THE FLOW OF SCHEDULED AIR TRAFFIC (II)

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Abstract

Numerical (IBM punched card) and analytical methods are employed to analyze three problems involving scheduled air traffic.

First, the effect upon the stack-delay and total time-keeping error statistics of a single en route control point is investigated under two conditions: (a) the control point reschedules the aircraft; (b) the control point attempts to keep each plane on its original schedule. In both cases the frequency of long stack delays is reduced significantly, when the traffic is heavy, even with moderate amounts of control. When the traffic is light, the control has relatively little effect. In case (a) the total time-keeping error statistics are not greatly altered by the control, since stack delay is effectively traded for added artificial en route delay. In case (b), however, the en route-deviation distribution is effectively narrowed by the control, thus reducing the total time-keeping error as well as the stack delay.

Second, the effect of a less rigid schedule (without en route control) is determined. The results are found to approximate those which would be produced by equivalent en route deviation statistics acting upon a rigid schedule.

Finally, the stacking caused by a sudden shutdown of the terminal is illustrated. This problem has been idealized severely, but does include the effect of a delayed feedback which eventually stops the flow of traffic to the terminal. Long stack delays, which cannot be accounted for in terms of reasonable en route-deviation distributions, may be caused by the shutdown; but the conditions required would be expected to occur rather rarely.
THE FLOW OF SCHEDULED AIR TRAFFIC (II)

I. Introduction

In a previous report (1), the group represented by the authors dealt with the effects of random en route time-keeping errors upon the flow of a traffic of aircraft which had originally been set up to follow a "proper" schedule (defined below). It was assumed that no en route control was available. The present report deals with several topics, the first of which concerns the effect of two simple types of en route control, the "One-Point Rescheduling Control", and the "One-Point On-Time Control". The two types of control were mentioned briefly in reference 1, and the relationship was shown between the results of the calculations carried out there and problems of en route control. Those results will form the basis for estimating the effectiveness of the en route control procedures.

The scheduling method for the present investigation of en route control follows that described in the aforementioned previous studies. Briefly, time is quantized in units of the minimum safe landing interval, \( t_0 \), and for such a proper schedule not more than one aircraft is scheduled for each time unit. The traffic parameter \( \epsilon \) is defined as before, namely as the ratio of the actual average arrival rate at the airport to the maximum acceptance rate.

The second topic to be treated considers the effect of an alternative method of scheduling which appeared to be worth collateral investigation; it may be called "block" scheduling*. With this alternative and less rigid scheduling system, the time axis is divided into sections of \( S' \) units each, called blocks. The number of aircraft scheduled to arrive within each such time block varies randomly between zero and \( S' + 1 \), in such a manner, however, that the total number of aircraft scheduled over a long period of time, comprising many blocks, yields a prescribed value of the traffic parameter \( \epsilon \). In contrast to a proper schedule, the planes within each block are scheduled at random times, so it becomes possible for several of them to be scheduled for simultaneous arrival; in no case, however, will more than \( S' + 1 \) planes be scheduled to arrive together. The congestion at the terminal which results when block-scheduled aircraft also suffer random en route time-keeping errors, without benefit of any en route control, is investigated here by a numerical procedure which is entirely analogous to that used in the previous computations (1).

Neither the random en route time-keeping errors alone nor even the superimposed effect of block scheduling appears to be primarily responsible for the rather large stacking delays sometimes observed in practice. One of the most obvious possible causes of such large delays would be a sudden reduction in the maximum acceptance rate of the landing strip, caused for example by a deterioration in weather conditions around the terminal zone. The resulting delays produced by such a change, including

* This problem was suggested by Dr. D. Ewing of the ANDB.
the effect of a delayed "feedback" which reduces the average rate of arrivals some time after the landing strip closes down, is the subject of the third topic in the present report. The treatment is in this instance purely analytical. Subject to the validity of the rather severe idealizations made in order to solve this problem analytically, the analysis does indicate that very long delays can be produced, but only under somewhat unusual conditions. As a result of one of the principal simplifying assumptions made, namely that the traffic flow may be regarded for this particular problem as a continuous process instead of a discrete one, this section has been entitled "Continuous-Flow Calculations".

The remaining content of this report is consequently divided as follows:

II. One-Point Rescheduling Control
III. One-Point On-Time Control
IV. Block Scheduling
V. Continuous-Flow Calculations
VI. Conclusions.
II. One-Point Rescheduling Control

2.1 Outline of Method

Without en route control, a plane flies from its origin A (Fig. 2.1) to its destination C, subject to a prescribed random en route-deviation distribution. It is assumed, in other words, that once the planes have started on their flight it is impossible to control them while en route.

Suppose now that a control point is established halfway along the route, as shown at B in Fig. 2.1. The object of the control point is to try to reduce the size of the stacks that form at the airport C by instructing the aircraft to speed up or slow down during the portion BC of their flight. The aircraft are originally scheduled in some proper sequence, and leave from point A accordingly. It is then assumed that during the portion AB of the flight the aircraft are subject to random en route errors from a deviation distribution of spread S. Thus the arrival sequence at B is no longer the original properly-scheduled one; instead, the aircraft may arrive in bunches.

The control point might then adopt either of two procedures in its effort to reduce the congestion at the destination C. It may compare the actual arrival time of a plane with its scheduled arrival time (at B), and then try to correct the difference between the two times. This is the "on-time" method of control, which is discussed in the next section. Alternatively, the control point may ignore the original schedule, and just try to control the planes so as to space them one unit apart at C. This is the rescheduling procedure, in the sense that the original schedule is not generally maintained. When bunches of aircraft arrive together at B, the control procedure is assumed to reschedule them into their originally scheduled order, insofar as this is possible with a limited amount of control. The limit to the amount of control depends upon the permissible speed variations of the aircraft, and it is quite possible that this will not be sufficient to compensate fully for all flight errors occurring during the leg AB.

During the second portion of the flight, from B to C in Fig. 2.1, the planes are again assumed to experience a random en route deviation arising from a deviation distribution of spread S. Thus the congestion at C may be said to depend upon two factors: the possible failure to compensate completely for the errors of the first portion of the flight, and the actual en route errors of the second portion.

For the reasons described in reference 1, the problem of determining the stack delays at the destination is not amenable to analytic treatment. The method of approach adopted is, as before, numerical computation on the IBM punched-card machines. For this purpose, a sample size of 1000 planes is again chosen, and the en route deviations are all taken to be delays (1). The control at B is therefore also assumed to introduce only delays. The maximum amount of time by which the control can delay a plane is
assumed to be \( n \) units. The basic unit of time is the same as in reference 1, namely \( t_0 \) (the minimum safe landing interval). For the present problem, the en route-deviation distributions for the two portions of the flight are both taken to be rectangular in shape, and of equal spread \( S \). Furthermore, they are assumed to be independent of each other, and care must be taken in the use of the random number table (2) to insure that this is so.

The IBM machines have accordingly been programmed for the following parameters.

<table>
<thead>
<tr>
<th>Deviation Distribution</th>
<th>Spread ( S ) (on both flight portions)</th>
<th>Control ( n )</th>
<th>Traffic Parameter ( \epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>6</td>
<td>3</td>
<td>1.0, 0.9, 0.8</td>
</tr>
</tbody>
</table>

The initial part of the preparation of the problem for the IBM machines is exactly the same as described in reference 1. Thus the scheduled arrival time at \( B \) of the \( j^{th} \) plane in the original schedule is denoted by \( t_j \), a \( t_j \) table being prepared for each required value of \( \epsilon \). The random deviation \( r_j \) suffered by this plane between \( A \) and \( B \) is chosen from the tables (1) for the rectangular en route-deviation distribution with spread \( S = 6 \), keeping in mind that another similar but independent set of \( r_j' \) is required in this problem for the second portion of the flight.

The actual arrival time at \( B \) of the \( j^{th} \) plane in the original schedule is denoted by \( p_j \), whence

\[
p_j = t_j + r_j
\]  

(2.1)

The \( p_j' \)'s of all the planes (\( j = 1, 2, \ldots, 1,000 \)) are then rearranged into order \( k \) of actual arrival time \( p_k \) at point \( B \). Using the same convention adopted in reference 1, the order of increasing \( k \) is therefore the order of increasing magnitude of the \( p_j' \)'s, unless several of the \( p_j' \)'s are of equal size. Since the latter situation means that several planes arrived together at \( B \), they are handled according to their originally scheduled order; in such cases then, the order of increasing \( k \) is that of increasing magnitude of \( j \).

The attempt made by the control at \( B \) to reorganize the \( p_k \) sequence so that no two planes will arrive at \( C \) less than one unit apart in time is conditioned by the following assumptions:

(1) Any plane may only be requested to adjust its speed over leg BC so as to be delayed by an integral number of time units between 0 and \( n \) inclusive.

(2) In choosing this scheduled delay for any plane, the controller assumes that its orders will be carried out exactly over the leg BC. That is, no attempt is made to predict the residual random errors which might occur after it has given its orders.

(3) The control at \( B \) has only the following information about the aircraft at any time \( t \):

(a) The original schedules for all planes

(b) The actual arrival times at \( B \) of all planes which have already arrived there by time \( t \)
(c) Its own past history of delay orders

(d) The identification of each plane.

The assumptions listed under (3) help give to B the simplest possible control characteristics by stating effectively that it is cut off from all actual flight information except its own observations in its immediate vicinity. It should be added here that point B need not be a physical point in space containing control equipment, but might represent a particular single time during the flight of each plane when it is contacted by some controller (possibly located at point C actually). Then the assumptions under (3) above mean simply that the actual progress of any plane en route is not observed by the ground except at one single "check time". In the strictest sense, therefore, this problem represents "discrete control"(1). Obviously there are many possible ramifications of this one-point rescheduling problem, depending upon the amount of information possessed by the controller, the actual correlation between en route errors over legs AB and BC, and the assumptions made by the controller about this correlation in giving its orders. The case treated here is merely one of the simplest and probably least effective rescheduling control systems.

In the present case, then, let the new scheduled arrival time at C (but referred to B) for the \( k \)th plane which actually arrives at B be given by \( q_k^{(n)} \). The value assigned by B to \( q_k^{(n)} \) proceeds by recursion according to the following scheme.

If the actual arrival time \( p_k \) at B comes after the new scheduled time \( q_k^{(n)} \) of the previous plane \( q_{k-1}^{(n)} \), i.e. if \( p_k > q_{k-1}^{(n)} \), then

\[
q_k^{(n)} = p_k \quad (2.2)
\]

If the actual arrival time is less than \( n \) units before the new scheduled time of the previous plane, that is if \( 0 < q_{k-1}^{(n)} - p_k < n - 1 \), then

\[
q_k^{(n)} = q_{k-1}^{(n)} + 1 \quad (2.3)
\]

Finally, if the actual arrival time is \( n \) or more units before the new scheduled time of the previous plane, then the controller cannot fully correct for the difference, but merely exerts the maximum control; this is a delay of \( n \) units. This situation may be called "overloading" of the control. Hence for \( q_{k-1}^{(n)} - p_k \geq n \)

\[
q_k^{(n)} = p_k + n \quad (2.4)
\]

The process is started with the first new scheduled time equal to the first actual arrival time

\[
q_1^{(n)} = p_1 \quad (2.5)
\]

That is, the first plane is not ordered to alter its motion.

Thus using Eqs. 2.2, 2.3, or 2.4, according to the associated conditions, the set
of \( q_k^{(n)} \) may be formed to give the new schedule. Next, the random deviation \( r_k^i \) resulting from the leg BC is added to the \( q_k^{(n)} \) to give essentially the new arrival distribution \( p_k^i \) at C, but as before these times are referred to B. Thus

\[
p_k^i = q_k^{(n)} + r_k^i.
\] (2.6)

These \( p_k^i \) are again arranged in nondecreasing order \( p_k^i \) and the stack delay \( \tau_k^i \) found from the formulas given below (1).

\[
\tau_k^i = \tau_k^{i-1} + 1 - (p_k^i - p_{k-1}^i)
\] if

\[
\tau_k^{i-1} + 1 - (p_k^i - p_{k-1}^i) \geq 0
\] (2.7)

and

\[
\tau_k^i = 0
\] if

\[
\tau_k^{i-1} + 1 - (p_k^i - p_{k-1}^i) < 0.
\] (2.8)

The initial conditions at C are

\[
\begin{align*}
p_0^i &= 0 \\
\tau_0^i &= 0
\end{align*}
\] (2.9)

which are the starting conditions for a clear airport at C.

In carrying out the procedure given above on the IBM machines, a decision has to be made for each plane to determine whether Eq. 2.2, 2.3, or 2.4 applies. This is liable to make the computations unduly complicated for certain ranges of n-values, and a simpler alternative method, using an iterative procedure, is given below.

The \( p_k \) sequence is found as before, and then a quantity \( q_k^{(o)} \) is calculated according to the relation

\[
q_k^{(o)} = \begin{cases} 
  p_k & \text{if } p_k - q_k^{(o)} > 0 \\
  p_k + 1 & \text{if } q_k^{(o)} - p_k \geq 0
\end{cases}
\]

with

\[
q_1^{(o)} = p_1.
\] (2.10)

The process is now iterated by finding \( q_k^{(1)} \) according to the relation

\[
q_k^{(1)} = \begin{cases} 
  q_k^{(o)} & \text{if } q_k^{(o)} - q_k^{(1)} > 0 \\
  q_k^{(o)} + 1 & \text{if } q_k^{(1)} - q_k^{(o)} \geq 0
\end{cases}
\]

with

\[
q_1^{(1)} = q_1^{(o)}.
\] (2.11)
Thus in general, repeating the iteration process \( \nu \) times gives

\[
q^{(\nu)}_k = \begin{cases} 
q^{(\nu-1)}_k & \text{if } q^{(\nu-1)}_k - q^{(\nu)}_k > 0 \\
q^{(\nu-1)}_k + 1 & \text{if } q^{(\nu)}_k - q^{(\nu-1)}_k \geq 0
\end{cases}
\]

with

\[
q^1_1 = q^{(\nu-1)}_1 = p_1.
\]

This process is carried out for \( \nu = 0, 1, 2, \ldots n \) inclusive, until the final set of \( q^{(n)}_k \) is obtained. The procedure actually amounts to adding the control one unit at a time, up to the maximum of \( n \) units, thus producing the same set of \( q^{(n)}_k \) as in the first method.

The rest of the process for finding the stack delays is identical with the first method.

For relatively small values of \( n \) it would appear easier to make the \( n+1 \) simple decisions, according to Eq. 2.12, than the one complicated decision of Eqs. 2.2, 2.3, and 2.4. As results were required here for a maximum control of \( n = 3 \) units, the second method was used in programming the machines.

For the parameter values listed previously the main results obtained are as follows:

1. Frequency distributions of \( \tau'_\mu \)
2. Progressive distribution of \( \tau'_\mu \)
3. Average stack delay \( \bar{\tau}'_\mu \).

In addition, a "control parameter," \( C_k \), is defined as

\[
C_k = q^{(n)}_k - p_k
\]

with

\[
0 \leq C_k \leq n.
\]

This simply specifies the amount of control, i.e. the delay, that the control point B exerts on each plane. It is thus a measure of how much control is needed; hence the statistics of \( C_k \) give some idea of the degree of utilization of the control point. Consequently, additional results obtained are:

4. Frequency distribution of control parameter \( C_k \)
5. Progressive distribution of control parameter \( C_k \)
6. Average control parameter, \( \bar{C}_k \)
7. Frequency distribution of total time-keeping error

\[
d_j = \tau'_j + r'_j + C_j + r_j
\]

8. Progressive distribution of \( d_j \)
9. Average total time-keeping error \( \bar{d}_j \).

The total time-keeping error as defined in Eq. 2.14 is never negative, as \( r'_j \), \( r_j \) and \( C_j \) have been taken as delays only, and \( \tau'_j \) is always a true delay. In practice the \( r'_j \), \( r_j \) and \( C_j \) could represent either advances or delays. The distributions might well be
centered about zero as their midpoints, and the control might also allow equally either an increase or decrease in speed. Thus to define the algebraic total delay under these conditions a shift has to be made in the origin of the total time-keeping error distributions. The total algebraic delay, $e_j$, is then given by

$$ e_j = d_j - (S + \frac{r}{2}) \quad (2.15) $$

in which both of the en route-deviation distributions have been taken as symmetric about zero, and the control may go from $-(n/2)$ to $+(n/2)$.

### 2.2 Outline of Results

Figures 2.2 and 2.3 show the frequency distribution and progressive frequency distribution of the stack delay, $\tau'_k$, for a spread $S = 6$, maximum control $n = 3$, and $\varepsilon = 1.0$, $0.9$, and $0.8$. The frequency distribution and progressive distribution of the control parameter $C$ are given in Figs. 2.4, and 2.5, and the corresponding total delay distributions in Figs. 2.6 and 2.7. The average values of the stack delay, control parameter, and total time-keeping error, are given in Table 1.

<table>
<thead>
<tr>
<th>Traffic Parameter $\varepsilon$</th>
<th>Average Stack Delay $\tau'_k$</th>
<th>Average Control Parameter $C_k$</th>
<th>Average Total Time-keeping Error $\bar{d}_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>2.723</td>
<td>1.537</td>
<td>10.372</td>
</tr>
<tr>
<td>0.9</td>
<td>1.362</td>
<td>1.036</td>
<td>8.510</td>
</tr>
<tr>
<td>0.8</td>
<td>0.878</td>
<td>0.843</td>
<td>7.833</td>
</tr>
</tbody>
</table>

In order to gain some idea of the effect of control upon the congestion at the destination, it is instructive to compare the results obtained above with those which would be obtained if no control were used. In this case the total en route delay is comprised of two independent delays, each from a rectangular deviation distribution of spread $S$. Thus the total en route error will be the result of a triangular distribution of spread $2S$. Results for the no-control case for three types of distributions (rectangular, triangular, and parabolic) are given in reference 1. The stack-delay distributions for the no-control case with a triangular en route distribution of spread $S = 12$, for $\varepsilon = 1.0$, $0.9$, and $0.8$, are plotted on the stack-delay curves of Fig. 2.2. It is immediately evident, especially for $\varepsilon = 1.0$, that the effect of the control has been to reduce the frequencies of long stack delays, and to increase the frequencies of short stack delays. For an additional comparison, the stack-delay distributions for the no-control case with a parabolic en route-deviation distribution of spread $S = 12$, were also plotted in Fig. 2.2. These curves follow the corresponding one-point control curves fairly closely, although the control
Fig. 2.2a

Frequency distribution of stack delays.

Fig. 2.2b

Fig. 2.2c
Fig. 2.3
Progressive stack-delay frequencies for one-point control (S = 6, n = 3).

Fig. 2.4
Frequency distribution of control parameter Cn
(S = S = 6, n = 3).

Fig. 2.5
Progressive frequency distribution of control parameter Cn
(S = S = 6, n = 3).
Frequency distribution of total delays.
is still slightly more effective in reducing the number of long delays. The average delays for the above mentioned cases are given in Table 2, which shows that the introduction of the control considerably reduces the average stack delay under heavy traffic conditions, i.e. for values of the traffic parameter close to unity. For smaller values of $\epsilon$ the improvement is not as noticeable, and it is questionable whether the control would give any appreciable improvement for small values of $\epsilon$ (in the range $\epsilon \leq 0.5$, for example).

From the stack-delay results it thus appears that for $S = 6$ and $n = 3$, with rectangular en route-deviation distributions, the effect of the control is to give approximately the same results as the no-control case with a parabolic en route-deviation distribution of spread 12. For smaller values of $n$ the control is obviously going to have less effect, and the resulting congestion as $n$ is decreased should approach that of the no-control case for a triangular distribution of spread 12. On the other hand, if the amount of control is increased, the effect of flight errors during the first leg AB of

Table 2

<table>
<thead>
<tr>
<th>Traffic Parameter $\epsilon$</th>
<th>One-point Control $S = 6$ $n = 3$</th>
<th>No-control $S = 12$ Triangular</th>
<th>No-control $S = 12$ Parabolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>2.723</td>
<td>3.510</td>
<td>2.920</td>
</tr>
<tr>
<td>0.9</td>
<td>1.362</td>
<td>1.526</td>
<td>1.435</td>
</tr>
<tr>
<td>0.8</td>
<td>0.878</td>
<td>1.015</td>
<td>1.036</td>
</tr>
</tbody>
</table>

the journey will become less noticeable, until for $n = 6$ such errors could be compensated entirely. Under these conditions the stack delay at the destination would be due only to the rectangular en route-deviation distribution of spread 6 which arises along the leg BC.

The frequency distribution of the control parameter $C_k$ may be correlated roughly with the value of the traffic parameter and the spread $S$. For values of $\epsilon$ close to unity it is reasonable to expect groups larger than or equal to $S/2$ planes to occur fairly often (1) at point B, so that under such conditions the full control available in this case ($n = 3$) would be used quite frequently. It would then often be desirable to have a greater range of control. As the traffic density decreases, however, the probability of
large groups at B decreases rapidly, so the full amount of control needs to be used less frequently. The distributions of $C_k$ plotted in Fig. 2.4, bear out these ideas. For example, when $\epsilon = 0.8$ the most probable amount of control is zero and the full control (3 units) is only used on about 5 percent of the aircraft. As indicated by the words "advance" and "delay" on the figures, changes of origin to account for symmetric en route-deviation distributions and a symmetric range of control should be kept in mind here to clarify the interpretation.

The total delay distributions given in Figs. 2.6(a), (b), (c), together with the corresponding total delay distributions of the no-control cases with triangular en route-deviation distributions (1), show that for values of $\epsilon$ close to unity the introduction of the control does not significantly change the total delays experienced by the aircraft. The reason for this may be ascribed to the fact that although the effect of the control is to reduce the terminal stack delays, this reduction is effectively brought about by the use of additional artificial en route delay. As the value of $\epsilon$ is decreased, the effect of the control begins to appear as a decrease in the number of late arrivals, as compared with the no-control cases. This change is relatively small, even for $\epsilon = 0.8$, and although no further computations have been made, it would seem that as $\epsilon$ decreases still further towards zero the two total delay distributions should approach each other again.

While additional computations for the rescheduling control problem would supply more specific numerical data, the general trends to be expected are clear. The fact that the results can be bracketed between those for no control and full control narrows the choice considerably. Furthermore, comparisons with the on-time control cases will show little difference between the two methods insofar as terminal congestion is concerned. Further calculations were therefore deemed unnecessary in this problem.
III. One-Point On-Time Control

3.1 Outline of Method

The control procedure described in Part II made use of the control point B (Fig. 2.1) to reschedule the aircraft with the general object of merely spacing the planes one unit apart at the destination. Using the same control point, receiving only the same information, it is also possible to use the available control to try to put the aircraft back on their original schedules. This type of control is for obvious reasons designated as "on-time" control.

As in the rescheduling system, the extent of the maximum possible amount of control, relative to the flight deviations, determines the amount by which the stack delays at the destination may be reduced. For a very small amount of control, the congestion will not be greatly relieved, whereas with sufficient control to compensate fully for any flight errors occurring in the first leg of the journey (full control), the congestion will be due to the flight errors of the second leg alone. In these respects, that is with either full or no control, the on-time method and the rescheduling method should give the same statistical results as far as terminal stack delays are concerned. On the other hand, the actual arrival sequence at the destination would generally be different for the two types of full control, and this might be significant if the absolute maintenance of the original schedule or the total time in the air becomes important from the passenger or fuel reserve point of view. Most of the basic assumptions of Part II will again hold here. Thus the independent en route-deviation distributions on the two portions of the flight (AB and BC of Fig. 2.1) will be assumed to be rectangular, but now of different spreads $S_1$ and $S_2$ respectively. The maximum control available is $n$ units, and it is further assumed that the distributions represented by $S_1$, $S_2$ and $n$ are symmetrical about zero. The difference in spreads for the two portions of the flight might result, for example, from differences in the average flight time over legs AB and BC (3). Thus point B need not in this case be at the center of AC.

Figure 3.1(a) shows the en route-deviation distributions for the leg AB of the flight. Planes may arrive at B up to $1/2 S_1$ units early or late. The control point notes the deviations from the original schedule and applies as much correction as necessary, limited by the range $n$. With $n < S_1$, only a fraction of the number of planes can be ordered back on their original schedules. If no further deviations were experienced, the total deviation distribution would therefore be as shown in Fig. 3.1(b). However, on the second leg of the flight, BC, the planes may experience an error according to the deviation distribution of spread $S_2$ shown in Fig. 3.1(c). Thus the total error is due to two independent errors, arising from the distributions.

![Fig. 3.1](image-url)
of Figs. 3.1(b) and 3.1(c) respectively. The total error for the whole flight, AC, may thus be regarded as arising from an equivalent en route-deviation distribution obtained by the convolution of the two independent distributions in Figs. 3.1(b) and 3.1(c). The use of an equivalent en route distribution for the whole flight thus essentially reduces the one-point on-time control case to the no-control problem dealt with in reference 1. Once the equivalent en route-deviation distribution has been determined, therefore, the one-point on-time control problem may be solved by using the previous results.

3.2 Convolution Methods

Let the deviation distribution of Fig. 3.1(b) be designated by $P_1(r)$, and the distribution of Fig. 3.1(c) by $P_2(r)$. Then to carry out the convolution, $P_2(r)$ is shifted by $m$ units to become $P_2(r+m)$, as shown in Fig. 3.2. A single point $r = m$ on the convolved distribution is now obtained by taking the product of the two distributions $P_1(r)$ and $P_2(r+m)$. Thus if $p_1, \eta$ and $p_2, \eta$ represent the discrete probabilities of delay $\eta$ units in $P_1(r)$ and $P_2(r)$ respectively, the convolved distribution $P_{eq}(m)$ is given by

$$P_{eq}(m) = \sum_{j=0}^{\infty} p_1, j \cdot p_2, (j+m) \tag{3.1}$$

where

$$m = 0, 1, 2, \ldots, \left(\frac{S_1 + S_2 - n}{2}\right)$$

and obviously $P_{eq}(-m) = P_{eq}(m)$. Clearly it is not necessary to consider $n > S_1$, since no corrections beyond the on-time condition will be exercised. Equation 3.1 allows the equivalent en route-deviation distribution to be determined for any specified $P_1(r)$ and $P_2(r)$. The calculation is simplified when the two independent flight distributions are assumed to be rectangular, but there is still a considerable amount of mechanical labor involved. A somewhat simpler method, using equivalent continous characteristics instead of discrete distributions is given below.

Suppose it is desired to obtain the convolution of the two continuous distributions $P_1(t)$ and $P_2(t)$ shown in Fig. 3.3(a) and (b). Both distributions are of the rectangular type, but Fig. 3.3(a) has, in addition, a delta function of magnitude $k$ at the origin. The equivalent en route-deviation distribution is given by

$$P_{eq}(r) = \int_{-\infty}^{\infty} P_1(t) P_2(t+r) \, dt \tag{3.2}$$
It is easily seen that for \( b > a \) Eq. 3.2 gives the symmetrical \( P_{eq}(r) \) distribution shown in Fig. 3.4. The problem now remains to relate the continuous distributions to the discrete distributions, so that the continuous \( P_{eq}(r) \) curve forms the envelope of the discrete \( P_{eq}(m) \) distribution. One immediately obvious point concerns the relation of the spread \( S \) and the probability of delay. For rectangular distributions in the continuous case, a spread of \( S \) units gives a uniform delay probability of \( 1/S \). For the discrete case, the same spread \( S \) gives a uniform delay probability of \( 1/(S+1) \). Furthermore the extreme delays, at \( \pm (b+a) \) in Fig. 3.4, have zero probability of occurrence in the continuous case, whereas in the discrete case these extreme delays must have a nonzero probability value. These difficulties may be overcome by the simple expedient of using a larger spread for the continuous distributions. Thus if spreads \( S_1 \) and \( S_2 \) are specified for the two legs of the flight in the discrete case, the continuous distributions used to obtain \( P_{eq}(r) \) are taken to have spreads of \( S_0 = S_1 + 1 \) and \( S'_0 = S_2 + 1 \) respectively. The final distribution will now be too wide by two units, so the two extreme units each having zero probability are removed from \( P_{eq}(r) \), giving a finite probability for the actual extreme discrete delays (or advances).

The parameters of Fig. 3.3(a), (b) now become

\[
\begin{align*}
a &= \frac{S_1 + 1 - n}{2} = \frac{S_0 - n}{2} \\
Q_1 &= S_1 + 1 = S_0 \\
k &= 1 - \left( \frac{S_1 + 1 - n}{S_1 + 1} \right) = \frac{n}{S_0} \\
b &= \frac{S_2 + 1}{2} = \frac{S'_0}{2} \\
Q_2 &= S_2 + 1 = S'_0
\end{align*}
\]

where \( S_1 \) and \( S_2 \) are the spreads of the discrete en route-deviation distributions of the two legs AB, BC (Fig. 2.1), and \( n \) is the control available at point B.

With the parameters given by Eq. 3.3, the equivalent continuous en route-deviation distributions are given in Figs. 3.5(a), (b), (c), depending upon the relative magnitudes of
a, b and n of Fig. 3.3. The procedure for determining the equivalent en route distribution may thus be outlined as follows.

Given $S_1$, $S_2$, and $n$

1. Form $S_0 = S_1 + 1$
   
   $S'_0 = S_2 + 1$

2. Check $S'_0 > S_0 - n$
   
   $S'_0 = S_0 - n$

3. Draw the envelope given by the appropriate figure chosen in (2).

4. Remove one unit from each end of the figure, leaving $S'_0 + S'_0 - n - 2 = S_1 + S_2 - n$
   
   spaces, and $S_1 + S_2 - n + 1$ discrete integer abscissae.

5. At each such discrete integer abscissa draw in a vertical line (ordinate) to touch the envelope.

Fig. 3.5
Equivalent en route-deviation distributions, $P_{eq}(r)$.

These lines form the discrete distribution, $P_{eq}(m)$. It is essential in applying the above continuous method that $n$, $S_1$, and $S_2$ all be even numbers, or else $n$ and $S_1$ be odd numbers, with $S_2$ even. Otherwise, while $P_{eq}(m)$ will be a discrete distribution, $m$ will take on both integer and noninteger values which are all multiples of 1/2. Moreover,
the envelope of the discrete distributions may not be given correctly by the continuous equivalent curves in such cases. These situations are inconvenient, though of course possible. For the purpose of a general investigation, however, there is really no need to consider them, since an inconvenient combination can always be bracketed by two more convenient ones.

As an example of the above procedure, consider the case \( S_1 = 6, S_2 = 6, n = 2 \). Then \( S_0 = 7, S'_0 = 7 \) and \( S'_0 > S_0 - n \). Hence Fig. 3.5(a) is chosen as the appropriate model. The dotted envelope shown in Fig. 3.6 is now drawn according to the specifications of Fig. 3.5(a), and the \( S_1 + S_2 - n + 1 = 11 \) ordinates are drawn in (omitting the two end portions) to form the equivalent en route-deviation distribution. Figures 3.7 and 3.8 show the equivalent distributions for the same \( S_1 \) and \( S_2 \), but with \( n = 0 \) and \( n = 4 \) respectively. For no control (\( n = 0 \)) the convolution of the two independent rectangular distributions obviously gives a triangular distribution of spread 12. For \( n = 4 \), when the control is approaching full control, the equivalent distribution is quite rectangular in shape. The equivalent discrete distributions shown in the figures were also checked directly on a discrete basis, according to Eq. 3.1.

3.3 Use of Equivalent En Route-Deviation Distributions.

As mentioned previously, once the equivalent distribution has been determined the results for the no-control cases described in reference 1 may be applied to determine the congestion at the terminal. As a simple illustration, the equivalent distribution shown in Fig. 3.6 may be regarded as approximately parabolic in shape, with a spread \( S = 10 \). For an assumed traffic parameter \( \epsilon = 0.95 \), (Fig. 2.14, Ref. 1) gives the average delay as approximately 1.7 units, and shows that about 32 percent of the aircraft will be delayed by at least 3 units (Fig. 4.10, Ref. 1). With no control, and the same individual en route-deviation distributions on the legs AB and BC, the equivalent distribution is given by Fig. 3.7. Application of the no-control results for the triangular case, with a spread \( S = 12 \), then gives an average delay of 2 units and approximately 37 percent of the aircraft delayed by at least 3 units. Thus the introduction of quite a small amount of control reduces the average delay by about 15 percent. For smaller values of the traffic parameter the difference would not be so noticeable.

Table 3 gives results for values of \( n \) ranging from zero to full control, namely \( n = 0, 2, 4, 6 \) for \( \epsilon = 1.0, 0.95, 0.8 \) and 0.5. The equivalent en route distributions for \( n = 2 \) and 4, shown in Figs. 3.6 and 3.8, were taken to be approximately parabolic in form, with spreads of 10 and 8 respectively. For full control (\( n = 6 \)) the distribution is of course rectangular, with a spread of 6.

The results listed in Table 3 show an interesting trend in the average stack delay. For values of \( \epsilon \) close to unity, a small amount of control is almost as effective as full control in reducing the average stack delay. The reduction in the number of planes delayed by at least 3 units (arbitrarily chosen) increases continuously with increasing control, except for small \( \epsilon \) values, where the introduction of any amount of control has
little effect. This is to be expected from the results given in reference 1, where it is shown that for small values of the traffic parameter the congestion is very insensitive to changes in either the form or the spread of the en route-deviation distribution.

It is hardly necessary to point out that the identification of the en route-deviation distributions in Figs. 3.6 and 3.8 with the "parabolic" case of reference 1 is rather crude. More refined techniques for identification are discussed in that reference, but did not seem justified here merely for the presentation of an illustrative example.

Figs. 3.6, 3.7 and 3.8
Equivalent total en route-deviation distribution; n = 2, 0, 4 respectively.

Table 3
One-Point On-Time Control Case, $S_1 = S_2 = 6$.
Comparison of Results for Varying Amounts of Control.

<table>
<thead>
<tr>
<th>Traffic Parameter $\epsilon$</th>
<th>Average Stack Delay, $\overline{\tau}$</th>
<th>Percent of Planes Delayed by at Least 3 Units</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n = 0$</td>
<td>$n = 2$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.35</td>
<td>0.35</td>
</tr>
<tr>
<td>0.8</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>0.95</td>
<td>2.0</td>
<td>1.7</td>
</tr>
<tr>
<td>1.00</td>
<td>3.5</td>
<td>2.9</td>
</tr>
</tbody>
</table>
IV. Block Scheduling

4.1 Outline of Method

In this method of scheduling, the time axis is broken up into blocks, each of duration $S'$ units. Within each block, the total number of planes scheduled to leave is specified, but the actual time within the block at which each must leave is not fixed in detail by the schedule. The over-all average traffic density (over many blocks) is still defined by the traffic parameter $\epsilon$, and within any block no more than $S' + 1$ planes are scheduled. The situation thus merely amounts to specifying that a certain number of planes should arrive at the destination within a certain time. This by itself, in the absence of further deviations, would lead to the possibility of small stacks occurring at the destination. In addition, however, the effects of the actual en route flight errors give rise to further terminal congestion. The problem of calculating the stack delays thus produced is readily carried out on a numerical basis, using IBM punched-card techniques.

In order to set up the problem for the IBM machines, it is necessary to specify the starting time of each aircraft, add a random en route delay, calculate the new arrival sequence, and then finally obtain the stack delays. Once the starting time has been specified, the problem is exactly the same as for the no-control case (1). The only real innovation arises in determining when the aircraft commence their flight. That is, the writing of the original block schedule constitutes the only new element in this problem.

Figure 4.1 is a time diagram of some of the sequences used for constructing a block schedule. Time is measured in $t$ units, and the $t_j$ sequence previously used as the proper schedule for a specified $\epsilon$ is indicated as well. Thus a proper schedule for any desired $\epsilon$ forms the starting point for the block schedule construction with the same $\epsilon$. The total time interval is now broken up into successive, noncontiguous blocks of length $S'$ units each, starting at $t_1 = 1$. The first time in each block is called the block mark, $B_\mu$, so that for a block length of $S'$ units the $B_\mu$ are as follows

$$
B_1 = 1 \\
B_2 = S' + 2 \\
B_3 = 2S' + 3 \\
\vdots \\
B_\mu = (\mu-1)(S' + 1) + 1 \\
\vdots \\
B_\ell = (\ell-1)(S' + 1) + 1 \\
$$

(4.1)

$B_\ell$ is the value of the last block mark needed in order to accommodate 1000 planes. Thus $\ell$ is the nearest integer to $1000/(S' + 1)$. This is liable to introduce a slight change in the actual value of $\epsilon$, since previously, for the proper schedule

$$
\epsilon_{\text{exact}} = \frac{1000}{t_{1000}}
$$

(4.2)
whereas now

$$\epsilon_{\text{exact}} = \frac{1000}{I(S' + 1)}$$  \hspace{1cm} (4.3)

The difference between the values of $\epsilon$ given by Eqs. 4.2 and 4.3 will not be more than about 1 percent, while neither value will differ from the nominal one by more than 3 percent (for the 1000 plane samples employed here).

Now that the blocks have been established, the aircraft specified by the $t_j$ sequence are assigned to the particular block within whose limits they fall. Thus, with reference to Fig. 4.1, aircraft 1 to 6 inclusive are assigned to block 1, aircraft 7 to 11 inclusive to block 2, and so on. Hence each block now contains a number of aircraft less than or equal to $S' + 1$, with the over-all traffic density equal to $\epsilon$.

It is now necessary to take the aircraft contained in each block and distribute them in a random manner throughout the block. This is accomplished by first assigning to each plane a new time $t'_j$, where

$$t'_j = B \mu \text{ for } B \mu < t_j < B_{\mu+1}$$

$$t'_j = B_I \text{ for } B_I < t'_j$$  \hspace{1cm} (4.4)

In this manner all the planes in a given block are placed at the beginning of that block. To each $t'_j$ is now added a random number $r'_j$, taken from a rectangular distribution of spread $S'$, so that the planes contained within each block are placed at random throughout the block interval. The sequence thus obtained may be regarded as the actual take-off sequence $t''_j$ of the aircraft, which then experience an en route delay, exactly as in the case of no-control with proper scheduling. If the actual en route delay is taken to be $r_j$, from a rectangular distribution of spread $S$, then $p'_j$, the actual arrival time of the $j$th scheduled aircraft, is given by

$$p'_j = t'_j + r'_j + r_j = t''_j + r_j$$  \hspace{1cm} (4.5)

It is important to note that $r_j$ and $r'_j$ are independent random sequences.

It is instructive to observe that the resulting arrival sequence given by Eq. 4.5 may be looked upon in a new light. Instead of describing it as the result of block scheduling $t''_j$ followed by a single random en route delay $r_j$, it may also be characterized as periodic scheduling at the block marks $B_{\mu}$, in random groups of 0 to $S' + 1$ planes each, followed by two independent en route delays $r'_j$ and $r_j$, both from rectangular...
distributions with spreads $S'$ and $S$ respectively.

Once the $p_j'$s have been determined, the stack delays are calculated by the method mentioned in Part II and described fully in reference 1.

In the cases where the two spreads $S'$ and $S$ are equal, the procedure for obtaining the $p_j'$s may be somewhat simplified. As the two distributions are assumed to be independent, the effect of using two random numbers from the two rectangular distributions of spread $S$ is equivalent to using one random number from a triangular distribution of spread $2S$. The computations in reference 1 made use of several such triangular distributions, so these results were used where possible. In the other cases, where two different random sets had to be used, care was taken in the use of the random number table (2) to see that the random number sequences used were indeed independent.

The difference between the actual arrival time, $p_j'$, and the original proper scheduled time, $t_j$, is a measure of what might be termed an "equivalent" en route delay. Its statistics represent an artificial en route-deviation distribution which, starting from a proper schedule, would have produced the same terminal congestion as was produced by the actual situation. Now it is desirable to keep all numbers positive for machine calculations. With block scheduling, the quantity $(p_j' - t_j)$ can become negative, i.e. the new actual time of arrival can occur before the original proper scheduled time. This is readily seen from Fig. 4.1 if $t_j$ is chosen at the end of a block; for example $t_{17} = 21$. This plane will have $t_{17}' = 15$, according to Eq. 4.4, and if the sum of the two random numbers (each of which is always $> 0$) satisfies

$$r_{17}' + r_{17} < 6$$

then

$$p_{17}' = t_{17}' + r_{17}' + r_{17} < 21$$

and

$$(p_{17}' - t_{17}) < 0.$$  

Furthermore it is obvious in general that the maximum magnitude of a negative value for $(p_j' - t_j)$ is $S'$, occurring when $t_j = B_j + S'$, where $B_j \leq t_j < B_{j+1}$, and $r_j' = r_j = 0$. Thus in order to avoid dealing with negative quantities, the equivalent en route delay $a$ is defined by the following equations

$$a = S' + p_j' - t_j' \quad (4.6)$$

where

$$0 \leq a \leq 2S' + S \quad (4.7)$$

The total time-keeping error, $d_j$, is defined as previously

$$d_j = r_j + \tau_j \quad (4.8)$$

For an $r_j$ distribution ranging from 0 to $S$, the total time-keeping error is always positive. If it is again assumed that the actual $r_j$ distribution ranges from $-S/2$ to $S/2$, 

-22-
permitting both delays and advances, then the total delay, \( e_j \), may be obtained from the
distribution by a simple shift of the origin

\[
e_j = d_j - \frac{S}{2}.
\]  

(4.9)

The four values of the traffic parameter used in the computations were \( \epsilon = 1.0, 0.95, 0.9, 0.8 \). In all cases the initial stack at the terminal was made zero, that is \( \tau_0 = 0 \), so that the system was never saturated (1). The values of \( S' \) and \( S \) used with the above
values of \( \epsilon \) are given in Table 4. The results derived directly from the computations

<table>
<thead>
<tr>
<th>Table 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values of ( S' ) and ( S ) Used for Block Scheduling Computations.</td>
</tr>
<tr>
<td>( S )</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>6</td>
</tr>
</tbody>
</table>

for the listed parameters were as follows:

1. Frequency distribution of \( \tau_j \)
2. Progressive distribution of \( \tau_j \)
3. Average stack delay \( \bar{\tau}_j \)
4. Frequency distribution of \( d_j \)
5. Progressive distribution of \( d_j \)
6. Average total time-keeping error \( d_j \).

In addition, for all four values of the traffic parameter, but only for \( S' = 6, S = 3, 6, 9, 12 \):

7. Frequency distribution of \( a_j \)
8. Average equivalent en route delay, \( \bar{a}_j \).

4.2 Outline of Results

Some typical results of the machine calculations are shown in Figs. 4.2 - 4.8. The
stack-delay distributions for \( S = 6, S' = 6; S = 12, S' = 12, \) and \( S = 24, S' = 18 \), are given
in Figs. 4.2 - 4.4. Figure 4.5 shows the progressive stack-delay distribution for
\( S = 6, S' = 6 \), while the total delay distributions for \( S = 6, S' = 6 \) and \( S = 12, S' = 12 \), are
given in Figs. 4.6 and 4.7 respectively. Finally, Fig. 4.8 gives the equivalent en route-
delay distributions for \( S' = 6, S = 3, 6, 9, 12 \).

As mentioned previously (Sec. 4.1), for the cases where \( S = S' \) a single triangular
en route-deviation distribution of spread \( 2S \) may be used instead of two separate and
independent rectangular distributions. This means that for these cases the only differ-
ence between the block scheduling case and the no-control case with proper scheduling
lies in sending groups of aircraft off at every block mark \( B_{\mu} \) (Sec. 4.1, Eq. 4.5ff)
Fig. 4.2a

Fig. 4.2b

Fig. 4.2c

Fig. 4.2d

Frequency distribution of stack delays.
Frequency distribution of stack delays.
Fig. 4.4
Frequency distribution of stack delays ($S = 24, S' = 18$).

Fig. 4.5
Progressive stack-delay frequencies ($S = 6, S' = 6$).

Fig. 4.6
Frequency distribution of total delays ($S = 6, S' = 6$).

Fig. 4.7
Frequency distribution of total delays ($S = 12, S' = 12$).
Fig. 4.8a  
(S = 3, S' = 6, ε = 1.0).

Fig. 4.8b  
(S = 6, S' = 6, ε = 1.0).

Fig. 4.8c  
(S = 9, S' = 6, ε = 1.0).

Fig. 4.8d  
(S = 12, S' = 6, ε = 1.0).

Frequency distribution of equivalent en route delay
instead of sending them off individually. It might be expected that this difference in procedure would not have a very great effect upon the congestion at the terminal airstrip, and to check this the stack-delay distributions for the corresponding no-control proper scheduling cases (taken from Ref. 1) were also plotted in Figs. 4.2 and 4.3. Table 5 gives the corresponding average delays. The two sets of results are undoubtedly very similar. In general, however, it appears that the block scheduling results show

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>Block Scheduling ( S = S' = 6 )</th>
<th>Proper Scheduling Triangular ( S = 12 )</th>
<th>Block Scheduling ( S = S' = 12 )</th>
<th>Proper Scheduling Triangular ( S = 24 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>3.552</td>
<td>3.510</td>
<td>4.742</td>
<td>4.470</td>
</tr>
<tr>
<td>0.95</td>
<td>1.745</td>
<td>1.950</td>
<td>2.712</td>
<td>2.680</td>
</tr>
<tr>
<td>0.9</td>
<td>1.327</td>
<td>1.526</td>
<td>1.839</td>
<td>2.211</td>
</tr>
<tr>
<td>0.8</td>
<td>0.939</td>
<td>1.015</td>
<td>1.330</td>
<td>1.299</td>
</tr>
</tbody>
</table>

slightly less congestion than the equivalent proper scheduling results. As a further check, the stack-delay distributions from reference 1 for parabolic en route errors (spread = 2S) were compared with Figs. 4.2 and 4.3. These distributions, however, were not in close agreement with the block scheduling results; they gave considerably less congestion. Hence it appears, at least in those cases where block scheduling results may be compared directly with the equivalent proper scheduling results, that dispatching aircraft in groups does not give materially different results from dispatching them individually. Any difference that does exist is apparently in favor of less congestion for the group scheduling; so it would appear that the use of the no-control proper scheduling results to estimate the effects of block scheduling would be on the pessimistic side. It must be remembered however that the block scheduling actually involves real en route flight errors of spread S, while the error in the equivalent proper scheduling case is from a distribution of spread 2S. If the two methods of scheduling were compared for the same spread of the flight error distribution alone, then the block scheduling method would be the one to give by far the worst congestion.

An obvious extension of the above method of comparison could be made for \( S \neq S' \) by use of the convolution methods described in Part III. The convolution of the two rectangular distributions of spread \( S' \) and \( S \) would give an equivalent en route-deviation distribution which could then be treated by the methods given in reference 1. It is to be anticipated, on the basis of the cases studied here, where \( S = S' \), that the congestion arising from applying this convolved distribution to a proper schedule will differ very little from that resulting when it is applied to aircraft scheduled in groups at the block marks (Sec. 4.1, Eq. 4.5 ff). Thus it is probable that the net effect on congestion of block scheduling in blocks of length \( S' \), followed by en route errors distributed with
spread S, is the same as (or only slightly better than) the result of applying to a proper schedule the equivalent en route-deviation distribution obtained by convolving the statistics characterized by spread S and S'.

The stack-delay distributions of Figs. 4.2 - 4.4 give the frequencies with which various stacks occurred in a sample of 1000 flights. Consequently a stack with a probability of occurrence of approximately one part in a thousand may or may not show up in the results. It is very unlikely that stacks with even smaller probabilities will occur at all in the sample, but it is of interest to know the maximum stack that may be expected. This may be deduced in the following simple manner. It has been proved (1) that the maximum number of planes in the air over the terminal resulting from an en route-deviation distribution of spread A is just A+1, provided the original schedule was a proper one. This result was demonstrated for any shape of deviation distribution. It was also pointed out in the present report, in connection with Eqs. 4.6 and 4.7, that the block-scheduling procedure, with block-size S', followed by an en route-deviation distribution of spread S, is equivalent to a proper scheduling procedure followed by some equivalent en route-delay distribution (a) of spread 2S' + S. Even though the shape of this equivalent deviation distribution cannot be known a priori, its existence alone immediately proves that the maximum number of planes which can be in the air over the terminal at any time is 2S' + S + 1. Since one of these planes is always in the process of landing, the maximum stack delay is

\[ \tau_{\text{max}} = 2S' + S \]  

in t units, and this is also equal to the maximum number of planes waiting to land (i.e. in the "stack"). The probability of occurrence of this maximum stack is of course extremely small, but it does set an upper bound to the size of the stacks that may occur. For example, for S = 6 and S' = 6, then \( \tau_{\text{max}} = 18 \), while the largest delay to show up in the numerical calculations, for \( \epsilon = 1 \), was \( \tau = 9 \). Similarly, for S = 24 and S' = 18, \( \tau_{\text{max}} = 60 \), while the maximum that actually occurred was \( \tau = 16 \).

The total delay distributions given in Figs. 4.6 and 4.7 clearly show that for \( \epsilon = 1.0 \) the effect of the stack delay far overcomes the effect of aircraft arriving before their scheduled times. This is exactly the same effect that was noticed in reference 1, and might indeed be expected from the previous discussion of the "equivalence" of the block scheduling method and the proper scheduling method. As \( \epsilon \) decreases and the stacking becomes less severe, the early arrivals are subject to smaller delays, and thus more planes experience a total negative delay.

The equivalent en route delay \( a \), as defined by Eqs. 4.6 and 4.7, obviously has, for a given S' and S, the same distribution for all values of \( \epsilon \). The distributions shown in Fig. 4.8 are for \( \epsilon = 1.0 \) only, but the distributions for the other values of \( \epsilon \) are practically identical with these, discrepancies being due only to using different sets of 1000 random numbers for the schedules in each case. The \( a \)-distributions are those which, if applied to a properly scheduled sequence of aircraft, would give the same congestion
as the block scheduling method used. It is not hard to show that for symmetrical \( r_j \) and \( r_j \) distributions, the \( \alpha \)-distribution should be symmetric about \( \alpha = S' + 1/2S \). The \( a_j \)'s are all positive quantities, by virtue of Eq. 4.6. If the \( S' \) that was originally added for this purpose to \( (p_j' - t_j) \) is subtracted again, the origin for the \( \alpha \)-distribution is moved \( S' \) units to the right. If then the actual flight errors, \( r_j \), are assumed to be either positive or negative, with equal probability, the origin must be shifted an additional \( S/2 \) units to the right. When this is done with the \( \alpha \)-distributions of Fig. 4.8, the new origin in each case is clearly at the center of symmetry of the distribution.
V. Continuous Flow Calculations

5.1 Outline of Method

The previous delay calculations, in this report and in reference 1, give estimates of the stack delays caused by random flight deviations from some originally prepared schedule, either with or without the benefit of en route control. One of the main points brought out by these calculations is that the average stack delay experienced by the aircraft does not appear to be as serious as was at first anticipated, even under fairly heavy traffic conditions. Moreover, the maximum possible delays, while large perhaps, occur extremely infrequently. The fact remains, however, that in practice aircraft sometimes do suffer relatively long delays, either being rerouted to a different airport, or else spending a long time in a stack. Since random flight deviations appear to be only a minor factor in causing such delays, some other delaying effect must be at work. One such effect may be the change in the effective minimum safe landing time to brought about by a change in weather conditions at the receiving airport. Under VFR conditions, it may well happen that to is of the order of one or two minutes. The change to instrument flight rules, using perhaps ILS or GCA, invariably is accompanied by an increase in the minimum safe landing interval, because of safety requirements. This means that the airport capacity may be reduced severely, and unless the rate at which aircraft arrive is reduced immediately, congestion is bound to occur.

This fact then forms the basis for an investigation of the delays caused by sudden changes in the landing rate at the airport. In order to carry out this investigation analytically, the following rather drastic simplifying assumptions are made: (1) The flow of aircraft is assumed to be continuous, rather than discrete. Time is no longer either quantized or normalized, unless specifically stated to the contrary. (2) The aircraft are assumed to arrive on time; there are no random en route deviations (no bunching). (3) The flight time of the aircraft, T, units, from the take-off point (or an equivalent rerouting point) to their arrival at the control zone of the destination, is assumed to be the same for all aircraft. Further detailed assumptions about specific conditions are made during the progress of the work, and are best explained as they arise.

The general pattern of events is visualized as follows. Aircraft approach the receiving airport at a steady rate, and are landed without any delay. The maximum landing rate at the airport is then changed abruptly, and continues at its new reduced rate for a time T_o, called the airport shutdown time. At the end of this T_o period the airport reverts back to its original maximum landing rate. The problem now depends upon the behavior of the incoming stream of aircraft during this period. Two major possibilities present themselves. The incoming rate may remain constant, regardless of the behavior of the acceptance rate. This may be termed the "no-feedback" case. Alternatively, the receiving airport may send a message to the take-off point, or an equivalent rerouting point, stating the change in landing conditions. This of course is called the "feedback" case.
The action taken at the take-off point upon receipt of the feedback message offers many possibilities, and initially two were considered. In one, the flow of aircraft was completely stopped; in the other, the flow was partially stopped. This latter action gave rise to undue analytical complication without adding much new information, and was consequently discarded in favor of the complete cessation of traffic flow.

It is further assumed that once a plane has taken off on its flight, of duration $T_f$, it is no longer possible to effect any control over it. Thus although the flow of traffic is curtailed at its source at the moment when the receiving airport shuts down, the aircraft continue to arrive at the destination at their previous rate for a time $T_f$ after the beginning of the shutdown. At the end of the shutdown period, the receiving airport informs the sending airport of its return to normal conditions, and the flow of traffic commences again. Then only after another delay of $T_f$ does the receiving airport return completely to its original state of operation.

The notation to be used is as follows:

- $a(t)$: maximum acceptance rate at receiving airport
- $r(t)$: incoming rate at receiving airport
- $\epsilon = \frac{r}{a} \leq 1$: traffic parameter before and after shutdown period only
- $k$: airport shutdown factor, $0 \leq k \leq 1$
- $T_o$: time of shutdown of airport (arbitrary units)
- $T_f$: time from take-off to arrival at control zone
- $T$: total time during which a nonzero stack exists
- $S(t)$: stack size
- $\frac{T_o}{T_f}$: dimensionless parameter
- $D$: total aggregate delay; equal to the product of the average delay and the total number of planes delayed.

All times are in the same units as $T_o$, but the units are arbitrary. Note that the traffic parameter $\epsilon$ is defined above as the ratio of the incoming rate to the maximum acceptance rate under normal conditions, i.e. before and after the airport changes its acceptance rate. Actually $\epsilon$ will only be used here at times when $r(t)$ and $a(t)$ are constants, so it will not be taken as a function of time.

The numerical quantities of primary interest in these problems will be: (1) the average stack delay over the period during which a nonzero stack exists; (2) the maximum stack delay suffered by any plane, and (3) the aggregate total delay (in plane-hours for example) suffered by all delayed planes as a result of the airport shutdown.

5.2 The No-Feedback Case

Figure 5.1 shows the time variation of the acceptance rate and the arrival rate, together with the corresponding stack size. In period A the system is stable ($r = \epsilon a; \epsilon < 1$) and has zero stack. At the beginning of period B the acceptance rate drops suddenly from $a$ to $ka$ and, provided $k < \epsilon$, a stack begins to form at a rate $(r - ka)$. Thus
at the end of period B there is a maximum stack $S_{\text{max}}$, where

$$S_{\text{max}} = T_o(r - ka) = T_o a(\varepsilon - k) . \quad (5.1)$$

The airport now returns to full operation and, consequently, has an excess landing rate of $(a - r) = (1 - \varepsilon)a$ with which to reduce the stack. Therefore

$$T_s = T_o + \frac{T_o a(\varepsilon - k)}{a - r}$$

or

$$T_s = T_o \left(\frac{1 - k}{1 - \varepsilon}\right) . \quad (5.2)$$

For $\varepsilon = 1$, Eq. 5.2 indicates that $T_s$ is infinite, i.e. the stack never clears. The reason for this may be seen from Fig. 5.1, since for $\varepsilon = 1$, $r = a$. Thus during period B the stack builds up to $T_o a(1-k)$, but when the airport resumes full operation again, it is saturated. It has no excess landing capacity since $(a-r) = 0$; consequently the stack remains at its maximum height.

The average stack over the entire period $T_s$ during which a nonzero stack exists is easily seen from Fig. 5.1 to be

(a) $\bar{S}(t) = \frac{1}{2} a(\varepsilon - k) T_o$ for $\varepsilon < 1$

(b) $\bar{S}(t) = a(1-k) T_o$ for $\varepsilon = 1$ \quad (5.3)

In order to calculate the average delay time it is necessary to use the fact that, upon arriving at the control zone, a plane enters the stack and cannot land until all of the previous aircraft in the stack have landed. That is, planes are landed from the stack on a first-come first-served basis. Figure 5.1 shows a plot of stack height against time, in which the time origin is taken at the beginning of the shutdown period.

Consider a group of $(\alpha dt)$ planes arriving in time interval $[t_1, t_1 + dt)$, when the stack height is $a(\varepsilon-k)t_1$. It is assumed that $t_1 < T_o$, as shown in the figure. The planes then in the stack will be landed at a rate $ka$ until time $T_o$, hence the group which arrived at $t_1$ will be delayed an aggregate time

$$dD = \frac{a(\varepsilon-k)t_1}{ka} \int_a^1 \epsilon \alpha dt = \left(\frac{\epsilon}{k} - 1\right) t_1 \alpha dt$$

provided that

$$0 \leq t_1 \leq \frac{k}{\epsilon} T_o . \quad (5.5)$$

Note that for the stack to build up at all
On the other hand if \( t_1 > \frac{k}{E} T_0 \), then there will be \( [a(\epsilon-k) t_1 - (T_0 - t_1) k a] \) planes still left to land at the end of the \( T_0 \) period. These will land at the rate \( a \), so the aggregate delay of the group is then given by

\[
dD = \left( (T_0 - t_1) + [(\epsilon-k) t_1 - (T_0 - t_1) k] \right) \epsilon a dt
\]

or

\[
dD = \left[ T_0 (1-k) - t_1 (1-\epsilon) \right] \epsilon a dt
\]

when

\[
0 \leq k \leq t_1 \leq T_0 \quad (5.8)
\]

Finally consider a group of \( \epsilon a dt \) planes arriving in time interval \( (t_2, t_2 + dt) \), with \( t_2 \geq T_0 \). The stack height at this moment is \( [T_0 a(1-k) - t_2 a(1-\epsilon)] \), so the aggregate delay of the group is given by

\[
dD = \left[ T_0 (1-k) - t_2 (1-\epsilon) \right] \epsilon a dt, \text{ for } t_2 \geq T_0 \quad (5.9)
\]

Equation 5.9 is identical with Eq. 5.7 apart from the different subscripts attached to the time variable. Hence the total aggregate delay suffered by all of the planes arriving during the stack period \( T_s \) is given by integrating Eqs. 5.4, 5.7 and 5.9 over their corresponding periods of validity (Eqs. 5.5, 5.8, 5.9). Therefore

\[
D = \text{Total Aggregate Delay} = \int_0^{T_0} \left( \frac{\epsilon}{k} - 1 \right) t \epsilon a dt + \int_{k T_0}^{T_0 (1-k) \over 1-\epsilon} \left[ T_0 (1-k) - t (1-\epsilon) \right] \epsilon a dt
\]

\[
= \frac{T_0^2 a}{2} \frac{(1-k)(\epsilon-k)}{(1-\epsilon)}
\]

Equation 5.12 gives the average delay experienced by those planes which arrive during the period \( T_s \) when the stack is not zero. The derivation of \( \bar{\tau} \) by the above method is quite straightforward, but tends to be rather unwieldy, particularly in more complicated cases involving the feedback. It is much more convenient to compute \( \bar{\tau} \) from the
ratio of the average stack height $S(t)$ to the average landing rate $a(t)$ or the average arrival rate $r(t)$, all averages being taken over the period $T_s$ during which the stack is never zero. That $\overline{S}$ can very generally be computed correctly in this manner may be demonstrated in the following way.

Consider a stack beginning at $t = 0$ and ending at $t = T_s$, as shown in Fig. 5.2. The variation of $S(t)$ is quite arbitrary, but is subject to the conditions

(a) $S(0) = S(T_s) = 0$

(b) $S(t) > 0$ for $0 < t < T_s$.

Using the notation defined previously, and Eq. 5.13, it is clear that

$$S(t) = \int_0^t \left[ r(\xi) - a(\xi) \right] d\xi$$

for $0 \leq t \leq T_s$. (5.14)

Equation 5.14 puts certain general restrictions upon $r(t)$ and $a(t)$, since they must lead to a stack which obeys Eq. 5.13. Granting these general restrictions, Eq. 5.14 is valid regardless of the detailed form of either $r(t)$ or $a(t)$. In particular, since $S(T_s) = 0$

$$\int_0^{T_s} a(\xi)d\xi = \int_0^{T_s} r(\xi)d\xi.$$ (5.15)

That is, for such a stack variation, the total number of arriving planes equals the total number landed.

The group of planes $r(\eta) \, d\eta$ which arrives in the interval between $t = \eta$ and $t = \eta + d\eta$ (Fig. 5.2) will have to wait until $t = \rho$ to land, because time is required to clear the stack $S(\eta)$. The time $\rho$ is therefore fixed by the condition

$$S(\eta) = \int_\eta^\rho a(\xi)d\xi.$$ (5.16)

which states simply that all the planes in the stack at $t = \eta$ must have landed [at rate $a(t)$] by the time $t = \rho$. From Eq. 5.14 for $S(\eta)$, Eq. 5.16 becomes

$$\int_0^\eta r(\xi)d\xi = \int_0^\rho a(\xi)d\xi.$$ (5.17)
and this relation defines \( \rho \) as a single-valued function of \( \eta \). Thus the delay of a plane is a function of the time it arrives, as might be expected.

Now the average delay \( \bar{\tau} \) is the total aggregate delay \( D \) divided by the total number of planes delayed

\[
\bar{\tau} = \frac{\int_0^T (\rho-\eta) r(\eta) d\eta}{\int_0^T r(\eta) d\eta}
\]

or in view of Eq. 5.15

\[
\bar{\tau} = \frac{\int_0^T (\rho-\eta) r(\eta) d\eta}{\int_0^T a(\eta) d\eta}
\]

By an integration by parts

\[
\int_0^T \rho r(\eta) d\eta = \left[ \rho \left( R(0) + \int_0^\eta r(\xi) d\xi \right) \right]_{\eta=0}^{\eta=T} - \int_0^T \left( R(0) + \int_0^\eta r(\xi) d\xi \right) \frac{d\rho}{d\eta} d\eta
\]

in which \( R(0) \) represents the total number of arrivals up to \( t = 0 \). Moreover because of Eqs. 5.13(a) and 5.17

\[
\begin{align*}
\rho &= 0 \text{ when } \eta = 0 \\
\rho &= T_s \text{ when } \eta = T_s
\end{align*}
\]

This simply means that planes which arrive when the stack is zero are not delayed at all. In addition, Eq. 5.17 can be used directly to eliminate

\[
\int_0^\eta r(\xi) d\xi
\]

from the second term on the right side of Eq. 5.20. A change of integration variable from \( \eta \) to \( \rho \) in that term then yields

\[
\int_0^T \rho r(\eta) d\eta = T_s \int_0^T r(\xi) d\xi - \int_0^T \left[ \int_0^\rho a(\xi) d\xi \right] d\rho
\]

From similar reasoning

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\begin{equation}
\int_0^{T_s} \eta r(\eta) d\eta = T_s \int_0^{T_s} r(\xi) d\xi - \int_0^{T_s} \left\{ \int_0^{\eta} r(\xi) d\xi \right\} d\eta . \tag{5.23}
\end{equation}

Recognition that \( \rho \) in the second integral on the right side of Eq. 5.22 is merely functioning there as a variable of integration leads to the desired alternative form of Eq. 5.19 (or Eq. 5.18)

\begin{equation}
\bar{\tau} = \frac{\int_0^{T_s} \left\{ \int_0^{\eta} [r(\xi) - a(\xi)] d\xi \right\} d\eta}{\int_0^{T_s} a(\eta) d\eta} = \frac{S(t)}{\bar{a}(t)} = \frac{S(t)}{r(t)} \tag{5.24}
\end{equation}

in which use has been made of Eqs. 5.14, 5.15 and the rather obvious definitions

(a) \( S(t) = \frac{1}{T_s} \int_0^{T_s} S(\eta) d\eta \)

(b) \( \bar{a}(t) = \frac{1}{T_s} \int_0^{T_s} a(\eta) d\eta \)

(c) \( \bar{r}(t) = \frac{1}{T_s} \int_0^{T_s} r(\eta) d\eta \). \tag{5.25}

As an example of the simplicity afforded by the result of Eq. 5.24, the average stack delay for the case previously treated may be found in this alternative manner. From Fig. 5.1 and Eq. 5.25

\( \bar{a}(t) = \frac{T_o ka + (T_s - T_o) a}{T_s} \)

which by Eq. 5.2 becomes

\( \bar{a}(t) = \epsilon a = r(t) \). \tag{5.26}

This incidentally checks Eq. 5.15. Use of Eqs. 5.3, 5.24 and 5.26 then yields directly

\( \bar{\tau} = \left( 1 - \frac{k}{\epsilon} \right) \frac{T_o}{2} \) \tag{5.27}

which is indeed identical with Eq. 5.12.

Now the maximum delay experienced by any one plane must be obtained from the
delay equations 5.4, 5.7 and 5.9. A brief examination of these will show that the maximum delay occurs at \( t = \left( \frac{k}{\epsilon} \right) T_o \), and has the value

\[
\tau_{\text{max}} = \left( 1 - \frac{k}{\epsilon} \right) T_o .
\]  

(5.28)

Observe that the maximum delay does not occur when the stack is largest.

In order to make comparisons with the feedback cases, discussion of the results found above will be postponed until the next section.

5.3 The Feedback Case

When feedback is introduced into the problem, the calculations fall into two major groups, defined by \( T_o > T_f \) and \( T_o < T_f \). As an illustration of the procedure, consider \( T_o > T_f \). Figure 5.3 shows the corresponding acceptance rate, arrival rate, and stack height for prescribed values of \( k \) and \( \epsilon \) in the ranges

\[
0 < k < \epsilon < 1 .
\]

During period A the system is undisturbed \((r = \epsilon a; \epsilon < 1)\) and there is no stack. At the beginning of period B the acceptance rate drops to \( ka \), and a stack immediately begins to form at a rate \( a(\epsilon - k) \), provided \( k < \epsilon \). At time \( T_f \) after the change in acceptance rate, the arrival rate becomes zero, and, in consequence, the stack no longer increases but begins to decrease at a rate \( ka \). After a further period of \((T_o - T_f)\) the airport resumes its normal acceptance rate. As no more planes will arrive until a time \( T_f \) has elapsed, the remaining portion of the stack decreases with the clearance rate \( a \).

Thus at time \( T_f \) the stack height is

\[
S(T_f) = T_f a(\epsilon - k)
\]  

and at time \( T_o \)

\[
S(T_o) = T_f a \left( \epsilon - k \frac{T_o}{T_f} \right) .
\]  

(5.30)

Note that \( S(T_o) \) is nonnegative only if \((\epsilon/k) \geq (T_o/T_f)\); if \((\epsilon/k) < (T_o/T_f)\), the stack clears entirely during period B. This is shown in Fig. 5.4.

Returning to the case \((\epsilon/k) \geq (T_o/T_f) \geq 1\), the time taken to clear the stack left at time \( T_o \) is simply

\[
\frac{S(T_o)}{a} = T_f \left( \epsilon - k \frac{T_o}{T_f} \right) .
\]
Therefore

\[ T_s = T_o + T_f \left( \frac{T_o}{T_f} \right) \]

for

\[ 1 \leq \frac{T_o}{T_f} \leq \frac{\epsilon}{k} \]  

(5.31)

The average stack over the period \( T_s \) may be computed in a straightforward manner, with the result

\[ S = \frac{aT_o (1-k) \left( 2\epsilon - k \frac{T_o}{T_f} \right) - \frac{T_f}{T_o} (1-\epsilon)}{2 \left[ \epsilon + \frac{T_o}{T_f} (1-k) \right]} \]

for

\[ 1 \leq \frac{T_o}{T_f} \leq \frac{\epsilon}{k} \]  

(5.32)

When \( 1 < \frac{\epsilon}{k} \leq \frac{T_o}{T_f} \) (Fig. 5.4), the corresponding expressions for \( T_s \) and \( S \) are

\[ T_s = T_f \left( \frac{\epsilon}{k} \right) \]

and

\[ S = \frac{1}{2} T_f a(\epsilon-k) \]  

(5.33)

both for

\[ 1 < \frac{\epsilon}{k} \leq \frac{T_o}{T_f} \]  

(5.34)

It is evident that in every case \( T_s \leq T_o + T_f \).

The average delay, \( \tau \), may be calculated by exactly the same means used in the no-feedback case (Eq. 5.24). The average landing rate \( \bar{a}(t) \) over the interval \( T_s \), when \( \frac{\epsilon}{k} \geq \frac{T_o}{T_f} \geq 1 \) is given by

\[ \bar{a}(t) = \frac{kaT_o + a(T_s - T_o)}{T_s} = \frac{a\epsilon}{\epsilon + \frac{T_o}{T_f} (1-k)} \]  

(5.35)

The average stack delay is therefore

\[ \tau = \frac{T_o}{2\epsilon} \left[ (1-k)(2\epsilon - kx) - \frac{x}{k} (1-\epsilon) \right] \]

for

\[ 1 \leq x \leq \frac{\epsilon}{k} \]  

(5.36)
in which the convenient notation

\[ x = \frac{T_o}{T_f} \]  

has been introduced. In all cases thus far, \( x \geq 1 \).

The aggregate delay \( D \) of all planes is just \( \tau \) times the total number of planes \( (N) \) which are delayed (i.e. the total number which arrive during \( T_s \)). By reason of Eqs. 5.15, 5.35, and 5.31

\[ N = T_s \bar{a}(t) = kaT_o + a(T_s - T_o) = a \epsilon T_f \]  

and therefore

\[ D = \frac{aT_o T_f}{2} \left[ (1-k)(2 \epsilon - kx) - \frac{\epsilon}{x} (1-\epsilon) \right] \]  

for \( 1 < x \leq \frac{\epsilon}{k} \) (5.39)

In order to find the maximum delay, the delay of planes arriving at all possible times within \( T_s \) must first be considered. Thus in Fig. 5.3 a group of planes arriving in time interval \( (t_1, t_1 + dt) \), with \( t_1 \leq T_f \), finds a stack of \( a(\epsilon - k)t_1 \) planes waiting to land, causing a delay for any plane in the group of

\[
\begin{align*}
(a) \quad \tau &= \left( \frac{\epsilon - k}{k} \right) t_1 \\
(b) \quad 0 \leq t_1 \leq T_f
\end{align*}
\]  

(5.40)

But Eq. 5.40 is only valid if, in addition

\[ \left( \frac{\epsilon - k}{k} \right) t_1 \leq T_o - t_1 \]  

or

\[ 0 \leq t_1 \leq \frac{k}{\epsilon} T_o \]  

(5.41)

The real limiting condition on \( t_1 \) in Eq. 5.40(a) therefore depends upon whether \( T_f < (k/\epsilon)T_o \) or \( T_f \geq (k/\epsilon)T_o \). The latter case will be considered for illustration here. When, therefore, \( T_f \geq t_1 > (k/\epsilon)T_o \), the group of planes under consideration will not have landed by time \( T_o \). The number of planes still due to land before them at this time is

\[ a(\epsilon - k)t_1 - (T_o - t_1)ak = a(\epsilon t_1 - kT_o) \]

Because these planes now land at rate \( a \), the total delay of the group is
Now it is clear that the maximum delay will occur for some group of planes arriving at or before $T_f$, during which time the stack takes on its largest value and the landing rate remains small. Thus the maximum delay of any one plane may be obtained from either Eq. 5.40 or 5.42, with $t_1 = (k/\epsilon)T_o$. Its value is

$$\tau = T_o(1-k) - t_1(1-\epsilon)$$

for

$$T_f > t_1 > \frac{k}{\epsilon}T_o$$

(5.42)

The calculations for the case where $1 < (\epsilon/k) \leq x$ and the stack therefore clears completely during period B (Figs. 5.3 and 5.4) follow essentially the same procedure as outlined above.

In the same manner, for $T_f > T_o$ (i.e. $x < 1$) the calculations split up into two portions, depending upon when the stack finally clears. The complete results are summarized in Table 6, which gives the length of the stack period, the maximum stack size, the maximum delay, the average delay, and the aggregate delay. More will be said about the aggregate delay later.

It is essential to keep in mind that no stack forms at all unless $\epsilon > k$, and this condition is therefore implicit in all formulas of Table 6. In no case can $\epsilon \leq k$.

From Table 6 the average delay, $\overline{\tau}$, is in general seen to be a function of the variables $T_o$, $T_f$, $\epsilon$, and $k$. It is convenient to assume that $T_o$ is fixed (i.e. a normalizing factor) and to specify $T_f$ by using the variable $x = T_o/T_f$. For a fixed value of $k$, a family of curves showing $\overline{\tau}$ vs. $x$ for various values of $\epsilon$ as parameter may be prepared. This was done for $\epsilon = 1.0, 0.95, 0.90, 0.80, 0.70, 0.60, 0.50, \text{and } k = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$. Figures 5.5, 5.6 and 5.7 show these families of curves for $k = 0, 0.3$ and 0.6 respectively. The maximum delays are shown similarly in Fig. 5.8 for the representative cases $k = 0, 0.3; \epsilon = 1.0, 0.8, 0.6, 0.5$.

Figure 5.5, for $k = 0$, is peculiar in the fact that all of the curves, regardless of $\epsilon$, approach $\overline{\tau}/T_o = 1$ as $x$ increases. This is to be expected, since for a fixed $T_o$ the only way in which $x (= T_o/T_f)$ can increase is for $T_f$ to decrease. For a very small $T_f$, the planes that are delayed must have arrived at the very beginning of the $T_o$ interval, and as they cannot begin to land until the end of the $T_o$ interval, the average delay must be approximately $T_o$. Clearly the maximum delay is also $T_o$ (Fig. 5.8).

The initial portions of all the $\overline{\tau}$ curves, with the exception of those for $\epsilon = 1.0$, are horizontal, i.e. show no variation with $x$. Further, from Table 6 it may be seen that the average delays represented by those horizontal portions of the curves are identical.
<table>
<thead>
<tr>
<th>Without Feedback</th>
<th>With Feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stack Time $T_s$</td>
<td>$T_o \left(\frac{1-k}{1-\epsilon}\right)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{\epsilon T_o}{k x}$ for $1 &lt; \frac{\epsilon}{k} \leq x$</td>
</tr>
<tr>
<td>Maximum Stack $S_{max}$</td>
<td>$a(\epsilon-k)T_o$</td>
</tr>
<tr>
<td></td>
<td>$(1 - \frac{k}{\epsilon})T_o$ for $1 \leq x \leq \frac{\epsilon}{k}$</td>
</tr>
<tr>
<td></td>
<td>$\left(\frac{\epsilon}{k} - 1\right)\frac{T_o}{x}$ for $1 &lt; \frac{\epsilon}{k} \leq x$</td>
</tr>
<tr>
<td>Average Delay $\bar{\tau}$</td>
<td>$\frac{T_o}{2} \left(1 - \frac{k}{\epsilon}\right)$ for $\epsilon &lt; 1$</td>
</tr>
<tr>
<td></td>
<td>$\frac{T_o}{2\epsilon} \left[(1-k)(2\epsilon - kx) - \frac{\epsilon}{x}(1-\epsilon)\right]$ for $1 &lt; \frac{\epsilon}{k} \leq x$</td>
</tr>
<tr>
<td>Aggregate Delay $D$ also $T_s \times S(t)$</td>
<td>$aT_o^{2}(\epsilon-k)(1-k)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{2}{2(1-\epsilon)}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{\epsilon T_o}{2x} \left[(1-k)(2\epsilon - kx) - \frac{\epsilon}{x}(1-\epsilon)\right]$ for $1 &lt; \frac{\epsilon}{k} \leq x$</td>
</tr>
</tbody>
</table>

Table 6
Comparison of Results for Feedback and No-feedback Cases.

In All Cases $\epsilon > k$; Never $\epsilon \leq k$. 
Fig. 5.6 Normalized average delay vs. x; shutdown factor \( k = 0.3 \).

Fig. 5.8 Normalized maximum delay vs. x.

Fig. 5.5 Normalized average delay vs. x; shutdown factor \( k = 0 \).

Fig. 5.7 Normalized average delay vs. x; shutdown factor \( k = 0.6 \).
with the average delays for the no-feedback case. The physical reason for this is very simple. For a small enough value of \( x \) \( [x < \frac{(1-\epsilon)}{(1-k)}] \), i.e. a large enough value of \( T_f \), the effect of the change in the sending rate is not noticeable at the receiving airport for some appreciable time. When the change finally becomes evident, the airport has been back to its normal receiving rate for a time \( (T_f - T_o) \), and hence will have cleared the stack that built up during the \( T_o \) period. As all the averages are taken over the stack period, the above case differs in no respect from the case without feedback. Consequently it is not surprising that the average delays should be the same. Essentially the same type of reasoning may be applied to explain the maximum delay curves (Fig. 5.8).

The behavior of the \( \overline{T} \) curves for slightly larger values of \( x \) is rather more surprising since it appears that the average delay with feedback may be greater than the average delay without feedback. This occurs because the average is taken over the stack time. For the feedback cases, the stack time \( T_s \) is never greater than for the corresponding no-feedback case. Figure 5.9 is drawn for a value of \( T_f \) slightly greater than \( T_o \), and shows stack height versus time plots both with and without feedback (Figs. 5.9(c) and 5.9(d) respectively). These two figures are identical up to the point \( P \), after which the stack height with feedback decreases more rapidly than the stack height without feedback. The stack times on the two cases are \( T_{s1} \) and \( T_{s2} \), with \( T_{s1} < T_{s2} \), and it is fairly obvious that the average stack over \( T_{s1} \) will be greater than the average stack over \( T_{s2} \). Moreover the average landing rate is less over \( T_{s1} \) than over \( T_{s2} \), so in this instance the average delay with feedback is surely greater than without feedback (Eq. 5.24).

As \( x \) increases, i.e. \( T_f \) decreases, the maximum stack with feedback becomes smaller, and \( T_{s1} \) decreases as well. Figure 5.9d remains the same of course, so it is apparent that at some value of \( x \) the two average delays will be the same once more. This actually occurs when \( x = \frac{(T_o/T_f)}{(\epsilon/k)} \). As \( x \) is increased beyond this value the average delay for the feedback case decreases towards zero, as does the maximum delay (when \( k > 0 \)).

One important effect of the feedback system may be appreciated by considering the aggregate delays. Although, as pointed out above, the introduction of feedback can give rise to greater average delays, the number of aircraft delayed is reduced; so in comparing the two systems it would seem logical to compare the total (or aggregate) delay experienced by all the delayed aircraft. From a consideration of the
expressions for the aggregate delays given in Table 6, it can be seen that the aggregate delays for the feedback cases are always less than for the no-feedback cases, except when \( x = T_o/T_f \leq (1-\epsilon)/(1-k) \), in which case they are the same.

It is clear on the other hand that appreciable improvements in \( \bar{T} \), \( S_{\text{max}} \), and \( \tau_{\text{max}} \) result from the feedback only when the time lag \( T_f \) is sufficiently small compared to \( T_o \), \( [x > (\epsilon/k)] \). This conclusion is of course merely a statement of what common sense would dictate.
VI. Conclusions

The one-point rescheduling and on-time control procedures considered in Parts II and III appear to have very similar effects upon terminal congestion. When the traffic density and en route-deviation distributions are specified, a given maximum available amount of control can apparently reduce terminal stacking by only a certain amount, regardless of the detailed method of application. In the case of one-point control, it is obvious that the two methods must yield the same stack statistics when either no control or full control is available. The work presented in Parts II and III indicates that, even for intermediate amounts, the two methods are also nearly identical in this respect.

The controls show greatest success under heavy traffic conditions, when a moderate amount can decrease significantly the frequency of long stack delays. Thus when $\epsilon = 1.0$, $S = 6$ (on both legs of the flight) and $n = 3$, the average stack delays, with one-point rescheduling and on-time controls, are 2.72 and 2.9 units respectively. With no control, but otherwise under the same conditions, the average stack delay is 3.51 units; and, with full control, an average stack delay of about 2 units would be expected (1). For small values of $\epsilon$ (approximately $\epsilon < 0.8$) even full control has relatively little effect upon the terminal congestion.

There are, however, certain other respects in which the rescheduling and on-time control methods may differ materially. First, the rescheduling procedure exerts an advance order on the planes more frequently than a delay order (Fig. 2.4), while the on-time procedure clearly exerts about equal numbers of orders both ways. Since aircraft operating at higher engine speeds consume disproportionately more fuel, and since the total delays have similar frequency distributions in both control methods, the rescheduling control may be at a slight disadvantage in this respect. On the other hand, this excess of advance orders is not a necessity. It is possible to assume a shift of origin in the en route-control characteristic, and have the control point break up the bunches of aircraft by ordering only true delays. In this case the action of the control becomes strictly a process of spreading out the potential terminal stack delays so they occur instead along the route. Nevertheless, the congestion at the terminal is reduced and this may indeed still be beneficial, quite aside from the delay question (1).

Second, the rescheduling control evidently forces some of the planes to deviate widely from their absolute schedules. Even if the rescheduling orders were carried out perfectly on the second half of the flight, certain planes might have become interchanged in sequence during the first half. While there would be no undesirable stacking at the terminal, the individual planes might, therefore, come in significantly off their original schedules. This might be undesirable from the passengers' point of view.

Third, the choice between on-time and rescheduling controls is influenced by an additional factor. The decisions involved in performing rescheduling are somewhat more complicated than those for on-time control, and this showed up clearly in programming the IBM machines. If more information had been used, the process would
have become even more complicated. This complication, along with the other factors already considered, seems to favor on-time discrete control. Before passing final judgement, however, further study should be given to the problems of exactly what information is needed for both types of control in a practical situation and exactly how seriously the absolute schedule is upset by the rescheduling. Probably more than one control point should be considered in this connection.

It should be clear that the above comments would lose their significance if the controls were exerted continuously during the flight; there would then be little practical difference between the two kinds. Only when discrete control is considered, with sufficient flight time between successive control points (if there are several) to allow moderately serious random bunching of aircraft between them, do the two varieties of control become distinct.

The possibilities for multipoint discrete control can be exploited, using either control method, by noting that if the control points are close enough together so each one can exert full control, the total en route-deviation statistics for the journey will have a spread equal to that of the last leg alone. With reference to Fig. 2.1 for example, the control point B could be placed as near to C as possible, consistent with the requirement that it have full control. The spread of the en route-deviation distribution can be assumed proportional to flight time, so that the spread per hour of flight is a constant. The time which can be made up or lost by deliberate speed variations is also proportional to the flight time, the proportionality factor being the fractional speed variation allowed within engine efficiency limitations. It is reasonable, though not essential, to suppose that the random en route-error spread per unit flight time is less than the maximum percentage speed variation obtainable by deliberate changes of throttle setting, so that perhaps point B could be moved more than halfway toward C and still be able to exercise full control. A second control point B' might then be placed between B and C, such that \( \frac{B'C}{BC} = \frac{BC}{AC} \), and so on. In this way, by bunching the control points closer together as the destination is approached, a considerable reduction in the spread of the en route errors could be accomplished. Thus, if the original spread over AC were S, two control points might possibly reduce this to less than \( \frac{S}{4} \). Since control "points" are merely places (or times) at which a plane en route is contacted, the scheme just outlined implies that closer contact and control are required in areas near the terminal than in those far away. This conclusion has also been reached previously (1) from a different point of view; namely, that the terminal represents the point of highest space-density of aircraft. These two lines of reasoning are independent, even though they lead to the same conclusion.

Since the terminal congestion caused by en route flight errors is apparently not very serious, unless (a) the traffic density is very high and (b) the minimum landing interval is small compared to the absolute (unnormalized) flight errors (1), a very few discrete control points might well be sufficient. Present day airports for example receive a major proportion of flights which take only about two hours en route, and the minimum
safe landing interval apparently cannot be pushed much below 3 minutes (3, 4). In a flight of two hours, a total spread of 45 minutes seems sufficiently pessimistic, so \( S = 45/3 = 15 \). This could probably be reduced to about 8 with a single control point, and to 3 or 4 with two such points. In the light of the results of reference 1, these figures imply a reduction in terminal congestion to the point where it is almost negligible.

The block scheduling described in Part IV has been treated in sufficient detail to indicate the extent to which its effects may be approximated by an equivalent proper scheduling method, so as to utilize the results of reference 1. Thus Fig. 4.2(a) shows the relatively good agreement between the stack-delay frequency distribution caused by the block scheduling method with \( S = 6, S' = 6, \epsilon = 1.0 \), and that resulting from the proper scheduling method, with triangular deviation distribution of spread \( S'' = S + S' = 12, \epsilon = 1.0 \). The latter stack-delay distribution is slightly narrower and more peaked than that arising from the block scheduling, which may be expected (1) from a consideration of the equivalent en route-delay distribution of Fig. 4.8(b). This is the en route-deviation distribution that, used with proper scheduling, would produce the same stacking statistics as the block scheduling. It will be noticed that this distribution is not triangular with a spread of 12, but instead has relatively small "tails," with a total spread of 18. This would account for the fact that the results obtained from using the triangular distribution do not quite fit the actual one. Nevertheless, a reasonable approximation to the effect of block scheduling may always be obtained in this way. Block scheduling with deviation distributions having spreads \( S' \) and \( S \) is approximately equivalent to proper scheduling with a total deviation distribution given by the convolution of the two. This provides a rapid estimate (using reference 1) of the effect of block scheduling, although such an estimate will probably yield a range of stack delays which is a little too small.

The final section, on the continuous flow calculations, is intended to provide an explanation for some of the long stack delays that are so improbable from the point of view of random flight deviations. The simplifications and assumptions necessary to arrive at a simple solution make the problem somewhat artificial; for example, the assumption of a constant traffic parameter and a sudden change in landing conditions. In practice, the change in landing conditions would probably occur gradually, and the traffic flow to the airport could be influenced to some extent by radio instructions from it. The results obtained nevertheless suggest some interesting points. As an example, consider an airport operating at a traffic parameter \( \epsilon = 0.9 \), which shuts down, for two hours, to three tenths of its normal acceptance rate, i.e. \( k = 0.3 \). With the feedback system, and a flight time of two hours, \( x = T_o/T_f = 1 \). The average delay is \( 0.53 \times 2 = 1.06 \) hours, (Fig. 5.6). If the normal acceptance rate is 12 planes per hour, (i.e. minimum landing interval = 5 minutes) then the aggregate delay \( D \) is given from the formulas in Table 6, for \( x = T_o/T_f = 1 \), as 23 plane hours. With no feedback, the average delay per plane is only \( 0.335 \times 2 = 0.670 \) hours.
same curve; see p. 43), but the aggregate delay (Table 6) is 101 plane hours. The maximum delay in this case is the same with and without feedback, and has the value 1.33 hours. The maximum stack, also the same for both cases, is about 14 planes. With feedback, the stack lasts 3.2 hours; without feedback it lasts 14 hours. Thus the average stack height is approximately 7 planes in both cases. It is clear that in this case the feedback helps principally by reducing the stack time, and therefore also the aggregate delay.

Evidently, under appropriate conditions, rather large delays can result from airport shutdown - even if the feedback is operating. At high traffic densities and large spreads the same might be said about en route errors; namely large delays can be produced under appropriate conditions. The point is that the normal range of spreads and traffic densities would very rarely produce these long delays (1), while the shutdown situations which can produce them appear to be more common. It is obvious on the other hand that if $T_f$ can be made much less than $T_o$ by rerouting to alternate airports those planes which are already en route, the long delays can be avoided. Alternatively, if some means were available for predicting the shutdown sufficiently in advance, $T_f$ could effectively be reduced to zero. Thus if take-offs could be stopped at a time $T_f$ in advance of the actual shutdown, the incoming rate at the terminal would become zero just as the acceptance rate dropped. Of course really adequate blind-landing facilities are better from all points of view, since they effectively prevent shutdown and leave the normal flow of traffic uninterrupted.

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