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Mathematical Programming Models for Determining Freshman Scholarship Offers

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Mathematical Programming Models for Determining Freshman Scholarship Offers*

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1. Introduction

This paper is concerned with the problem of deciding how much scholarship aid to offer to each financially needy applicant who has been admitted to the freshman class of an institution of higher learning. It will be assumed that the admission decisions that determine the members of the admitted freshman class are made prior to all decisions regarding scholarship awards and do not take into account the financial resources available to any applicant. It will also be assumed that the preferred solution to the problem, that of offering each needy admitted applicant as much scholarship aid as he requires, is infeasible.

The problem of determining scholarship offers for admitted freshman applicants with financial need (hereafter usually referred to as "needy admits") is faced annually by student aid administrators at most colleges and universities throughout the country. The particular models described in this paper were developed to assist the administrators of the Student Aid Center at the Massachusetts Institute of Technology; however, the results presented should be of more general interest and use.

The group of freshman admits who have been classified as financially needy constitute the total population of admitted applicants to which our

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models pertain. Admits who are considered for scholarship aid on grounds other than financial need are not included. We shall assume that an admitted applicant's financial need is basically determined by two estimates: (i) an estimate of a reasonable school and personal expense budget for the normal academic year and (ii) an estimate of the money the admit and his parents can be expected to contribute toward meeting his expenses. The expense budget, hereafter called the standard budget, is determined by adding up the costs of tuition, room and board, medical insurance, an allowance for books and materials, and a second allowance for miscellaneous expenses. The estimated contribution expected from the admit and his parents is based primarily on an assessment of the "Parents Confidential Statement" of personal finances (submitted through the College Scholarship Service of the College Entrance Examination Board) and on an estimate of the summer earnings capability of the admit. An admit's need, if any, is defined as the difference between the standard budget and the expected contribution from him and his parents.

After the financial condition of each member of the admitted freshman class has been determined, the class of all admits can be divided into two groups: the need group and the no-need group. The scholarship offers made to members of the need group will provide an amount of aid equal to some fraction of their need. And theoretically, at least, these fractional values are limited only in that they must lie between 0 and 1. In cases where need is not met in full, we assume that low cost loan funds are available to cover the difference between need and offered scholarship aid. A freshman scholarship offer policy is defined as a set of university scholarship offers, one

to each member of the need group, where each partial scholarship is supplemented to the extent of need by an offer of a low cost loan.

In addition to receiving a university scholarship offer, a needy admit may also be offered useable financial aid from a non-university source, e.g. from the National Merit Scholarship Program. If this happens, and the admit decides to matriculate, he is expected to accept the outside award. However, in cases where the outside award is smaller than the university offer, a scholarship supplement will be granted by the university to make up the difference.

At the point in time when a scholarship offer policy must be settled on, the policy administrators will have some information about the amounts of aid that will be offered to members of the need group from non-university sources. In addition, these administrators should have a backlog of information on non-university scholarships offered to needy admits in past years. We shall assume that on the basis of this current and historical information on outside support, the policymakers can accurately estimate the amount of non-university financial aid that will be offered to each needy admit, prior to their determination of a freshman scholarship offer policy.

The success of any scholarship offer policy is measured, a posteriori, in terms of the number of needy admits that enroll, their quality, and the amount of scholarship aid that they accepted. The number that enroll will usually be compared with a target figure and if the two numbers are close to each other the policy can be considered a success with regard to yield.

Since we have assumed that all admission decisions precede financial aid considerations it follows that the quality (i.e. the academic fitness

and scholastic and professional promise) of all needy admits has been judged admissible. However, financial aid administrators may still want to take into account differences in quality within the need group in developing a scholarship offer policy. Thus, we shall assume that criteria exist by which the members of the need group can be ranked from first to last with regard to quality. Given this ranking an administrator might then assess the aggregate quality of the enrolling needy admits in terms of the number of enrollees from among the higher ranking members of the need group.

The monetary success of a scholarship offer policy can be measured by comparing the number of dollars actually spent on aid with the number of dollars budgeted. If the budget figure is exceeded by a non-trivial amount the implemented policy must be judged a financial failure--even though it may very well have been an outright success when judged by yield or quality.

In developing models to assist scholarship aid administrators in choosing an offer policy we shall be guided by the yield, quality, and cost measures used in an a posteriori policy evaluation. Our models, however, must take into account the fact that there is no way to predict with certainty how any needy admit will respond to a particular scholarship offer, and, hence, there is no way, a priori, to precisely forecast the results of any policy. Therefore, the best we can do is to develop a careful description of the scholarship offer and acceptance process that includes the probabilistic aspects of the situation. This description, or descriptive model, can then be used as a basis for developing optimization models by which to evaluate alternative policies.

The next section is devoted to the development of a descriptive model. A general optimization model is presented in section 3. In section 4 a special case of this model is formulated. The special case has the form of a quadratic programming problem, and, consequently it is amenable to solution using a computer. In the fifth section a particular version of the quadratic program is formulated in which an upper bound is placed on the loan component included in any scholarship-loan "package" offered to a needy admit. The last section is devoted to a brief discussion of possibilities for future research efforts.

2. A Descriptive Model

Consider, now, a need group consisting of m admits. Suppose that the admits are ordered as to their quality by the index i, $(i=1,2,\ldots,m)$, such that the quality of admit $i=1$ is superior or equal to the quality of admit $i=2$, the quality of admit $i=2$ is superior or equal to the quality of admit $i=3$, and so on. Furthermore, let the demonstrated need of admit i be denoted by $n_i B$, $(0<n_i \leq 1)$, where B is the amount of the standard budget.

In addition, let $x_i B$, $(0 \leq x_i \leq n_i)$, represent the amount of scholarship aid that is expected to be offered to admit i from non-university sources. And let $y_i B$, $(x_i \leq y_i \leq n_i)$, denote the total amount of scholarship aid offered to admit i by the university. Then if

$$(2.1) \qquad y_i = x_i + u_i, \qquad (i=1,2,\ldots,m) ,$$

it follows that $u_i B$ represents the amount of scholarship aid from university sources that will be awarded to admit i if he enrolls. Any vector

$$\underline{y} = (y_1, y_2, \ldots, y_m) , \qquad (x_i \leq y_i \leq n_i) ,$$

will describe a possible scholarship offer policy.

Finally, let $\tilde{z}_i$, $(i=1,2,\ldots,m)$, denote a set of indicator random variables, where $(\tilde{z}_i=1)$ represents the event that admit i decides to enroll and $(\tilde{z}_i=0)$ the event that he decides not to enroll; and let the conditional probability function,

$$f_i(z_i) = P(\tilde{z}_i = z_i | x_i, y_i, n_i, B), \qquad (i=1,2,\ldots,m).$$

Then $f_i(1)$ is the probability that admit i decides to enroll given that he is offered x_iB dollars of scholarship aid from outside sources, y_iB dollars of aid from the university, he is ith ranked with regard to quality, and his demonstrated need is n_iB.

Now for any set of estimates of non-university scholarship offers $\{x_1B, x_2B, \ldots, x_mB\}$, which we denote more simply by the vector $\underline{x} = (x_1, x_2 \ldots, x_m)$, and any freshman scholarship offer policy $\underline{y}$ (where, then, by (2.1), $\underline{y} = \underline{x}+\underline{u}$) we can characterize the number of needy admits that can be expected to enroll by the equation,

$$E\left(\sum_{i=1}^{m} \tilde{z}_i \,\middle|\, \underline{x},\underline{y}\right) = \sum_{i=1}^{m} f_i(1). \tag{2.2}$$

Moreover, letting α be any number between 0 and 1, and defining $[\alpha m]$ as the largest integer less than or equal to αm, we can characterize the number of needy admits that can be expected to enroll from among those in the top $100\alpha\%$ of the need group ordered with regard to quality, i.e. from among the admits with quality indices $1, 2, \ldots, [\alpha m]$ by the equation,

$$E\left(\sum_{i=1}^{[\alpha m]} \tilde{z}_i \,\middle|\, \underline{x},\underline{y}\right) = \sum_{i=1}^{[\alpha m]} f_i(1). \tag{2.3}$$

And, finally we can describe the expected amount of money that will be required from university sources to support the freshman scholarship program by the equation,

$$E\left(\sum_{i=1}^{m} (y_i - x_i) B \tilde{z}_i \,\middle|\, \underline{x},\underline{y}\right) = B \sum_{i=1}^{m} u_i f_i(1). \tag{2.4}$$

The equations (2.2), (2.3), and (2.4) are viewed collectively as a descriptive model of the scholarship offer and acceptance process that relates the basic inputs,

(i) the quality ranking index i,

(ii) the needs $\underline{n} = (n_1, n_2, \ldots, n_m)$

(iii) the outside aid offers $\underline{x} = (x_1, x_2, \ldots, x_m)$

(iv) the conditional probability functions $\underline{f} = (f_1(1), f_2(1), \ldots, f_m(1))$

(v) the standard budget B

and (vi) the parameter α ,

to expected quantity, quality, and cost related outputs, for a given vector of policy variables $\underline{y}$.

3. A General Optimization Model

We now proceed from the descriptive model to the formulation of a general optimization model which, at least theoretically, provides a basis for determining an optimal scholarship offer policy. Such an optimal policy is expected to satisfy the following conditions:

(i) The expected number of enrolling needy admits must equal T,

(ii) The top 100α% of the need group ordered by quality is expected to yield at least βT enrollees, ($[\alpha m] \geq \beta T$),

(iii) Given the above two constraints, the expected amount of university scholarship money required to support the freshman scholarship program is to be minimized.

The general optimization model (and problem) corresponding to these conditions is stated as follows: Given the quality ranking index i, and the vectors and scalers $\underline{n}$, $\underline{x}$, $\underline{f}$, B, and α, find a vector $\underline{u}$, and, implicitly, a scholarship offer policy $\underline{y} = \underline{x}+\underline{u}$, to

$$\text{minimize:} \quad S = B \sum_{i=1}^{m} u_i f_i(1)$$

$$(3.1) \qquad \text{subject to:} \quad \sum_{i=1}^{m} f_i(1) = T$$

$$\sum_{i=1}^{[\alpha m]} f_i(1) \geq \beta T$$

$$u_i \leq n_i - x_i, \quad (i=1,2,...,m)$$

$$u_i \geq 0, \qquad (i=1,2,...,m).$$

The general model (3.1) is conceptually useful, however, it cannot be manipulated mathematically until a functional form for the vector $\underline{f}$ of conditional probability functions is postulated. In the next section we argue for the reasonableness of a linear form which allows (3.1) to be reduced to a quadratic program.

4. A Quadratic Programming Formulation

Recall that the conditional probability function $f_i(1)$ was defined as the probability that needy admit i decides to enroll, given that he is offered x_iB dollars in scholarship aid from non-university sources, y_iB dollars in aid from the university of which then he receives u_iB, he is i^{th} ranked with regard to quality, and his demonstrated need is n_iB. In general we would expect this function to look like the one pictured in Figure 4.1.

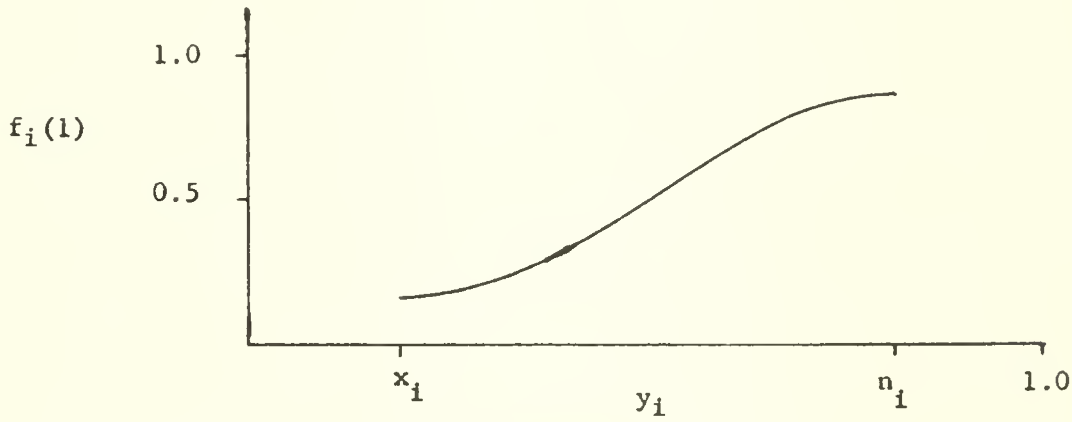

Figure 4.1

Moreover, under the assumption that low cost loan funds are available to supplement a scholarship offer that does not meet total need, we would further expect that the probability function would increase at a relatively constant rate. Hence, it does not seem unreasonable to postulate a linear form for $f_i(1)$ as a satisfactory approximation to its actual shape. Thus, we shall let,

$$f_i(1) = P(\tilde{z}_i=1|x_i,y_i,n_i,B) = c_i+b_iy_i \qquad (x_i \le y_i \le n_i)$$

$$= c_i+b_i(x_i+u_i)$$

$$= (c_i+b_ix_i) + b_iu_i,$$

or, alternatively,

$$f_i(1) = a_i+b_iu_i, \qquad (0 \le u_i \le n_i-x_i) \tag{4.1}$$

where $a_i = c_i+b_ix_i$.

We further assume that values of a_i and b_i can be estimated for all members of the need group. Note that a_i can be defined as the probability that admit i decides to enroll given that he is offered a low cost loan to cover his entire need over and above any scholarship aid from outside sources. The parameter b_i estimates the increase in $f_i(1)$ given a unit increase in the university scholarship aid parameter u_i.

The substitution of the linear form (4.1) into the general model (3.1) results in the following quadratic program: Given the quality ranking index i, and the vectors and scalers $\underline{n}$, $\underline{x}$, $\underline{a}$, $\underline{b}$, B, and α, find a vector $\underline{u}$, and consequently, a scholarship offer policy $\underline{y}$ to,

$$\text{minimize:} \quad S = B\left[\sum_{i=1}^{m} a_iu_i + \sum_{i=1}^{m} b_iu_i^2\right]$$

subject to: $\sum_{i=1}^{m} b_i u_i = T - \sum_{i=1}^{m} a_i$

(4.2) $$\sum_{i=1}^{[\alpha m]} b_i u_i \geq \beta T - \sum_{i=1}^{[\alpha m]} a_i$$

$$u_i \leq n_i - x_i, \qquad (i=1,2,\ldots,m)$$

$$u_i \geq 0, \qquad (i=1,2,\ldots,m).$$

Computer programs to solve problems of the form (4.2) can be readily developed from standard quadratic programming codes. Hence, if the assumptions and objectives underlying the formulation of the quadratic program are accepted, the program can be used to explicitly determine a mathematically optimal solution to the question of how much scholarship aid to offer to each needy freshman admit.

The quadratic program (4.2) can also be used as an experimental tool to determine the relationship between the cost, S^*, of an optimal policy and the value of a decision parameter (T, α, or β). For example, suppose α and β are fixed, and (4.2) is solved repeatedly for a range of values of T. As a result of these computations a convex function $S^*(T)$ over the domain $\sum_{i=1}^{m} a_i \leq T \leq \sum_{i=1}^{m} (a_i + b_i)$ could be plotted. Such a graph could be used to determine a reasonable (S^*, T) combination in cases where a value for T was not fixed in advance. Similar analyses could be carried out for the functions $S^*(\alpha)$ and $S^*(\beta)$.

5. A Loan Limit Version of the Quadratic Program

In formulating the quadratic program (4.2) no upper limit was placed on the size of the loan that might be offered to a needy admit. Suppose we now let r_i denote the fraction of the standard budget B offered to needy admit i in the form of a loan. Furthermore, let r denote the maximum fraction that can be offered to any needy admit. Then we have the constraint,

(5.1) $$0 \le r_i \le r, \qquad (i=1,2,...,m),$$

and the relationship between r_i and u_i,

(5.2) $$u_i = n_i - x_i - r_i, \qquad (i=1,2,...,m.$$

When the equations (5.2) are substituted into the quadratic program (4.2) and the relations (5.1) are accounted for, the following program can be viewed as a loan limit version of (4.2): Given the quality ranking index i, and the vectors and scalers $\underline{n}$, $\underline{x}$, $\underline{a}$, $\underline{b}$, B, α, and r, find the loan policy $\underline{r} = (r_1, r_2, ..., r_m)$ to,

$$\text{minimize:} \quad S = B\left[\sum_{i=1}^{m} a_i(n_i-x_i-r_i) + \sum_{i=1}^{m} b_i(n_i-x_i-r_i)^2\right]$$

(5.3)
$$\text{subject to:} \quad \sum_{i=1}^{m} b_i(n_i-x_i-r_i) \ge T - \sum_{i=1}^{m} a_i$$

$$\sum_{i=1}^{[\alpha m]} b_i(n_i-x_i-r_i) \ge \beta T - \sum_{i=1}^{[\alpha m]} a_i$$

$$r_i \le \min(r, n_i - x_i), \quad (i=1,2,...,m)$$

$$r_i \ge 0, \qquad (i=1,2,...,m).$$

(Note that the yield equation in (5.3) has been written as an inequality. This has been done to increase the number of feasible solutions to the program.)

The program (5.3) presents experimental possibilities similar to those of (4.2). For example, (5.3) could be solved repeatedly for different values of r, thus supplying data for a graph to relate r to the cost of an optimal policy.

6. Discussion

The models presented in this paper are intended to assist student aid administrators in determining reasonable and feasible freshman scholarship offer policies. It is expected that they will serve best as experimental tools to investigate more fully the relationships between input parameters and output variables. Furthermore, they will point up the marginal trade-offs between expected cost, yield, and quality.

The models do not take into account the possibility that some scholarship funds may be restricted, and thus only offered to needy admits with special qualifications. However, it is not expected that this complication will seriously affect the usefulness of the models. Moreover, the models are limited in the sense that only two types of financial aid are considered, scholarships and loans. In the next stage of this research effort we shall expand our analysis to include other forms of aid such as term-time student employment. We shall also carry our analysis in another direction towards a study of the total undergraduate scholarship aid problem where we shall consider all four undergraduate classes rather than just the freshman class.

In addition, we plan to undertake empirical studies to evaluate the practicability of implementing models such as (4.2) and (5.3).