production. Thus a complete resolution of uncertainties via communication of information might not be possible, or could be very expensive. In other words, the convexity of survey costs may result in a mixed use of two alternatives. We conclude that determining optimal use of inventory requires detailed operational analysis as well as the static design strategy examined by Milgrom and Roberts. Hence, in order to answer the make-to-order versus make-to-stock question in many productive organizations, dynamic models are necessary.

Why do firms make to stock instead making purely to order? Because of the obvious costs of holding inventory, we may phrase the question more directly: why hold inventories? The traditional economic justification for inventories are decreasing costs or increasing returns to scale (the presence of ordering cost or set-up cost) in procurement and production, seasonality (the anticipated variability in requirements or supplies), and uncertainty (the stochastic variability in requirements or supplies).

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Most inventory control policies are obtained by analyzing tradeoffs among the set-up costs (adjustment costs), holding costs, and stockout or lost sale costs (shortage costs). But there are other reasons besides. If firms compete on speed of delivery, then production lead times may necessitate holding inventory. First, if the production lead time is sufficiently long, then the early realization of sales itself may become a reason to have inventory. Second, the characteristics of the customer are important. For example, if customers are impatient, then holding inventories may help to reduce the customer waiting time and to increase the number of sales that are completed. Finally, the existence and characteristics of competing firms may also be a decisive factor for the firm's use of inventory. For example, suppose there are rival firms in a market competing on the quality of service, particularly, the speed of delivery. The firm which can deliver the product the earliest completes the sale with a waiting customer. Then holding inventory can be used as a tactic to facilitate prompt delivery as opposed to other measures such as increasing the capacity (shortening the production lead time). Certainly, these effects occur only when there is lead time or demand arrival uncertainty because no matter what the lead time, if we know far enough in advance what the requirements will be, we can schedule deliveries accordingly as long as the capacity allows. Traditional inventory models lump these economic effects of timing uncertainty into *costs of shortage* (see Scarf (1963)). However, this simplification makes *costs of shortage* the least computable costs among those that concern inventory control. For the reasons mentioned above, obviously, no one knows how to do it *correctly* even in a very rough sense without some kind of model of the economic environment in which the productive organization operates. Therefore, incorporating the behavior of other agents in the economy is an important task for modeling the strategic use of inventory.

In this paper, we work toward a better understanding of economic issues regarding make-to-order versus make-to-stock, particularly taking into account the characteristics of customers and competing firms. We first set up a continuous time, stochastic control model to study the stocking policy of a single firm, and show that the optimal mix between make-to-order and make-to-stock operations can be determined in our setting. In this case, the firm simply sets an inventory limit via a newsboy-like formula. The optimal inventory limit increases as the demand rate is higher, the average production lead time is longer, the sale's contribution is larger, and the holding cost is lower. In the case that the firm can backlog all of its demand, the possibility of holding inventory is purely due to the time value of money. We also study the case in which customers are characterized by differences in patience and show how increasing impatience shifts the firm's optimal policy toward produce-to-stock regime. These comparative statics results are consistent with and illuminate a variety of observed behavior of productive organizations, for example, the Burger King Restaurant mentioned above. The model is then extended to an n-firm market game in which firms compete for orders in the aspect of early delivery, as in an *oligopoly racing market*. It is noteworthy that

this mode of competition is very common for microchip producers in the semiconductor industry[*]. In that industry, customers often place duplicate orders with a number of suppliers and buy from the first one that can deliver. Duplicate orders are then cancelled so that only the winner of the "race" makes a sale.

The analysis shows that competition can induce inventory holding just as other economic reasons like economics of scale, seasonality, or uncertainty. Firms are more likely to hold inventory when the number of competitors increases (competition is intensified). We also discuss the analytical and numerical methods to solve for the equilibrium in a duopoly racing market. Allowing duplicate orders increases the buyer's welfare and decreases the producer's welfare. Hence, duplicate ordering and induced delivery-time competition may or may not be socially desirable.

## 2. Formulation of the Model

We consider a firm that produces and sells a homogeneous good. Demand arrives in a random fashion. Orders that occur when the products are available in inventory are fulfilled immediately; others must wait for processing. The firm processes orders on a first-come-first-served basis. The cumulative input (demand) and output (production) are represented by two increasing non-negative integer-valued stochastic processes $A = \{A(t), t \geq 0\}$ and $B = \{B(t), t \geq 0\}$, where $A(t)$ and $B(t)$ denote the amount of demand and the amount of production in the time interval $[0, t]$. Then the order waiting and inventory level at time $t$ can be represented by

$$Z(t) = x + A(t) - B(t), \qquad (2.1)$$

where $x$ is the amount of order waiting at time zero if $x \geq 0$ and $-x$ is the amount of initial inventory if $x < 0$. Similarly, $Z(t)$ is the amount of order waiting at $t$ if it is positive and $-Z(t)$ is the amount of inventory at $t$ if $Z(t) < 0$. We assume that

$A, B$ are independent Poisson processes with random intensities $\{\alpha_t, t \geq 0\}$ and $\{\beta_t, t \geq 0\}$.

$$(2.2)$$

That is, $\{A(t) - \int_0^t \alpha_s ds, t \geq 0\}$ and $\{B(t) - \int_0^t \beta_s ds, t \geq 0\}$ are martingales. In the model, $\beta_t$ is the actual production rate at time $t$ which is controllable by the firm. The maximum rate that the firm can achieve is $\mu$. A *feasible operating policy* is then defined as a stochastic process $\beta$ ($\{\beta_t, t \geq 0\}$) that satisfies the following:

(2.3) $\beta$ is left continuous and have right-hand limits,

(2.4) $\beta$ is adapted with respect to $Z$,

(2.5) $0 \leq \beta_t \leq \mu$ for all $t \geq 0$.

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[*] I am indebted to Professor R. Leachman of University of California at Berkeley for conversations on this topic.

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On the other hand, characteristics of process $\alpha$ reflects the action of customers. For example, if we assume that $\alpha_t = \lambda, t \geq 0$, i.e., $A$ is a Poisson process with constant intensity $\lambda$, then demands that arrive will wait indefinitely no matter what the firm's operating policy is. Obviously this is not realistic for many cases, as we allow customers change their behavior based on the historical information. The process $\alpha$ must satisfy the conditions:

(2.6) $\alpha$ is left continuous and have right-hand limits,

(2.7) $\alpha$ is adapted with respect to $Z$,

(2.8) $0 \leq \alpha_t \leq \lambda$ for all $t \geq 0$.

Essentially, these conditions assume that customers behave according to the historical information and that the maximum demand is characterized by a constant arrival rate $\lambda$. For further mathematical justification of the formulation, see Bremaud (1981) and Li (1986).

Note that the monotone increasing assumption of process $A$ implies there is no customer renege, i.e., a customer can not cancel the order once it places the order with the firm. However, this assumption will be relieved in Section 5. where reneging is endogenized in an equilibrium model.

To complete our formulation, we specify the cost structure as follows. The contribution of each order fulfilled is $p$ dollars. A physical holding cost of $h$ dollars per unit time is incurred for each unit of product held in inventory. Assume that the firm earns interest at rate $r > 0$, compounded continuously, on the funds which are required for production operations. Production is planned over an infinite time horizon. Given an initial order waiting or inventory of $x$, the expected firm profit is

$$\pi(x) \equiv E_x \{ \int_0^\infty e^{-rt} [1_{(0,\infty)}(Z(t))p\,dB(t) + 1_{(-\infty,0)}(Z(t))p\,dA(t) - hZ(t)^- \, dt] \}, \qquad (2.9)$$

where $E_x$ denotes the expectation conditional on $Z(0) = x$, $1_S(\cdot)$ is an indicator function for set $S$, and $Z(t)^- \equiv \max(0, -Z(t))$. Note that the firm is making to order when $Z > 0$ and is making to stock when $Z < 0$. Thus, a completion of the processing generates a sale when $Z > 0$, an order arrival generates a sale when $Z < 0$, and neither of them generates any sale when $Z = 0$. A realization of processes $A$, $B$, and $Z$ is constructed in Figure 1.

The problem of the firm is to choose a control process $\beta$ to maximize the expected profit (2.9) such that assumption (2.1)-(2.2), and feasibility constraints (2.3)-(2.5) are satisfied.

## 3. The Single-Firm Problem

We first consider the case that that customers have no alternative and definitely need the processing service from the firm, i.e., $\alpha_t = \lambda$, for $t \geq 0$. In the situation like that, make-to-order business seems quite attractive since backlogged demand is not lost and there is no waiting cost. However, the firm still could hold inventory as shown later, trading off the holding cost for a quicker completion of sales.

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The form of the optimal policy is rather simple, namely, a *barrier policy*. That is, the firm operates at its full capacity until process $Z$ hits a lower barrier $-b$ ($b \geq 0$) and resumes operation until the inventory level $b$ is depleted by one unit. Under a barrier policy with parameter $b$,

$$\beta_t = \mu 1_{(-b,\infty)}(Z(t_-)),$$

and we denote the value function by $v^b(x)$ where the superscript $b$ will be used only when it is necessary.

**Proposition 3.1.** *The value function under a barrier policy with parameter $b$ ($b > 0$) is of the form,*

$$v(x) = \begin{cases} G\rho^x + \frac{\mu}{r}p, & \text{if } x \geq 0; \\ E\rho_1^x + F\rho_2^x + \frac{\lambda}{r}p + (\frac{\lambda-\mu}{r} + x)\frac{h}{r}, & \text{if } x \leq 0; \end{cases} \tag{3.1}$$

*where $\rho = \rho_1$, $\rho_1$ and $\rho_2$ are the two roots of the quadratic equation*

$$\rho = \frac{\lambda}{\lambda + \mu + r} \cdot \rho^2 + \frac{\mu}{\lambda + \mu + r} \tag{3.2}$$

*with $0 < \rho_1 < 1, \rho_2 > 1$, for $r > 0$. More explicitly,*

$$\rho_1 = \frac{\lambda + \mu + r - \sqrt{(\lambda + \mu + r)^2 - 4\lambda\mu}}{2\lambda}, \tag{3.3}$$

$$\rho_2 = \frac{\lambda + \mu + r + \sqrt{(\lambda + \mu + r)^2 - 4\lambda\mu}}{2\lambda}, \tag{3.4}$$

$$E \equiv \frac{d(\rho, \rho_2)\mu h - a(\rho_2)\lambda((\mu - \lambda)(1 - \rho) - r)\rho_2^{-b}(p + \frac{h}{r})}{re(b)}, \tag{3.5}$$

$$F \equiv \frac{a(\rho_1)\lambda((\mu - \lambda)(1 - \rho) - r)\rho_1^{-b}(p + \frac{h}{r}) - d(\rho, \rho_1)\mu h}{re(b)}, \tag{3.6}$$

$$\begin{aligned} G &\equiv E + F + \frac{\lambda - \mu}{r}(p + \frac{h}{r}) \\ &= \frac{\mu^2(\rho_1^{-1} - \rho_2^{-1})h - \lambda((\lambda - \mu)(1 - \rho) + r)(a(\rho_1)\rho_1^{-b} - a(\rho_2)\rho_2^{-b})(p + \frac{h}{r})}{re(b)} \\ &\quad + \frac{\lambda - \mu}{r}(p + \frac{h}{r}), \end{aligned} \tag{3.7}$$

$$a(\rho_i) \equiv \lambda(1 - \rho_i) + r, \qquad d(\rho, \rho_i) \equiv \lambda(1 - \rho) + \mu(1 - \rho_i^{-1}) + r, \quad i = 1, 2, \tag{3.8}$$

$$e(b) \equiv a(\rho_1)d(\rho, \rho_2)\rho_1^{-b} - a(\rho_2)d(\rho, \rho_1)\rho_2^{-b}. \tag{3.9}$$

*Proof.* By the Markov property of process $Z$ under a barrier policy, we can write down the system of difference equations that the value functions $v$ should satisfy, i.e.,

$$v(x) = \frac{\lambda}{\lambda + \mu + r}v(x + 1) + \frac{\mu}{\lambda + \mu + r}v(x - 1) + \frac{\mu}{\lambda + \mu + r}p, \text{ for } x \geq 1, \tag{3.10}$$

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$$v(0) = \frac{\lambda}{\lambda + \mu + r} v(1) + \frac{\mu}{\lambda + \mu + r} v(-1), \tag{3.11}$$

$$v(x) = \frac{\lambda}{\lambda + \mu + r} v(x + 1) + \frac{\mu}{\lambda + \mu + r} v(x - 1) + \frac{\lambda p + xh}{\lambda + \mu + r}, \text{ for } -b + 1 \leq x \leq -1, \tag{3.12}$$

$$v(-b) = \frac{\lambda}{\lambda + r} v(-b + 1) + \frac{\lambda p - bh}{\lambda + r}. \tag{3.13}$$

We show only the derivation of equation (3.12) as an example. The remaining equations can be obtained in a similar fashion. Suppose $Z(0) = x$ and $T$ is the first jump time of $Z$. Under a barrier policy, $T$ is exponential with parameter $\lambda + \mu$ for $-b + 1 \leq x \leq -1$. If a demand comes before a completion of production, the firm fulfils it at time $T$, get $p$ dollars, and the inventory level decreases by one unit, i.e., $Z(T) = x + 1$. Otherwise, the firm has no gain but increases the inventory by one unit by time $T$, i.e., $Z(T) = x - 1$. In either case, the firm holds $|x|$ unit of inventory for $T$ length of time. Thus,

$$v(x) = E_x[e^{-rT} 1_{\{x+1\}}(Z(T))]v(x + 1) + E_x[e^{-rT} 1_{\{x-1\}}(Z(T))]v(x - 1)$$
$$+ E_x[e^{-rT} 1_{\{x+1\}}(Z(T))]p + E_x[\int_0^T e^{-rt} dt]xh.$$

Equation (3.12) follows by noticing that

$$E_x[e^{-rT} 1_{\{x+1\}}(Z(T))] = \lambda E_x[\int_0^T e^{-rt} dt],$$

$$E_x[e^{-rT} 1_{\{x-1\}}(Z(T))] = \mu E_x[\int_0^T e^{-rt} dt],$$

$$E_x[\int_0^T e^{-rt} dt] = E_x[\frac{1 - e^{-rT}}{r}] = \frac{1}{\lambda + \mu + r}.$$

where the first two equations come from Lemma 3.5. in Li (1986) taking $f(\cdot) = 1_{\{x+1\}}(\cdot)$ and $f(\cdot) = 1_{\{x-1\}}(\cdot)$ respectively.

It can be verified that the general solution of the above difference equation must have the form of that defined in (3.1) (see Levy and Lessman (1961)). The remaining problem is to determine $G, E, F$ using the boundary conditions (3.11), (3.13) and the equation

$$G + \frac{\mu}{r} p = E + F + \frac{\lambda}{r} p + (\frac{\lambda - \mu}{r}) \frac{h}{r}. \tag{3.14}$$

which follows from the fact that $v(0)$ satisfies both forms in (3.1).

∎

From the proof of the above proposition, we can see that the computation of the value functions under a barrier policy requires solving a system of difference equations. This is the standard technique we will use throughout the paper. The following proposition shows there is a unique optimal barrier policy.

**Proposition 3.2.** *There exists an optimal barrier policy with the inventory limit $b^*$, that is uniquely determined by the condition*

$$k(b^* + 1) \leq \frac{h/r}{p + h/r}, \text{ and } k(b^*) > \frac{h/r}{p + h/r}, \tag{3.15}$$

*where*

$$k(b) \equiv \frac{a(\rho_1)a(\rho_2)((\lambda - \mu)(1 - \rho) + r)(\rho_2^{-1} - \rho_1^{-1})}{r[a(\rho_1)d(\rho, \rho_2)(1 - \rho_1)\rho_2^b - a(\rho_2)d(\rho, \rho_1)(1 - \rho_2)\rho_1^b]}. \tag{3.16}$$

*And $k(b)$ is decreasing in $b$ with $k(\infty) = 0$.*

*Proof.* Compute the difference

$$v^b(x) - v^{b-1}(x) = (G(b) - G(b-1))\rho_1^x = C(k(b) - \frac{h/r}{p + h/r})\rho_1^x.$$

where

$$C \equiv \frac{\mu^2(\rho_1^{-1} - \rho_2^{-1})[a(\rho_1)d(\rho, \rho_2)(1 - \rho_1)\rho_1^{-b} - a(\rho_2)d(\rho, \rho_1)(1 - \rho_2)\rho_2^{-b}](p + h/r)}{e(b)e(b-1)} > 0.$$

So as a function of $b$, the value function is strictly increasing for $b \leq b^*$ and strictly decreasing for $b > b^*$ since it can be verified that $k$ is strictly decreasing to zero.

∎

Note that the optimal barrier policy determined above is also optimal among all adapted policies and that we have actually solved the intensity control problem formulated in the preceding section (see Li (1984) for a proof). Surprisingly, the solution is very simple and is similar to that of a newsboy problem which fixes a fractile of the demand distribution equal to a critical ratio (a function of the overage cost and the underage cost). Here, the firm simply fixes a inventory limit to balance a decreasing function $k$ and a ratio. To get the economic intuition of formula (3.15), we can rewrite (3.15) as follows either through a policy improvement logic as in Chapter 3.3. of Li (1984) or through direct verification. Let

$$T(y) \equiv \inf\{t \geq 0 : Z(t) = y\},$$
$$\theta(x, y) \equiv E_x[e^{-rT(y)}],$$
$$\Delta v(x) \equiv v(x) - v(x + 1).$$

Then, (3.15) is equivalent to

$$\theta(-(b+1), \infty)\Delta v(\infty) - \Delta v(-(b+1)) \leq 0, \text{ and } \theta(-b, \infty)\Delta v(\infty) - \Delta v(-b) > 0,$$

or simply,

$$\Delta v(-(b+1)) \geq 0, \text{ and } \Delta v(-b) < 0,$$

since $\Delta v(\infty) = 0$. Therefore, the firm builds up inventory just to the point where marginal profit equals zero.

Direct observation gives the following comparative statics result.

**Corollary 3.1.** *The optimal inventory limit $b^*$ increases as $p$ increases or $h$ decreases.*

In this paper we want to study the rationale behind make-to-stock and make-to-order. Therefore we are more interested in the conditions under which the boundary is drawn between make-to-order and make-to-stock.

**Proposition 3.3.** *The firm holds inventory if and only if*

$$g(\rho_1, \lambda, \mu, r) > \frac{h}{p}, \tag{3.17}$$

*where*

$$g(\rho, \lambda, \mu, r) \equiv \lambda \left(1 - \frac{\mu(1-\rho)}{\lambda(1-\rho) + r}\right). \tag{3.18}$$

*Proof.* Using the same technique as in the proof of Proposition 3.1., we can compute the value functions for the policies of zero inventory and only one unit of inventory. That is, for $x \geq 0$,

$$v^0(x) = \frac{\mu}{r}p - \frac{\mu p}{\lambda(1-\rho_1)+r}\rho_1^x. \tag{3.19}$$

$$v^1(x) = \frac{\mu}{r}p - \frac{\mu((\mu+r)p + h)}{\lambda(\lambda+r)(1-\rho_1) + (\lambda+\mu+r)r}\rho_1^x. \tag{3.20}$$

Then,

$$v^1(x) - v^0(x) = \frac{\mu p \rho_1^x}{\lambda(\lambda+r)(1-\rho_1) + (\lambda+\mu+r)r}\left(g(\rho_1, \lambda, \mu, r) - \frac{h}{p}\right).$$

So, $v^1(x) > v^0(x)$ if and only if (3.17) holds. We conclude the proof by observing the monotonicity property of $v^b$ proved in Proposition 3.1. ∎

Condition (3.17) implies that whether the firm would like to hold inventory or solely make to order depends on the parameters of the model: the demand rate ($\lambda$), the processing capacity ($\mu$), the interest rate ($r$), the holding cost ($h$), and the contribution of a sale ($p$). In particular,

**Corollary 3.2.** *The firm tends to hold inventory if the demand rate, $\lambda$, is higher, the processing capacity, $\mu$ is lower, the sale's contribution, $p$, is bigger, or the holding cost, $h$, is smaller.*

The proof follows from the simple lemmas stated below.

**Lemma 3.1.** *Function $\rho(\lambda, \mu, r)$ is increasing in $\mu$, and decreasing in $\lambda$ and $r$, where*

$$\rho(\lambda, \mu, r) \equiv \rho_1 = \frac{\lambda + \mu + r - \sqrt{(\lambda+\mu+r)^2 - 4\lambda\mu}}{2\lambda}. \tag{3.21}$$

*More explicitly,*

$$\frac{\partial \rho}{\partial \lambda} = -\frac{\rho(1-\rho)}{\mu\rho^{-1} - \lambda\rho}, \quad \frac{\partial \rho}{\partial \mu} = \frac{1-\rho}{\mu\rho^{-1} - \lambda\rho}, \quad \frac{\partial \rho}{\partial \lambda} = -\frac{\rho}{\mu\rho^{-1} - \lambda\rho}.$$

**Lemma 3.2.** *Function* $g(\rho, \lambda, \mu, r)$ *is increasing in* $\rho$, $\lambda$ *and* $r$, *and decreasing in* $\mu$. *More explicitly,*

$$\frac{\partial g}{\partial \rho} = \frac{\lambda \mu r}{(\lambda(1-\rho)+r)^2}, \quad \frac{\partial g}{\partial \lambda} = 1 - \frac{\mu r(1-\rho)}{(\lambda(1-\rho)+r)^2},$$

$$\frac{\partial g}{\partial \mu} = -\frac{\lambda(1-\rho)}{\lambda(1-\rho)+r}, \quad \frac{\partial g}{\partial r} = \frac{\lambda \mu(1-\rho)}{(\lambda(1-\rho)+r)^2}.$$

**Lemma 3.3.** *Function* $g(\rho(\lambda, \mu, r), \lambda, \mu, r)$ *is increasing in* $\lambda$ *and decreasing in* $\mu$.

*Proof.* Using the result in the above two lemmas and the chain rule of differentiation, we can calculate

$$\frac{\partial g}{\partial \lambda} = 1 - \frac{\mu(1-\rho)}{\lambda(1-\rho)+r} \cdot \frac{r\mu\rho^{-1}}{(\lambda(1-\rho)+r)(\mu\rho^{-1}-\lambda\rho)}$$

$$> 1 - \frac{\mu(1-\rho)}{\lambda(1-\rho)+r} > 0,$$

since

$$\frac{r\mu\rho^{-1}}{(\lambda(1-\rho)+r)(\mu\rho^{-1}-\lambda\rho)} < 1.$$

The fact that

$$\frac{\mu(1-\rho)}{\lambda(1-\rho)+r} < 1, \text{ and } \frac{r\mu\rho^{-1}}{(\lambda(1-\rho)+r)(\mu\rho^{-1}-\lambda\rho)} < 1,$$

can be verified directly. The second assertion of the lemma can be similarly proved.

∎

To develop an economic intuition for Corollary 3.2, we first notice that the restriction to barrier policies makes the firm's problem an optimal stopping problem. That is, the firm only needs to decide when to turn off the production and when to restart it again. Keeping this time effect in mind, the economic intuition becomes clear. What happens when the firm increase its inventory? On the one hand, the firm can capture more current demands (since there is no loss of demand, the gain simply comes from an earlier realization of sales). On the other hand, there are more goods tied up in inventory. The optimal policy balances the marginal benefit of the former effect and the marginal cost of the latter. Clearly, a higher contribution margin, $p$, is in favor of inventory since it works for the former effect without affecting the latter. Here we want emphasize that it is the contribution margin, not the price, that is in favor of inventory. A higher price may work in either ways. For example, a higher price may imply a higher cost and then a higher holding cost. A higher holding cost certainly implies lower inventory. An increase in the demand rate implies that the inventory, if there is any, will be cleared more quickly and the expenditure on inventory is reduced by holding the goods for a shorter period of time. An increase in the processing capacity will help complete sales earlier while orders are waiting, and will build up inventory faster and hence make a product stay in stock for a longer time while no order is waiting. Therefore, a higher demand rate justifies more inventory and a higher production rate reduces the need for inventory. In fact, those effects holds also for the optimal inventory limit as stated below.

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**Corollary 3.3.** *The optimal inventory limit $b^*$ increases as $\lambda$ increases or $\mu$ decreases.*

We omit the proof here since it consists only of straightforward computations.

## 4. The Single-Firm Problem with Impatient Customers

In the previous section, we looked at the stocking policies of a firm that can backlog all unfulfilled demand. The customer orders may therefore wait for processing for a long time. In reality, a customer who has waited or expects to wait for a long time may want cancel the order, or does not place the order in the first place, and turns to other alternatives. We now look at a formulation in which customers have limited patience. Suppose that each customer who comes in with an order can see the number of backorders the firm currently has and leaves if the number of orders waiting exceeds certain parameter $c$, $(c \geq 0)$. If the backlog is less than $c$, the customer issues the order and waits for processing. In other words, the arrival of demand will be turned off if the order waiting hits an upper barrier $c$. Mathematically, we assume,

$$\alpha_t = \lambda 1_{(-\infty, c)}(Z(t_-)).$$

Here, the parameter $c$ can be viewed as a measurement of the patience of customers. The larger $c$ is, the longer customers are willing to wait, and the more patient they are. As a matter of fact, the problem we solved with perfect backlogging is a special case by setting $c = \infty$. Again, we can prove that the optimal policy of the firm with impatient customers is still a barrier policy with certain parameter $b$, the optimal inventory limit.

**Proposition 4.1.** *The value function under a barrier policy with parameter $b$ $(b > 0)$ is of the form,*

$$v(x) = \begin{cases} G\rho_1^x + H\rho_2^x + \frac{\mu}{r}p, & \text{if } x \geq 0; \\ E\rho_1^x + F\rho_2^x + \frac{\lambda}{r}p + (\frac{\lambda - \mu}{r} + x)\frac{h}{r}, & \text{if } x \leq 0; \end{cases} \tag{4.1}$$

*where $\rho_1$ and $\rho_2$ are defined in (3.3) and (3.4), and $G$, $H$, $E$, and $F$ are unique solution of the following linear equations,*

$$G + H + \frac{\mu}{r}p = E + F + \frac{\lambda}{r}p + \frac{\lambda - \mu}{r} \cdot \frac{h}{r}, \tag{4.2}$$

$$G\rho_1^c + H\rho_2^c = \frac{\mu}{\mu + r}(G\rho_1^{c-1} + H\rho_2^{c-1} + \frac{\mu}{r}p) + \frac{\mu}{\mu + r}p. \tag{4.3}$$

$$G + H + \frac{\mu}{r}p = \frac{\lambda}{\lambda + \mu + r}(G\rho_1 + H\rho_2 + \frac{\mu}{r}p)$$

$$+ \frac{\mu}{\lambda + \mu + r}(E\rho_1^{-1} + F\rho_2^{-1} + \frac{\lambda}{r}p + (\frac{\lambda - \mu}{r} - 1)\frac{h}{r}), \tag{4.4}$$

$$E\rho_1^{-b} + F\rho_2^{-b} = \frac{\lambda}{\lambda + r}(E\rho_1^{-b+1} + F\rho_2^{-b+1}) + \frac{\mu}{\lambda + r} \cdot \frac{h}{r}. \tag{4.5}$$

We omit the proof of the proposition which is very similar to that of Proposition 3.1. Also we do not list the values of $G$, $H$, $E$, $F$ here to avoid complex expressions. Parallel to the results from proceeding section, we have

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**Proposition 4.2.** *There exists an optimal barrier policy with inventory limit $b^*$, that is uniquely determined by the condition*

$$k_1(b^* + 1, c) \leq \frac{h/r}{p + h/r}, \quad \text{and } k_1(b^*, c) > \frac{h/r}{p + h/r}, \tag{4.6}$$

*where*

$$k_1(b, c) \equiv \frac{a^*(\rho_1)a^*(\rho_2)(\rho_1^{-1} - \rho_2^{-1})((\lambda - \mu)(1 - \rho_2) + r)a_*(\rho_1)\rho_1^c - ((\lambda - \mu)(1 - \rho_1) + r)a_*(\rho_2)\rho_2^c)}{r(a_*(\rho_1)d(\rho_2, \rho_1)(1 - \rho_2)\rho_1^{b+c} + a_*(\rho_2)d(\rho_1, \rho_2)(1 - \rho_1)\rho_2^{b+c})}, \tag{4.7}$$

*where*

$$a_*(\rho_i) \equiv \lambda(1 - \rho_i) + r, \qquad a^*(\rho_i) \equiv \mu(1 - \rho_i^{-1}) + r, \tag{4.8}$$

*and $d(\rho_i, \rho_j)$ is defined in (3.8). And $k_1(b, c)$ is decreasing in $b$ with $k_1(\infty, c) = 0$.*

*Proof.* Similar to the proof of Proposition 3.2. ∎

**Corollary 4.1.** *The optimal inventory limit $b^*$ determined by (4.6) is an nonincreasing function of the patience measure $c$.*

*Proof.* Denote the denominator of the fractional expression of $k_1(b, c)$ in (4.7) by $e_1(b, c)$. We have

$$k_1(b, c) - k_1(b, c - 1)$$
$$= \frac{a_*(\rho_1)a_*(\rho_2)\rho_1^c\rho_2^c(\rho_1^{-1} - \rho_2^{-1})}{e_1(b, c)c_1(b, c - 1)}$$
$$\cdot (((\lambda - \mu)(1 - \rho_1) + r)d(\rho_2, \rho_1)(1 - \rho_2)\rho_1^b + ((\lambda - \mu)(1 - \rho_2) + r)d(\rho_1, \rho_2)(1 - \rho_1)\rho_2^b)$$
$$< 0.$$

That is, $k_1(b, \cdot)$ is decreasing. Condition (4.6) together with the fact that $k_1(\cdot, c)$ and $k_1(b, \cdot)$ are decreasing implies that $b^*(c)$ is decreasing in $c$. ∎

**Proposition 4.3.** *The firm holds inventory if and only if*

$$g(\phi(c), \lambda, \mu, r) > \frac{h}{p}, \tag{4.9}$$

*where function $g$ is defined in (3.18), and*

$$\phi(c) \equiv \frac{\rho_2 a_*(\rho_1)\rho_1^c - \rho_1 a_*(\rho_2)\rho_2^c}{a_*(\rho_1)\rho_1^c - a_*(\rho_2)\rho_2^c}. \tag{4.10}$$

*Proof.* Compute the value functions for the policies of zero inventory and only one unit of inventory. That is, for any $x \geq 0$,

$$v^0(x) = \frac{\mu}{r}p - \frac{(a_*(\rho_1)\rho_1^c\rho_2^x - a_*(\rho_2)\rho_2^c\rho_1^x)\mu p}{a^*(\rho_2)a_*(\rho_1)\rho_1^c - a^*(\rho_1)a_*(\rho_2)\rho_2^c}, \tag{4.11}$$

$$v^1(x) = \frac{\mu}{r}p - \frac{(a_*(\rho_1)\rho_1^c\rho_2^x - a_*(\rho_2)\rho_2^c\rho_1^x)\mu((\mu + r)p + h)}{\psi(c)}, \tag{4.12}$$

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where

$$\psi(c) \equiv (\lambda(\lambda + r)(1 - \rho_2) + (\lambda + \mu + r)r)a_*(\rho_1)\rho_1^c - (\lambda(\lambda + r)(1 - \rho_1) + (\lambda + \mu + r)r)a_*(\rho_2)\rho_2^c.$$

Then,

$$v^1(x) - v^0(x) = \frac{\mu p(a_*(\rho_1)\rho_1^c\rho_2^z - a_*(\rho_2)\rho_2^c\rho_1^z)}{\psi(c)}(g(\phi(c), \lambda, \mu, r) - \frac{h}{p}).$$

So, $v^1(x) > v^0(x)$ if and only if (4.9) holds.

∎

**Lemma 4.1.** *Function $\phi(c)$ is decreasing in $c$, and*

$$\lim_{c \to \infty} \phi(c) = \rho_1$$

.

*Proof.* Simply notice that

$$\phi(c) - \phi(c-1) = \frac{a_*(\rho_1)a_*(\rho_2)\rho_1^{c+1}\rho_2^{c+1}(\rho_1^{-1} - \rho_2^{-1})^2}{(a_*(\rho_1)\rho_1^c - a_*(\rho_2)\rho_2^c)(a_*(\rho_1)\rho_1^{c-1} - a_*(\rho_2)\rho_2^{c-1})} < 0.$$

∎

**Corollary 4.2.** *Function $g(\phi(c), \lambda, \mu, r)$ is decreasing in $c$, and hence, the firm tends to hold inventory if customers are less patient, that is, $c$ is smaller.*

*Proof.* The first assertion follows from Lemma 3.2. and 4.1., and second is a corollary of Proposition 4.3.

∎

With impatient customers, in addition to the two effects of holding inventory mentioned in Section 3 (earlier realization of demands and incurrence of holding cost), there is a third effect: a reduction of lost sales. Hence, facing less patient customers, a situation of having more demands lost in the long run, the firm tends to hold more inventory.

What happens to a demand lost to the firm? In reality, it does not actually disappear; rather, it goes to other alternatives. One can not precisely capture the cost of lost sales to the firm without expanding the scope of a single firm setting to equilibrium models to study the behavior of customers as well as rival firms.

## 5. Competitive Stocking

In this section, we extend the single-firm model to a multi-firm market setting and study the impact of competition on inventory policies. The competition we introduce here is only on the

timely delivery rather than price, product quality, or other aspects of the market. Our goal is to show that competition may breed a demand for produce-to-stock.

Assume there are $n$ firms in the market and each has a maximum processing rate $\mu$. We consider a particular form of competition - order racing. Industry demand arrives in a Poisson fashion with intensity $\lambda$. The demands that occur while the products are available in the inventory of exactly one firm are fulfilled immediately by that firm. If several firms have non-empty inventory, then each will complete a sale with equal probability. If the product is not immediately available in any of the firms, then the buyer places the same order with each firm, but completes the sale only with the firm which finishes the order first. Orders with rest of the firms are cancelled. Assume there is no penalty for order cancellation. It is noteworthy that this is a common occurrence in the semi-conductor industry. Production yields of chips are very unpredictable. A great deal of reprocessing is required and it is often difficult to meet delivery schedules of the buyers who are primarily the assemblers of electronic equipment. As a result, assembly plants often issue the same orders to several suppliers and later cancel the orders with the late suppliers.

We first look for the necessary and sufficient condition under which firms will have inventory in the equilibrium, or equivalently, the condition under which there is an equilibrium in which no firm holds inventory.

**Proposition 5.1.** *For any $i$, $i = 1, \ldots, n$, suppose firms $j$, $j \neq i$. do not hold inventory. Then firm $i$'s best response is an barrier policy with inventory limit $\bar{b}^*$, that is uniquely determined by the condition*

$$\bar{k}(\bar{b}^* + 1) \leq \frac{h/r}{p + h/r}, \text{ and } \bar{k}(\bar{b}^*) > \frac{h/r}{p + h/r}, \tag{5.1}$$

*where*

$$\bar{k}(b) \equiv \frac{a(\rho_1)a(\rho_2)((\lambda - \mu)(1 - \eta_n) + r)(\rho_2^{-1} - \rho_1^{-1})}{r[a(\rho_1)d(\eta_n, \rho_2)(1 - \rho_1)\rho_2^b - a(\rho_2)d(\eta_n, \rho_1)(1 - \rho_2)\rho_1^b]}. \tag{5.2}$$

$$\eta_n \equiv \rho(\lambda, n\mu, r) = \frac{\lambda + n\mu + r - \sqrt{(\lambda + n\mu + r)^2 - 4n\lambda\mu}}{2\lambda}, \tag{5.3}$$

*And $\bar{k}(b)$ is decreasing in $b$ with $\bar{k}(\infty) = 0$.*

*Proof.* Suppose firm $i$ plays a barrier policy with parameter $b$ ($b > 0$). Then the payoff function to $i$, $v_i$'s, satisfy the system of equations (3.10) - (3.13) except equation (3.10) altered to be

$$v_i(x) = \frac{\lambda}{\lambda + n\mu + r}v(x + 1) + \frac{\mu}{\lambda + n\mu + r}v(x - 1) + \frac{\mu}{\lambda + n\mu + r}p, \text{ for } x \geq 1, \tag{5.4}$$

Therefore, the solution is of the same form as in (3.1) - (3.9) but $\rho$ is replaced by $\eta_n$ wherever it appears. The rest of the proof is just a repetition of the proof of Proposition 3.2. ∎

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**Proposition 5.2.** *There cannot be an equilibrium in which no inventory is held if and only if*

$$g(\eta_n, \lambda, \mu, r) > \frac{h}{p}, \qquad (5.5)$$

*where the function $g$ is defined in (3.18).*

*Proof.* Using the same argument as in Proposition 3.3. again with $\rho$ replaced by $\eta_n$ wherever it appears, we can show that firm $i$ has incentive to deviate from zero inventory arrangement if and only if (5.5) holds. The assertion of the proposition follows directly.

∎

We call the gaming situation formulated above an *Oligopoly Racing Market*. We would like to compare firms' stocking policy in a racing market with that in a *Monopoly Market* and that in a *Demand Sharing Market*. By *Monopoly Market*, we mean one firm in the market alone. By Proposition 3.3., the monopoly holds inventory if and only if

$$g(\rho_1, \lambda, \mu, r) > \frac{h}{p}. \qquad (5.6)$$

In a *Demand Sharing Market*, placing duplicate orders is somehow prohibited and $n$ firms share the demand in the following fashion. Each time a demand occurs, the probability for each firm to get the order is $1/n$, which is independent of the arrival process $A$. The decomposition of a Poisson process implies that the demand stream each firm faces is an Poisson Process with rate $\lambda/n$ which is independent of the others. Thus, each firm's decision problem is reduced to a single-firm decision problem with a Poisson demand of rate $\lambda/n$. The demand sharing firms hold inventory if and only if

$$g(\rho(\lambda/n, \mu, r), \lambda/n, \mu, r) > \frac{h}{p}. \qquad (5.7)$$

The *Demand Sharing Market* represents the situation in which duplicate order is not allowed, a customer arrives places an order to a firm on a purely random basis without any strategic consideration, and hence, there is no delivery-time competition at all.

**Proposition 5.3.**

$$g(\rho_1, \lambda, \mu, r) > g(\rho(\lambda/n, \mu, r), \lambda/n, \mu, r), \qquad (5.8)$$

$$g(\rho(\lambda/n, \mu, r), \lambda/n, \mu, r) > g(\rho(\lambda/(n+1), \mu, r), \lambda/(n+1), \mu, r), \qquad (5.9)$$

*for $n \geq 2$, and $g(\rho(\lambda/n, \mu, r), \lambda/n, \mu, r) \to 0$ as $n \to \infty$.*

*That is, the firm as a monopoly is more likely to hold inventory than in a demand sharing multi-firm market and the likeliness of holding inventory decreases as the number of firms increases in a demand sharing market.*

*Proof.* A direct corollary of Lemma 3.3.

∎

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**Proposition 5.4.**

$$g(\eta_n, \lambda, \mu, r) > g(\rho_1, \lambda, \mu, r),$$ (5.10)

$$g(\eta_{n+1}, \lambda, \mu, r) > g(\eta_n, \lambda, \mu, r),$$ (5.11)

for $n \geq 2$, and $g(\eta_n, \lambda, \mu, r) \rightarrow \lambda$ as $n \rightarrow \infty$.

*That is, firms are more likely to have inventory in an oligopoly racing market than in a monopoly market and the likeliness of holding inventory increases as the competition is intensified (the number of firms increases).*

*Proof.* Note that by Lemma 3.1.,

$$\eta_{n+1} = \rho(\lambda, (n+1)\mu, r) > \eta_n = \rho(\lambda, n\mu, r) > \rho(\lambda, \mu, r) = \rho_1.$$

So, the first two assertions in the proposition follows directly from 3.2. The limiting result is easy to verify.

∎

To explain the word "likely" used in the above propositions, we provide the following examples. Suppose the parameters of the model, $\lambda, \mu, r, p$, and $h$, are so chosen that

$$g(\rho(\lambda/m, \mu, r), \lambda/m, \mu, r) = \frac{h}{p}.$$

Then, by conditions (5.5), (5.6), (5.7) and the above propositions, no inventory is held in any *demand sharing market* in which the number of firms is greater than or equal to $m$, while there is inventory in any other *demand sharing market*, in the *monopoly market*, or in any *oligopoly racing market*. Suppose that the parameters are such that

$$g(\rho_1, \lambda, \mu, r) = \frac{h}{p}.$$

Then, no inventory held in any of the *demand sharing markets* or *monopoly market*, but there is inventory in any of the *oligopoly racing markets*. Suppose that the parameters are such that

$$g(\eta_m, \lambda, \mu, r) = \frac{h}{p}.$$

Then, no inventory held in any *demand sharing market*, the *monopoly market*, or any *oligopoly racing market* in which the number of firms is less than or equal to $m$, but firms hold inventory in any other *oligopoly racing market* which is bigger.

The inventory that occurs in the third example is solely induced by competition since a firm wouldn't have any inventory in a *monopoly market* or a *demand sharing market*. The basic conclusion we can draw here is that inventory holding can be used as part of a firm's competitive strategy. We call the inventory of this kind, the *Competitive Stock*, analogous to the common terms *safety stock* and *cycle stock*.

To further illustrate *competitive stocking*, we compare the stocking policies with and without competition when the discount rate is small.

**Proposition 5.5.** *For $n\mu > \lambda$ and $n \geq 1$,*

$$\lim_{r \downarrow 0} g(\rho(\lambda/n, \mu, r), \lambda/n, \mu, r) = 0. \tag{5.12}$$

*For $n\mu > \lambda$ and $n \geq 2$,*

$$\lim_{r \downarrow 0} g(\eta_n, \lambda/n, \mu, r) = \lambda(1 - \frac{1}{n}). \tag{5.13}$$

*Proof.* By L'hospital's rule,

$$\lim_{r \downarrow 0} \frac{r}{1 - \rho(\lambda, \mu, r)} = \mu - \lambda, \tag{5.14}$$

if $\mu > \lambda$. This implies that

$$\lim_{r \downarrow 0} \frac{r}{1 - \rho(\lambda/n, \mu, r)} = \mu - \lambda/n, \text{ and } \lim_{r \downarrow 0} \frac{r}{1 - \eta_n} = n\mu - \lambda,$$

if $n\mu > \lambda$. Therefore,

$$\lim_{r \downarrow 0} g(\rho(\lambda/n, \mu, r), \lambda/n, \mu, r) = \lim_{r \downarrow 0} \lambda \left( 1 - \frac{\mu}{\frac{\lambda}{n} + \frac{r}{1-\rho(\lambda/n,\mu,r)}} \right) = 0.$$

$$\lim_{r \downarrow 0} g(\eta_n, \lambda/n, \mu, r) = \lim_{r \downarrow 0} \lambda \left( 1 - \frac{\mu}{\lambda + \frac{r}{1-\eta_n}} \right) = \lambda(1 - \frac{1}{n}).$$

∎

The assumption that the joint capacity of firms is greater than the demand rate, $n\mu > \lambda$, is necessary to avoid having the order waiting blow up to infinity in the long run. The proposition says that in that case, no inventory will be held in a monopoly market or a demand sharing multi-firm market when the interest rate is sufficient small, but this is not the case in an oligopoly racing market. In the presence of competition, the firms always have incentive to hold inventory no matter how little discounting there is as long as $\lambda(1 - 1/n) > h/p$.

Finally, we observe that customers do have incentive to issue multiple orders instead of splitting their orders in a demand sharing market. Assume that customers are better off by spending less time from order to its completion. Also assume no inventory held in both oligopoly racing and demand sharing cases, i.e., condition, $g(\eta_n, \lambda, \mu, r) \leq h/p$, holds. Then, application of the simple queueing formula implies that the average time a customer waits in an oligopoly racing market is strictly less than that in a demand sharing market, that is

$$W^o - W^s = \frac{1}{n\mu - \lambda} - \frac{1}{\mu - \lambda/n} = -\frac{n-1}{n\mu - \lambda} < 0,$$

for $n\mu > \lambda$, and $n \geq 2$, where $W^o$ and $W^s$ are average waiting times in an oligopoly racing market and in a demand sharing market respectively. From the social welfare point of view, allowing duplicate orders increases the buyer's welfare.

However, the increase in the buyer's welfare is achieved by scarifying the producer's welfare. To see this, we again assume no inventory holding is optimal in both oligopoly racing and demand sharing cases and let $v^o$ and $v^e$ be the expected profits of a firm in a oligopoly racing market and in a demand sharing market. That is,

$$v^o(x) = \frac{\mu}{r}p - \frac{\mu p}{\lambda(1 - \rho(\lambda, n\mu, r)) + r}\rho(\lambda, n\mu, r))^x,$$

$$v^e(x) = \frac{\mu}{r}p - \frac{\mu p}{\lambda(1 - \rho(\lambda/n, \mu, r)) + r}\rho(\lambda/n, \mu, r))^x,$$

where the function $\rho$ is defined in (3.21). And

$$v^e(x) - v^o(x)$$

$$= \frac{\mu p}{\lambda(1 - \rho(\lambda, n\mu, r)) + r}((\rho(\lambda, n\mu, r)^x - \rho(\lambda/n, \mu, r)^x)$$

$$+ \frac{\lambda\rho(\lambda/n, \mu, r)^x}{\lambda(1 - \rho(\lambda/n, \mu)) + r}(\rho(\lambda, n\mu, r) - \rho(\lambda/n, \mu, r))) > 0$$

since

$$\rho(\lambda, n\mu, r) - \rho(\lambda/n, \mu, r)$$

$$= \frac{\lambda + n\mu + r - \sqrt{(\lambda + n\mu + r)^2 - 4n\lambda\mu}}{2\lambda} - \frac{\lambda + n\mu + nr - \sqrt{(\lambda + n\mu + nr)^2 - 4n\lambda\mu}}{2\lambda}$$

$$= \frac{(n - 1)r(\rho(\lambda, n\mu, r) + \rho(\lambda/n, \mu, r))}{\sqrt{(\lambda + n\mu + nr)^2 - 4n\lambda\mu} + \sqrt{(\lambda + n\mu + r)^2 - 4n\lambda\mu}} > 0,$$

for $n \geq 2$. In the case that there are inventories in the equilibrium of an oligopoly racing market (remember that there is no inventory held in a demand sharing market when the interest rate is small), the holding costs are the further losses to the producer's welfare.

In summary, allowing duplicate ordering increases the buyer's welfare and decreases the producer's welfare. From a policy-making point of view, a more careful study of the situation (the preferences of buyers and producers, the parameters of the market, etc.) is needed in order to decide whether duplicate orders should be allowed or not. Sometimes, a cancellation fee may be appropriate as a measure to deter duplicate orders or as a welfare transfer mechanism.

In the remainder of the section, we shall discuss briefly how to obtain a solution to a *duopoly racing game*. We restrict the strategies of the players to be the barrier policies. Note that the best response to a barrier policy is a barrier policy if firms observe their own inventory level instantaneously but not the other firms' inventory level. Suppose firm $i$ employs a barrier policy with $b_i$, $i = 1, 2$, in a two-player game. The payoffs to firm 1, denoted by $v_1$, satisfy the following equations. Note that the states are the number of orders waiting if $Z > 0$, or the inventory levels of both firms if there is no order waiting. If there are orders waiting, then firms race for the order, and whoever generates an output first gets the first order in line. Thus,

$$v_1(z) = \frac{\lambda}{\lambda + 2\mu + r}v_1(z + 1) + \frac{2\mu}{\lambda + 2\mu + r}v_1(z - 1) + \frac{\mu}{\lambda + 2\mu + r}p, \tag{5.15}$$

for $z \geq 1$ where $z$ is the number of order waiting. If there is no order waiting, then firms are operating in the make-to-stock regime, and $v_1(x, y)$ is the payoff to firm 1 if its own inventory level is $-x$ and the rival's inventory level is $-y$ for $x < 0, y \leq 0$ or $x \leq 0, y < 0$. Thus,

$$v_1(0) = \frac{\lambda}{\lambda + 2\mu + r} v_1(1) + \frac{\mu}{\lambda + 2\mu + r}(v_1(-1, 0) + v_1(0, -1)), \qquad (5.16)$$

$$v_1(0, y) = \frac{\lambda}{\lambda + 2\mu + r} v_1(0, y + 1) + \frac{\mu}{\lambda + 2\mu + r}(v_1(-1, y) + v_1(0, y - 1)), \qquad (5.17)$$

for $-b_2 < y < 0$,

$$v_1(0, -b_2) = \frac{\lambda}{\lambda + \mu + r} v_1(0, -b_2 + 1) + \frac{\mu}{\lambda + \mu + r} v_1(-1, -b_2), \qquad (5.18)$$

$$v_1(x, 0) = \frac{\lambda}{\lambda + 2\mu + r} v_1(x + 1, 0)$$
$$+ \frac{\mu}{\lambda + 2\mu + r}(v_1(x - 1, 0) + v_1(x, -1)) + \frac{\lambda p + x h}{\lambda + 2\mu + r}, \qquad (5.19)$$

for $-b_1 < x < 0$,

$$v_1(-b_1, 0) = \frac{\lambda}{\lambda + \mu + r} v_1(-b_1 + 1, 0) + \frac{\mu}{\lambda + \mu + r} v_1(-b_1, -1) + \frac{\lambda p - b_1 h}{\lambda + \mu + r}, \qquad (5.20)$$

$$v_1(x, y) = \frac{\lambda/2}{\lambda + 2\mu + r}(v_1(x + 1, y) + v_1(x, y + 1))$$
$$+ \frac{\mu}{\lambda + 2\mu + r}(v_1(x - 1, y) + v_1(x, y - 1)) + \frac{\lambda p/2 + x h}{\lambda + 2\mu + r}, \qquad (5.21)$$

for $-b_1 < x < 0, -b_2 < y < 0$,

$$v_1(-b_1, y) = \frac{\lambda/2}{\lambda + \mu + r}(v_1(-b_1 + 1, y) + v_1(-b_1, y + 1))$$
$$+ \frac{\mu}{\lambda + \mu + r} v_1(-b_1, y - 1) + \frac{\lambda p/2 - b_1 h}{\lambda + \mu + r}, \qquad (5.22)$$

for $-b_2 < y < 0$,

$$v_1(x, -b_2) = \frac{\lambda/2}{\lambda + \mu + r}(v_1(x + 1, -b_2) + v_1(x, -b_2 + 1))$$
$$+ \frac{\mu}{\lambda + \mu + r} v_1(x - 1, -b_2) + \frac{\lambda p/2 + x h}{\lambda + \mu + r}, \qquad (5.23)$$

for $-b_1 < x < 0$,

$$v_1(-b_1, -b_2) = \frac{\lambda/2}{\lambda + r}(v_1(-b_1 + 1, -b_2) + v_1(-b_1, b_2 + 1)) + \frac{\lambda p/2 - b_1 h}{\lambda + r}. \qquad (5.24)$$

In order to get the value functions for each player, we have to solve a second order, two variable difference equation on an rectangle with specified boundary conditions (see Figure 2.). The general solution for (5.15) is simple,

$$v_1(z) = A\eta^z + \frac{\mu}{r} p,$$

where

$$\eta = \frac{\lambda + 2\mu + r - \sqrt{(\lambda + 2\mu + r)^2 - 8\lambda\mu}}{2\lambda},$$

and $A$ is a constant to determine. The general solution to (5.21) is

$$v_1(x, y) = \int_\tau B(\tau)\tau^x \xi(\tau)^y d\tau + \frac{\lambda p/2}{r} + (x + \frac{\lambda/2 - \mu}{r})\frac{h}{r},$$

where $\tau$ and $\xi$ satisfy the following relations,

$$\lambda + 2\mu + r = \frac{\lambda}{2}(\tau + \xi) + \mu(\tau^{-1} + \xi^{-1}),$$

and $B$'s are parameters to determine.

Theoretically, one may pick up proper base $\{\tau_1, \tau_2, \ldots\}$ and determine the constants by the boundary conditions (5.16) - (5.20) and (5.22) - (5.24). Numerically, system of difference equations of this kind can be solved by two-color multigrid method (Gauss-Seidel) very efficiently (see Kuo (1987)). After the payoffs to each player are obtained, searching for an equilibrium is routine (see Lemke and Howson (1964)). We do not intend to discuss in depth the equilibria of the stocking game here. More technical readers may find the Appendix interesting.

## 6. Conclusion

In this paper, we model formally the role of inventory in the competition of timing. It has long been recognized that uncertainty justifies safety stock, but not much has been done to expand the scope of inventory models to include the interaction between a productive organization and other agents in the economy. With lead time uncertainty, we identify three important factors in the decision on make-to-order and make-to-stock: discounting, customer characteristics and competition. Though the Poisson assumptions may not fit exactly in many real-life situations, the qualitative results obtained should prevail. We also show that allowing duplicate orders increase the buyer's surplus while decreases the producer's surplus. Thus, our analysis provides a basis to study the welfare and policy implications under the competition of this kind. The possible extension of the work would be to include price competition as well. In that case, we conjecture that firms would use pricing to mitigate the inventory effect induced by delivery-time competition. Our model also provides a framework to further study the other strategic interaction among buyers and suppliers, for example, the dynamic selection of suppliers, bribing for an early service, etc.

## Appendix

## Alternate Stocking and a Folk Theorem in the Stocking Game

We shall look at a modified version of a *duopoly racing game*, namely, *alternate stocking* in this section. In the setting of the *alternate stocking game*, whenever there is no order waiting, only one firm is allowed to make to stock, and the opportunity of stocking is given to the firms alternately. To make the game symmetric, we assign equal probability for each player to be the starter of stocking. Denote by $v_i^1(x)$ the payoff to player $i$ if $Z(t) = x$ and player $i$ can hold inventory the first time $Z = 0$ and $v_i^2(x)$ that if player $i$ can hold inventory the second time $Z = 0$. Assume that player $i$ adopts a barrier policy with parameter $b_i$, $i = 1, 2$.

**Proposition A.1.** *The value functions under barrier policies with parameters $b_i$ ($b_i > 0$) $i = 1, 2$, are.*

$$v_i^1(x) = \begin{cases} G_1(b_i, b_j)\eta^x + \frac{\mu}{r}p, & \text{if } x \geq 0; \\ E_1(b_i, b_j)\rho_1^x + F_1(b_i, b_j)\rho_2^x + \frac{\lambda}{r}p + (\frac{\lambda - \mu}{r} + x)\frac{h}{r}, & \text{if } x \leq 0; \end{cases} \quad (A.1)$$

$$v_i^2(x) = \begin{cases} G_2(b_i, b_j)\eta^x + \frac{\mu}{r}p, & \text{if } x \geq 0; \\ E_2(b_i, b_j)\rho_1^x + F_2(b_i, b_j)\rho_2^x. & \text{if } x \leq 0; \end{cases} \quad (A.2)$$

*for $i = 1, 2$, and $j \neq i$, where $\eta \equiv \rho(\lambda, 2\mu, r)$, $\rho_1$ and $\rho_2$ are the two roots of the quadratic equation (3.2).*

$$E_1(b_i, b_j) \equiv \frac{1}{g(b_i, b_j)}[((\lambda + \mu + r)h_1(b_j) - (\lambda\eta + \mu\rho_2^{-1})h_2(b_j))\mu\frac{h}{r}$$
$$+ a(\rho_2)\rho_2^{-b_i}(h_1(b_j)c_1 - h_2(b_j)c_2)]. \quad (A.3)$$

$$F_1(b_i, b_j) \equiv -\frac{1}{g(b_i, b_j)}[((\lambda + \mu + r)h_1(b_j) - (\lambda\eta + \mu\rho_1^{-1})h_2(b_j))\mu\frac{h}{r}$$
$$+ a(\rho_1)\rho_1^{-b_i}(h_1(b_j)c_1 - h_2(b_j)c_2)]. \quad (A.4)$$

$$E_2(b_i, b_j) \equiv \frac{a(\rho_2)\rho_2^{-b_j}}{g(b_i, b_j)}[h_2(b_i)c_1 - h_1(b_i)c_2 - (\lambda + \mu + r)(\rho_1^{-1} - \rho_2^{-1})\mu^2\frac{h}{r}], \quad (A.5)$$

$$F_2(b_i, b_j) \equiv -\frac{a(\rho_1)\rho_1^{-b_j}}{g(b_i, b_j)}[h_2(b_i)c_1 - h_1(b_i)c_2 - (\lambda + \mu + r)(\rho_1^{-1} - \rho_2^{-1})\mu^2\frac{h}{r}], \quad (A.6)$$

$$G_1(b_i, b_j) \equiv E_2 + F_2$$
$$= \frac{h_1(b_j)[(\lambda + \mu + r)(\rho_1^{-1} - \rho_2^{-1})\mu^2\frac{h}{r} + h_1(b_i)c_2 - h_2(b_i)c_1]}{(\lambda + \mu + r)g(b_i, b_j)} \quad (A.7)$$

$$G_2(b_i, b_j) \equiv E_1 + F_1 + \frac{\lambda - \mu}{r}(p + \frac{h}{r})$$
$$= \frac{(\lambda + \mu + r)(\rho_1^{-1} - \rho_2^{-1})h_2(b_j)\mu^2\frac{h}{r} + h_1(b_i)(h_2(b_j)c_2 - h_1(b_j)c_1)}{(\lambda + \mu + r)g(b_i, b_j)} \quad (A.8)$$
$$+ \frac{\lambda - \mu}{r}(p + \frac{h}{r}),$$

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$$h_1(b) \equiv (\lambda + \mu + r)(a(\rho_1)\rho_1^{-b} - a(\rho_2)\rho_2^{-b}), \tag{A.9}$$

$$h_2(b) \equiv (\lambda\eta + \mu\rho_2^{-1})a(\rho_1)\rho_1^{-b} - (\lambda\eta + \mu\rho_1^{-1})a(\rho_2)\rho_2^{-b}, \tag{A.10}$$

$$g(b_i, b_j) \equiv h_1(b_i)h_1(b_j) - h_2(b_i)h_2(b_j), \tag{A.11}$$

$$c_1 \equiv \frac{1}{r}[\lambda(\lambda + \mu\eta + r)p + (\lambda + \mu + r)(\lambda - \mu)h/r], \tag{A.12}$$

$$c_2 \equiv \frac{1}{r}[\lambda(\lambda + \mu(2 - \eta))p + ((\lambda + \mu)(\lambda - \mu) - \mu r)h/r]. \tag{A.13}$$

*Proof.* For notational simplicity, we only calculate the value functions for firm 1. Similar to the proof of Proposition 3.1., we can write down the system of difference equations that the value functions $v_1^i$ should satisfy, i.e.,

$$v_1^i(x) = \frac{\lambda}{\lambda + 2\mu + r}v_1^i(x+1) + \frac{2\mu}{\lambda + 2\mu + r}v_1^i(x-1) + \frac{\mu}{\lambda + 2\mu + r}p, \text{ for } x \geq 1, \tag{A.14}$$

$$v_1^i(0) = \frac{\lambda}{\lambda + \mu + r}v_1^j(1) + \frac{\mu}{\lambda + \mu + r}v_1^i(-1), j \neq i \tag{A.15}$$

$$v_1^1(-1) = \frac{\lambda}{\lambda + \mu + r}v_1^2(0) + \frac{\mu}{\lambda + \mu + r}v_1^1(-2) + \frac{\lambda p - h}{\lambda + \mu + r}. \tag{A.16}$$

$$v_1^1(x) = \frac{\lambda}{\lambda + \mu + r}v_1^1(x+1) + \frac{\mu}{\lambda + \mu + r}v_1^1(x-1) + \frac{\lambda p + xh}{\lambda + \mu + r}. \tag{A.17}$$
for $-b_1 + 1 \leq x \leq -2$,

$$v_1^1(-b_1) = \frac{\lambda}{\lambda + r}v_1^1(-b_1 + 1) + \frac{\lambda p - b_1 h}{\lambda + r}, \tag{A.18}$$

$$v_1^2(-1) = \frac{\lambda}{\lambda + \mu + r}v_1^1(0) + \frac{\mu}{\lambda + \mu + r}v_1^1(-2), \tag{A.19}$$

$$v_1^2(x) = \frac{\lambda}{\lambda + \mu + r}v_1^2(x+1) + \frac{\mu}{\lambda + \mu + r}v_1^2(x-1). \text{ for } -b_2 + 1 \leq x \leq -2, \tag{A.20}$$

$$v_1^2(-b_2) = \frac{\lambda}{\lambda + r}v_1^2(-b_2 + 1). \tag{A.21}$$

It can be verified that the general solution of the above difference equations must have the form of that defined in (A.1) and (A.2). Parameters $G_i, E_i, F_i$ are determined by the boundary conditions (A.15), (A.18), (A.21) and the relations,

$$G_1 + \frac{\mu}{r}p = E_2 + F_2, \tag{A.22}$$

$$G_2 + \frac{\mu}{r}p = E_1 + F_1 + \frac{\lambda}{r}p + \frac{\lambda - \mu}{r} \cdot \frac{h}{r}, \tag{A.23}$$

imposed by equations (A.16) and (A.19).

∎

The following two propositions show that there is a symmetric stationary Nash equilibrium in the *alternate stocking game.*

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**Proposition A.2.** *Given that firm 2 adopts a barrier policy with inventory limit $b_2$, the best response of firm 1 is a barrier policy with inventory limit $b_1(b_2)$ such that*

$$k(b_1 + 1, b_2) \leq 0, \text{ and } k(b_1, b_2) > 0, \tag{A.24}$$

*where*

$$k(b_1, b_2) \equiv r a(\rho_1) a(\rho_2)(\rho_1 - \rho_2)(c_1 - \gamma(b_2)c_2)$$
$$- \mu h \rho_1^{b_1} \rho_2^{b_1}((h_1(b_1) - h_1(b_1 - 1)) - \gamma(b_2)(h_2(b_1) - h_2(b_1 - 1))), \tag{A.25}$$
$$\gamma(b) \equiv \frac{h_2(b)}{h_1(b)}. \tag{A.26}$$

*Proof.* Using the value functions obtained in Proposition A.1., we have

$$v_1^{b_1, b_2}(x) - v_1^{b_1-1, b_2}(x) = \frac{1}{2} \sum_{j=1}^{2} [v_1^{j, b_1, b_2}(x) - v_1^{j, b_1-1, b_2}(x)]$$

$$= \frac{1}{2} \sum_{j=1}^{2} [(E_j(b_1, b_2) + F_j(b_1, b_2)) - (E_j(b_1 - 1, b_2) + F_j(b_1 - 1, b_2))] \eta^x$$

$$= C \cdot k(b_1, b_2),$$

where the positive constant

$$C \equiv \frac{\mu(h_1(b_2) + h_2(b_2))h_1(b_2)(\rho_1^{-1} - \rho_2^{-1})\rho_1^{-b_1}\rho_2^{-b_1}}{2r g(b_1, b_2) g(b_1 - 1, b_2)}.$$

Note that for $b > 2$,

$$h_1(b) - h_2(b) = a(\rho_1)d(\eta, \rho_2)\rho_1^{-b} - a(\rho_2)d(\eta, \rho_1)\rho_2^{-b} > 0, \tag{A.27}$$

where functions $a$ and $d$ are defined as in (3.8), and also the deference in (A.27) increases to infinity as $b$ increases. Therefore,

$$0 < \gamma(b) < 1, \text{ and } \gamma(b) \downarrow \frac{\lambda \eta + \mu \rho_2^{-1}}{\lambda + \mu + r}, \text{ as } b \uparrow \infty. \tag{A.28}$$

Then the only term in $k(b_1, b_2)$ which depends on $b_1$,

$$\rho_1^{b_1} \rho_2^{b_2}(h_1(b_1) - h_1(b_1 - 1)) - \gamma(b_2)(h_2(b_1) - h_2(b_1 - 1))$$
$$= a(\rho_1)d(\eta\gamma(b_2), \rho_2\gamma(b_2))(1 - \rho_1)\rho_2^{b_1} - a(\rho_2)d(\eta\gamma(b_2), \rho_1\gamma(b_2))(1 - \rho_2)\rho_1^{b_2}. \tag{A.29}$$

increases to infinity as $b_1$ increases. This implies that $k(b_1, b_2) \downarrow -\infty$ as $b_1 \uparrow \infty$ since expression (A.29) has a negative sign in $k(b_1, b_2)$. Hence, $b_1(b_2)$ does the job.

∎

**Proposition A.3.** *There is a stationary symmetric Nash equilibrium in which each firm sets its inventory limit $b^*$ where $b^*$ is determined by*

$$k(b^* + 1, b^*) \leq 0, \text{ and } k(b^*, b^*) > 0. \tag{A.30}$$

*Proof.* Suppose $b_2 = \infty$, meaning firm 2 sets no upper barrier for holding inventory. Then firm 1's best response is determined by

$$k(b_1 + 1, \infty) \le 0, \text{ and } k(b_1, \infty) > 0,$$

where,

$$k(b_1, \infty) \equiv ra(\rho_1)a(\rho_2)(\rho_1 - \rho_2)(c_1 - \gamma(\infty)c_2)$$
$$- \mu h \rho_1^{b_1} \rho_2^{b_1}((h_1(b_1) - h_1(b_1 - 1)) - \gamma(\infty)(h_2(b_1) - h_2(b_1 - 1)),$$

where $\gamma(\infty) \equiv (\lambda\eta + \mu\rho_2^{-1})/(\lambda + \mu + r)$ as in (A.28), and $k(b_1, \infty)$ decreases to negative infinity as $b_1$ increases to infinity. That implies $b_1(\infty)$ is bounded from above, and hence, there exists a $b^*$ such that (A.30) holds. The symmetry of the game implies the assertion in the proposition.

∎

However, there are many other possible equilibria for this game. In particular, the stocking game described in the paper is a strategic rivalry in a long term relationship which, as recognized by many game theorists, may differ from that of a one-shot game. As Aumann and Shapley (1976), Rubinstein (1977), Fudenberg and Maskin (1986) and others have shown, any individually rational outcome, i.e., an outcome that Pareto dominates the minimax point, can arise as a Nash equilibrium in infinitely repeated games with sufficiently little discounting. This assertion constitutes the well-known "Folk Theorem" for repeated games. Though the stocking game we studied is not a repeated play of a one-shot game, and as a matter of fact, it is a stochastic game of timing, we shall show that the phenomenon addressed by the Folk Theorem is still true.

For simplicity and illustration, we only prove a weak version of the Folk Theorem. That is, for the case that $\lambda < \mu$, we show that any outcome that Pareto dominates the zero inventory outcome, can arise as a Nash equilibrium in the alternate stocking game with sufficient little discounting. Denote by $v^0$ the payoff when no firm holds inventory, i.e., $v^0 \equiv v_t^{0,0}(0)$, and $v^\infty$ the payoff to one firm that employs a barrier policy with parameter $b$ when it is the rival's turn to hold inventory and it sets no upper limit for the inventory, i.e., $v^\infty \equiv v^{2,b,\infty}(0)$.

**Lemma A.1.**

$$v^0 = \frac{\lambda\mu(1 - \eta)}{r(\lambda(1 - \eta) + r)}p, \tag{A.31}$$

$$v^\infty = \frac{(\lambda + \mu + r)(\rho_1^{-1} - \rho_2^{-1})\gamma(\infty)\mu^2\frac{h}{r} + h_1(b)(\gamma(\infty)c_2 - c_1)}{(\lambda + \mu + r)(h_1(b) - \gamma(\infty)h_2(b))}$$
$$+ \frac{\lambda}{r}p + \frac{\lambda - \mu}{r}\frac{h}{r}, \tag{A.32}$$

where $\gamma(\infty) \equiv (\lambda\eta + \mu\rho_2^{-1})/(\lambda + \mu + r)$. For $\mu > \lambda$,

$$\lim_{r\downarrow 0} rv^0 = \frac{\lambda}{2}, \text{ and } \lim_{r\downarrow 0} rv^\infty = -\frac{2\lambda^2}{(\lambda + \mu)(2\mu - \lambda)}h. \tag{A.33}$$

*Proof.* Equation (A.31) follows from (3.19) by replacing $\rho_1$ with $\eta$, and

$$v^\infty = \lim_{b'\uparrow\infty}\left(G_2(b,b') + \frac{\mu}{r}p\right).$$

The limit in (A.33) can be obtained by noticing the facts in the following lemma.

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**Lemma A.2.** *For* $\mu > \lambda$, $\rho_1 \to 1$, $\rho_2 \to \mu/\lambda$,

$$\frac{1-\rho_1}{r} \to \frac{1}{\mu-\lambda}, \text{ and } \frac{1-\eta}{r} \to \frac{1}{2\mu-\lambda}.$$

*as* $r \to 0$.

*Proof.* By L'Hospital's rule.

∎

In the Folk Theorem, the individually rational outcome arises as an equilibrium supported by certain penalizing strategy. We will see that the strategy of setting infinite inventory is sufficient to make the outcome that dominates the zero inventory outcome be an equilibrium outcome.

**Proposition A.4.** *Suppose* $\mu > \lambda$. *There exists* $r^* \in (0,\infty)$ *such that, for all* $r \in (0, r^*)$ *and for any feasible payoff pairs* $(v_1, v_2)$ *such that* $v_i^2(0) \geq v^0$, $i = 1, 2$, *there exists a Nash equilibrium of the stocking game in which firm* $i$'s *payoff is* $v_i$ *when the discount rate is* $r$.

*Proof.* Suppose that $b_i$ is the strategy employed by firm $i$ which results in the payoff $v_i$, $i = 1, 2$. The strategies that support the equilibrium are as follows. Each firm play $b_i$ as long as the rival is doing so, and once firm $i$ deviates from $b_i$, say, to $b_i'$, the other firm will punish it by playing $b_j' = \infty$. What we would like to show is that no firm wants to deviate when $r$ is sufficient small.

Without loss of generality, our discussion concerns firm 1 only. Suppose it is firm 1's turn. Denote by $v_1^1(0)$ and $\bar{v}_1^1(0)$ the payoffs to firm 1 if it stay with $b_1$ and if it deviates to $b_1'$ respectively when $Z = 0$ and it is firm 1' turn to hold inventory. Then,

$$v_1^1(0) = \frac{\lambda}{\lambda+\mu+r}\left(\frac{2\mu}{\lambda(1-\eta)+2\mu+r}v_1^2(0) + \frac{\lambda(1-\eta)+r}{\lambda(1-\eta)+2\mu+r}\cdot\frac{\mu}{r}p\right)$$

$$+ \frac{\mu}{\lambda+\mu+r}\left(\frac{\lambda(a(\rho_1)\rho_1^{-b_1+1} - a(\rho_2)\rho_2^{-b_1+1})(v_1^2(0) - (\frac{\lambda}{r}p + \frac{\lambda-\mu}{r}\frac{h}{r})) + (\rho_1^{-1} - \rho_2^{-1})\mu^2\frac{h}{r}}{(\lambda+\mu(1-\rho_2^{-1})+r)a(\rho_1)\rho_1^{-b_1+1} - (\lambda+\mu(1-\rho_1^{-1})+r)a(\rho_2)\rho_2^{-b_1+1}}\right),$$

$$\bar{v}_1^1(0) = \frac{\lambda}{\lambda+\mu+r}\left(\frac{2\mu}{\lambda(1-\eta)+2\mu+r}v^\infty + \frac{\lambda(1-\eta)+r}{\lambda(1-\eta)+2\mu+r}\cdot\frac{\mu}{r}p\right)$$

$$+ \frac{\mu}{\lambda+\mu+r}\left(\frac{\lambda(a(\rho_1)\rho_1^{-b_1'+1} - a(\rho_2)\rho_2^{-b_1'+1})(v^\infty - (\frac{\lambda}{r}p + \frac{\lambda-\mu}{r}\frac{h}{r})) + (\rho_1^{-1} - \rho_2^{-1})\mu^2\frac{h}{r}}{(\lambda+\mu(1-\rho_2^{-1})+r)a(\rho_1)\rho_1^{-b_1'+1} - (\lambda+\mu(1-\rho_1^{-1})+r)a(\rho_2)\rho_2^{-b_1'+1}}\right).$$

Using the above expression and Lemma A.2., we have for $\mu > \lambda$,

$$\lim_{r\downarrow 0} r(v_1^1(0) - \bar{v}_1^1(0))$$

$$= \lim_{r\downarrow 0} r(v_1^2(0) - v^\infty) \tag{A.34}$$

By assumption and Lemma A.1.,

$$\lim_{r\downarrow 0} r(v_1^2(0) - v^\infty) \geq \lim_{r\downarrow 0} r(v^0 - v^\infty) = \frac{\lambda}{2}p + \frac{2\lambda^2}{(\lambda + \mu)(2\mu - \lambda)}h > 0,$$

for any $b_1'$. Therefore, there exists a $r^*$ such that for $r \in (0, r^*)$, $v_1^1(0) \geq \bar{v}_1^1(0)$, for any $b_1' \neq b_1$.

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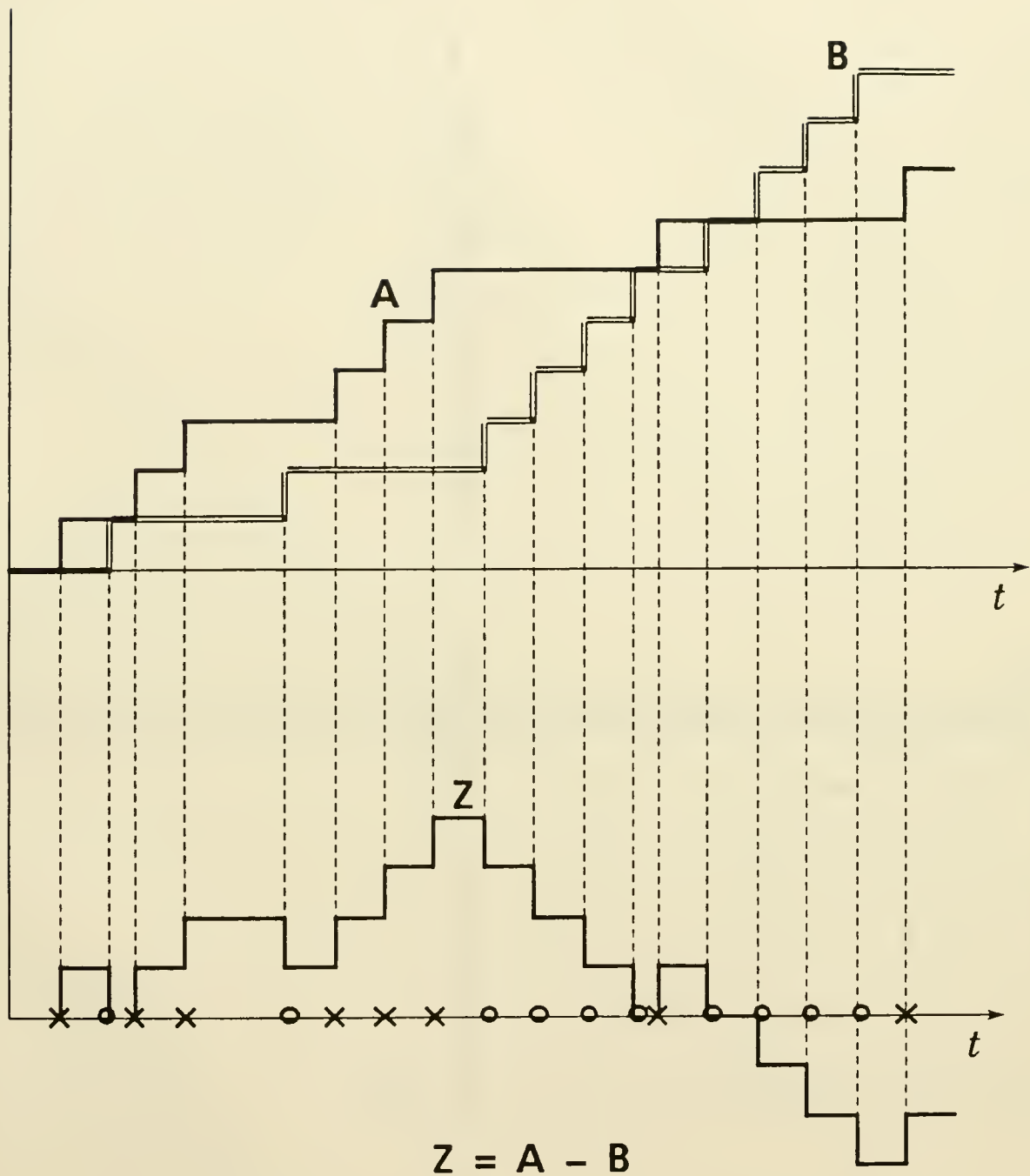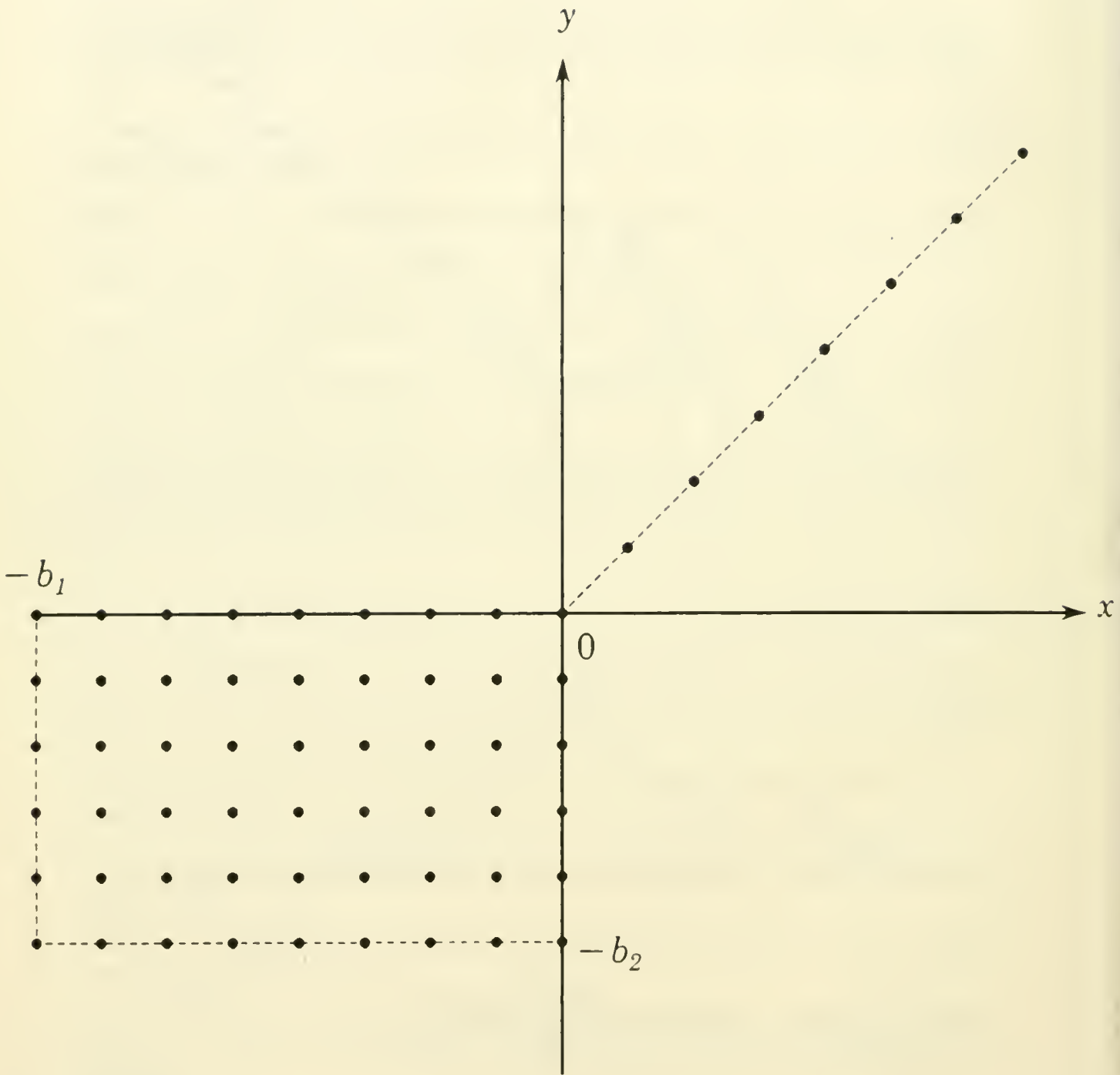
figure 1

figure 2