

SYNTHESIS OF RC TRANSFER FUNCTIONS AS UNBALANCED TWO TERMINAL-PAIR NETWORKS

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TWO TERMINAL-PAIR NETWORKS

Benjamin J. Dasher

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Abstract

A new general method for the synthesis of RC transfer functions is presented. Based on an extension of Brune's procedure for the synthesis of driving-point impedances, it makes use of canonical sections for controlling transmission zeros in much the same way as canonical sections are used by Darlington in synthesizing lossless two terminal-pair networks.

The characteristic process of the method makes use of the following theorem, proof of which comprises the principal result of the report.

Given a function $F_1(\lambda)$ of suitable degree, realizable as an RC impedance or admittance, it is always possible to design an unbalanced two terminal-pair network that: (a) produces a transmission zero or single pair of conjugate zeros at any given point or points, respectively, not in the right half-plane; and (b) produces the prescribed function $F_1(\lambda)$ when terminated in a second function $F_2(\lambda)$, also RC, the degree of which is less than the degree of $F_1(\lambda)$.

The networks obtained by the new method take the form of ladder networks cascaded with bridge sections similar to the well-known twin-T null network. The "zero sections" are obtained one at a time, each by a single application of the theorem, and this process is repeated until the original function is completely developed.

When used in combination with existing synthesis techniques the method provides a highly flexible procedure which allows limited control over impedance levels and insertion loss.

SYNTHESIS OF RC TRANSFER FUNCTIONS AS UNBALANCED TWO TERMINAL-PAIR NETWORKS

I. INTRODUCTION

A. Brief Statement of the Problem

This report deals with a new method for the synthesis of RC transfer functions* as unbalanced two terminal-pair networks. The method is based on an extension of Brune's procedure (1) for the synthesis of driving-point impedances and makes use of canonical sections for controlling transmission zeros in much the same way as canonical sections are used by Darlington (2) in synthesizing lossless two terminal-pair networks.

The characteristic process of the method makes use of the following theorem, proof of which comprises the principal result of the report.

Given a function $F_1(\lambda)$ of suitable degree, realizable as an RC impedance or admittance, it is always possible to design an unbalanced two terminal-pair RC network that 1) produces a transmission zero or single pair of conjugate zeros at any given point or points, respectively, not in the right half-plane; and 2) produces the prescribed function $F_1(\lambda)$ when terminated in a second function $F_2(\lambda)$, also RC, the degree of which is less than the degree of $F_1(\lambda)$.

In case the desired zero lies on the negative-real axis, including the origin and infinity, the zero-shifting technique introduced by Cauer (3) and applied to RC networks by Guillemin (4) is used without change. Complex or imaginary zeros, i.e. zeros lying in the left half-plane or on the imaginary axis, respectively, are realized by means of bridge sections similar to the well-known twin-T null network (5). Thus, the networks obtained by the new method take the form of ladder networks cascaded with bridge sections. The "zero sections" are obtained one at a time, each by a single application of the theorem, and this process is repeated until the original function is completely developed.

The typical procedure for realizing a prescribed transfer function consists of two parts. First, a driving-point function is derived that is compatible with the given transfer function. Second, the driving-point function, which may be either an impedance or an admittance, is developed in such a way that the zeros of the transfer function of the resulting two terminal-pair network coincide with those given. A number of methods are already available for accomplishing the first part of the procedure. The purpose of this report is to provide a method for accomplishing the second part.

*The term transfer function is used to include all possible combinations of output-input ratios, such as E_2/I_1 , E_2/E_1 , I_2/I , etc.

B. Origin and History

In a large number of practical applications of electrical networks, the transfer functions of two terminal-pair networks are of chief interest. The design of networks to produce transfer functions having prescribed behavior is a relatively new art. This is especially true in the case of the insertion loss methods employed by Darlington (2) which are based upon the theory of network synthesis as developed principally by Cauer (3), Brune (1), Gewertz (6), and Guillemin (7). The most common form taken by a network designed according to these methods is a network of pure reactances terminated at one or both ends in a resistance. Not only is this the configuration that yields most readily to synthesis theory but it is usually the most desirable form where efficient power transfer is required along with a specified frequency behavior. There are, however, many applications in which efficiency is of secondary interest and only the relative power transfer as a function of frequency is of primary importance. In addition, there are many situations in which the quality of available components does not allow the design to be carried out on the basis of lossless reactive elements. This difficulty becomes more serious as the lower limit of important frequencies is reduced. Consequently, the question of designing "lossy" networks is of great importance. At very low frequencies, RC networks offer attractive possibilities for use in all kinds of frequency selective circuits. Many applications require the network to be unbalanced, i. e. the input and output must have a common terminal.

Guillemin (6) has shown that it is possible to design a passive network containing only resistances and capacitances which produces any specified minimum-phase insertion-loss characteristics to within any specified tolerance, except, of course, a characteristic calling for a pole on the finite part of the imaginary axis. (The restriction to minimum-phase functions is not essential; but a function having zeros in the right half-plane is realizable in unbalanced form only if the polynomial defining the totality of its zeros has all positive coefficients.) The design is accomplished by first constraining the poles of the transfer function to lie on the negative-real axis; then a configuration of zeros can be found to satisfy the given requirements. When the resulting function is expressed as a transfer admittance, the corresponding network may be realized as a number of ladder networks connected in parallel at their sending and receiving ends.

Realizing the network as a group of paralleled ladders, although straightforward, leaves much to be desired from both the practical and theoretical points of view. Ordinarily, it results in the use of a much larger number of elements than would be expected for a system function of the same complexity. Also, this particular procedure, involving the use of admittance functions, is suited for use only with low-impedance sources. The difficulty of the large number of elements can be avoided in some degree through the use of isolating vacuum-tube amplifiers. This scheme avoids rather than solves the theoretical problem and, furthermore, it does not always furnish a satisfactory solution to practical problems.

An RC transfer function may always be realized by means of a symmetrical lattice (9). Bower and Ordung (10) have described lattice synthesis in some detail and have considered methods for obtaining certain practical results; specifically, a "maximum level of transmission." Their results are practically useful, but the method is fundamentally the same as that already described by Guillemin (9).

Both the parallel-ladders and the lattice methods furnish solutions to the RC transfer function problem. The chief disadvantage of the former method is the requirement of an excessive number of elements; of the latter, that one cannot be sure of reducing the lattice to an unbalanced form. Another disadvantage, shared by both, is that changing a single element may adversely affect the network performance over the whole frequency range of importance. For example, if a network is supposed to produce nulls at several different frequencies, it is practically always necessary to provide adjustments to compensate for inevitable small errors in element values. When parallel ladders and lattices are used, adjustments at the different frequencies cannot be independent, with the result that tedious alignment procedures are required. On the other hand, when each null is produced by a separate section of the network, the adjustments may be virtually independent.

All of the difficulties just mentioned are partially removed by a method of sectioning, or partitioning, the network given by Weinberg (11). This device, which will be considered in detail in section V, permits the network to be realized as two sections in cascade and it is a substantial improvement over methods yielding only one section. A similar procedure for obtaining two sections in cascade has been described recently by Ordung, Hopkins, Krauss, and Sparrow (12). However, neither of these methods has been extended to include partitioning to more than two sections.

It is evident that the limitations ascribed to existing procedures arise from the necessity of having an unbalanced structure that produces complex or imaginary zeros. The synthesis procedure to be presented herein leads directly to an unbalanced network in which the zeros are independently produced by sections in cascade in the desired manner.

C. Introduction to the New Procedure

Since the new procedure is most conveniently regarded as an extension of Brune's method for synthesizing a driving-point impedance, this method will be reviewed briefly.

Starting with a positive-real (p-r) function having neither poles nor zeros on the imaginary axis, Brune proceeds to find and remove the smallest minimum attained by its real part along the imaginary axis. The remainder is p-r and is a pure reactance at the frequency where the minimum occurs. On removing this reactance, the second remainder has a zero of impedance that is removed as a pole of the corresponding admittance. The reciprocal of the third remainder is an impedance having a pole at infinity. This pole is removed, completing a cycle and leaving a p-r remainder.

The "Brune section" of reactances thus obtained is a T section containing one

negative element. The section as a whole, however, is realizable by means of a condenser and a pair of close-coupled coils.

The significant features of this procedure as it relates to the present problem are

1. It makes use of a zero-shifting technique to force a zero on the imaginary axis.
2. The zero shifting is done in two stages, i.e. removal first of a resistance and then a reactance; the remainder after removing the resistance must be p-r.
3. The process is essentially a continued-fraction development.
4. The elements as first obtained are not all realizable but the section is realizable as a two terminal-pair.
5. The residue condition, $k_{11}k_{22} - k_{12}^2 = 0$ is satisfied with the equality sign for both the z- and the y-functions of the two terminal-pair.
6. Neither the poles nor the zeros of the driving-point functions of the Brune section are necessarily included in either of the impedances which it terminates.
7. Zeros of the transfer functions of the Brune section are present in the transfer functions of the two terminal-pair networks in which it is included.

Brune dealt only with the synthesis of driving-point impedances and, consequently, had no occasion to control the locations of the transmission zeros of his networks. Nevertheless, removing a resistance corresponding to the minimum of the real part is really a zero-shifting step. In itself, it produces no zero (except in special cases), but it leaves a function such that a zero may be formed subsequently without destroying the p-r characteristics of the network. In other words, removing the resistance amounts to more than disposing of a part of the network; it does, and must, guarantee that the derivative of the remaining function will be real and positive after removing the reactive, thus permitting a pole to be removed in the next step.

Brune also gives an alternate procedure in which a pair of real zeros is obtained. In this case, the residues in the corresponding poles must be real and negative. Darlington, through the use of the "type-D" section, obtained zeros at complex λ -values. He then found a way to insure at the outset that the derivatives at all of the zeros, and hence the residues in the corresponding poles, would behave properly, thereby eliminating the intermediate "preparatory" steps and consigning all losses to the terminations.

The point of view taken in the discussion above is somewhat different from that taken by Brune and Darlington. However*, when dealing with RC functions it is not enough to control the behavior of the real part along the imaginary axis because the RC character as well as the p-r character must be preserved, and this fact tends to emphasize the derivative rather than the real part. In addition, no way has been found to eliminate the preparatory steps or to reduce them to a single operation as in the case of Darlington's networks. Instead, one faces an essentially new problem at the

* The fact that with RC synthesis one cannot employ ideal transformers or avail oneself of a technique equivalent to close-coupling, renders the RC problem more difficult.

end of each cycle. Hence, the RC procedure is more closely related to Brune's problem than to Darlington's, although the three have much in common.

It was noted in item 5 above that the Brune section of reactances permits the residue condition $k_{11}k_{22} = k_{12}^2$ to be satisfied with the equality sign for both the z- and the y-functions. By analogy with the close-coupled coils, any two terminal-pair for which the above residue condition is fulfilled with the equality sign for both the z- and the y-functions, may also be called close-coupled, or more briefly, compact. A compact section is completely specified when one transfer function and one driving-point function are known. It has the important property that neither the poles nor the zeros of its driving-point functions are necessarily present in the impedances which it terminates. For example, the input impedance of the Brune section has a pole at infinity and a pole at the origin, but if a resistance is connected to its output terminals the input impedance of the resulting network does not have a pole at either point.

It is necessary for the elementary zero sections to be compact in order that they may be truly canonical forms. Otherwise, a Brune section, for example, could be used only if one of the associated functions had a pole at, say, infinity. The analogous situation in the RC case will become clear in section III where the canonical sections will be studied in detail. Just as two-terminal impedances consisting of R, L, C, and various combinations of these elements in series or in parallel comprise the elementary building blocks, into which any driving-point function may be decomposed, so compact sections of various types comprise the elementary building blocks of transfer functions.

The RC synthesis procedure was not arrived at through a direct adaptation of the Brune method, but the salient feature, involving the close-coupled sections, and in particular the decomposition of the sections into elements some of which are nonphysical, follows exactly the same pattern.

The design of an RC zero section may begin with either an admittance or an impedance function, depending on considerations to be given later. However, it is appropriate to assume that an admittance, Y_{11} , is given and that a transmission zero is desired at $\lambda = \lambda_0 = -\alpha_0 + j\beta_0$. The sequence of operations is as follows:

1. Remove an admittance, Y_0 , which is part of Y_{11} such that the remainder, Y_1 , and its derivative, Y_1' , satisfy certain requirements (to be stated later) at $\lambda = \lambda_0$.
2. Remove a second admittance y_1 to satisfy the relation, $Y_1(\lambda_0) = y_1(\lambda_0)$.
3. The remainder in operation 2, Y_2 , has a zero at $\lambda = \lambda_0$ (actually a conjugate pair of zeros at λ_0 and $\bar{\lambda}_0$) which is removed as a pole of its reciprocal. The impedance removed in this step is $1/y_{12}$, which is nonphysical.
4. The reciprocal of the impedance remaining in operation 3 has the same pole (on the negative-real axis) as y_1 . This is removed as an admittance, y_2 , to complete a cycle. The final remainder is both p-r and RC.

The three admittances, y_1 , y_{12} , and y_2 , constitute the π -section representation of a compact section. Thus, the y-matrix of a physically realizable two terminal-pair

that produces the required transmission zero and is compatible with the given function, has been determined.

Proof of the validity of the above procedure together with other necessary details will be given in the sections to follow.

II. GENERAL PROPERTIES OF RC FUNCTIONS

A. Driving-Point Functions

The general properties of RC functions are well known. They will be stated here for convenient reference and in addition several special properties will be derived. It is ordinarily more convenient to deal with admittance functions rather than with impedance functions; however the results are readily modified to fit the impedance case.

There will be frequent occasion to make reference to the partial-fraction expansion of Z_{11} and to the expression obtained by expanding Y_{11}/λ in partial fractions and multiplying the result by λ . Since these expressions, which place the poles of the functions in evidence, lead directly to Foster's canonical networks, they will be referred to hereafter as Foster expansions.

The Foster expansion of the most general RC driving-point admittance may be written as

$$Y_{11} = k_{\infty}\lambda + k_0 + \sum_{v=1}^n \frac{k_v \lambda}{\lambda + \sigma_v}. \quad (1)$$

In this expression, the k 's must all be real and positive and the σ_v 's must all be real and positive. Excepting k_{∞} , the k 's are the residues in the poles of Y_{11}/λ ; $k_{\infty} = \lim_{\lambda \rightarrow \infty} (Y_{11}/\lambda)$. Poles and zeros of Y_{11} all lie on the negative-real λ -axis where they mutually separate each other. The Foster expansion for an RC impedance is

$$Z_{11} = k_{\infty} + \frac{k_0}{\lambda} + \sum_{v=1}^n \frac{k_v}{\lambda + \sigma_v}. \quad (2)$$

Here $k_{\infty} = \lim_{\lambda \rightarrow \infty} Z_{11}$, and the remaining k 's are the residues in the corresponding poles.

Concerning the behavior of Y_{11} and Z_{11} at the origin and at infinity, the following property is noted for future reference.

If Y_{11} and $Z_{11} = 1/Y_{11}$ represent the same function, then between them, Z_{11} and Y_{11} contain two and only two of the four possible coefficients k_{∞} and k_0 in Eqs. 1 and 2. Thus, if Y_{11} has a pole at infinity, Z_{11} must be zero there; if k_{∞} and k_0 are both present in the expansion of Y_{11} , then both are missing in the expansion of $Z_{11} = 1/Y_{11}$, and so on.

Again, referring to Eq. 2, the function

$$\lambda Z_{11} = k_{\infty}\lambda + k_0 + \sum_{v=1}^n \frac{k_0 \lambda}{\lambda + \sigma_v} \quad (3)$$

is a function that has precisely the same properties as Y_{11} . Actually, λZ_{11} has a network representation as an impedance containing only resistance and inductance. Such a network may be derived from an RC network by replacing every capacitor with a resistor and every resistor with an inductor. Similarly, an RC impedance may be derived from an RL impedance by the inverse operation.

Returning to Eq. 1, if one substitutes $\lambda = -\sigma + j\omega$ and separates the real and imaginary parts, one has

$$\text{Rl } [Y_{11}] = g = -\sigma k_{\infty} + k_0 + \sum_{v=1}^n \frac{k_v (\omega^2 + \sigma^2 - \sigma_v \sigma)}{(\sigma_v - \sigma)^2 + \omega^2} \quad (4)$$

and

$$\text{Im } [Y_{11}] = b = k_{\infty} \omega + \sum_{v=1}^n \frac{k_v \sigma_v \omega}{(\sigma_v - \sigma)^2 + \omega^2}. \quad (5)$$

From these equations it is evident that for $\pi/2 < \arg \lambda < \pi$ $\text{Rl } [Y_{11}]$ may be positive, negative or zero while the $\text{Im } [Y_{11}]$ is always positive. For $\lambda = j\omega$, $\sigma = 0$ and both $\text{Rl } [Y_{11}]$ and $\text{Im } [Y_{11}]$ are positive, and for $\lambda = -\sigma$, $\omega = 0$, $\text{Im } [Y_{11}]$ is identically zero.

It is known from the general theory of positive-real functions that $|\arg [Y_{11}]| \leq |\arg \lambda|$ for $0 \leq |\arg \lambda| \leq \pi/2$. It is not difficult to show that in the case of an RC admittance, a much stronger relation holds, namely,

$$\arg [Y_{11}] \leq \arg \lambda \quad \text{for } 0 \leq \arg \lambda \leq \pi. \quad (6)$$

In these inequalities, the equality sign holds only if it holds identically.

Finally, consider a single term of Eq. 1, such as

$$y_v = \frac{k_v \lambda}{\lambda + \sigma_v} \quad (7)$$

and differentiate with respect to λ :

$$y'_v = \frac{d}{d\lambda} y_v = \frac{k_v \sigma_v}{(\lambda + \sigma_v)^2}. \quad (8)$$

On substituting $\lambda = -\sigma + j\omega$ in Eqs. 7 and 8, there results

$$b_v = \text{Im } [y_v] = \frac{k_v \sigma_v \omega}{(\sigma_v - \sigma)^2 + \omega^2} \quad (9)$$

and

$$\omega |y'_v| = \frac{k_v \omega \sigma_v}{(\sigma_v - \sigma)^2 + \omega^2}. \quad (10)$$

Thus

$$b_v = \omega |y_v| \quad (11)$$

at every point.

For the general admittance of Eq. 1, one has

$$b = k_\infty \omega + \sum_{v=1}^n b_v \quad (12)$$

and

$$\omega |Y'_{11}| = \omega \sqrt{(g')^2 + (b')^2} = |k_\infty \omega + \sum_{v=1}^n (g'_v + jb'_v)|. \quad (13)$$

Since

$$|Y'_{11}| \leq \sum |Y'_v| \quad (14)$$

the relation

$$b - \omega |Y'_{11}| \geq 0 \quad (15)$$

is seen to hold whenever λ lies anywhere in the left half-plane or on the imaginary axis excluding the points zero and infinity. The equality sign in Eq. 15 holds only for $Y_{11} = y_v$, i.e. when the function comprises only a single term.

B. Realizability of Transfer Functions

It is shown in the general theory (4) of network synthesis that if Y_{11} , Y_{12} , and Y_{22} are the short-circuit admittance parameters of a physically realizable RC two terminal-pair, the residue condition,

$$k_{11} k_{22} - k_{12}^2 \geq 0 \quad (16)$$

must hold at every pole, and the same is true for a family of impedance functions. When transformers, both real and ideal, are excluded from the networks, as they naturally must be, in the present problem, an additional restriction applies at infinity (where Y may have a pole) and at the origin (where the corresponding Z may have a pole). In the vicinity of these points the network behaves like either a combination of capacitors or a combination of resistors. Thus, it is clear that

$$\begin{aligned} Y_{12}(\infty) &\leq \text{the smaller of } Y_{11}(\infty) \text{ and } Y_{22}(\infty) \\ Z_{12}(\infty) &\leq \text{the smaller of } Z_{11}(\infty) \text{ and } Z_{22}(\infty) \end{aligned} \quad (17)$$

and

$$\begin{aligned} Y_{12}(0) &\leq \text{the smaller of } Y_{11}(0) \text{ and } Y_{22}(0) \\ Z_{12}(0) &\leq \text{the smaller of } Z_{11}(0) \text{ and } Z_{22}(0) \end{aligned} \quad (18)$$

must hold. Notice that no effort is made to control both Y_{11} and Y_{22} . In addition, the transfer function that serves as a starting point for a particular problem is usually specified only to within a constant multiplier. The resulting networks are usually such that relations 16, 17, and 18 are satisfied with the equality sign for the poles that are common to Y_{11} , Y_{12} , and Y_{22} or Z_{11} , Z_{12} , and Z_{22} , as the case may be. However, if the development proceeds from the "11" end of the network, then the "22" functions may contain poles not found in either of the other two. The question as to whether these "extra" poles are necessary, or whether they might be eliminated, for example, by a suitable choice of the zeros of Z_{11} when deriving it from Z_{12} , is an open one. Be that as it may, normal procedures introduce at most one "extra" pole, and where this occurs it will be regarded as normal.

Since RC driving-point functions must have only simple poles and zeros on the negative-real axis, admissible transfer functions may have only simple poles on the negative-real axis. Zeros may be real, pure imaginary, or complex, but may not lie in the right half-plane. A summary of methods for deriving appropriate associated driving-point functions will be given in a later section.

C. Zero Shifting on the Negative-Real Axis

As a matter of convenience, transmission zeros on the negative-real axis, including the origin and infinity, should be disposed of first.

Control of transmission zeros is based on the following simple principles.

1. If, in a ladder development, a shunt branch has a pole of admittance at $\lambda = \lambda_0$ that is not a pole of either of the adjoining transfer admittances; or a series branch has a pole of impedance at $\lambda = \lambda_0$ that is not a pole of either of the adjoining transfer impedances, then λ_0 will be a transmission zero.
2. If $F_1(\lambda)$ and $f_1(\lambda)$ are rational functions, and if $F_1(\lambda_0) - f_1(\lambda_0) = 0$, then $(\lambda - \lambda_0)$ is a factor of $[F_1(\lambda) - f_1(\lambda)]$.

Item 1 must be modified slightly when the branch in question is at one pair of terminals of the network, in which case one must consider whether it is "current" or "voltage" that is supposed to be transmitted. For example, a shunt capacitance at either end introduces a zero of Z_{12} at infinity, but not of Y_{12} . In this connection it should be noted that throughout this report the internal impedances of the generator and the load are considered as integral parts of the networks, except where explicitly stated otherwise. This point of view is taken simply because there is no specific reason for any other. It neither restricts the results nor offers any outstanding advantage over the more usual concept of a system comprising a generator, load, and coupling network.

Regarding item 2, the manipulation must be carried out in such a way that all the functions involved are both p-r and RC. Suppose the given function is Y_1 and a zero is required at $\lambda = -\sigma_0$.

- a. Compute $Y_1(-\sigma_0)$.

- b. Find a y_1 such that $y_1(-\sigma_0) = Y_1(-\sigma_0)$. y_1 must be part of Y_1 .
- c. Compute $Y_2 = Y_1 - y_1$.
- d. Remove the pole of $Z_2 = 1/Y_2$ at $\lambda = -\sigma_0$.

It may not be possible to find a suitable y_1 in step b. In this case, one inverts Y_1 and carries out the analogous steps with Z_1 . One of these alternatives will always succeed.

As an example, consider the function

$$Y_1 = \frac{\lambda + 1}{\lambda + 2} = \frac{1}{2} + \frac{\frac{1}{2}\lambda}{\lambda + 2} \quad (19)$$

and assume a zero is desired at $\lambda = -3/2$.

$$a. \quad Y_1(-3/2) = -1. \quad (20)$$

b. Let

$$y_1 = \frac{G\lambda}{\lambda + 2} \quad (21)$$

$$y_1(-3/2) = -3G = -1$$

$$G = \frac{1}{3} \quad \text{and} \quad y_1 = \frac{\frac{1}{3}\lambda}{\lambda + 2}.$$

c.

$$Y_2 = Y_1 - y_1 = \frac{\frac{2}{3}\lambda + 1}{\lambda + 2} \quad (22)$$

$$Z_2 = \frac{3}{2} \frac{\lambda + 2}{\lambda + \frac{3}{2}}. \quad (23)$$

d. Separate the pole at $-1/2$:

$$Z_2 = \frac{\frac{3}{4}}{\lambda + \frac{3}{2}} + \frac{3}{2}. \quad (24)$$

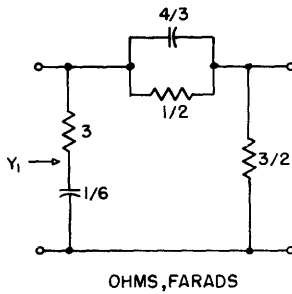


Fig. 1

Network having a transmission zero at $\lambda = -3/2$.

The network is shown in Fig. 1. The transfer impedance of this two terminal-pair is

$$Z_{12} = \frac{\lambda + \frac{3}{2}}{\lambda + 1}. \quad (25)$$

Further details of this process may be found elsewhere (13). However, for the sake of completeness, especially in view of its relation to the more difficult problem considered later, some further consideration will be given to the question of finding a suitable y_1 in step b by means of a

single operation on Y_1 or Z_1 .

In the first place, there may or may not be a unique solution, but in certain special cases there is a unique solution which always exists.

- (1) A zero of Z may be produced at infinity always and only by removing a series resistance.
- (2) The largest zero of Z , i.e. corresponding to the largest σ , may be moved out an arbitrary distance always and only by removing a series resistance.
- (3) A zero of Y may be produced at the origin always and only by removing a shunt conductance.
- (4) The smallest zero of Y may be moved an arbitrary distance toward the origin always and only by removing a shunt conductance.

Consideration of the Foster expansions of Z and Y shows that for any point on the negative-real axis at least one of them can be separated into two groups of terms of which one is positive and the other negative. By reducing the residues of one or more terms in the group that contributes the larger magnitude at the point in question, a zero can always be produced. Thus, it is possible to shift a zero to any desired point through a single operation with either Z or Y , but not necessarily with both; i.e. there may be no choice.

It is usually possible to effect the shift by removing part of a single pole (or a single resistance). In fact, experience suggests that this will always be the case, but no formal proof is available.

III. THE CANONICAL SECTIONS

A. Basic Requirements

The zero-shifting procedure described in the previous section is restricted to the negative-real axis only because every component network is required to be realizable as an RC two-terminal network. The analogous limitation in the case of LC networks is the restriction of zeros to the imaginary axis. It was pointed out in section I-C that Darlington overcame this limitation through the use of close-coupled coils employing negative mutual inductance. In the RC case, of course, negative mutual inductance cannot be used. But suitable networks can be found which do have the close-coupled property and which do provide the necessary negative mutual impedances. Thus, through a combination of Cauer's zero-shifting technique and the "transformed" Brune-Darlington sections a very useful RC procedure is obtained.

The close-coupled RC sections are not themselves essentially new. Their incorporation in the synthesis procedure is new, however; and since an understanding of the whole process is greatly facilitated by a thorough familiarity with the zero sections, they will now be considered in detail.

The basic requirements imposed on the sections are:

1. The element values should be obtainable directly from the function matrix.

2. They must be compact. This is necessary for several reasons. First, it insures the realizability of the simplest possible system function. Second, it leads to mutual equivalence of sections whether derived on the basis of impedance or admittance functions. Finally, a noncompact section embedded in a network, i.e. not appearing at one of its terminals, must introduce a transmission zero at some point on the negative-real axis in addition to the desired conjugate pair. (In special cases, some elements may be saved by removing the compact requirement.)
3. Of the parameters necessary to specify the section completely, as many as possible should be independent in order that the section itself may be as simple as possible.

B. Properties of the Canonical Sections

An asymmetric form of the twin-T null network meets the requirements outlined above in the case of imaginary zeros. This network is shown in Fig. 2 in which the notation for the elements has been chosen in such a way that the short-circuit admittance functions appear in their simplest form. This network is obtained by bisecting the symmetrical form of the twin-T in the manner indicated in Fig. 3. All admittances to the right of the dotted line are then multiplied by $1/a$. This modification leaves the short-circuit transfer admittance unchanged except for a constant multiplier, but it modifies the driving-point admittances in that their zeros are shifted. Straightforward analysis of the circuit in Fig. 2 yields the following equations:

$$y_{12} = \frac{c_o(\lambda^2 + \omega_o^2)}{\lambda + \sigma_o} \quad (26)$$

$$y_{11} = \frac{c_o \left[\lambda^2 + (1+a) \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right) \lambda + \omega_o^2 \right]}{\lambda + \sigma_o} \quad (27)$$

$$y_{22} = \frac{c_o \left[\lambda^2 + \left(1 + \frac{1}{a} \right) \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right) \lambda + \omega_o^2 \right]}{\lambda + \sigma_o} \quad (28)$$

The notation in these equations is consistent with the following scheme, which is used throughout.

1. Lower-case letters refer to quantities related directly to the zero sections. Lower-case letters having double subscripts always refer to short-circuit admittance parameters or open-circuit impedance parameters, respectively.
2. Capital letters refer to the over-all network. When these appear with double subscripts they refer to the short-circuit admittance parameters or to the

open-circuit impedance parameters of an entire network which may comprise several sections.

3. λ -values at which y_{12} and z_{12} are zero are defined as $\lambda_o = -\alpha_o + j\beta_o$, $\bar{\lambda}_o = -\alpha_o - j\beta_o$, and $\alpha_o^2 + \beta_o^2 = \omega_o^2$. When $\alpha_o = 0$, ω_o^2 will be used rather than β_o^2 .
4. The internal pole of y_{12} is designated σ_o ; likewise the internal pole of z_{12} .
(Although, for the same section, z_{12} and y_{12} have different internal poles, it is unnecessary to talk about z and y simultaneously.)

In order to identify Eqs. 26, 27, and 28 with a continued-fraction development, make the Foster expansions as follows:

$$y_{12} = c_o \left[\lambda + \frac{\omega_o^2}{\sigma_o} - \frac{\left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right) \lambda}{\lambda + \sigma_o} \right] \quad (29)$$

$$y_{11} = c_o \left[\lambda + \frac{\omega_o^2}{\sigma_o} + \frac{a \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right) \lambda}{\lambda + \sigma_o} \right] \quad (30)$$

$$y_{22} = c_o \left[\lambda + \frac{\omega_o^2}{\sigma_o} + \frac{\frac{1}{a} \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right) \lambda}{\lambda + \sigma_o} \right] \quad (31)$$

Next compute

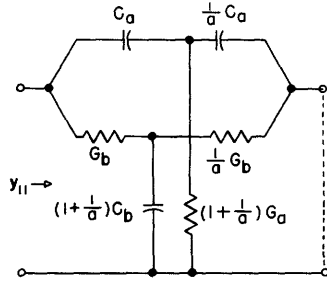
$$y_1 = y_{11} - y_{12} = \frac{c_o(1+a) \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right) \lambda}{\lambda + \sigma_o} \quad (32)$$

$$y_2 = y_{22} - y_{12} = \frac{c_o \left(1 + \frac{1}{a} \right) \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right) \lambda}{\lambda + \sigma_o} \quad (33)$$

Taken together, y_{12} , y_1 and y_2 comprise the π -section representation of the twin-T in Fig. 2. The π section is illustrated in Fig. 4. Note that y_{12} contains the negative elements c_3 and G_3 and is not physically realizable. But so far as its external behavior is concerned this circuit is completely equivalent to the one in Fig. 2.

Before proceeding to the networks and equations for use with complex zeros, it is desirable to verify the fact that this network meets the requirements set forth above.

Examination of the Foster expansions of Eqs. 29, 30, and 31 shows that the section is Y-compact; that is, the residue condition, $k_{11}k_{22} - k_{12}^2 \geq 0$, is satisfied with the



$$\begin{aligned} C_a &= (1+a) C_0 \\ C_b &= (1+a) C_0 \frac{\omega_0^2}{\sigma_0^2} \\ G_a &= (1+a) C_0 \sigma_0 \\ G_b &= (1+a) C_0 \frac{\omega_0^2}{\sigma_0} \end{aligned}$$

Fig. 2

Asymmetric form of the twin-T null network.

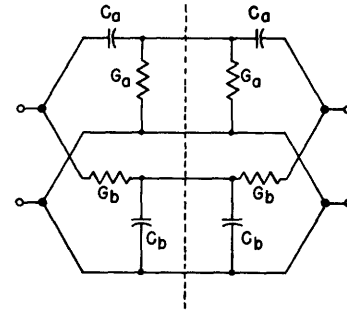
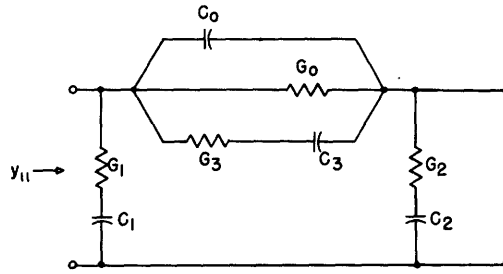


Fig. 3

Symmetrical form of the twin-T.



$$y_1: \begin{cases} G_1 = C_0(1+a)(\sigma_0 + \frac{\omega_0^2}{\sigma_0}) \\ C_1 = \frac{G_1}{\sigma_0} \end{cases} \quad y_2: \begin{cases} G_2 = C_0(1+\frac{1}{a})(\sigma_0 + \frac{\omega_0^2}{\sigma_0}) \\ C_2 = \frac{G_2}{\sigma_0} \end{cases}$$

$$y_{12}: \begin{cases} C_0 = C_0 \\ G_0 = C_0 \frac{\omega_0^2}{\sigma_0} \\ G_3 = -C_0(\sigma_0 + \frac{\omega_0^2}{\sigma_0}) \\ C_3 = -\frac{G_3}{\sigma_0} \end{cases}$$

Fig. 4

π -section representation of the network in Fig. 2.

equality sign at all poles of y , and the analogous relation is true for the constant terms. Also, the special conditions on the behavior at infinity and at the origin (Eqs. 17 and 18) are satisfied. Straightforward analysis shows that the section is also z -compact. Therefore, requirements 1 and 2 are met.

With respect to requirement 3 it is found that the network is completely specified by only four parameters, which are independent, whereas there are six elements in the network. Nothing is really sacrificed, however. The most general transfer function realizable by means of the configuration in Fig. 2 has the form

$$y_{12} = \frac{c(\lambda + a_1)(\lambda^2 + b_1\lambda + b_o)}{(\lambda + a_o)(\lambda + a_2)}. \quad (34)$$

One possible degree of freedom is used up by requiring one cancellation in Eq. 34 (which, incidentally, also precludes the use of this network for complex zeros) and another is used up by the compact requirement. Thus, only in special cases is it advantageous to choose more than four parameters independently. Finally, the required functions cannot be obtained by any RC network of simpler configuration.

The normal procedure will lead directly to values of σ_o , c_o , and \underline{a} , in that order, where ω_o^2 is known in advance. The element values may then be determined from the formulas given in Fig. 2. The canonical section thus becomes a building block that has a one-to-one correspondence with the admittance functions expressed by Eqs. 29, 30, and 31. In fact, only y_{12} (Eq. 29) and y_1 (Eq. 32) are necessary to specify the section completely.

It is ordinarily convenient to depend on the formulas for computing actual element values. It is not essential, however, because a very simple method may be used to determine them from y_{12} and \underline{a} . One first synthesizes a symmetric lattice using Eq. 29 and Eq. 30 with \underline{a} set equal to unity. After reducing the lattice to unbalanced form the final network is obtained immediately by introducing the parameter \underline{a} and adjusting the scale factor.

C. A Numerical Example

It is worthwhile to anticipate the general procedure and show by means of a numerical example how the design of a zero section may be carried out in a special case. The situation is special in that step 1 of the general procedure outlined in section I is unnecessary. Assume as given

$$Y_{12} = \frac{h(\lambda^2 + 1)}{2\lambda^2 + 15\lambda + 5} \quad (35)$$

and

$$Y_{11} = \frac{14\lambda^2 + 18\lambda + 2}{2\lambda^2 + 15\lambda + 5}. \quad (36)$$

Since $y_{12}(\lambda_o) = 0$, the admittance as seen at the input terminals of the network is simply $y_1(\lambda_o)$. Thus, $Y_1(\lambda_o) = y_1(\lambda_o)$.

Substituting $\lambda = \lambda_o = j$ in Eq. 36 gives

$$Y_1(j) = g + jb = 1 + j. \quad (37)$$

Making the same substitution in Eq. 33 gives

$$y_1(j) = c_o(1+a)\left(j + \frac{1}{\sigma_o}\right). \quad (38)$$

Equating the real and imaginary parts, respectively, of Eqs. 37 and 38 results in

$$\frac{c_o(1+a)}{\sigma_o} = 1 \quad (39)$$

$$c_o(1+a) = 1. \quad (40)$$

Dividing Eq. 40 by Eq. 39 now yields

$$\sigma_o = 1 \quad (41)$$

and therefore

$$y_1 = \frac{2\lambda}{\lambda + 1} \quad (42)$$

y_1 is next removed.

$$Y_2 = Y_1 - y_1 = \frac{14\lambda^2 + 18\lambda + 2}{2\lambda^2 + 15\lambda + 5} - \frac{2\lambda}{\lambda + 1} = \frac{(\lambda^2 + 1)(10\lambda + 2)}{(\lambda + 1)(2\lambda^2 + 15\lambda + 5)}. \quad (43)$$

Now that σ_o is known, y_{12} is known except for c_o . Since $Z_2 = 1/Y_2$ has a pole at $\lambda = j$, invert Y_2 and remove $1/y_{12}$. Thus

$$Z_3 = Z_2 - \frac{1}{y_{12}} = \frac{(\lambda + 1)(2\lambda^2 + 15\lambda + 5)}{(\lambda^2 + 1)(10\lambda + 2)} - \frac{\lambda + 1}{c_o(\lambda^2 + 1)}. \quad (44)$$

Setting $\lambda = j$ in Eq. 44 gives

$$c_o = \frac{2}{3}. \quad (45)$$

This manipulation is equivalent to computing the residue in the pole at $\lambda = j$ of the function $\lambda Z_3/(\lambda + 1)$. Note, for future reference, that this residue must be real and positive in order that Z_3 may be p-r, but it is not guaranteed to be so by any operation so far performed.

Completing the operation indicated in Eq. 44 with the help of Eq. 45 gives

$$Z_3 = \frac{\lambda + 1}{5\lambda + 1}. \quad (46)$$

Substituting Eq. 45 in Eq. 40 gives

$$a = \frac{1}{2}. \quad (47)$$

From Eq. 33, one has

$$y_2 = \frac{4\lambda}{\lambda + 1} \quad (48)$$

and removing y_2 from $1/Z_3$ finally yields

$$Y_4 = Y_3 - y_2 = \frac{5\lambda + 1}{\lambda + 1} - \frac{4\lambda}{\lambda + 1} = 1. \quad (49)$$

Substituting the appropriate data into the formulas of Fig. 2 enables the element values to be computed. These are

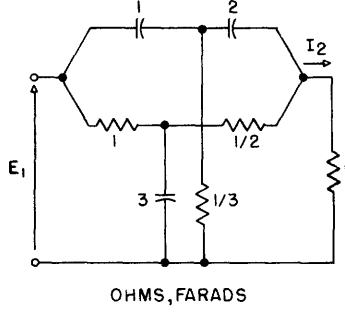


Fig. 5

Network realization of the functions of Eqs. 35 and 36.

$$\begin{aligned}
 C_a &= 1 \\
 \frac{1}{a} C_a &= 2 \\
 \left(1 + \frac{1}{a}\right) G_a &= 3 \\
 G_b &= 1 \\
 \frac{1}{a} G_b &= 2 \\
 \left(1 + \frac{1}{a}\right) C_b &= 3.
 \end{aligned} \tag{50}$$

The final network with element values given in farads and ohms is shown in Fig. 5.

D. Complex Zeros

Complex zeros present no fundamentally new problems. The general comments in the preceding section apply equally well to both imaginary and complex zeros. All that remains is to show that suitable canonical sections do exist and to list the associated equations for reference.

The standard form to be used for the transfer admittance is

$$y_{12} = \frac{c_o(\lambda^2 + 2a_o\lambda + \omega_o^2)}{\lambda + \sigma_o}. \tag{51}$$

The elements of the matrix of a compact network may be found by making the Foster expansion of Eq. 51 and associating appropriate driving-point admittances with the result.

$$y_{12} = c_o \left[\lambda + \frac{\omega_o^2}{\sigma_o} - \frac{\lambda \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2a_o \right)}{\lambda + \sigma_o} \right] \tag{52}$$

$$y_{11} = c_o \left[\lambda + \frac{\omega_o^2}{\sigma_o} + \frac{a\lambda \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2a_o \right)}{\lambda + \sigma_o} \right] \tag{53}$$

$$y_{22} = c_o \left[\lambda + \frac{\omega_o^2}{\sigma_o} + \frac{\frac{1}{a}\lambda \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2a_o \right)}{\lambda + \sigma_o} \right]. \tag{54}$$

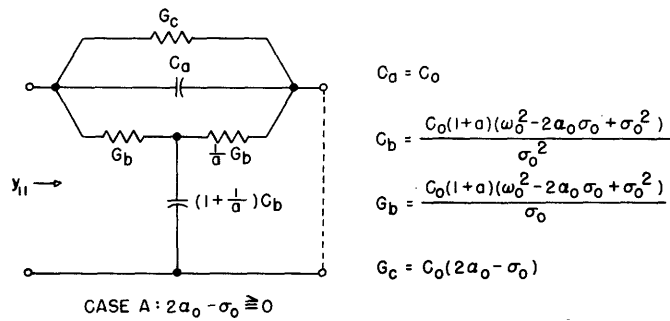


Fig. 6
Canonical section used for Case A.

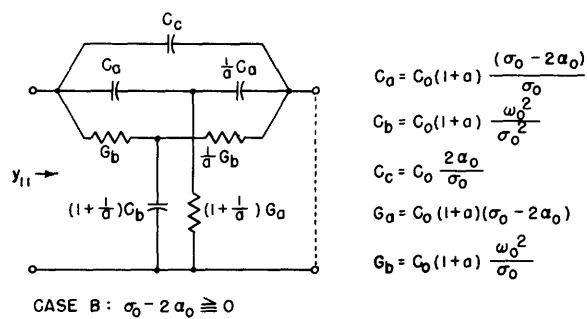


Fig. 7
Canonical section used for Case B.

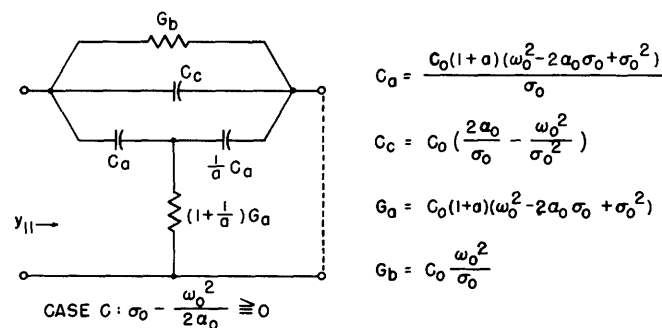


Fig. 8
Canonical section used for Case C.

From these equations,

$$y_1 = c_o(1+a) \frac{\left(\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2a_o\right) \lambda}{\lambda + \sigma_o} \quad (55)$$

and

$$y_2 = c_o \left(1 + \frac{1}{a}\right) \frac{\left(\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2a_o\right) \lambda}{\lambda + \sigma_o} \quad (56)$$

are obtained immediately. Note that the expression $(\sigma_o + \omega_o^2/\sigma_o - 2a_o)$ is always positive because it is equivalent to

$$\left[\frac{(\sigma_o - a_o)^2 + \beta_o^2}{\sigma_o} \right].$$

Unfortunately, there does not appear to be a ready-made network for this case as there was for imaginary zeros. The bridged-T networks commonly used for complex zeros do not permit ω_o and σ_o to be chosen independently, while a number of other potentially useful configurations fail to have the compact property. The importance of the compact property cannot be overemphasized. Without it, there is not a true correspondence between the network and the matrix, any more than there is a true correspondence between an ideal transformer and a practical transformer. Consequently it is necessary to use two, and desirable to use three different networks corresponding to three situations that may be encountered, as follows:

$$\text{Case A: } 2a_o - \sigma_o \geq 0$$

$$\text{Case B: } \sigma_o - 2a_o \geq 0$$

$$\text{Case C: } \sigma_o - \frac{\omega_o^2}{2a_o} \geq 0.$$

These sections are shown in Figs. 6, 7, and 8, respectively. The formulas required for computing the element values from ω_o , σ_o , a , and a_o are given also.

In view of the fact that Case A and Case B are mutually exclusive, except when they degenerate to identical circuits, Case C might be dispensed with. It is retained because it affords a saving of components in those instances where it may replace Case B.

E. Networks Derived on Impedance Basis

Design of a zero section is ordinarily preceded by the removal of a shunt or a series branch from the network. It often happens that there is no choice as to which is required, and when a series branch must be removed, it is necessary to deal with the

impedance rather than the admittance function. It is then convenient to represent the zero sections by T sections, and to describe them by means of the open-circuit impedance parameters. These are readily obtained from the admittance parameters by means of the relations,

$$\left. \begin{aligned} z_{12} &= \frac{y_{12}}{|y|} \\ z_{11} &= \frac{y_{22}}{|y|} \\ z_{22} &= \frac{y_{11}}{|y|} \\ |y| &= y_{11}y_{22} - y_{12}^2 \end{aligned} \right\} \quad (57)$$

Applying these transformations to the appropriate equations gives the following results.

In the case of imaginary zeros, $\lambda_o = +j\omega_o$:

$$z_{12} = \frac{a\sigma_o}{c_o(\sigma_o^2 + \omega_o^2)(1+a)^2} \left[1 + \frac{\sigma_o}{\lambda} - \frac{\sigma_o + \frac{\omega_o^2}{\sigma_o}}{\lambda + \frac{\omega_o^2}{\sigma_o}} \right] \quad (58)$$

$$z_{11} = \frac{a\sigma_o}{c_o(\sigma_o^2 + \omega_o^2)(1+a)^2} \left[1 + \frac{\sigma_o}{\lambda} + \frac{\frac{1}{a} \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right)}{\lambda + \frac{\omega_o^2}{\sigma_o}} \right] \quad (59)$$

$$z_{22} = \frac{a\sigma_o}{c_o(\sigma_o^2 + \omega_o^2)(1+a)^2} \left[1 + \frac{\sigma_o}{\lambda} + \frac{a \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right)}{\lambda + \frac{\omega_o^2}{\sigma_o}} \right] \quad (60)$$

In the case of complex zeros, $\lambda_o = -\alpha_o + j\beta_o$:

$$z_{12} = \frac{a\sigma_o}{c_o(\sigma_o^2 - 2\alpha_o\sigma_o + \omega_o^2)(1+a)^2} \left[1 + \frac{\sigma_o}{\lambda} - \frac{\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2\alpha_o}{\lambda + \frac{\omega_o^2}{\sigma_o}} \right] \quad (61)$$

$$z_{11} = \frac{a\sigma_o}{c_o(\sigma_o^2 - 2a_o\sigma_o + \omega_o^2)(1+a)^2} \left[1 + \frac{\sigma_o}{\lambda} + \frac{\frac{1}{a} \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2a_o \right)}{\lambda + \frac{\omega_o^2}{\sigma_o}} \right] \quad (62)$$

$$z_{22} = \frac{a\sigma_o}{c_o(\sigma_o^2 - 2a_o\sigma_o + \omega_o^2)(1+a)^2} \left[1 + \frac{\sigma_o}{\lambda} + \frac{a \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2a_o \right)}{\lambda + \frac{\omega_o^2}{\sigma_o}} \right]. \quad (63)$$

Notice that one and the same zero section is arrived at regardless of whether one operates on the impedance basis or on the admittance basis.* It is, therefore undesirable to have two sets of notation for the elements, but, on the otherhand, Eqs. 58-63 are awkward to handle. The difficulty is easily resolved. Let

$$z_{12} = R_\infty \frac{[\lambda^2 + \omega_o^2]}{\lambda(\lambda + \sigma_o)} = R_\infty \left[1 + \frac{\omega_o^2}{\sigma_o\lambda} - \frac{\sigma_o + \frac{\omega_o^2}{\sigma_o}}{\lambda + \sigma_o} \right] \quad (64)$$

and equate the coefficients in Eq. 58 to those in Eq. 64, and similarly for z_{11} and z_{22} . Thus, using an obvious notation, one has for $\lambda_o = +j\omega_o$

$$c_o = \frac{a_z \sigma_{oz}}{R_\infty(\sigma_{oz}^2 + \omega_o^2)(1 + a_z)^2} \quad (65)$$

and for $\lambda_o = -a_o + j\beta_o$

$$c_o = \frac{a_z \sigma_{oz}}{R_\infty(\sigma_{oz}^2 - 2a_o\sigma_{oz} + \omega_o^2)(1 + a_z)^2} \quad (66)$$

and in both cases

$$\sigma_{oy} = \frac{\omega_o^2}{\sigma_{oz}} \quad (67)$$

$$a_y = \frac{1}{a_z}. \quad (68)$$

*An exactly analogous situation exists for Brune networks.

F. A Dual Procedure

It is now possible to obtain for impedance functions a procedure that is the dual of the procedure for admittance functions.

To this end, multiply Eq. 64 through by λ to obtain

$$\lambda z_{12} = R_{\infty} \left[\lambda + \frac{\omega_o^2}{\sigma_o} - \frac{\left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right) \lambda}{\lambda + \sigma_o} \right]. \quad (69)$$

The corresponding driving-point functions are

$$\lambda z_{11} = R_{\infty} \left[\lambda + \frac{\omega_o^2}{\sigma_o} + \frac{a \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right) \lambda}{\lambda + \sigma_o} \right] \quad (70)$$

and

$$\lambda z_{22} = R_{\infty} \left[\lambda + \frac{\omega_o^2}{\sigma_o} + \frac{\frac{1}{a} \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} \right) \lambda}{\lambda + \sigma_o} \right]. \quad (71)$$

The above manipulation is equivalent to replacing every resistance in the RC network by an inductance and every capacitance by a resistance. The resulting RL network is the dual of the RC network.

Corresponding to Eqs. 61-63 are the following:

$$\lambda z_{12} = R_{\infty} \left[\lambda + \frac{\omega_o^2}{\sigma_o} - \frac{\left(\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2a_o \right) \lambda}{\lambda + \sigma_o} \right] \quad (72)$$

$$\lambda z_{11} = R_{\infty} \left[\lambda + \frac{\omega_o^2}{\sigma_o} + \frac{a \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2a_o \right) \lambda}{\lambda + \sigma_o} \right] \quad (73)$$

$$\lambda z_{22} = R_{\infty} \left[\lambda + \frac{\omega_o^2}{\sigma_o} + \frac{\frac{1}{a} \left(\sigma_o + \frac{\omega_o^2}{\sigma_o} - 2a_o \right) \lambda}{\lambda + \sigma_o} \right]. \quad (74)$$

The procedure to be used for the impedance basis is now quite clear. Instead of operating on the RC network, one multiplies Z_{RC} by λ to obtain Z_{RL} and designs a

T-section representation for the RL network. The elements of the RL network need not be determined; all that is really required are the values of R_∞ , σ_{OZ} , and a_z , and from these the RC section may be designed directly.

The advantages of using the RL network are that it simplifies the determination of Z_O , the impedance removed in step 1 of the general procedure, and it makes it unnecessary to develop two separate methods. Of equal interest is the fact that the whole treatment may be applied to both RC and RL networks.

G. Summary

The primary objective has been to establish a one-to-one correspondence, between the y- or z-parameter matrices on the one hand, and the physical networks on the other, with respect to the canonical sections which serve as the building blocks for the general procedure. Because of the unique properties of compact sections only one physical network is required for each type of zero section regardless of whether it is designed on the impedance or on the admittance basis. Moreover, the remarkable dual properties of RC admittances and RL impedances permit a single procedure to suffice for both impedance and admittance functions. Hereafter, the transition from a z- or a y-matrix to a physical zero section will be regarded as completely automatic.

IV. THE GENERAL PROCEDURE

A. Outline of the Proof of the Theorem

The proof of the general theorem stated on page one will be completed in this section.

The zero-shifting procedure described in section II evidently suffices to produce transmission zeros at any point on the negative-real axis, including the origin and infinity. What remains is to show how this idea may be applied to the design of zero sections of the types treated in the preceding section. When this has been done, methods will be available for producing zeros at any point not in the right half-plane, each independent of the other, in any desired sequence, with each zero placed in evidence by a separate three-terminal network.

In connection with the illustrative example of section III-C it was pointed out that nothing in the procedure used there would guarantee a positive, real value for c_O . This is the crux of the whole problem. Consequently, the present development begins by finding the necessary and sufficient conditions under which a positive, real c_O may be found. This leads to certain restrictions on the admittance function of the network at the input terminals of the zero section. Next, it is found that these restrictions guarantee the realizability of the section as a whole, which in turn guarantees a p-r, RC remainder after removing the section. Finally, it is proved that it is always possible to manipulate any RC driving-point function of appropriate degree in such a way that the restrictions are met.

B. The Necessary Conditions

Consider the network of Fig. 9. Here y_1 and y_{12} represent two branches of the π -section equivalent of a zero section, and Y_3 represents the remainder of the zero section in parallel with the rest of the network. Since y_{12} is zero for $\lambda = \lambda_0$, Y_1 itself reduces to y_1 at this point. Thus

$$Y_1(\lambda_0) - y_1(\lambda_0) = 0. \quad (76)$$

Assuming that y_1 has been determined from Eq. 76, compute

$$Y_2 = Y_1 - y_1 \quad (77)$$

and then invert and remove y_{12} :

$$Z_3 = \frac{1}{Y_2} - \frac{1}{y_{12}} = \frac{y_{12} - Y_2 + y_1}{Y_2 y_{12}}. \quad (78)$$

Since all of the admittance functions are expressed in terms of polynomials having real coefficients, if Eq. 76 is satisfied for $\lambda_0 = -\alpha_0 + j\beta_0$, it is satisfied also for $\bar{\lambda}_0 = -\alpha_0 - j\beta_0$, and therefore Y_2 contains the factor $(\lambda^2 + 2\alpha_0\lambda + \omega_0^2)$ in its numerator. There will necessarily be one cancellation of this factor in the expression on the right in Eq. 78, but removal of y_{12} implies two cancellations. It follows that

$$y'_{12}(\lambda_0) - Y'_1(\lambda_0) + y'_1(\lambda_0) = 0 \quad (79)$$

in which the primes indicate differentiation with respect to λ , must hold. Moreover, since all the functions are assumed to be "real," Eq. 79 guarantees that c_0 will be real.

Equations 76 and 79 are equivalent to the following pair

$$Y_1(\lambda_0) = y_{11}(\lambda_0) \quad (80)$$

$$Y'_1(\lambda_0) = y'_{11}(\lambda_0) \quad (81)$$

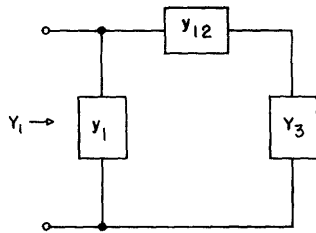


Fig. 9
Ladder development of
a zero section.

which as a pair of simultaneous equations represent necessary conditions for realization of the section. Strictly speaking, equations similar to 80 and 81, but with λ_0 replaced by $\bar{\lambda}$ are required also. However, since in these equations λ_0 is never real, the second pair of equations may be replaced by the requirement of real coefficients. In due course, it is found that Eqs. 80 and 81 imply a condition on the residues in the poles of $1/y_{12}$ at λ_0 and $\bar{\lambda}_0$.

C. The Conditional Equation for Imaginary Zeros

It is now necessary to apply these results to the detailed structure of the bridge, thereby identifying σ_o , a , and c_o with the values assumed by Y_1 and Y'_1 at λ_o . It would be permissible to derive the results for complex zeros and treat imaginary zeros as a special case. However, the manipulation in the latter case is less involved, yet it demonstrates the method equally as well as the former. Only the equivalent results for complex zeros need be given.

Let

$$Y_1(\lambda_o) = g + jb$$

$$Y'_1(\lambda_o) = g' + jb'$$

and

$$\lambda_o = j\omega_o.$$

Substitution for λ_o in y_{11} (Eq. 30) and its derivative yields

$$y_{11}(j\omega_o) = c_o(1+a) \frac{\omega_o^2}{\sigma_o^2} + jc_o(1+a) \omega_o \quad (82)$$

and

$$y'_{11}(j\omega_o) = \frac{c_o \left[\sigma_o^2 + \omega_o^2 + a(\sigma_o^2 - \omega_o^2) \right]}{\sigma_o^2 + \omega_o^2} - j \frac{2c_o a \omega_o \sigma_o}{\sigma_o^2 + \omega_o^2}. \quad (83)$$

Substituting these results in Eqs. 80 and 81, respectively, gives

$$g + jb = c_o(1+a) \frac{\omega_o^2}{\sigma_o^2} + jc_o(1+a) \omega_o \quad (84)$$

and

$$g' + jb' = c_o \frac{\left[\sigma_o^2 + \omega_o^2 + a(\sigma_o^2 - \omega_o^2) \right]}{\sigma_o^2 + \omega_o^2} - j \frac{2c_o a \omega_o \sigma_o}{\sigma_o^2 + \omega_o^2}. \quad (85)$$

From Eqs. 84 and 85, a set of four simultaneous equations is obtained, namely

$$g = c_o(1+a) \frac{\omega_o^2}{\sigma_o^2} \quad (86)$$

$$b = c_o(1+a) \omega_o \quad (87)$$

$$g' = c_o \frac{\left[\sigma_o^2 + \omega_o^2 + a(\sigma_o^2 - \omega_o^2) \right]}{\sigma_o^2 + \omega_o^2} \quad (88)$$

$$b' = \frac{-2c_o a \omega_o \sigma_o}{\sigma_o^2 + \omega_o^2}.$$

Inasmuch as these four equations contain only three unknowns, a solution exists only when a certain relation exists among the variables. The required relation may be found by eliminating c_o , \underline{a} , and σ_o from the original equations.

Multiply Eq. 88 by ω_o and subtract the result from Eq. 87 to obtain

$$b - g'\omega_o = \frac{2c_o a \omega_o^3}{\sigma_o^2 + \omega_o^2} \quad (90)$$

and divide Eq. 90 by Eq. 89 to obtain

$$\frac{b - g'\omega_o}{-b'\omega_o} = \frac{\omega_o}{\sigma_o}. \quad (91)$$

Dividing Eq. 86 by Eq. 87 now yields

$$\frac{g}{b} = \frac{\omega_o}{\sigma_o} \quad (92)$$

and elimination of σ_o from these last two equations gives for the final result

$$\frac{g}{b} = \frac{b - g'\omega_o}{-b'\omega_o}. \quad (93)$$

Equation 93 expresses a condition on $Y_1(\lambda_o)$ and $Y_1'(\lambda_o)$ which must be satisfied in order for Eqs. 86-89 to yield solutions for c_o , \underline{a} , and σ_o . Noting that both g and b are positive and finite for $\lambda = j\omega$, it may be seen that positive, finite values will be obtained for c_o and σ_o , provided a positive, finite value is found for \underline{a} .

To investigate this situation, divide Eq. 88 by Eq. 89 and solve for \underline{a} to obtain

$$a = \frac{-b'(\sigma_o^2 + \omega_o^2)}{2\omega_o g' \sigma_o + b'(\sigma_o^2 - \omega_o^2)}. \quad (94)$$

Next, form the function,

$$w(a) = \frac{1 - a}{1 + a} = \frac{g' + b' \frac{\sigma_o}{\omega_o}}{g' - b' \frac{\sigma_o}{\omega_o}}. \quad (95)$$

If $w(a) \pm 1 = 0$ has no roots for $0 < \omega < \infty$, then \underline{a} remains positive and finite over this range. Setting the right member of Eq. 95 equal to -1 yields, after substitution for ω_o/σ_o from Eq. 91,

$$b^2 - (g'\omega_o)^2 = (b'\omega_o)^2 \quad (96)$$

or

$$b = \omega_o \sqrt{(g')^2 + (b')^2} = \omega_o |Y'|. \quad (97)$$

It was shown in section II, however, that

$$b - \omega_o |Y'| > 0 \quad \text{for } 0 < \omega < \infty,$$

and therefore Eq. 97 has no solution over the range in question. Likewise, setting $w(a) = +1$ leads to

$$(b'\omega_o)^2 + (b - g'\omega_o)^2 = 0 \quad (98)$$

and here again there is no root over the range in question. Consequently, a positive, finite value for \underline{a} is assured.

To summarize what has been accomplished so far, reference is made again to Fig. 9 and the discussion in section IV-B. It has been demonstrated that (for $\lambda_o = j\omega_o$) the operations indicated by Eqs. 76 and 78 may be used to determine the parameters of a realizable section if, and only if, $Y_1(\lambda_o)$ and $Y'_1(\lambda_o)$ satisfy the relation given by Eq. 93.

D. The Conditional Equation for Complex Zeros

Before continuing, the pertinent results for $\lambda_o = -a_o + j\beta_o$ will be stated. The following equations correspond to Eqs. 86-89:

$$g = c_o(1+a) \frac{(a_o^2 + \beta_o^2 - a_o\sigma_o)}{\sigma_o} \quad (99)$$

$$b = c_o(1+a) \beta_o \quad (100)$$

$$g' = c_o \frac{\left\{ (\sigma_o - a_o)^2 + \beta_o^2 + a \left[(\sigma_o - a_o)^2 - \beta_o^2 \right] \right\}}{(\sigma_o - a_o)^2 + \beta_o^2} \quad (101)$$

$$b' = \frac{-2c_o a \beta_o (\sigma_o - a_o)}{(\sigma_o - a_o)^2 + \beta_o^2}. \quad (102)$$

The conditional equation corresponding to Eq. 93 is

$$\frac{g}{b} = \frac{1 + \left(\frac{\beta_o}{a_o} \right) \left(\frac{b - g'\beta_o}{b'\beta_o} \right)}{\frac{b - g'\beta_o}{b'\beta_o} - \frac{\beta_o}{a_o}}. \quad (103)$$

In this case,

$$\sigma_o = \frac{a_o^2 + \beta_o^2}{a_o \left(1 + \frac{g \beta_o}{b a_o} \right)} \quad (104)$$

and since g is not necessarily positive, a question arises as to whether σ_o is always positive and finite. Clearly, there is no difficulty when g is positive. When g is negative, it is required that

$$\frac{g \beta_o}{b a_o} < 1. \quad (105)$$

This inequality is assured by the fact that $\arg(Y_1) < \arg(\lambda)$ over the whole left half-plane, exclusive of the negative-real axis, and therefore σ_o is always positive and finite.

The parameter \underline{a} is given by

$$a = \frac{-b' \left[(\sigma_o - a_o)^2 + \beta_o^2 \right]}{2\beta_o g' (\sigma_o - a_o) + b' \left[(\sigma_o - a_o)^2 - \beta_o^2 \right]}. \quad (106)$$

Here, as before, the function $w(a)$ is investigated:

$$w(a) = \frac{1 - a}{1 + a} = \frac{g' + b' \left(\frac{\sigma_o - a_o}{\beta_o} \right)}{g' - b' \left(\frac{\beta_o}{\sigma_o - a_o} \right)}. \quad (107)$$

Under the assumption that Eq. 103 is satisfied, Eq. 107 leads to the same criteria as before, that is, Eqs. 97 and 98. Since the inequalities implied by these equations are valid over the whole left half-plane, \underline{a} is guaranteed to be positive and finite.

E. Removing the Bridge

It is now apparent that imaginary zeros may be regarded simply as special cases of complex zeros, which is, of course, exactly as it should be. In view of this fact, the remainder of the present demonstration will be essentially confined to the complex zeros; its application to imaginary zeros is quite obvious.

The next step is to show that "removing" the bridge will leave a remainder of reduced degree that is both p-r and RC. As a preliminary, consider the behavior of y_1 and $1/y_{12}$ along the negative-real axis. y_1 itself is p-r, and has a zero at the origin and a simple pole at $\lambda = -\sigma_o$. It is finite and positive for large λ . $1/y_{12}$ has a simple zero at $\lambda = -\sigma_o$, a single maximum given by

$$\lambda = -\sigma_o + \sqrt{(\sigma_o - a_o)^2 + \beta_o^2} \quad (108)$$

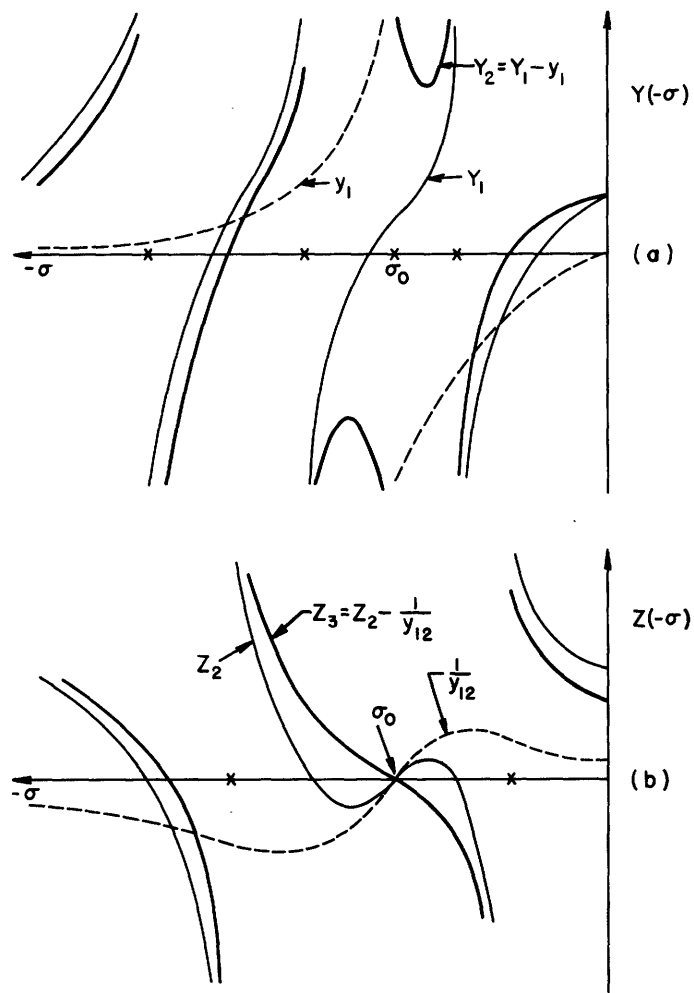


Fig. 10
The transition from Y_1 to Z_3 .

and a single minimum at

$$\lambda = -\sigma_0 - \sqrt{(\sigma_0 - a_0)^2 + \beta_0^2}. \quad (109)$$

It tends to zero as λ tends to infinity.

The sketches in Fig. 10 will help to make the argument clear. The light, solid lines in Fig. 10 show the behavior of a typical admittance, Y_1 ; the dotted lines show the behavior of y_1 ; and the heavy, solid lines show the remainder after removing y_1 , i.e. $Y_2 = Y_1 - y_1$. Since y_1 has a simple pole at $\lambda = -\sigma_0$, y_2 must have exactly one more pole and one more zero than Y_1 . It may be seen from a study of the figure that only one of the original zeros of Y_1 is removed from the negative-real axis; the remaining ones are only shifted. (The poles of Y_1 are not moved by removing y_1 .) Also, since y_1 has been chosen in such a way that two of the zeros of Y_2 comprise a conjugate pair, these are the only zeros of Y_2 that are not on the negative-real axis.

In Fig. 10b the operation of removing $1/y_{12}$ from $Z_2 = 1/Y_2$ is illustrated. Here, the light, solid lines show Z_2 ; the dotted lines show $1/y_{12}$; and the heavy, solid lines show Z_3 , the remainder after removing $1/y_{12}$. Z_2 and $1/y_{12}$ have a common zero at $\lambda = -\sigma_0$. The fact that the residue of y_{12} in this pole is less than the corresponding residue of y_2 , (compare Eqs. 52 and 55) guarantees that Z_2 and $1/y_{12}$ will intersect in the manner shown. It is clear from the diagram that the two zeros of Z_2 on either side of σ_0 are removed; the possibility of the occurrence of any other new zeros is eliminated by the assurance of a cancellation at this stage. Consequently, Z_3 is p-r and RC as evidenced by the arrangement of its poles and zeros. It contains a zero at $-\sigma_0$, and when this is removed from its reciprocal to complete the zero section, the final remainder must be RC. Moreover, the final remainder has two less poles and two less zeros than Y_1 .

It is now possible to state that Eq. 103 gives a condition on Y_1 that is both necessary and sufficient for the success of the whole process. Only a slight modification is necessary in order to determine the corresponding condition on Z_1 .

Instead of considering Z_1 directly, consider λZ_1 . According to the discussion in section III-F it is only necessary to regard λZ_1 as the impedance of an RL network that is the dual of the RC admittance. Then all of the preceding proof applies to λZ_1 ; or, merely changing the symbols according to the following definitions

$$\begin{aligned} (\lambda Z_1)_{\lambda=\lambda_0} &= r + jx \\ (\lambda Z_1)'_{\lambda=\lambda_0} &= r' + jx' \end{aligned}$$

Eq. 103 becomes

$$\frac{r}{x} = \frac{1 + \left(\frac{\beta_0}{a_0}\right) \left(\frac{x - r'\beta_0}{x'\beta_0}\right)}{\frac{x - r'\beta_0}{x'\beta_0} - \frac{\beta_0}{a_0}}. \quad (110)$$

Thus, if (λZ_1) and its derivative satisfy Eq. 110, a zero section may be designed on the impedance basis. It is not necessary, however, to design an RL network and transform it to RC. For, if (λZ_1) and $(\lambda Z_1)'$ satisfy Eq. 110, then Y_1 must satisfy Eq. 103 and the admittance basis may be used. Still another alternative is to proceed on the impedance basis using Z_1 itself and the T-section representation derived from Eqs. 61-63 (or Eqs. 58-60 when $\lambda_0 = j\omega_0$).

This seeming digression to the impedance case is in order because the final step of the proof reveals only that it is always possible to manipulate a given function so that one of the two conditional equations is satisfied a priori. Specifically, Eq. 93 may be used to guide the removal of a shunt branch in order to fulfill the conditions. However, it is very often necessary to remove a series branch instead, and when this is so, the procedure must be on the impedance basis, but necessarily only until the series branch has been determined.

F. The Proof Completed

Once more, the point of departure is the Foster expansion of Y_{11} ;

$$Y_{11} = k_\infty \lambda + k_0 + \sum_{v=1}^n \frac{k_v \lambda}{\lambda + \sigma_0}.$$

In terms of this expression, y , b , g' , and b' must be computed for an arbitrary point, $\lambda_0 = -a_0 + j\beta_0$, and these results substituted in Eq. 103. The details of this computation are extensive and will be omitted, but such partial results as are useful for future reference will be included. Also, in the subsequent discussion the term "residues" will be considered to include k_0 in the expression for Y_{11} . If this departure from rigorous terminology needs any justification, it may be pointed out that k_0 is, in fact, the residue in the pole at the origin of Y_{11}/λ .

To proceed, the following results are listed without comment.

$$g = -a_0 k_\infty + k_0 + \sum_{v=1}^n \frac{k_v (a_0^2 + \beta_0^2 - \sigma_v \sigma_0)}{(\sigma_v - a_0)^2 + \beta_0^2} \quad (111)$$

$$b = \beta_0 k_\infty + \sum_{v=1}^n \frac{k_v \sigma_v \beta_0}{(\sigma_v - a_0)^2 + \beta_0^2} \quad (112)$$

$$g' = k_\infty + \sum_{v=1}^n \frac{\sigma_v k_v [(\sigma_v - a_0)^2 - \beta_0^2]}{[(\sigma_v - a_0)^2 + \beta_0^2]^2} \quad (113)$$

$$b' = -2 \sum_{v=1}^n \frac{k_v \sigma_v \beta_0 (\sigma_v - a_0)}{[(\sigma_v - a_0)^2 + \beta_0^2]^2} \quad (114)$$

$$\frac{b/\beta_o - g'}{-b'} = \frac{\sum_{v=1}^n \frac{k_v \sigma_v \beta_o}{[(\sigma_v - a_o)^2 + \beta_o^2]^2}}{\sum_{v=1}^n \frac{k_v \sigma_v (\sigma_v - a_o)}{[(\sigma_v - a_o)^2 + \beta_o^2]^2}}. \quad (115)$$

Now, let

$$(\sigma_v - a_o)^2 + \beta_o^2 = \rho_v^2 \quad (116)$$

and substitute in Eq. 103 to obtain

$$\frac{k_\infty - \frac{k_o a_o}{a_o^2 + \beta_o^2} + \sum_{v=1}^n \frac{k_v (\sigma_v - a_o)}{\rho_v^2}}{\frac{k_o}{a_o^2 + \beta_o^2} + \sum_{v=1}^n \frac{k_v}{\rho_v^2}} = \frac{\sum_{v=1}^n \frac{k_v \sigma_v (\sigma_v - a_o)}{\rho_v^4}}{\sum_{v=1}^n \frac{k_v \sigma_v}{\rho_v^4}}. \quad (117)$$

Equation 117 yields the following results:

$$k_\infty \sum_{v=1}^n \frac{k_v \sigma_v}{\rho_v^4} - k_o \sum_{v=1}^n \frac{k_v \sigma_v^2}{\rho_v \omega_o^2} + \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \frac{k_j k_k (\sigma_j - \sigma_k)^2 (\sigma_j \sigma_k - \omega_o^2)}{\rho_j^4 \rho_k^4} = 0. \quad (118)$$

Equation 118 expresses in general form the condition under which it is possible to determine a zero section on the Y-basis. Although Y_{11} as originally given does not ordinarily satisfy this equation, it is possible either (a) to make $Y_{11} - Y_o$ satisfy it by suitable choice of Y_o , or (b) if this is not possible, to make $\lambda Z_{11} - \lambda Z_o$ satisfy an equation of exactly the same form by suitable choice of Z_o . Note that Eqs. 103 and 118 are equivalent, but whereas Eq. 103 is stated in terms of the values of Y_1 and its derivative at $\lambda = \lambda_o$, Eq. 118 places in evidence the contribution of each pole.

The Foster expansion of λZ_{11} is identical in form with the expansion of Y_{11} , and the expression corresponding to Eq. 118 for the impedance case differs from Eq. 118 only in that the σ_v 's are the poles of Z_{11} instead of the poles of Y_{11} , and the k 's are the residues in these poles. Furthermore, the k 's are the same whether one considers Z_{11} or λZ_{11} , and Eq. 118 therefore represents the conditional equation for either Y_{11} or Z_{11} according to whether the σ_v 's and the k 's are associated with Y_{11} or Z_{11} , respectively.

The situation at this point may appear awkward because the conditional equation represents a constraint on Y_1 , whereas Y_1 cannot be found until Y_o is determined. However, this is as it should be. The fundamental idea is to evaluate Eq. 118 for Y_{11} except that one or more of the k 's is assumed unspecified. If a solution to Eq. 118 can be obtained by making these k 's less than those of Y_{11} , the difference between the required values and those of Y_{11} may be assigned to Y_o . For example, suppose k_∞ is

left unspecified, but all other values pertaining to Y_{11} are substituted in Eq. 118. One then has an expression of the form

$$ak_{\infty 1} - b = 0. \quad (119)$$

The subscript "1" was added to show that the residue is associated with Y_1 . Equation 119 yields

$$k_{\infty 1} = \frac{b}{a} \quad (120)$$

and therefore, one has

$$Y_0 = \lambda \left(k_{\infty 11} - \frac{b}{a} \right). \quad (121)$$

It is evident that the choice of Y_0 may in some cases involve considerable experimentation, especially if one insists on finding the simplest configuration. However, it is not necessary that Y_0 be of simple configuration, but only that it be part of Y_{11} , and this last requirement can be met, as evidenced by the following considerations.

The coefficient of k_{∞} in Eq. 118 is always positive, the coefficient of k_0 is always negative, while the double summation may be positive or negative. Thus, if neither k_0 nor k_{∞} is missing, the equation takes the form

$$ak_{\infty} - bk_0 + c = 0 \quad (122)$$

and a solution can always be found by reducing some of the residues. If k_0 should be missing and c positive, then one considers Z_{11} instead of Y_{11} and since k_0 cannot be missing from both expressions, again a solution may be found. If all of the terms comprising c were positive in the expansion for Y_{11} , then at least some of them must be positive in the expansions for Z_{11} (because of the separation property of poles and zeros of Y_{11}), and so even though the sign of c might change, it will be true for either Z_{11} or Y_{11} that some positive and some negative terms will appear. By reducing some or all of the residues in the group of terms making the larger contribution, a solution can always be found without completely removing any one pole. It may be concluded, therefore, that in every case, a single operation on either Y_{11} or Z_{11} resulting in the removal of either a shunt or a series branch will leave a remainder that satisfies the conditional equation. The removed branch, however, does not introduce any transmission zeros.

It should not be assumed that the determination of Y_0 or Z_0 is necessarily a tedious operation. There are a number of factors that serve to guide the choice as to whether one should start with Z_{11} or Y_{11} , whether to remove a shunt resistance or an RC branch, etc. and these factors will be pointed out in the next section. Experience suggests that here, as in the case of real zeros, Y_0 and Z_0 need not be more complicated than single RC series or parallel configurations, but proof of this is lacking.

G. Example

As an example illustrating the general theorem, assume as given

$$Z_{11} = \frac{(\lambda + 0.5)(\lambda + 2)}{10(\lambda + 0.42020)(\lambda + 1)(\lambda + 2.3798)} \quad (123)$$

and

$$Z_{12} = \frac{h(\lambda^2 + 4.571)}{10(\lambda + 0.42020)(\lambda + 1)(\lambda + 2.3798)} \quad (124)$$

1. It may be seen by inspection of Z_{11} that the conditional equation will contain only negative terms and so it is not possible to proceed on the impedance basis. Accordingly, Z_{11} is inverted and the Foster expansion is made, yielding

$$Y_{11} = 10\lambda + 10 + \frac{\lambda}{\lambda + 0.5} + \frac{2\lambda}{\lambda + 2.0} \quad (125)$$

Here,

$$\begin{aligned} k_{\infty} &= 10 & k_1 &= 1.0 \\ k_0 &= 10 & k_2 &= 2.0 \\ \sigma_1 &= 0.5 & \omega_0 &= 2.138. \\ \sigma_2 &= 2.0 \end{aligned}$$

Substituting these values into the summations required in Eq. 118 leads to the following results:

$$\begin{aligned} \sum_{v=1}^2 \frac{k_v \sigma_v}{\rho_v^4} &= 0.075963 \\ \sum_{v=1}^2 \frac{k_v \sigma_v^2}{\rho_v^4 \omega_0^2} &= 0.026177 \\ \frac{1}{2} \sum_{j=1}^2 \sum_{k=1}^2 \frac{k_j k_k (\sigma_j \sigma_k)^2 (\sigma_j \sigma_k - \omega_0^2)}{\rho_j^4 \rho_k^4} &= 0.0094116. \end{aligned}$$

It is apparent that one or both of k_{∞} and k_0 must be reduced and the conditional equation therefore may be written as

$$0.075963 k_{\infty 1} = 0.026177 k_{01} + 0.0094116.$$

Further investigation reveals that only k_{∞} must necessarily be reduced, but if desired, k_{01} may be reduced, or even set equal to zero. Setting $k_{01} = k_{\infty 1}$ yields

$$k_{\infty 1} = k_{01} = 0.18904.$$

Therefore

$$Y_0 = (\lambda + 1)(10 - 0.18904) = 9.8110\lambda + 9.8110 \quad (125a)$$

and

$$Y_1 = 0.18904\lambda + 0.18904 + \frac{\lambda}{\lambda + 0.5} + \frac{2\lambda}{\lambda + 2}. \quad (126)$$

From this point, the design of the zero section is completely straightforward.

2. Substituting $\lambda = j\omega_0$ into Eq. 126 gives

$$Y_1(\lambda_0) = 2.2038 + j 1.6237 = g + jb.$$

Use of Eqs. 92 and 87 then yields

$$\sigma_0 = 1.5752$$

and

$$c_0(1+a) = 0.75945$$

whence

$$y_1 = \frac{3.4001\lambda}{\lambda + 1.5752}. \quad (127)$$

Removing y_1 leaves

$$Y_2 = Y_1 - y_1 = \frac{(\lambda^2 + 4.571)(0.18904\lambda^2 + 0.55935\lambda + 0.065146)}{(\lambda + 1.5752)(\lambda^2 + 2.5\lambda + 1)}.$$

3. Multiplying $1/Y_2$ by y_{12}/c_0 and substituting $\lambda = j\omega_0$ into the result yields*

$$\frac{1}{c_0} = 4.4695$$

and therefore

$$y_{12} = \frac{0.22374(\lambda^2 + 4.571)}{\lambda + 1.5752}. \quad (128)$$

The remainder, after removing $1/y_{12}$, is

$$Z_3 = \frac{(\lambda + 1.5752)(0.15507)}{0.18904\lambda^2 + 0.55935\lambda + 0.065146}. \quad (129)$$

4. Substituting the value found above for c_0 into Eq. 87 gives

$$a = 2.3943$$

and this result leads to

$$y_2 = \frac{1.4201\lambda}{\lambda + 1.5752}. \quad (130)$$

* Compare this procedure with the equivalent step in the example of section III-C.

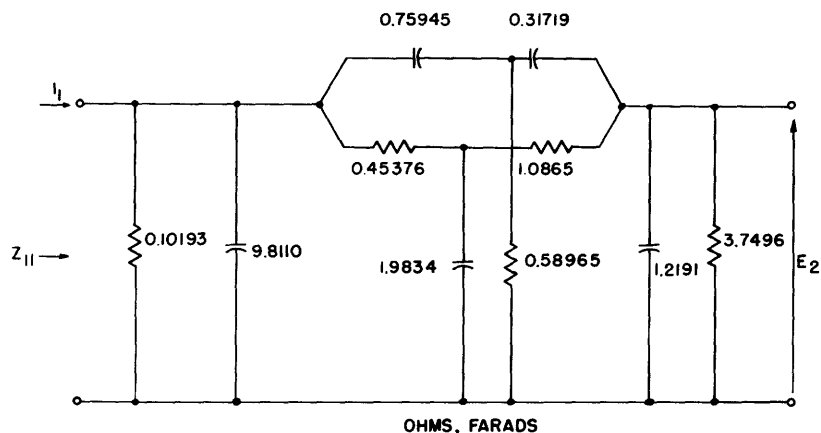


Fig. 11

The network designed in the example of section IV-G.

Finally removing y_2 from $1/Z_3$ yields

$$Y_4 = 1.2191\lambda + 0.2667. \quad (131)$$

The design of the zero section is completed by substituting the values of σ_o , c_o , and a , found above, into the formulas given in Fig. 2. The final network is shown in Fig. 11.

The choice of k_{o1} in step 1 above, when a choice exists, is entirely incidental to the synthesis procedure itself. This freedom is useful because it permits the designer to control the relative impedance levels at the input and output terminals of the bridge. This question will be considered in section V.

H. The General Procedure

The procedure in the general case may now be formulated. Assume as given a transfer function, $H(\lambda)$, having only simple poles on the negative-real axis including the origin and infinity and zeros anywhere in the left half-plane or on the imaginary axis. The zero may be of any order, but complex or imaginary zeros must occur in conjugate pairs.

1. Associate with $H(\lambda)$ an RC driving-point function, $F(\lambda)$. If $H(\lambda)$ is a transfer impedance or admittance, $F(\lambda)$ must have the same poles as $H(\lambda)$ but its zeros may be assigned arbitrarily within the requirement that $F(\lambda)$ be RC. If, on the other hand, $H(\lambda)$ is a transfer ratio, the zeros of $F(\lambda)$ must be the poles of $H(\lambda)$, and the poles of $F(\lambda)$ may be chosen arbitrarily within the RC requirement. The distinction between transfer admittance, transfer impedance, etc. is made by controlling the behavior of the terminating sections of the network.
2. Develop $F(\lambda)$ in unbalanced form. Realize real zeros of $H(\lambda)$ through the use of the zero-shifting procedure of section II and realize complex and imaginary

zeros of $H(\lambda)$ through the use of the method developed in this and the preceding section, treating zeros of multiple order as though they were distinct, that is, realizing them one at a time.

The material essential for the application of the new procedure to synthesis problems has now been presented. In the next section certain special techniques that have been found practically useful will be discussed briefly and additional illustrative examples will be given.

V. SPECIAL TECHNIQUES

A. Realizing a Single Pair of Complex or Imaginary Zeros

When $F(\lambda)$ is of the second degree only Eq. 118 reduces to a very simple form. It is then readily found that a section terminated in a short circuit is realizable if, and only if, the zeros of its input admittance are inverse to the "zero circle," i.e. the circle on which λ_0 lies. (This fact may alternatively be deduced directly from Eq. 27.) Likewise, a section terminated in an open circuit is realizable if, and only if, the zeros of its input impedance are inverse to the zero circle. In the first case, the input admittance necessarily has a pole at infinity; in the second case, the input impedance necessarily has a pole at the origin. It is not difficult to show that, given a function

$$F(\lambda) = \frac{a_2 \lambda^2 + a_1 \lambda + a_0}{b_2 \lambda^2 + b_1 \lambda + b_0} \quad (132)$$

it is always possible to shift the zeros and the poles to the proper positions to realize either an open-circuit or a short-circuit termination. The techniques involved are essentially the same as those used for zero shifting on the negative-real axis. Here, however, the fact that one may always remove a part of the residue in any pole without introducing a transmission zero is put to good use.

A special situation arises when the poles of $F(\lambda)$ are inverse to the zero circle, for obviously, in this case, the zeros cannot be. This difficulty is remedied by the simple expedient of making a preliminary shift of the poles and then proceeding as before.

The following example will illustrate the method. Consider the functions

$$Z_{11} = \frac{(\lambda+1)(\lambda+3)}{(\lambda+0.5)(\lambda+2)} = \frac{\lambda^2 + 4\lambda + 3}{\lambda^2 + 2.5\lambda + 1} \quad (133)$$

and

$$Z_{12} = \frac{h(\lambda^2 + 1)}{(\lambda+0.5)(\lambda+2)} \quad (134)$$

1. These functions fall under the special case, so one begins by shifting a pole of impedance to $\lambda = -0.25$ by removing from $1/Z_{11}$ a shunt conductance, $G_1 = 0.2121$. Designating the remainder Y_1 , one has

$$Y_1 = \frac{1}{Z_{11}} - 0.2121 = \frac{0.8858(\lambda + 0.25)(\lambda + 1.847)}{\lambda^2 + 4\lambda + 3}. \quad (135)$$

2. Invert Y_1 and remove enough of the residue in the pole at $\lambda = -0.25$ to shift the zero circle of Z_1 to $|\lambda| = 1$ as required by Z_{12} .

$$Z_2 = Z_1 - \frac{1}{c_2(\lambda + 0.25)} = \frac{1.269}{\lambda + 0.25} \frac{\lambda^2 + 4\lambda + 3}{\lambda + 1.846} - \frac{1}{c_2}.$$

c_2 is fixed by requiring that

$$3 - 1.846/c_2 = 1$$

from which one obtains

$$c_2 = 0.9231.$$

Completing the indicated operation then yields

$$Z_2 = \frac{1.269(\lambda^2 + 2.917\lambda + 1)}{(\lambda + 0.25)(\lambda + 1.846)}. \quad (136)$$

3. The last step is to shift a pole of impedance into the origin. This is done by inverting Z_2 and removing a shunt conductance of 0.3636 mhos. Inverting once more then gives

$$Z_3 = \frac{2.357(\lambda^2 + 2.917\lambda + 1)}{\lambda(\lambda + 1.393)}. \quad (137)$$

Since Z_3 is the open-circuit input impedance of the zero section, its transfer impedance is

$$z_{12} = \frac{2.357(\lambda^2 + 1)}{\lambda(\lambda + 1.393)}. \quad (138)$$

Equations 137 and 138 provide all the data necessary to complete the bridge design.

Strictly speaking, the only restriction on the location of the pole in the first shift is that it must be chosen so as to make b_0/b_2 differ from ω_0^2 . Otherwise, it may be adjusted to control other characteristics of the network, such as the spread of element values, and so on. The preliminary shift may be made for this purpose even though the special case does not appear. The original function sets certain limits on what can be done, but very often much freedom is available.

It is interesting to note that in the example given, the constant multiplier Z_{12} will be unity regardless of the location of the pole in the first shift, provided no additional unnecessary operations are performed. On the other hand, if the first shunt branch represents the internal impedance of the source, the transfer voltage ratio differs from Z_{12} only by a constant multiplier.

The magnitude of the constant multiplier is affected by the placement of the pole.

It will be a maximum when the resistance of the first shunt branch is a minimum. This occurs when a pole of impedance is shifted into the origin in the first step. It is then subsequently necessary to remove a series capacitance before making the zero. If this capacitance should be undesirable, one merely avoids the origin, thereby insuring that the capacitance will be shunted by a resistance.

In the above example, the maximum possible constant multiplier of the transfer voltage ratio is

$$K_{\max} = \frac{Z_{11}(\infty)}{Z_{11}(0)} = \frac{1}{3}.$$

This result is arrived at by noting that the first shunt branch is actually $Z_{11}(0)$, and $Z_{12}(\infty) = Z_{12}(0) = Z_{11}(\infty)$. In the example as given

$$K = 0.2299.$$

B. Constructing the Driving-Point Function

It has been emphasized at several points in the development of the procedure that it places no special demands on the driving-point functions because the preparatory steps are designed to meet the requirements of the zero sections in all cases. Assuming that the poles of $F(\lambda)$ are prescribed by $H(\lambda)$, its zeros may be chosen to satisfy some requirement not explicitly given by $H(\lambda)$. It is ordinarily desirable to maintain the constant multiplier on $H(\lambda)$ as large as possible, to maintain the number of elements as small as possible, and to maintain the spread of element values as small as possible. Very little of a precise nature is known about the relation between these factors and the choice of zero locations. Although the present procedure leads to substantially less elements than the parallel-ladders procedure, it may not lead to the smallest possible number of elements, even if the initial driving-point impedance is considered as part of the given data. On the other hand, it is apparent from the example of the previous section that some advantage may be gained by deliberately introducing excess elements. Moreover, it is evident that the constant multiplier on $H(\lambda)$ is not necessarily fixed by the choice of $F(\lambda)$ nor is it related in a simple way to the number of elements, even in a very simple case. Thus, neither of the first two items mentioned above plays a dominant role in the choice of the zeros of $F(\lambda)$.

A large spread of element values is undesirable not only because it may lead to impractical networks, but also because it results in a serious deterioration in the accuracy of computations as the development proceeds. In general, very close spacing of poles and close spacing between zeros and poles tend to produce large spreads in element values, and the converse is also true. One good way, therefore, to fix the zeros of $F(\lambda)$ is to interpolate them approximately midway between the poles. One additional internal zero may be required, since account must be taken of the fact that if a driving-point impedance has a zero at infinity, the associated transfer impedance must have a zero at infinity, and if a driving-point admittance has a zero at the origin,

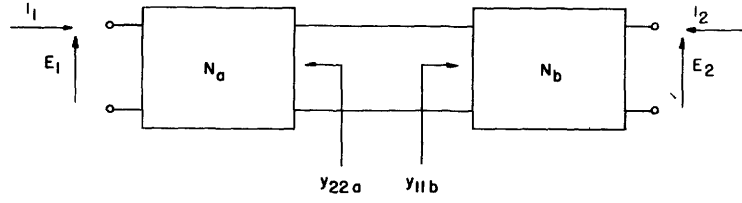


Fig. 12
Partitioned RC network.

the associated transfer admittance must likewise have a zero at the origin.

Another very useful way to fix the zeros is based on the partitioning theorem referred to in section I-B.

Consider a network comprising two sections in cascade as shown in Fig. 12. Let

$$Y_{12} = \left. \frac{-I_2}{E_1} \right|_{E_2=0}$$

then it is easy to show that

$$Y_{12} = \frac{y_{12a}y_{12b}}{y_{22a} + y_{11b}} \quad (139)$$

in which y_{12a} and y_{12b} are the similarly defined short-circuit transfer admittances of the sections N_a and N_b respectively, and y_{22a} and y_{11b} are the short-circuit input admittances indicated in the figure. Other expressions corresponding to Eq. 139 may be derived without difficulty, relating various transfer functions of the over-all network to those of the two sections. Use of the theorem reduces the solution of one complex problem to the solution of two simpler problems. If the original function has only two pairs of complex or imaginary zeros, these are assigned to opposite sections. If all other zeros are removed first, it is frequently possible to make use of the methods in section V-A to obtain the last sections thereby saving a great deal of computational labor.

The simplest way of using the theorem is as follows. Given

$$Y_{12} = \frac{P_{12}}{Q_{12}} = \frac{P_{12}}{(\lambda + \sigma_1)(\lambda + \sigma_2)(\lambda + \sigma_3)(\lambda + \sigma_4) + \dots (\lambda + \sigma_n)} \quad (140)$$

in which $\sigma_1 < \sigma_2 < \sigma_3 \dots < \sigma_n$ and P_{12} is a polynomial defining the zeros of Y_{12} . By taking alternate zeros of Q_{12} as the poles of both y_{22a} and y_{11b} , and the remainder as their zeros, Eq. 140 can be put in the following form:

$$Y_{12} = \frac{P_{12}}{\frac{[(\lambda + \sigma_2)(\lambda + \sigma_4) \dots (\lambda + \sigma_n)] [(\lambda + \sigma_2)(\lambda + \sigma_4) \dots (\lambda + \sigma_n)]}{(\lambda + \sigma_1)(\lambda + \sigma_3) \dots (\lambda + \sigma_1)(\lambda + \sigma_3) \dots} + \frac{2(\lambda + 2)(\lambda + \sigma_4) \dots (\lambda + \sigma_n)}{2(\lambda + \sigma_2)(\lambda + \sigma_4) \dots (\lambda + \sigma_n)}} \quad (141)$$

The identification of Eq. 141 with Eq. 139 is made complete by distributing the zeros of P_{12} between y_{12a} and y_{12b} .

One advantage of this procedure is that the poles and zeros of the component sections are separated from each other as far as possible. Two of its many variations are used in examples 1 and 2 of section V-D.

An interesting feature of the partitioning theorem in the form used above is that it provides a unique way of deriving the driving-point functions from the transfer function.

C. Controlling Terminations and Impedance Levels

When the development of the network begins at one end, the terminating impedance is fixed by first removing a shunt resistance or a series resistance, as required, and constructing $F(\lambda)$ accordingly. Thus, if a series resistance must be removed, Z_{11} is not permitted to have a zero at infinity, and if a shunt resistance must be removed, Z_{11} is not permitted to have a pole at the origin. Analogous statements govern the construction of Y_{11} . A saving of elements results when removing the first branch also serves as the preparatory step for making the first zero.

At the receiving end of the network the methods of section V-A may be used to provide either an open-circuit or a short-circuit termination. If some intermediate resistance is desired, one merely avoids both of the limiting cases.

When the development of the network begins at some interior point, as it does when the partitioning theorem is used, both ends of the over-all network are treated in the same manner as that described above for the receiving end. In this case the partitioning must take into account the requirements at the terminals. The addition of surplus factors to Y_{12} before partitioning is sometimes useful in this connection.

D. Examples

1. Example illustrating the partitioning theorem

It is desired to synthesize to within a constant multiplier the transfer impedance

$$Z_{12} = \frac{(\lambda^2 + 4.571)(\lambda^2 + 0.6285\lambda + 0.13)}{(\lambda + 0.4)(\lambda + 0.625)(\lambda + 1)(\lambda + 1.6)(\lambda + 2.5)}.$$

The termination at the receiving end of the network must be an open circuit; and at the sending end, a shunt resistance.

The form of the partitioning theorem appropriate to this problem is

$$Z_{12} = \frac{z_{12a}z_{12b}}{z_{22a} + z_{11b}} = \frac{P_{12}}{Q_{12}}.$$

In order to provide the "b" network (the section on the right) with a pole of impedance at the origin as required by the open-circuit termination, the factor representing the smallest zero of Q_{12} is separated into two components

$$\lambda + 0.4 = 0.2\lambda + 0.8(\lambda + 0.5)$$

and Q_{12} is expanded accordingly to give

$$Q_{12} = 0.2\lambda(\lambda + 0.625)(\lambda + 1)(\lambda + 1.6)(\lambda + 2.5) \\ + 0.8(\lambda + 0.5)(\lambda + 0.625)(\lambda + 1)(\lambda + 1.6)(\lambda + 2.5).$$

After dividing the numerator and the denominator of Z_{12} by

$$\lambda(\lambda + 0.625)^2(\lambda + 1.6)^2$$

the denominator appears as follows

$$\frac{0.2(\lambda + 1)(\lambda + 2.5)}{(\lambda + 0.625)(\lambda + 1.6)} + \frac{0.8(\lambda + 0.5)(\lambda + 1)(\lambda + 2.5)}{\lambda(\lambda + 0.625)(\lambda + 1.6)}.$$

The transmission zeros at infinity and at $+j 2.138$ are assigned to the "b" network, and since the constant at infinity appearing in the expression on the right serves no useful purpose, it is removed and associated with the "a" network.

The following identifications are then made.

$$z_{12a} = \frac{h(\lambda^2 + 0.6285\lambda + 0.13)}{\lambda^2 + 2.225\lambda + 1}$$

$$z_{22a} = \frac{\lambda^2 + 2.48\lambda + 1.3}{\lambda^2 + 2.225\lambda + 1}$$

$$z_{12b} = \frac{h(\lambda^2 + 4.571)}{\lambda(\lambda^2 + 2.225\lambda + 1)}$$

and

$$z_{11b} = \frac{1.42\lambda^2 + 2.6\lambda + 1}{\lambda(\lambda^2 + 2.225\lambda + 1)}.$$

The "b" network is disposed of first. A ladder development is begun by removing a shunt capacitance from $1/z_{11b}$, thereby realizing a transmission zero at infinity. The reciprocal of the remainder is the input impedance, Z_{11} , of a two terminal-pair network that must produce the imaginary zeros. This impedance is

$$Z_{11} = \frac{3.604(\lambda^2 + 1.831\lambda + 0.7042)}{\lambda^2 + 0.7507\lambda}$$

The zero circle of Z_{11} is now shifted in the manner of section V-A by removing a series resistance, $R_o = 3.0487$. (A shift in the opposite direction is made by removing part of the pole at the origin. The direction in which the shift must be made at this point can be determined from the zeros of z_{11b} inasmuch as it is not affected by removing the condenser.) The remainder is

$$Z_1 = Z_{11} - R_o = \frac{0.5552(\lambda^2 + 7.763\lambda + 4.571)}{\lambda(\lambda + 0.7507)}$$

Since $Z_1 \equiv z_{11}$, the following data, from which the element values for the bridge may be computed through the use of the formulas given in section III, are available.

$$R_\infty = 0.5552$$

$$\sigma_{oz} = 0.7507$$

$$\frac{\omega_o^2}{\sigma_{oz}} = 6.08922$$

$$(1 + a_z) \left(\sigma_{oz} + \frac{\omega_o^2}{\sigma_{oz}} \right) = 7.763.$$

The final results are shown in Fig. 13 where this portion of the network is designated N_b .

Returning to the "a" network, a variation of the preparatory step will be employed in which Eq. 103 is used directly for the determination of Y_o . Here,

$$z_{22a} \equiv Z_{11} = \frac{\lambda^2 + 2.48\lambda + 1.3}{\lambda^2 + 2.225\lambda + 1}$$

and it is noted that $\sigma_1 \sigma_2 > \omega_o^2$ for both Z_{11} and Y_{11} , and reference to Eq. 118 shows that the synthesis should be carried out on the admittance basis. Direct substitution of $\lambda_o = -0.3143 + j0.1768$ into Y_{11} and its derivative leads to

$$Y_{11}(\lambda_o) = g_{11} + jb_{11} = 0.6820 + j0.1003$$

and

$$\frac{b_{11} - g'_{11} \beta_o}{b'_{11} \beta_o} = -0.3898.$$

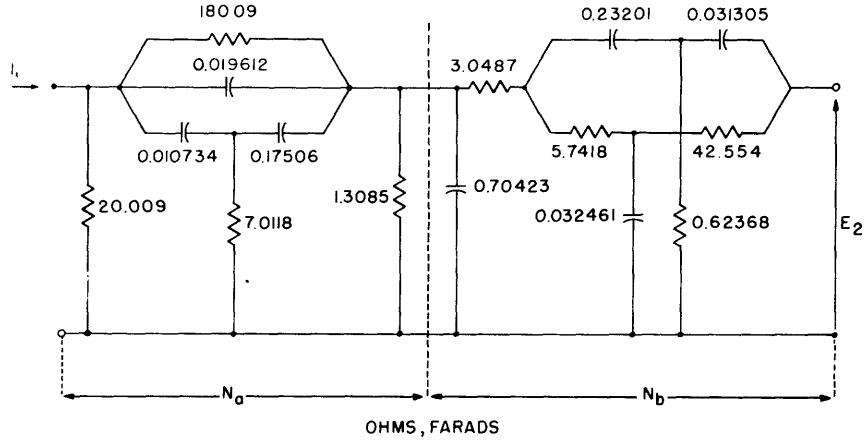


Fig. 13

Network representation of the function in example 1.

Since removing G_o does not change b_{11} , g'_{11} , and b'_{11} , Eq. 103 may be written as

$$\frac{g_{11} - G_o}{b_{11}} = \frac{g}{b} = \frac{1 + \left(\frac{\beta_o}{a_o}\right) \left(\frac{b_{11} - g'_{11} \beta_o}{b'_{11} \beta_o}\right)}{\frac{b_{11} - g'_{11} \beta_o}{b'_{11} \beta_o} - \frac{\beta_o}{a_o}}.$$

Making the appropriate substitutions into this expression gives

$$\frac{g}{b} = -0.8197$$

$$g_{11} - G_o = -0.08223$$

and

$$G_o = 0.6820 + 0.08223 = 0.7642.$$

Use of Eq. 104 now gives

$$\sigma_o = 0.7677$$

and Eq. 55 yields

$$y_1 = \frac{0.1751\lambda}{\lambda + 0.7676}.$$

Removing G_o and y_1 from Y_{11} enables c_o and then a to be determined in the succeeding step. These are,

$$c_o = 0.03279$$

and

$$a = 16.31.$$

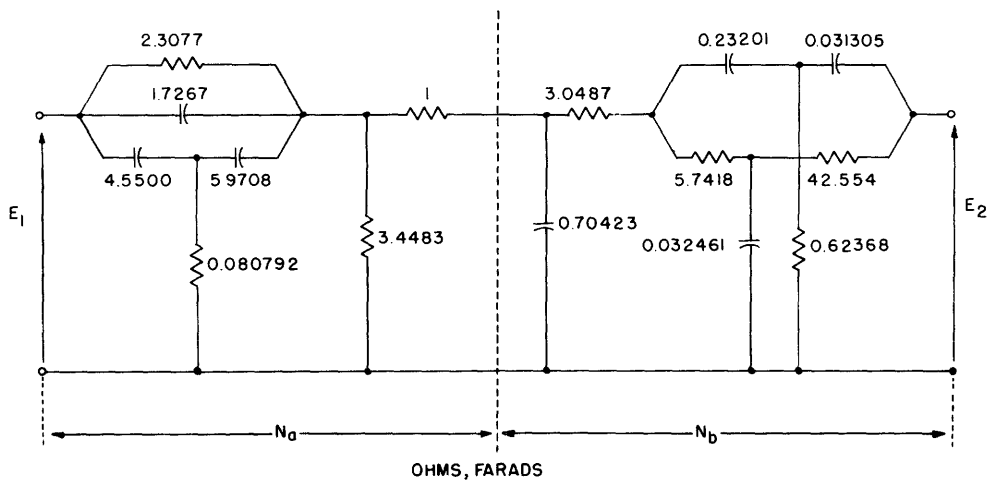


Fig. 14
Network representation of the function in example 2.

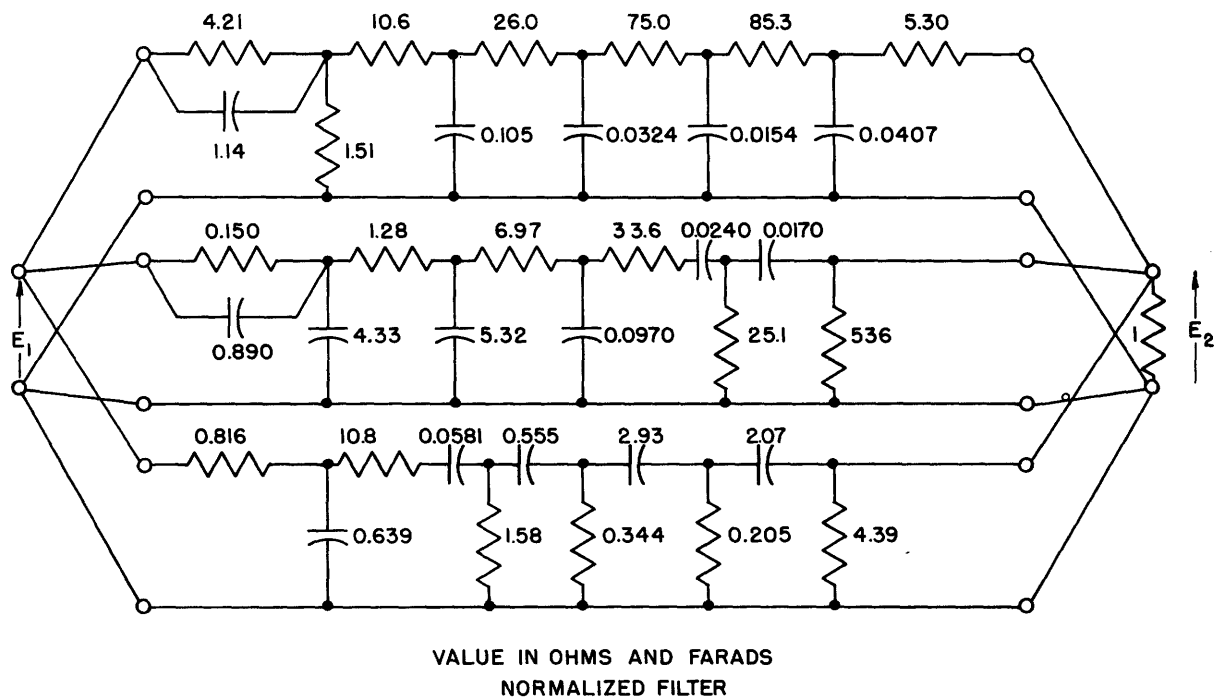


Fig. 15
Parallel-ladders realization of the function in example 2.

The remainder after removing the zero section is

$$Y_4 = 0.04998.$$

Since

$$\sigma_0 - \frac{\omega_0^2}{2a_0} = 0.5608 > 0$$

the zero section corresponds to Case C and the network has the configuration shown in Fig. 8. The portion of the complete network incorporating this section is designated N_a in Fig. 13.

2. Realization of a transfer voltage ratio

A modification of the procedure used in example 1 yields a network suitable for use when the transfer function represents a transfer voltage ratio. The modification consists of designing the "a" network for a short-circuit termination. The first step is to shift a zero of z_{22a} to infinity by removing a series resistance. Then the zero circle of the remaining admittance is shifted in the manner already described. The final result is shown in Fig. 14.

The transfer function given in example 1 was used by Patrick and Thomas (14) as the basis of a lowpass RC filter. As an illustration of what may be gained in a specific case through the use of the new procedure, the network designed by Patrick and Thomas using the parallel-ladders procedure is shown in Fig. 15. In this network the d-c output-input voltage ratio is 0.00105, whereas in the network of Fig. 14 the corresponding ratio is 0.594.

E. Example in which Two Cycles of the Procedure Are Used

A great deal of labor is involved in the use of Eq. 118 when the function is of high degree. However, it is often satisfactory to use Eq. 103 instead, in the manner

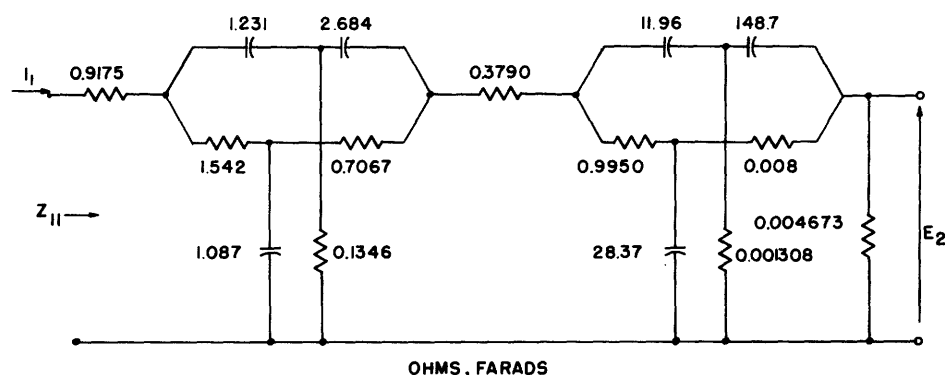


Fig. 16

Network representation of the function in section V-E.

described in the preceding section. A consideration of the distribution of the poles of $F(\lambda)$ in relation to the zeros of $H(\lambda)$ will normally allow a decision to be made as to whether to proceed on the z - or the y -basis, and it is usually worthwhile to attempt the removal of a resistance first. If this attempt fails, it is still possible to separate a pole from $F(\lambda)$ and to use Eq. 103 to determine its proper residue, which, if less than the original residue, permits Z_o or Y_o to be calculated. Since Eq. 118, and hence, Eq. 103, represents a linear function of any single residue, this procedure does not lead to additional complications.

Figure 16 shows the network representation of the functions,

$$Z_{11} = \frac{(\lambda + 0.4)(\lambda + 0.8)(\lambda + 1.2)(\lambda + 1.6)}{(\lambda + 0.2)(\lambda + 0.6)(\lambda + 1.0)(\lambda + 1.4)}$$

and

$$Z_{12} = \frac{1.963 \times 10^{-4}(\lambda^2 + 1)(\lambda^2 + 4)}{(\lambda + 0.2)(\lambda + 0.6)(\lambda + 1.0)(\lambda + 1.4)}.$$

The network was obtained by straightforward use of the method in the manner just described. The zero at $\pm j$ was produced first, and the remainder at the end of the first cycle was

$$Z_4 = \frac{0.3799(\lambda^2 + 2.067\lambda + 1.0)}{(\lambda^2 + 1.848\lambda + 0.7735)}.$$

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References

1. O. Brune: Synthesis of a Finite Two-Terminal Network Whose Driving-Point Impedance is a Prescribed Function of Frequency, J. Math. Phys. 10, 191-236, 1930-31
2. S. Darlington: Synthesis of Reactance Four Poles, J. Math. Phys. 18, 257-353, 1939
3. W. Cauer: Die Verwirklichung von Wechselstromwiderstanden vorgeschriebener Frequenzabhängigkeit, Arch. f. Elektrotechn, Bd. 17, 355-388, 1926-27
4. E. A. Guillemin: Notes for Subjects 6.561 and 6.562, M.I.T. 1950-51, unpublished
5. H. H. Scott: A New Type of Selective Circuit and Some Applications, Proc. I.R.E. 26, 226, 1938
6. C. M. Gewertz: Synthesis of a Finite Four-Terminal Network from its Prescribed Driving-Point Functions and Transfer Functions, J. Math. Phys. 12, 1-257, 1933
7. E. A. Guillemin: Communications Networks, Vol. II, John Wiley, New York 1935
8. E. A. Guillemin: Synthesis of RC Networks, J. Math. Phys. 28, 22-42, 1949
9. E. A. Guillemin: RC-Coupling Networks, Radiation Laboratory Report No. 43, M.I.T. Oct. 11, 1944
10. J. L. Bower, P. F. Ordung: The Synthesis of Resistor-Capacitor Networks, Proc. I.R.E. 38, No. 3, 263-269, 1950
11. L. Weinberg: New Synthesis Procedures for Realizing Transfer Functions of RLC- and RC-Networks, Technical Report No. 201, Research Laboratory of Electronics, M.I.T. 1951
12. P. F. Ordung, F. Hopkin, H. L. Krauss, E. L. Sparrow: Synthesis of Cascaded Three-Terminal RC Networks with Minimum Phase Transfer Functions, unpublished manuscript of the Dunham Laboratory of Electrical Engineering, Yale University, 1951
13. E. A. Guillemin: A Note on the Ladder Development of RC Networks, scheduled for publication, Proc. I.R.E. 1952
14. K. W. Patrick, V. E. Thomas: A Design Procedure for Linear, Passive RC Filter Networks, M.S. Thesis, Dept. of Electrical Engineering, M.I.T. 1946