WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

The Optimum Structure of Public Prices under Conditions of Risk

by

Marti G. Subrahmanyam*

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139

The Optimum Structure of Public Prices under Conditions of Risk

by

Marti G. Subrahmanyam*

* I wish to acknwledge the helpful comments of Professors Paul MacAvoy and Stewart Myers.

The Optimum Structure of Public Prices under Conditions of Risk

The theory of pricing of publicly produced goods has received considerable attention lately. Some authors (Baumol and Bradford, Diamond and Mirrlees, Dixit, Feldstein, Mohring) have been concerned with clarifying and extending the optimal pricing rules for public enterprises derived by Boiteux and Ramsey. Another line of enquiry (Boiteux, Brown and Johnson, Pressman, Turvey, Williamson) has been the consideration of pricing in the particular context of a public utility which faces varying demands in different periods. In the former approach, the emphasis has been on considering deviations from the marginal cost pricing rule in the presence of constraints on the profit to be made by the enterprise, distributional equity or other constraints on the optimization. The analysis of public utility pricing has been oriented towards deriving optimal pricing rules in the context of variations in demand between periods.

In both problems, it has been recognized that a key element of the analysis is the interrelationship between the demands for different commodities or of the same commodity in different periods. For example, the demand for the service provided by a public telephone company in each of two periods (known as the peak and off-peak periods) depends on the price charged in both periods. Similarly,

the demand for air-mail letters depends on the prices of both air and surface-mail letters. In the context of the peak-load pricing problem, Brown and Johnson introduced the notion of stochastic variation in the demand but did not explicitly recognize the possibility of interaction between the demands in the peak and off-peak periods. A similar consideration is relevant in the general case of public pricing of close substitutes, whose demands are subject to stochastic variations. It is proposed in this paper, to extend the notion of risk to the case of pricing of goods whose demands are interdependent.

The Objective Fuction

One of the problems in the area of pricing of publicly produced goods is the specification of an objective that is both theoretically acceptable and analytically tractable. While it has been generally recognized that some notion of value of the goods to the consumer less social costs should be optimized, there has been considerable debate on the choice of the specific measure to be used. One measure that has considerable analytical appeal is the notion of consumer's surplus[1] which was defined (by Marshall) as the difference between what the consumer would be willing to

1. For an excellent critique of this concept see Harberger.

pay for a good and what he actually pays for it. Considering both the producers and consumers as a group, the gross benefit would be the sum of the consumer's surplus and the total revenue of the producer, also known as the consumer's willingness to pay. Thus, the net benefit to the group (and to society as a whole) is the difference between the consumer's willingness to pay and the costs incurred by the producer with the proviso that the producer's costs reflect social costs (ie. there are no divergences between the prices of inputs and the social opportunity costs.) It is therefore assumed for purposes of this analysis, that the objective is to maximize the difference between the consumer's willingness to pay and the producer's total costs.

Although the use of the above procedure is fairly widespread, several theoretical and practical objections to its use in cost-benefit studies have been raised. In a recent article, Harberger lists the criticisms and provides fairly persuasive arguments against their validity. The major criticisms have been a) Consumer's surplus analysis is valid only when marginal utility of real income is constant b) Consumer's surplus analysis does not take account of changes in income distribution caused by the policy change being studied c) Consumer's surplus analysis is partial equilibrium in nature and

does not take account of the general equilibrium consequences of the actions whose effects are being studied and d) Consumer's surplus analysis is valid for small changes but not for large ones.

The measurement of consumer's surplus will be conceptually accurate if the Hicks-Slutsky compensated demand curve is used for analysis so that only pure substitution effects are included. It has been argued by Mishan that even in the presence of income effects, "the difference that arises from using constant real income as against constant money income in the statistical derivation of a demand curve for a single good, is likely to be too slight relative to the usual order of statistical error to make the distinction significant in any cost-benefit study." Mishan qualifies this statement by saying that if alterations take place in the prices of closely related goods, the measure of consumer's surplus has to be suitably adjusted. Harberger provides a more convincing theoretical justification when he says that comparability of consumer's surplus measures does not require the constancy of the marginal utility of real income but only 'well-behavedness' ie. when real income falls by ΔY as a consequence of a change in one policy variable, its marginal utility should change by the same amount

as occurs when real income falls by ΔY as a consequence of a change in some other policy variable.

The criticism regarding changes in income distribution is valid. Several authors included lump sum transfers in their analysis to counteract such effects. Recently, Feldstein has proposed that the effects be explicitly incorporated in the analysis. The other two criticisms are valid in general in practical studies, though not at a theoretical level for the general equilibrium effects and size effects can be explicitly incorporated if necessary. However, to the extent that partial equilibrium analysis provides us with explicit benchmark relations, it would be more fruitful to employ it, unless the problem under consideration has widespread economy-wide effects.

Several mathematical definitions of consumer's surplus exist in the leterature[2]. The most popular definition has been the area under the demand curve above the price line as proposed by Marshall and most of the above discussion pertains to this concept. If P* is the price actually paid by consumers of a good, corresponding to an equilibrium quantity Q* , the consumer would be willing to pay a price P(Q) – P* for the

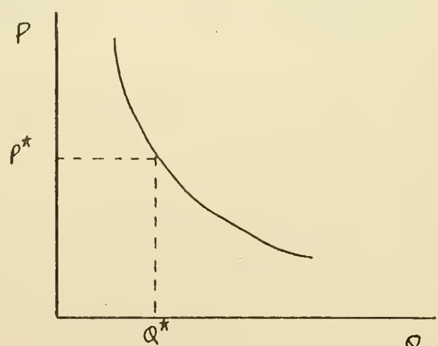


Fig. 1

good rather than go without it for every quantity Q between 0 and Q*. Thus, the summation of the minute changes in the satisfaction of the consumer is represented by the Marshallian consumer's surplus.

If P(Q) is the inverse demand function, the consumer's surplus is given by

$$(1) \qquad S = \int_0^{Q^*} (P(Q) - P^*)\, dQ = \int_0^{Q^*} P(Q)\, dQ - P^* Q^*$$

Alternatively,

$$(2) \qquad S = \int_{P^*}^{\infty} Q(P)\, dP$$

where Q(P) is the demand function. In the case of goods with independent demands, the total surplus is obtained by summing over the consumer's surplus for the different goods.

$$(3) \qquad S = \sum_{i=1}^{n} S_i = \sum_{i=1}^{n} \int_{P_i^*}^{\infty} Q_i(P_i)\, dP_i = \sum_{i=1}^{n} \int_0^{Q_i^*} P_i(Q_i)\, dQ_i - P_i^* Q_i^*$$

When the demands are interdependent, the total surplus is given by[3]

$$(4) \qquad S = \oint_{(0,0)}^{(Q_1^*, Q_2^*)} P_1\, dQ_1 + P_2\, dQ_2 - \left[P_1^* Q_1^* + P_2^* Q_2^*\right]$$

where P_1 and P_2 are each functions of both Q_1 and Q_2 ie. $P_1 = P_1(Q_1, Q_2)$ and $P_2 = P_2(Q_1, Q_2)$ and S is the total

3. See Hotelling, Maass et. al. Pressman.

consumer's surplus between $(Q_1,Q_2) = (0,0)$ and $(Q_1,Q_2) = (Q_1^*,Q_2^*)$. In terms of the price variable,

(5)
$$S = \oint_{(P_1^*,P_2^*)}^{(\infty,\infty)} Q_1\,dP_1 + Q_2\,dP_2 = \oint_{(P_1',P_2')}^{(P_1'',P_2'')} Q_1\,dP_1 + Q_2\,dP_2$$

where $Q_1 = Q_1(P_1,P_2)$ and $Q_2 = Q_2(P_1,P_2)$ and S is the total consumer's surplus between $(P_1,P_2) = (P_1',P_2')$ and $(P_1,P_2) = (P_1'',P_2'')$. It is to be noted that the concept of consumer's surplus carries over into the two good interdependent demand case, ie. what the consumer is willing to pay for both goods rather than go without them, the only change being in the definition of the limits of integration. Presumably, in order to estimate consumer's surplus in this context, some notion of the substitutability of the two goods between themselves is relevant.

Pressman clarifies that the necessary condition for an optimum (the first derivatives vanish at all critical points) is satisfied and the integration is path independent if the following conditions are satisfied,

(6)
$$\frac{\partial Q_2}{\partial P_1} = \frac{\partial Q_1}{\partial P_2}$$

ie. if and only if the rate at which the quantity of the first good demanded changes when the price of the second good changes equals the rate at which the quantity of the second good demanded changes when the price of the first good changes. This relation is not a general property of

all 'normal' goods and holds only if the two demand functions are of the Hicks-Slutsky income compensated type or if the income elasticity of demand for both goods happens to be unity.[4]

The Peak-Load Problem

The problem of peak load pricing in the context of independent demands was first attempted by Houthakker and was extended (especially into practice) by Boiteux and other economists of the French school.[5] A geometric analysis of the problem for linear costs with a capacity constraint was provided by Steiner. Williamson provided a welfare justification for the earlier analysis and extended to the case of peak and off peak pricing with unequal periods and linear costs, using more rigorous analysis. The solution was extended to the case of interdependent demands between periods by Pressman in a more general framework. Brown and Johnson introduced the concept of risk in the context of peak-load pricing

4. The cross effects $-\frac{\partial Q_1}{\partial r_2}$ and $\frac{\partial Q_2}{\partial r_1}$ - can be separated into two distinct components, a substitution effect and an income effect. By the symmetry of the problem, since the order of differentiation of the cost function with respect to the individual prices of the two goods is immaterial, the two substitution effects are equal. The income effect reflects the change in purchases of a good with a change in the price, which causes a change in the real income of the consumer. The real income changes when the budget of the consumer is fixed and he can buy more of the commodities whose price has fallen and less of commodities whose price has risen. However, when the budget is not fixed, price changes cause no appreciable change in the effective buying power of the consumer i.e. the

and considered a stochastic element of demand in their analysis. However, they considered explicitly only a single period demand function which varied stochastically between the peak and off-peak periods and stated that the result could be generalized to several periods when the demands and independent.

The present study extends the analysis to the case of interdependence between the peak and off-peak demands. Contrary to the expectation of Brown and Johnson that their results " can be easily extended to include cases in which random shocks exist in the peak-load and off-peak-load cycle" where the results are "qualitatively comparable to the simpler situation," the present analysis leads to significantly different results. In fact, only in the particular case when the peak and off-peak periods occur for equal durations are the prices in both periods equal to the operating cost per unit. When the peak period occurs less than (more than) half the time both peak and off-peak prices are greater than (less than) the operating cost per unit.

same amount of each commodity can be purchased, thus the income effects will be negligible. For a more detailed discussion, see McFadden and Winter.

5. A summary of their work is available in Dreze. Several of the articles on peak-load pricing are reprinted in Nelson.

<u>The Riskless Model</u>

In the case of a single period problem, let the demand function be represented by $Q = Q(P)$, where the function is single valued, monotone decreasing and continuously differentiable. The total surplus obtained is given by the sum of the consumer's and producer's surplus ie.

$$T = \int_{P^*}^{\infty} Q(P)\,dP + P^* Q^* \tag{7}$$

The total welfare is obtained by subtracting the total costs to the producer,

$$W = \int_{P^*}^{\infty} Q(P)\,dP + P^* Q^* - C(Q^*) = \int_{0}^{Q^*} P(Q^*)\,dQ - C(Q^*) \tag{8}$$

Assuming that the total costs are separable into fixed (capacity) costs per unit[6] β and variable (operating) costs per unit b, the cost function can be written as $C(Q) = (b+\beta)Q$, assuming that the plant capacity is completely utilized.[7] Maximizing with respect to the ~~price~~ quantity Q^*, we obtain

$$P^* = (b+\beta) \tag{9}$$

6. Following Williamson, short run marginal costs are defined as the operating costs of supplying the incremental unit of output (at levels of operation less than full capacity) over a whole demand cycle, namely b per unit per cycle. Similarly, the incremental capacity costs are defined at the rate of β per unit per cycle so that long run marginal costs are $b+\beta$ per unit per cycle

7. Marginal operating costs are constant at b per unit per

This result can be explained by reference to Fig. 2. The optimum levels of price and quantity are obtained by the intersection of the demand function DD and the long run marginal cost curve LRMC. The short run marginal cost curve SRMC is at a level b, the operating costs per unit, upto the optimum quantity Q* when it becomes vertical.

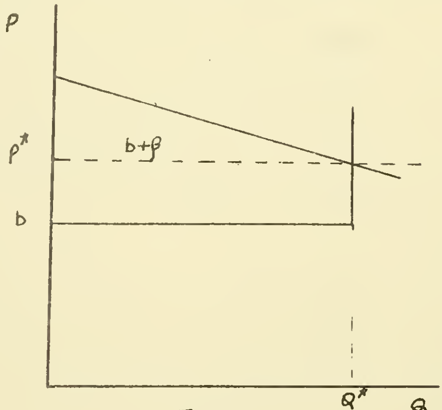


Fig 2

The solution was extended by Williamson[8] to the case of changing but independent demand functions within a planning period from peak to off-peak times. If the off-peak and peak demand functions are given by $Q_1(P_1)$ and $Q_2(P_2)$ respectively and the fraction of the total time for which the peak demand occurs is w, the welfare function becomes,

$$W = (1-\omega)\int_{P_1^*}^{\infty} Q_1(P_1)\,dP_1 + \omega\int_{P_2^*}^{\infty} Q_2(P_2)\,dP_2 + (1-\omega)P_1^* Q_1(P_1^*)$$

$$(10) \qquad + \omega P_2^* Q_2(P_2^*) - (1-\omega)Q_1(P_1^*).b - \omega Q_2(P_2^*).b$$

$$- \beta Q_2(P_2^*)$$

8. Williamson's analysis was an extension of the work done by Boiteux, Houthakker, Hirshleifer and Steiner. However, it was more general and provided a link between geometric and algebraic methods.

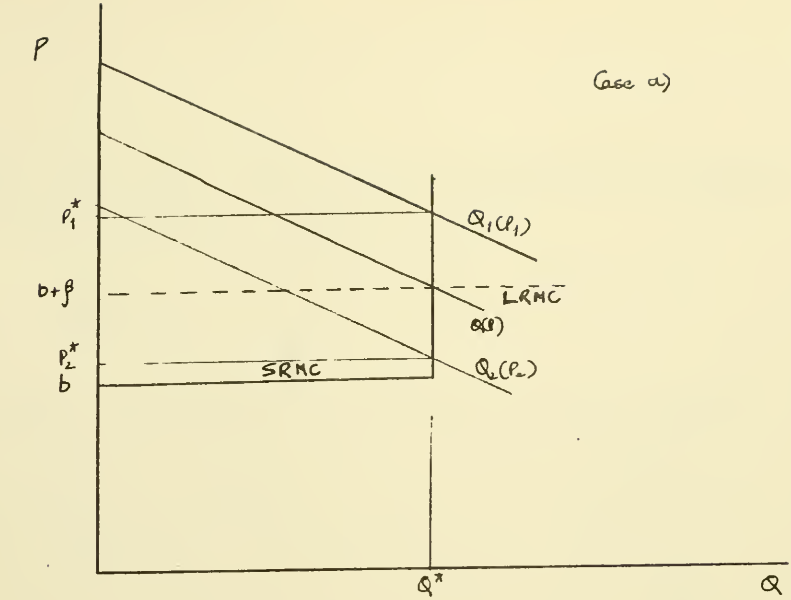
cycle, so long as output is less than capacity. When capacity is reached, a sharp kink develops and the marginal operating costs become effectively infinite.

In the above equation[8], the capacity costs per unit β are associated with the quantity $Q_2(P_2^*)$ since capacity is assumed to be perfectly divisible. There is no reason to have a capacity greater than the peak demand $Q_2(P_2^*)$ since this is the maximum possible demand; if the is multiplied by a quantity less than $Q_2(P_2^*)$, the entire capacity costs would not be covered.

Williamson distinguishes two cases, a) when the plant is used to capacity in both periods and b) when the plant is underutilized in the off-peak period. Geometrically, in the first case, the off-peak demand curve intersects the SRMC curve in the vertical section while in the second case, the off-peak demand curve intersects the SRMC curve in its horizontal section. In case a), the quantities in both periods are set equal to Q, the

8'. Equation 10 can be written in terms of the inverse demand curves for ease in further analysis ie.

$$(10a) \quad W = (1-\omega)\int_0^{Q_1^*} P_1(Q_1)\,dQ_1 + \omega\int_0^{Q_2^*} P_2(Q_2)\,dQ_2 - b\left[(1-\omega)Q_1(P_1^*) + \omega\cdot Q_2(P_2^*)\right] - \beta\, Q_2(P_2^*)$$


P
Case a)
P_1^*
$b+\beta$
P_2^*
b
SRMC
$Q_1(P_1)$
LRMC
$Q(P)$
$Q_2(P_2)$
Q^*
Q

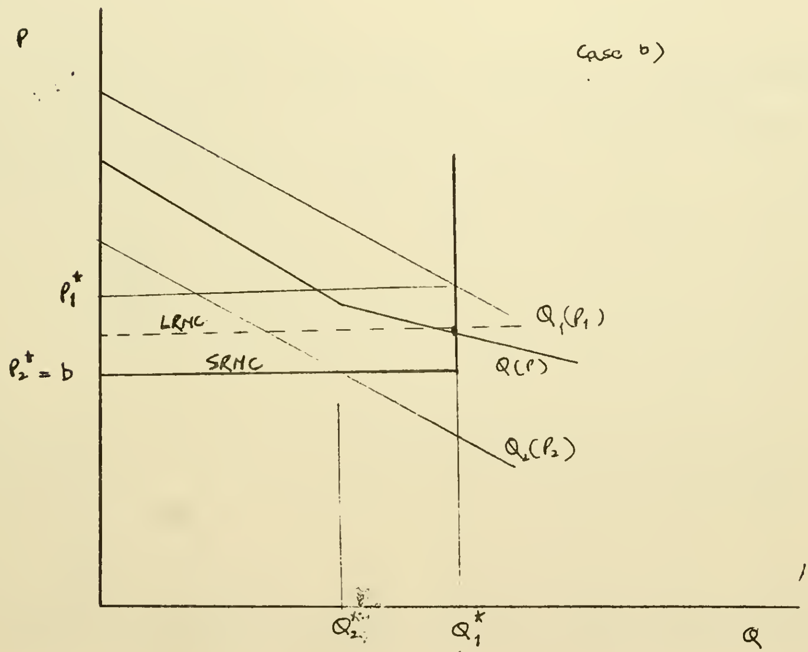

P
Case b)
P_1^*
LRMC
$P_2^* = b$
SRMC
$Q_1(P_1)$
$Q(P)$
$Q_2(P_2)$
Q_2^*
Q_1^*
Q

Fig 3

optimal plant size. After using the relationship $Q_1 = Q_2 = Q$ and differentiating the welfare function,

$$\text{(11)} \qquad \frac{\partial W}{\partial Q} = (1-\omega) P_1^* + \omega P_2^* - b[1-\omega+\omega] - \beta = 0$$

Equilibrium prices P_1^* and P_2^* clear the market in each of the sub-periods. The demand for capacity curve used by Williamson is represented by $Q(P)$ and determines the optimal capacity Q^*[9]. The amount by which revenues in the off-peak period fail to cover pro-rata total costs is precisely offset by the revenues in the peak period in excess of pro-rata costs.

In the second case, the welfare equation is differentiated partially with respect to the quantities in the two periods Q_1 and Q_2. The equilibrium conditions are

$$\text{(12)} \qquad \begin{aligned} \frac{\partial W}{\partial Q_1} &= (1-\omega) P_1 - (1-\omega) b_1 = 0 \\ \frac{\partial W}{\partial Q_2} &= \omega P_2 - \omega b - \beta = 0 \\ P_1 &= b \quad ; \quad P_2 = b + \frac{\beta}{\omega} \end{aligned}$$

The off-peak price P_1 is set equal to the marginal operating costs per unit b and no capacity costs are levied since the

9. The demand for capacity curve is obtained by adding vertically to the SRMC the vertical summation of the weighted demand for

capacity is underutilized. The peak price includes the operating costs as well as the capacity costs β spread over the peak phase of the cycle.

Thus, in general, the optimal price in each period is given by the intersection of the SRMC and the sub-period demand curve. Further, in a fully adjusted two-period model, the peak-load price is always above the LRMC, while the off-peak-load price is always below the LRMC, in the riskless case; only when the off-peak load fails to use the plant to capacity when priced at the SRMC, does the peak load bear the entire burden of the capacity costs.

Brown and Johnson introduced a stochastic demand element into a single demand function and determined the optimum price P and capacity Z before the production period begins and the actual demand is known. They concluded that the optimal price under risky conditions will always be lower than under the riskless case and with linear demand, the optimal output will generally be higher than in the riskless model. Thus, the enterprise fails to recover its capacity costs unless the risk can be diversified away through perfect risk markets.

capacity curves, which are constructed by taking the vertical difference between the periodic-load curve and the SRMC and multiplying it by the fraction of the cycle for which the particular curve arises.

The Risky Demand Model

Consider the case of a public enterprise which faces different demands during peak and off-peak periods[10]. Suppose that although the patterns of demand in the two periods are known, there is a great deal of random variation about the known patterns in the two periods. It is reasonable to assume that there is a certain interrelationship between the demands in the two periods i.e. $\frac{\partial Q_1}{\partial P_2^*} > 0$ $\frac{\partial Q_2}{\partial P_1^*} > 0$ where Q_1 and Q_2 are the demands in the off-peak and peak-periods respectively and P_1^* and P_2^* are the respective prices.

In order to simplify the analysis, it is assumed that the peak demand occurs for a fraction w of the total planning period, while the off-peak demand occurs during the rest of the planning cycle, 1-w. The manager of the enterprise is faced with the problem of choosing the optimum prices P_1^* and P_2^* in the two periods as well as the optimum plant size Z. The choice of a particular price for the peak period not only affects the quantity demanded in that period but also the off-peak period demand. If the peak period price is chosen at a level without regard to its effect on the off-peak demand, it may happen that there is a shift in demand from one period to the next, which now

10. A similar analysis can be carried out in the pricing of different goods with interdependent demands. However, general results do not emerge in that case without making resyrictive assumptions regarding the nature of the costs.

renders the price chosen for the peak-period non-optimal. It is therefore imperative to choose the prices in the two periods with due consideration being given to the interaction between the respective demands.

The riskless demands in the two periods are given by

$$(13) \qquad Q_1 = Q_1(P_1^*, P_2^*) \qquad Q_2 = Q_2(P_1^*, P_2^*)$$

The prices corresponding to zero demands in the two periods are finite i.e. in

$$(13a) \qquad Q_1(P_1^*, P_2^*) = 0 \quad \text{and} \quad Q_2(P_1^*, P_2^*) = 0$$

$$P_1^* ; P_2^* < \infty$$

The Additive Case

It is assumed that the stochastic element in the demand enters as follows

$$(14) \qquad D_1 = Q_1(P_1^*, P_2^*) + \tilde{v}_1$$

$$D_2 = Q_2(P_1^*, P_2^*) + \tilde{v}_2$$

where the probability density functions $\phi(\tilde{v}_1)$ and $\phi(\tilde{v}_2)$ of the disturbances $\tilde{v}_1$ and $\tilde{v}_2$ have the following properties.

$$E(\tilde{v}_1) = E(\tilde{v}_2) = 0$$

$$Var(\tilde{v}_1) < \infty \quad , \quad Var(\tilde{v}_2) < \infty$$

(15) $$\int_{-\infty}^{a_1} \phi_1(\tilde{v}_1)\, dv_1 = \Phi_1(a_1), \int_{-\infty}^{a_2} \phi_2(\tilde{v}_2)\, dv_2 = \Phi_2(a_2)$$

$$Cov(\tilde{v}_1, \tilde{v}_2) = 0$$

Since the capacity Z is chosen before the value of the random disturbances $\tilde{v}_1$ and $\tilde{v}_2$ are known, the actual demands D_1 and D_2 could be greater than or less than Z depending on the size of the disturbance terms. Thus, the actual sales in the two periods M_1 and M_2 are given by

(16) $$M_1 = \begin{cases} Q_1(P_1^*, P_2^*) + \tilde{v}_1 & , \quad \tilde{v}_1 \leq Z - Q_1(P_1^*, P_2^*) \\ Z & , \quad \tilde{v}_1 > Z - Q_1(P_1^*, P_2^*) \end{cases}$$

$$M_2 = \begin{cases} Q_2(P_1^*, P_2^*) + \tilde{v}_2 & , \quad \tilde{v}_2 \leq Z - Q_2(P_1^*, P_2^*) \\ Z & , \quad \tilde{v}_2 > Z - Q_2(P_1^*, P_2^*) \end{cases}$$

The sales in a particular period will be given by D_1 (or D_2) if it is less than the capacity Z. If the disturbance $\tilde{v}_1$ (or $\tilde{v}_2$) results in a demand D_1(or D_2) which is greater than Z only Z will be sold.

As in the case of the riskless model, it is assumed that the total costs are separable into fixed (capacity costs) per unit β and variable (operating) costs per unit b. The only change is that the capacity costs are known with certainty to be βZ while the operating costs are dependent on the probability distribution of the

stochastic demand. The maximand in the risky model is similar to that in the riskless case with the difference that expected values are used instead of certain benefits and costs. Thus net social welfare is given by

W = E(Consumer's Willingness to Pay)
- E(Sales).Operating Costs per Unit
- E(Capacity Costs)

This equation assumes that expected values of social benefits and costs are equivalent to their respective expected utilities to society. This assumption is valid in cases such as the present analysis, where the objective is the prescription of optimal social policy and the risk of system ruin due to the failure of a particular enterprise is negligible. Further, it is also assumed that society's portfolio of projects is suitably diversified so that explicit allowance for risk is unnecessary.[11]

If it can be assumed that the demand is always less than capacity in both periods, the consumer's surplus in the riskless case with interdependent demands will be

$$(17) \quad S = \oint_{(p_1^*,p_2^*)}^{(p_1',p_2')} (1-w)\, Q_1(p_1^*,p_2)\, dp_1 + w\, Q_2(p_1,p_2)\, dp_2$$

11. For a discussion see Samuelson, Vickrey.

where P_1^* and P_2^* are the prices charged for off-peak and peak loads respectively; P_1' and P_2' are the maximum prices of the off-peak and peak-loads respectively; $Q_1(P_1,P_2)$ and $Q_2(P_1,P_2)$ are the off-peak and peak-demands respectively (as functions of prices P_1 and P_2 in the off-peak and peak periods respectively.).

When the stochastic variations in the demands are introduced into the analysis, the consumer's surplus becomes

$$S = \int_{-\infty}^{+\infty} \phi_1(\tilde{v}_1) \int_{-\infty}^{+\infty} \phi_2(\tilde{v}_2) \oint_{(P_1^*,P_2^*)}^{(P_1',P_2')} \Big[(1-\omega)\big(Q_1(P_1,P_2)+\tilde{v}_1\big)\,dP_1 + \omega\big(Q_2(P_1,P_2)+\tilde{v}_2\big)\,dP_2\Big]\,dv_2\,dv_1 \qquad (18)$$

where the demand terms include the stochastic elements $\tilde{v}_1$ and $\tilde{v}_2$ and are integrated over the range of prices and then over the entire range of variation of the stochastic terms. The upper limit of integration is given by the prices in the two periods corresponding to zero demands in the whole cycle,

$$\begin{aligned} Q_1(P_1',P_2') + \tilde{v}_1 &= 0 \\ Q_2(P_1',P_2') + \tilde{v}_2 &= 0 \\ (P_1',P_2') &= \left(Q_1^{-1}(-\tilde{v}_1,-\tilde{v}_2),\ Q_2^{-1}(-\tilde{v}_1,-\tilde{v}_2)\right) \end{aligned} \qquad (19)$$

As indicated earlier, the line integral can be written as the sum of two definite integrals when the cross derivatives

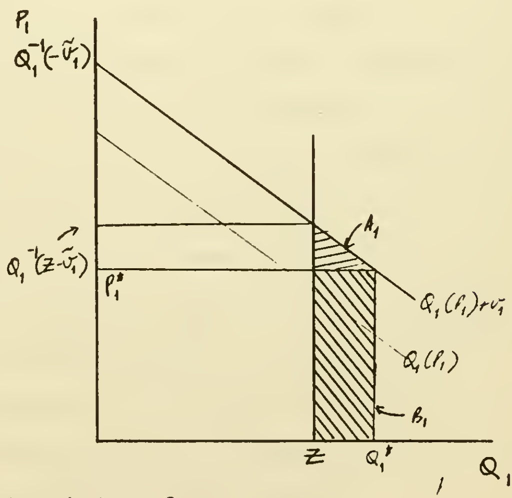
$\frac{\partial Q_1}{\partial P_2}$ and $\frac{\partial Q_2}{\partial P_1}$ are equal. Further, since the stochastic terms in the two periods are uncorrelated by assumption, the consumer's willingness to pay can be written as

$$T = (1-\omega)\int_{-\infty}^{+\infty} \phi_1(\tilde{v}_1) \int_{P_1^*}^{P_1'} (Q_1(P_1, P_2^*) + \tilde{v}_1)\, dP_1\, dv_1$$

(2-0) $$+ \quad \omega \int_{-\infty}^{+\infty} \phi_2(\tilde{v}_2) \int_{P_2^*}^{P_2'} (Q_2(P_1', P_2) + \tilde{v}_2)\, dP_2\, dv_2$$

$$+ \ (1-\omega)\, P_1^* Q_1(P_1^*, P_2^*) + \omega\, P_2^* Q_2(P_1^*, P_2^*)$$

However, there are cases when the demands D_1 and D_2 in the two periods are greater than capacity Z and consequently, a reduction in the consumer's willingness to pay takes place. This occurs for large positive values of $\tilde{v}_1$ and $\tilde{v}_2$. If the analysis were restricted to a single period's demand and price, the reduction in consumer's willingness to pay would be as shown in Fig. 4. When $D_1 = Q_1(P_1) + \tilde{v}_1$ is greater than the capacity Z, the reduction in willingness to pay is given by the sum of the two areas A_1 and B_1. In this case, the areas A_1 and B_1 are given by the following integrals.



$$E(A) = \int_{Z-Q_1(P_1)}^{\infty} \phi_1(\tilde{v}_1) \int_{P_1^*}^{Q_1^{-1}(Z-\tilde{v}_1)} (Q_1(P_1) + \tilde{v}_1 - Z)\, dP_1\, dv_1$$

(21)

$$E(B) = \int_{Z-Q_1(P_1)}^{\infty} \phi_1(\tilde{v}_1) \cdot P_1 \left(Q_1(P_1) + \tilde{v}_1 - Z\right) dv_1$$

The assumption made here is that in the event of a shortage, consumers with a higher willingness to pay are serviced first. Only then will the total willingness to pay be equal to the area under the demand curve upto the capacity constraint. If an alternative assumption is made regarding the distribution of output in the event of a shortage, the value of consumer's surplus will have to be adjusted accordingly; the shadow price of an extra unit of output obtained through additional capacity would be higher and would yield an even larger value of Z. In the above integrals, the lower limit of integration of the error termsis obtained from the fact that the above adjustments arise only when the error term more than makes up the difference between the capacity Z and the riskless demand $Q_1(P_1)$ ie. $\tilde{v}_1 = Z - Q_1(P_1)$. Thus the integration of $\tilde{v}_1$ proceeds between $Z - Q_1(P_1)$ and infinity. The limits of integration of the price in the equation for A_1 are from the price actually charged P_1^* to the price at which the demand including the stochastic term just becomes equal to the capacity Z, by setting $Q_1(P_1) + \tilde{v}_1 = Z$ and solving for $P_1 = Q_1^{-1}(Z-\tilde{v}_1)$.

The notion of losses in consumer's surplus can now be extended to the two good case. It is clear that as in the definition of consumer's surplus for interdependent goods, the losses A and B will be line integrals which can be written as the sum of two definite integrals when the cross derivates are equal and the stochastic elements in the two periods are not correlated. Thus the areas A and B are given by

(22)

$$E(A) = (1-\omega)\int_{Z-Q_1(P_1^*,P_2^*)}^{\infty}\phi_1(\tilde{v}_1)\int_{P_1^*}^{P_1''}\left(Q_1(P_1,P_2^*)+\tilde{v}_1-Z\right)dP_1\,dv_1$$

$$+\ \omega\int_{Z-Q_2(P_1'',P_2^*)}^{\infty}\phi_2(\tilde{v}_2)\int_{P_2^*}^{P_2''}\left(Q_2(P_1'',P_2)+\tilde{v}_2-Z\right)dP_2\,dv_2$$

$$E(B) = (1-\omega)\int_{Z-Q_1(P_1^*,P_2^*)}^{\infty}\phi_1(\tilde{v}_1)\left[Q_1(P_1^*,P_2^*)+\tilde{v}_1-Z\right]P_1\,dv_1)$$

$$+\ \omega\int_{Z-Q_2(P_1^*,P_2^*)}^{\infty}\phi_2(\tilde{v}_2)\left[Q_2(P_1^*,P_2^*)+\tilde{v}_2-Z\right]P_2\,dv_2$$

The distinction between these areas and the single price-demand case in equation (21) is that the quantity variables Q_1 and Q_2 are functions of both P_1^* and P_2^* in the two-good case. Further, the upper limits of integration of the error terms are obtained by setting the demands in both periods (including the stochastic elements) equal to the capacity Z.

(23)

$$Q_1(P_1'',P_2'')+\tilde{v}_1 = Z \quad ; \quad Q_2(P_1'',P_2'')+\tilde{v}_2 = Z \quad ,$$

$$(P_1'',P_2'') = \left[Q_1^{-1}((Z-\tilde{v}_1),(Z-\tilde{v}_2)),\ Q_2^{-1}((Z-\tilde{v}_1),(Z-\tilde{v}_2))\right]$$

The expected sales in the whole planning cycle is given by the expected values of the demands in the two periods weighted by the fraction of the total period for which each period occurs after subtracting the range when actual demand exceeds capacity.

$$E(s) = (1-\omega) Q_1 + \omega Q_2 - (1-\omega) \int_{Z-Q_1}^{\infty} \phi_1(\tilde{v}_1)\big(\tilde{v}_1 - (Z-Q_1)\big)\,dv_1 \tag{24}$$
$$- \omega \int_{Z-Q_2}^{\infty} \phi_2(\tilde{v}_2)\big(\tilde{v}_2 - (Z-Q_2)\big)\,dv_2$$

The expected costs are obtained by multiplying the expected sales by b, the marginal operating costs per unit and adding the capacity costs βZ. The net social welfare is written as

$$W = (1-\omega) \int_{-\infty}^{+\infty} \phi_1(\tilde{v}_1) \int_{\rho_1^*}^{\rho_1'} Q_1(\rho_1, \rho_2^*)\, d\rho_1\, dv_1 + \omega \int_{-\infty}^{+\infty} \phi_2(\tilde{v}_2) \int_{\rho_2^*}^{\rho_2'} Q_2(\rho_1', \rho_2)\, d\rho_2\, dv_2$$
$$+ (1-\omega) \rho_1^* Q_1 + \omega \rho_2^* Q_2$$
$$- (1-\omega) \int_{Z-Q_1}^{+\infty} \phi_1(\tilde{v}_1) \left\{ \left[\int_{\rho_1^*}^{\rho_1''} Q_1(\rho_1, \rho_2^*) + \tilde{v}_1 - Z \right] d\rho_1 + \rho_1^* \left[Q_1 + v_1 - Z \right] \right\} dv_1 \tag{25}$$
$$- \omega \int_{Z-Q_2}^{+\infty} \phi_2(\tilde{v}_2) \left\{ \left[\int_{\rho_2^*}^{\rho_2''} Q_2(\rho_1'', \rho_2) + \tilde{v}_2 - Z \right] d\rho_2 + \rho_2^* \left[Q_2 + v_2 - Z \right] \right\} dv_2$$
$$- b \left[(1-\omega) Q_1 + \omega Q_2 - \left\{ (1-\omega) \int_{Z-Q_1}^{\infty} \phi_1(\tilde{v}_1) \left[\tilde{v}_1 - (Z-Q_1) \right] dv_1 + \omega \int_{Z-Q_2}^{\infty} \phi_2(\tilde{v}_2) \left[\tilde{v}_2 - (Z-Q_2) \right] d \right. \right.$$
$$- \beta Z$$

Differentiating this expression [12] with respect to the prices in the two periods P_1^* and P_2^*, and the capacity Z, the first order conditions for a welfare maximum are obtained as follows.

$$\frac{\partial W}{\partial P_1^*} = (1-\omega) P_1^* \frac{\partial Q_1}{\partial P_1^*} \Phi_1 [Z-Q_1] + \omega P_2^* \frac{\partial Q_2}{\partial P_1^*} \Phi_2 [Z-Q_2]$$

$$- b \left\{ (1-\omega) \frac{\partial Q_1}{\partial P_1^*} \Phi_1 [Z-Q_1] + \omega \frac{\partial Q_2}{\partial P_1^*} \Phi_2 [Z-Q_2] \right\} = 0$$

$$\frac{\partial W}{\partial P_2^*} = (1-\omega) P_1^* \frac{\partial Q_1}{\partial P_2^*} \Phi_1 [Z-Q_1] + \omega P_2^* \frac{\partial Q_2}{\partial P_2^*} \Phi_2 [Z-Q_2]$$

$$- b \left\{ (1-\omega) \frac{\partial Q_1}{\partial P_2^*} \Phi_1 [Z-Q_1] + \omega \frac{\partial Q_2}{\partial P_2^*} \Phi_2 [Z-Q_2] \right\} + K = 0$$

(26) $$K = (1-\omega) \int_{P_1^*}^{P_1'} \frac{\partial Q_1}{\partial P_2^*} (P_1, P_2^*) \, dP_1 - (1-\omega) \int_{Z-Q_1}^{\infty} \phi_1(\tilde{v}_1) \int_{P_1^*}^{P_1''} \frac{\partial Q_1}{\partial P_2^*} (P_1, P_2') \, dP_1 \, dv_1$$

$$+ \omega Q_2 (P_1^*, P_2^*) - \omega Q_2 (P_1', P_2^*)$$

$$+ \omega \int_{Z-Q_2}^{\infty} \phi_2(\tilde{v}_2) \left[Q_2 (P_1'', P_2^*) - Q_2 (P_1^*, P_2^*) \right] dv_2$$

$$\frac{\partial W}{\partial Z} = (1-\omega) \int_{Z-Q_1}^{\infty} \phi_1 (\tilde{v}_1) (P_1'' - b) \, dv_1$$

$$+ \omega \int_{Z-Q_2}^{\infty} \phi_2(\tilde{v}_2) (P_2'' - b) \, dv_2 - \beta = 0$$

12. In the above and later equations, the demands Q_1 and Q_2 are evaluated at the prices P_1^* and P_2^*, unless otherwise specified.

The term K in the expression for $\frac{\partial W}{\partial P_2^*}$ can be rewritten as

$$(27a)\quad \begin{aligned} K = & (1-w)\left[\int_{P_1^*}^{P_1'} \frac{\partial Q_1}{\partial P_2^*}(P_1, P_2^*)\,dP_1\right] \\ & -(1-w)\left[1-\Phi_1(Z-Q_1)\right]\left[\int_{P_1^*}^{P_1''} \frac{\partial Q_1}{\partial P_2^*}(P_1, P_2')\,dP_1\right. \\ & + w\left[Q_2(P_1^*, P_2^*) - Q_2(P_1', P_2^*)\right] \\ & + w\left[1-\Phi_2(Z-Q_2)\right]\left[Q_2(P_1'', P_2^*) - Q_2(P_1^*, P_2^*)\right] \end{aligned}$$

But since Q_1 and Q_2 are the off peak and peak demands respectively, $Q_1 < Q_2$ or $(Z-Q_1) > (Z-Q_2)$. Assuming that the distributions of the stochastic elements in the two periods are similar, $\Phi(\alpha) = \Phi_1(\alpha) = \Phi_2(\alpha)$, we have[13]

$$\begin{aligned} K = & (1-2w)\left[Q_2(P_1', P_2^*) - Q_2(P_1^*, P_2^*)\right] \\ & -(1-2w)\left[Q_2(P_1'', P_2^*) - Q_2(P_1^*, P_2^*)\right] \\ & +(1-2w)\left[\Phi(Z-Q_1)\right]\left[Q_2(P_1'', P_2^*) - Q_2(P_1^*, P_2^*)\right] \\ & - w\left[\Phi(Z-Q_2)\right]\left[Q_2(P_1'', P_2^*) - Q_2(P_1^*, P_2^*)\right] \end{aligned}$$

$$(27b)\quad \begin{aligned} = & (1-2w)\left[Q_2(P_1', P_2^*) - Q_2(P_1^*, P_2^*)\right] \\ & +\left[(1-w)(\Phi(Z-Q_1)) - w(\Phi(Z-Q_2))\right] \\ & \left[Q_2(P_1'', P_2^*) - Q_2(P_1^*, P_2^*)\right] \end{aligned}$$

13. This can be done when $P_1' = Q_1^{-1}[-\tilde{\sigma}_1, -\tilde{\sigma}_2] < \infty$, $P_1'' = Q_2^{-1}[Z-\tilde{\sigma}_1, Z-\tilde{\sigma}_2] < \infty$ for then the respective integrals will converge. In this case,

$$\int_{P_1^*}^{P_1'} \frac{\partial Q_1(P_1, P_2^*)}{\partial P_2^*}\,dP_1 = \int_{P_1^*}^{P_1'} \frac{\partial Q_2}{\partial P_1^*}(P_1, P_2^*)\,dP_1 = \frac{\partial}{\partial P_1^*}\int_{P_1^*}^{P_1'} Q_2(P_1, P_2^*)\,dP_1$$

$$= \left[Q_2(P_1', P_2^*) - Q_2(P_1^*, P_2^*)\right]$$

Similarly, $$\int_{P_1^*}^{P_1''} \frac{\partial Q_1(P_1, P_2^*)}{\partial P_2^*}\,dP_1 = \left[Q_2(P_1'', P_2^*) - Q_2(P_1^*, P_2^*)\right]$$

When $w < \frac{1}{2}$, we can write

$$K = (1-2w)\left[Q_2(\rho_1', \rho_2^*) - Q_2(\rho_1'', \rho_2^*)\right]$$

(27c)
$$+ \left[(1-2w)\,\Phi(z-Q_1) + w \int_{z-Q_2}^{z-Q_1} \phi(\tilde{v})\,d\tilde{v}\right]$$
$$\left[Q_2(\rho_1'', \rho_2^*) - Q_2(\rho_1^*, \rho_2^*)\right]$$

Here $\rho_1' > \rho_1''$ since $\rho_1' = Q_1^{-1}[-\tilde{v}_1, -\tilde{v}_2]$ and $\rho_1'' = -Q_1^{-1}[z-\tilde{v}_1, z-v_2]$.

Hence $Q_2(\rho_1', \rho_2^*) > Q_2(\rho_1'', \rho_2^*)$ since $\frac{\partial Q_2}{\partial \rho_1} > 0$. Similarly, $Q_2(\rho_1'', \rho_2^*) > Q_2(\rho_1^*, \rho_2^*)$. Since $2w < 1$, $\Phi(z-Q_1) > 0$ and $\int_{z-Q_2}^{z-Q_1} \phi(\tilde{v})\,dv > 0$ as $Q_2 > Q_1$, K is positive when $w < \frac{1}{2}$.

When $w > \frac{1}{2}$, K can be written as

$$K = (1-2w)\left[Q_2(\rho_1', \rho_2^*) - Q_2(\rho_1'', \rho_2^*)\right]$$

(27d)
$$+ (1-2w)\left[\Phi(z-Q_2) + (1-w)\int_{z-Q_2}^{z-Q_1} \phi(\tilde{v})\,dv\right]$$
$$\cdot \left[Q_2(\rho_1'', \rho_2^*) - Q_2(\rho_1^*, \rho_2^*)\right]$$

By an argument similar to that in the previous case, K is negative when $w > \frac{1}{2}$. Thus, we can distinguish three cases a) $w < \frac{1}{2}$ b) $w > \frac{1}{2}$ c) $w = \frac{1}{2}$, for which the value of K is positive, negative and zero respectively,

When $K \neq 0$, we can write the simultaneous equations for P_1^* and P_2^* as follows[14];

$$(28)\quad \begin{aligned} (1-w)\, P_1^* \frac{\partial Q_1}{\partial P_1^*} \Phi_1 + w\, P_2^* \frac{\partial Q_2}{\partial P_1^*} \Phi_2 &= b\left[(1-w) \frac{\partial Q_1}{\partial P_1^*} \Phi_1 + w \frac{\partial Q_2}{\partial P_1^*} \Phi_2\right] \\ (1-w)\, P_1^* \frac{\partial Q_1}{\partial P_2^*} \Phi_1 + w\, P_2^* \frac{\partial Q_2}{\partial P_2^*} \Phi_2 &= b\left[(1-w) \frac{\partial Q_1}{\partial P_2^*} \Phi_1 + w \frac{\partial Q_2}{\partial P_2^*} \Phi_2\right] - K \end{aligned}$$

Thus, P_1 and P_2 can be written as[15]

$$(29)\quad \begin{aligned} P_1 &= b + \frac{w K \Phi_2 \frac{\partial Q_2}{\partial P_1^*}}{D_0} = b + k_1 \\ P_2 &= b - \frac{(1-w) K \Phi_1 \frac{\partial Q_1}{\partial P_1^*}}{D_0} = b + k_2 \end{aligned}$$

where k_1 and k_2 have the same sign as K. Hence, when $w < \frac{1}{2}$ (case a), $P_1 > b$ and $P_2 > b$, while for $w > \frac{1}{2}$ (case b), $P_1 < b$ and $P_2 < b$.[15a]

14. Φ_1 and Φ_2 refer to $\Phi_1[z - Q_1]$ and $\Phi_2[z - Q_2]$ respectively.

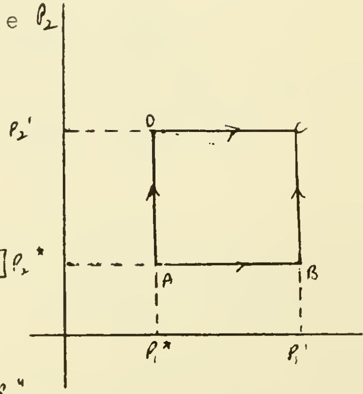
15. Since the own price effects $\frac{\partial Q_1}{\partial P_1^*}$ and $\frac{\partial Q_2}{\partial P_2^*}$ are negative and the cross price effects $\frac{\partial Q_1}{\partial P_2^*}$ and $\frac{\partial Q_2}{\partial P_1^*}$ are positive, but the own price effects have a greater absolute magnitude, the determinant

$$D_0 = \begin{vmatrix} (1-w) \frac{\partial Q_1}{\partial P_1^*} \Phi_1 & w \frac{\partial Q_2}{\partial P_1^*} \Phi_2 \\ (1-w) \frac{\partial Q_1}{\partial P_2^*} \Phi_1 & w \frac{\partial Q_2}{\partial P_2^*} \Phi_2 \end{vmatrix}$$

is positive.

13 (contd.) Otherwise, the convergence of the integrals will depend on the specific demand function used.

15a It is easily demonstrated that these results are not dependent on the particular path of integration chosen. Instead of integrating along ABC, we now proceed along ADC i.e. from (ρ_1^*, ρ_2^*) to (ρ_1^*, ρ_2') and then to (ρ_1', ρ_2'). In order not to complicate the notation, I interchange the peak and off-peak demands – w becomes the period of off peak demand and (1-w) the peak period, Q_1 the peak demand and Q_2 the off peak demand.



Equations (28) now becomes

$$\omega \rho_1^* \frac{\partial Q_1}{\partial \rho_1^*} \Phi_1 + (1-\omega) \rho_2^* \frac{\partial Q_2}{\partial \rho_1^*} \Phi_2 = b \left[\omega \frac{\partial Q_1}{\partial \rho_1^*} \Phi_1 + (1-\omega) \frac{\partial Q_2}{\partial \rho_1^*} \Phi_2 \right]$$

$$(28') \qquad \omega \rho_1^* \frac{\partial Q_1}{\partial \rho_2^*} \Phi_1 + (1-\omega) \rho_2^* \frac{\partial Q_2}{\partial \rho_2^*} \Phi_2 = b \left[\omega \frac{\partial Q_1}{\partial \rho_2^*} \Phi_1 + (1-\omega) \frac{\partial Q_2}{\partial \rho_2^*} \Phi_2 \right] \rho_2^* - K$$

where

$$K = \omega \int_{\rho_1^*}^{\rho_1'} \frac{\partial Q_1}{\partial \rho_2^*} (\rho_1, \rho_2^*) - \omega \int_{z-Q_1}^{\infty} \phi(v_1) \int_{\rho_1'}^{\rho_1''} \frac{\partial Q_1}{\partial \rho_2^*} (\rho_1, \rho_2')$$
$$+ (1-\omega) \left[Q_2(\rho_1^*, \rho_2^*) - Q_2(\rho_1', \rho_2^*) \right]$$
$$+ (1-\omega) \left[\int_{z-Q_2}^{\infty} \Phi_2(v_2) \right] \left[Q_2(\rho_1'', \rho_2^*) - Q_2(\rho_1^*, \rho_2^*) \right] dv_2$$

It can be shown that K is positive, negative, or zero when w is >, < or = 1. The solution to (28') is

$$(29') \qquad \rho_1^* = b + \frac{(1-\omega) K \frac{\partial Q_2}{\partial \rho_1} \Phi_2}{D_0} = b + k_1$$
$$\rho_2^* = b - \frac{\omega K \frac{\partial Q_1}{\partial \rho_2} \Phi_1}{D_0} = b + k_2$$

where k_1 and k_2 have the same sign as K. We conclude as before that when the peak period is less than/greater than/equal to half the total cycle, K is positive/negative, zero and hence ρ_1^*, ρ_2^* are greater than/less than/equal to b.

/

The simultaneous equations can be solved in Case c) when $w = \frac{1}{2}$, so that $K = 0$, to write

$$(30) \quad \begin{aligned} (1-\omega) P_1^* \frac{\partial Q_1}{\partial P_1^*} \Phi_1 + \omega P_2^* \frac{\partial Q_2}{\partial P_1^*} \Phi_2 &= b\left[(1-\omega)\frac{\partial Q_1}{\partial P_1^*}\Phi_1 + \omega\frac{\partial Q_2}{\partial P_1^*}\Phi_2\right] \\ (1-\omega) P_2^* \frac{\partial Q_1}{\partial P_2^*} \Phi_1 + \omega P_2^* \frac{\partial Q_2}{\partial P_2^*} \Phi_2 &= b\left[(1-\omega)\frac{\partial Q_1}{\partial P_2^*}\Phi_1 + \omega\frac{\partial Q_2}{\partial P_2^*}\Phi_2\right] \end{aligned}$$

which on solution yield[16]

$$(31) \quad P_1^* = b \quad , \quad P_2^* = b$$

Thus, the optimum conditions indiciate that the prices in both periods should be set equal to the marginal operating costs per unit, b. This is in contrast to the results obtained in the riskless case when the optimal off-peak and peak prices $P_1^* = b$ and $P_2^* = b + \beta/w$ respectively. The implication of this result is that in the presence of risk, the enterprise should bear the capacity costs for an optimal welfare solution. It is important to note that the Brown and Johnson results do not carry over to the case of interdependent demands, except for the case when the off-peak and peak periods occur for equal durations in the total cycle.

16. In the context of optimal pricing of two close substitutes produced by a public enterprise using the same capacity but with different marginal operating costs per unit b_1 and b_2 the result would be similar except that the prices would equal the respective marginal operating costs.

The Multiplicative Case

In the multiplicative case, the stochastic elements of the demands in the off-peak and peak-periods enter in a multiplicative fashion.

$$(32) \qquad D_1 = Q_1(P_1^*, P_2^*) \cdot \tilde{v}_1 \quad , \quad D_2 = Q_2(P_1^*, P_2^*) \cdot \tilde{v}_2$$

The assumptions regarding the random element made in the additive case carry over ie. variances are finite and there is no correlation between the disturbances in the two periods, but the expected values, $E(\tilde{v}_1)$ and $E(\tilde{v}_2)$ are now equal to 1.

The welfare function is obtained along lines similar to the additive case. The total willingness to pay, without considering the effects of demand exceeding capacity is given by[17]

$$(33) \qquad \begin{aligned} T = \; & (1-\omega) \int_{-\infty}^{+\infty} \phi_1(\tilde{v}_1) \int_{P_1^*}^{P_1'} Q_1(P_1, P_2^*) \cdot \tilde{v}_1 \, dP_1 \, dv_1 \\ & + \omega \int_{-\infty}^{+\infty} \phi_2(\tilde{v}_2) \int_{P_2^*}^{P_2'} Q_2(P_1', P_2^*) \cdot \tilde{v}_2 \, dP_2 \, dv_2 \\ & + (1-\omega) P_1^* \cdot Q_1(P_1^*, P_2^*) - \omega P_2^* \cdot Q_2(P_1^*, P_2^*) \end{aligned}$$

The losses of willingness to pay due to the constraints

17. P_1' and P_2' are obtained from $Q_1(P_1', P_2') \cdot \tilde{v}_1 = 0$

imposed by capacity are

$$E(A) = (1-\omega)\int_{Z/Q_1}^{\infty} \phi_1(\tilde{v}_1) \int_{P_1^*}^{P_1''} \left[Q_1(P_1, P_2^*).\tilde{v}_1 - Z\right] dP_1 \, dv_1$$

$$+ \quad \omega \int_{Z/Q_2}^{\infty} \phi_2(\tilde{v}_2) \int_{P_2^*}^{P_2''} \left[Q_2(P_1', P_2).\tilde{v}_2 - Z\right] dP_2 \, dv_2$$

(34)

$$E(B) = (1-\omega)\int_{Z/Q_1}^{\infty} \phi_1(\tilde{v}_1) \; P_1^*(Q_1.v_1 - Z)\, dv_1$$

$$+ \quad \omega \int_{Z/Q_2}^{\infty} \phi_2(\tilde{v}_2) \; P_2^*(Q_2.v_2 - Z)\, dv_2$$

The lower limits of integration of $\tilde{v}_1$ and $\tilde{v}_2$ are given by setting $Q_1(P_1^*, P_2^*).\tilde{v}_1 = Z$ and $Q_2(P_1^*, P_2^*).\tilde{v}_2 = Z$, so that the limits are Z/Q_1 and Z/Q_2 respectively. The upper limits of integration of the prices are obtained from the intersection of the demand curves and the capacity constraints.

$$Q_1(P_1'', P_2'').\tilde{v}_1 = Z \; ; \; Q_2(P_1'', P_2'').\tilde{v}_2 = Z;$$

(35)

$$(P_1'', P_2'') = \left[Q_1^{-1}(Z/\tilde{v}_1, Z/\tilde{v}_2),\; Q_2^{-1}(Z/\tilde{v}_1, Z/\tilde{v}_2)\right]$$

and $Q_2(P_1', P_2').\tilde{v}_2 = 0$. This happens only when $\tilde{v}_1$ and $\tilde{v}_2$ are zero, since Q_1 and Q_2 are > 0. Hence, $P_1' = Q_1^{-1}(0,0)$ and $P_2' = Q_2^{-1}(0,0)$.

The expected value of sales is given as before by the expectation of the respective demands weighted by the fractions of the total planning cycle, after subtracting the range when demands exceed capacity in the two periods.

$$
\begin{aligned}
E(S) = \; & (1-\omega)\, Q_1 + \omega\, Q_2 \\
& - (1-\omega) \int_{Z/Q_1}^{\infty} \phi_1(\tilde{v}_1)\,(Q_1 \cdot \tilde{v}_1 - Z)\, dv_1 \\
& + \omega \int_{Z/Q_2}^{\infty} \phi_2(\tilde{v}_2)\,(Q_2 \, \tilde{v}_2 - Z)\, dv_2
\end{aligned} \tag{36}
$$

The net social welfare function is given by

$$
\begin{aligned}
W = \; & (1-\omega) \int_{-\infty}^{+\infty} \phi_1(\tilde{v}_1) \int_{\rho_1^*}^{\rho_1'} Q_1(\rho_1, \rho_2^*) \cdot \tilde{v}_1 \, d\rho_1 \, dv_1 \\
& + \omega \int_{-\infty}^{+\infty} \phi_2(\tilde{v}_2) \int_{\rho_2^*}^{\rho_2'} Q_2(\rho_1', \rho_2) \cdot \tilde{v}_2 \, d\rho_2 \, dv_2 + (1-\omega)\rho_1^* Q_1 + \omega \rho_2^* Q_2 \\
& - (1-\omega) \int_{Z/Q_1}^{+\infty} \phi_1(\tilde{v}_1) \left\{ \int_{\rho_1^*}^{\rho_1'} [Q_1(\rho_1, \rho_2^*) \cdot \tilde{v}_1 - Z]\, d\rho_1 + \rho_1^* [Q_1 \cdot \tilde{v}_1 - Z] \right\} dv_1 \\
& - \omega \int_{Z/Q_2}^{+\infty} \phi_2(\tilde{v}_2) \left\{ \int_{\rho_2^*}^{\rho_2''} [Q_2(\rho_1'', \rho_2) \cdot \tilde{v}_2 - Z]\, d\rho_2 + \rho_2^* [Q_2 \cdot \tilde{v}_2 - Z] \right\} dv_2 \\
& - b \Big[(1-\omega) Q_1 + \omega Q_2 + (1-\omega) \int_{Z/Q_1}^{\infty} \phi_1(\tilde{v}_1) [Q_1 \, \tilde{v}_1 - Z]\, dv_1 \\
& \qquad + \omega \int_{Z/Q_2}^{\infty} \phi_2(\tilde{v}_2) [Q_2 \cdot \tilde{v}_2 - Z]\, dv_2 \Big] - \beta Z
\end{aligned} \tag{37}
$$

By differentiating with respect to the two prices and the capacity, and simplifying, results similar to the additive case are obtained[18].

(38)
$$(1-\omega) P_1^* \frac{\partial Q_1}{\partial P_1^*} \Phi_1 + \omega P_2^* \frac{\partial Q_2}{\partial P_1^*} \Phi_2 = b \left[(1-\omega) \frac{\partial Q_1}{\partial P_1^*} \Phi_1 + \omega \frac{\partial Q_2}{\partial P_1^*} \Phi_2 \right]$$

$$(1-\omega) P_1^* \frac{\partial Q_1}{\partial P_2^*} \Phi_2 + \omega P_2^* \frac{\partial Q_2}{\partial P_2^*} \Phi_2 = b \left[(1-\omega) \frac{\partial Q_1}{\partial P_2^*} \Phi_1 + \omega \frac{\partial Q_2}{\partial P_2^*} \Phi_2 \right] - K$$

where[19]

$$K = (1-2\omega) \left\{ \left[Q_2(P_1', P_2^*) - Q_2(P_1'', P_2^*) \right] + \int_{-\infty}^{Z/Q_1} \phi_1(\tilde{v}_1) \int_{P_1^*}^{P_1''} \frac{\partial Q_1(P_1, P_2^*)}{\partial P_2} \, dP_1 \right\}$$

$$\frac{\partial W}{\partial Z} = (1-\omega) \int_{Z/Q_1}^{+\infty} \phi_1(\tilde{v}_1)\,(P_1'' - b)\, dv_1 + \omega \int_{Z/Q_2}^{+\infty} \phi_2(\tilde{v}_2)\,(P_2'' - b)\, dv_2 - \beta = 0$$

As before, K is positive, negative or zero according as w is less than, greater than or equal to one half.

18. Here Φ_1 and Φ_2 stand for $\Phi_1(Z/Q_1)$ and $\Phi_2(Z/Q_2)$ respectively.

19. $P_1' = Q_1^{-1}(0,0)$ and $P_1'' = Q_1^{-1}(Z/v_1, Z/v_2)$.

Hence P_1 and P_2 can be solved from the above quations to obtain

(39) $$P_1 = b + k_1 \text{ and } P_2 = b + k_2$$

where k_1 and k_2 have the same sign as K and are equal to zero when K is equal to zero (ie. $w = \frac{1}{2}$). The conclusion is that even in the case when the stochastic element in the demand enters multiplicatively, prices in both periods are greater than (less than) marginal operating costs when the peak period arises less than (more than) half the time. Only in the case when both peak and off-peak periods occur for equal durations in the planning cycle is marginal cost pricing called for.

An Example

The conditions for optimality derived in the additive and multiplicative cases cannot be solved to obtain a general solution for the optimal capacity Z*, since the expressions involve integrals that depend on the probability density functions of the error terms $\phi_1(\tilde{v}_1)$ and $\phi_2(\tilde{v}_2)$ as well as the demand functions $Q_1(P_1, P_2)$ and $Q_2(P_1, P_2)$. In order to arrive at specific conclusions in an illustrative case, certain simplifying assumptions regarding the demand functions and the probability density

functions of the disturbance terms have to be made.

a) The off-peak and peak periods occur for equal halves of the total cycle so that $w = \frac{1}{2}$ which implies that the price in both periods is equal to the marginal operating costs per unit b.

b) The demand curves in the off-peak and peak periods are linear and are given by[20]

$$(39) \qquad \begin{aligned} Q_1 &= A_1 + B_1 P_1 + C\,P_2 \\ Q_2 &= A_2 + C\,P_1 + B_2 P_2 \end{aligned}$$

It will be noticed that in the above formulation, the cross derivatives $\frac{\partial Q_1}{\partial P_2}$ and $\frac{\partial Q_2}{\partial P_1}$ are equal so that the integrability conditions are satisfied.

c) The disturbances in the two periods are independent, identically distributed random variables with rectangular distributions ie. $\phi_1(\tilde{v}_1) = \phi_2(\tilde{v}_2) = 1/2\lambda$. [21]

The necessary condition for optimality in equation specifies that

$$(40) \qquad \frac{\partial W}{\partial z} = \frac{1}{2}\int_{z-Q_1}^{+\infty} \phi_1(\tilde{v}_1)(P_1'' - b)\, dv_1 + \frac{1}{2}\int_{z-Q_2}^{+\infty} \phi_2(\tilde{v}_2)(P_2'' - b)\, dv_2 - \beta = 0$$

20. It is assumed that the stochastic element enters additively.

21. To ensure that demand is always positive, $\lambda < A_1 + (B_1 + C)b$ and $\lambda < A_2 + (B_2 + C)b$, for then, when $P_1 = P_2 = b$, $D_1 = Q_1(P_1, P_2) + \tilde{v}_1$ and $D_2 = Q_2(P_1, P_2) + \tilde{v}_2$ are greater than or equal to zero.

which can be written as

$$\int_{(z-(A_1+B_1b+Cb))}^{+\infty} \frac{1}{2\lambda}\left[\frac{B_2(z-\tilde{v}_1-A_1)-C(z-\tilde{v}_2-A_2)}{D}\right]dv_1$$

(41)

$$+\int_{(z-(A_2+B_2b+Cb))}^{+\infty} \frac{1}{2\lambda}\left[\frac{B_1(z-\tilde{v}_2-A_2)-C(z-\tilde{v}_1-A_1)}{D}\right]dv_2 \quad - 2\beta = 0$$

(Noting that P_1'' and P_2'' are the prices at which the respective demand schedules cut the capacity constraint, we can write

(41a)

$$A_1 + B_1P_1'' + CP_2'' = z-\tilde{v}_1$$

$$A_2 + B_2P_2'' + CP_1'' = z-\tilde{v}_2$$

so that $$P_1'' = \frac{B_2(z-\tilde{v}_1-A_1)-C(z-\tilde{v}_2-A_2)}{D}$$

$$P_2'' = \frac{B_1(z-\tilde{v}_2-A_2)-C(z-\tilde{v}_1-A_1)}{D}$$

where $D = B_1B_2 - C^2$)

This can be simplified as follows

$$\frac{1}{2D\lambda}\left[\left(B_2z-B_2A_1-C(z-\tilde{v}_1-A_2)-bD\right)\tilde{v}_1 - \frac{B_2v_1^2}{2}\right]_{z-(A_1+B_1b+Cb)}^{\lambda}$$

$$+\frac{1}{2D\lambda}\left[\left(B_1z-B_1A_2-C(z-\tilde{v}_2-A_1)-bD\right)\tilde{v}_2 - \frac{B_1v_2^2}{2}\right]_{z-(A_2+B_2b+Cb)}^{\lambda}$$

$$-2\beta = 0$$

which on substituting yields

(42)
$$\frac{1}{2D\lambda}\Big[-\frac{B_2}{2}(Z-A_1-B_1b-Cb)^2+2C(Z-A_2-Cb-B_2b)(Z-A_1-B_1b-Cb)$$
$$+\lambda\{(B_2-C)(Z-A_1-B_1b-Cb)+(B_1-C)(Z-A_2-Cb-B_2b)\}$$
$$-\frac{(B_1+B_2)\lambda^2}{2}-\frac{B_1}{2}(Z-A_2-B_2b-Cb)^2\Big]-2\beta=0$$

Let

(42 a)
$$A_1+B_1b+Cb=M$$
$$A_2+Cb+B_2b=N$$

which leads to

$$\frac{1}{2D\lambda}\Big[-\frac{B_2}{2}(Z-M)^2+2C(Z-M)(Z-N)$$
$$+\lambda\{(B_2-C)(Z-M)+(B_1-C)(Z-N)\}$$
$$-\frac{(B_1+B_2)\lambda^2}{2}-\frac{B_1}{2}(Z-N)^2\Big]-2\beta=0$$

(43)
$$\frac{1}{2D\lambda}\Big[Z^2\left(-\frac{B_1}{2}-\frac{B_2}{2}+2C\right)+Z\left(MB_2+NB_1-2C(M+N)\right.$$
$$+\lambda(B_2-C)+B_1-C)+\left(-\frac{B_2M^2}{2}-\frac{B_1N^2}{2}+2CMN\right.$$
$$-\lambda(M(B_2-C)+N(B_1-C)-\frac{B_1+B_2}{2}\lambda^2-4D\lambda\beta\Big]$$
$$=0$$

Solving for Z and simplifying, the solution for Z is written as

(44)
$$Z^{*}=\bar{Z}+\lambda-2E\beta\pm2\sqrt{(-2\lambda E\beta)}$$

where $\bar{Z}$ is the value of the output in the riskless case and is given by

$$\bar{Z} = A_1 + B_1 b + C(b + 2\beta)$$

(45) or,

$$\bar{Z} = A_2 + B_2 (b + 2\beta) + Cb$$

and $$E = \frac{B_1 B_2 - 2C^2 - B_2 C}{B_1 + B_2 - 4C}$$

Thus,[22]

(46) $$Z^* - \bar{Z} = \lambda - 2\sqrt{(-2\lambda E\beta)} - 2E\beta$$

Multiplying both sides by λ which is > 0,

(47) $$\lambda(Z^* - \bar{Z}) = \left[\lambda - \sqrt{(-2\lambda E\beta)}\right]^2$$

Since the right hand side is ≥ 0, and $\lambda \geq 0$, either $Z^* - \bar{Z} = 0$ or $Z^* \geq \bar{Z}$. $Z^* - \bar{Z} = 0$ implies that $\lambda = \sqrt{(-2E\beta)}$. In general, the output under conditions of risk is greater than in the riskless case, if the demand functions are linear and the disturbances are identically distributed and follow a rectangular distribution, with the peak and off-peak periods occuring for equal portions of the planning cycle.

22. The negative root is chosen for examination since if output in this is case is greater than in the riskless model, it follows that the positive root would yield an output that is even greater. Hence the conclusion applies to both roots.

Conclusion

The peak load pricing problem - optimum pricing with a varying demand over the day, typically a peak and off peak period - has been extensively studied by several authors including Hirshleifer, Steiner, Williamson and Brown and Johnson amongst others. In the context of a riskless world, maximization of welfare yields a solution where net revenue is identically equal to zero. In the context of a two period problem, off peak period customers are charged only marginal operating costs per unit while peak period customers incur both operating costs as well as the entire capacity costs spread over the peak period output.

However, in the context of stochastic demand, Brown and Johnson concluded that welfare maximization implies marginal cost pricing throughout. They further showed that when the demand is linear and the stochastic term follows a rectangular distribution, the optimal capacity is greater than in the riskless case. This result was shown to be valid in a more general setting by Littlechild using the state-preference approach. Thus, in this situation, short run operating costs are covered but capacity costs are not and the public enterprise will have to be subsidized[23].

The incorporation of stochastic elements in the demand in both the off-peak and peak periods with an interdependence in the demands calls for different recommendations. The fraction of the total cycle for which the peak load occurs, becomes a critical element in the analysis and the present results coincide with the Brown and Johnson conclusions of marginal cost pricing only when the peak and off-peak periods occur for equal durations. In the case when the peak-load occurs less than half the time, both the off-peak-load and peak-load prices are greater than the marginal operating costs. Thus, at least a part of the capacity costs are recovered in this case. When the peak-load occurs for a major part of the total cycle, pricing below the marginal operating costs in both periods is called for. Hence, not even the variable operating costs are covered, not to speak of the capacity costs, in this case.

The results stated above are quite unusual and deserve further intuitive explanation. In general, it can be stated that if the demand is relatively high, the demand curve

23. In their reply to critics, Brown and Johnson concede that their analysis implicitly assumed that there was a probability, however small of the demand function with the stochastic term intersecting the SRMC on the horizontal section, which as in the riskless case, implies marginal cost pricing.

intersects the SRMC in its vertical section and pricing above the marginal operating costs is called for. In an intermediate range, pricing at the marginal operating costs is optimal and at a low enough value of demand, pricing below marginal costs may be justified, in the case of interdependent demands. The setting of the optimal capacity is dependent on the fraction of the total period for which the peak occurs. The greater this fraction, the greater is the weightage in favour of a higher optimal capacity, subject of course to the tradeoff between the incremental capacity costs and the gain in surplus.

If the peak-load occurs less than half the time, the optimal capacity is set relatively low. The off peak and peak load curves in this situation are more likely to intersect the SRMC on its vertical section involving pricing above the marginal operating costs b. However, when $w > \frac{1}{2}$ the peak capacity is set at a relatively high level, the peak and off-peak load curves intersect the SRMC on its horizontal segment. Further the tradeoff between consumer's surplus and revenue is such that a price below the marginal operating cost is warranted in both periods. A reduction of price in one period not only raises consumer's surplus in that period but also draws demand away from the other period and consequently raises that part of the consumer's surplus as well. Pricing at the marginal operating cost in both periods should be resorted to only ~~when the peak and off-peak periods should be resorted to only~~ when the peak and off-peak loads occur for equal durations.

Bibliography

1. Baumol, W.J. and Bradford, D.F. "Optimal Departures from Marginal Cost Pricing." The American Economic Review, Vol. 60 No. 3, June 1970, pp 265-83.

2. Boiteux, M. "Peak-Load Pricing." Journal of Business, Vol. 33 No. 1, April 1960, pp 157-79.

3. Boiteux, M. "On the Management of Monopolies subject to Budgetary Constraints." Journal of Economic Theory, Vol. 3 No. 3, Sept. 1971, 219-40.

4. Brown, G. Jr. and Johnson, M. "Public Utility Pricing and Output under Risk." The American Economic Review, Vol. 59 No. 1, March 1969, pp 119-28.

5. Brown, G. Jr. and Johnson, M. "Public Utility Pricing and Output under Risk: Reply." The American Economic Review, Vol. 60 No. 3, June 1970, pp 489-90.

6. Diamond, P.A. and Mirrlees, J.A. "Optimal Taxation and Public Production : I - Production Efficiency." The American Economic Review, Vol. 61 No. 1. March 1971, pp 8-27.

7. Dreze, J. "Post War Contributions of French Economists." The American Economic Review, Vol. 54. No. 3. Part 2, June 1964.

8. Dixit, A.K. "On the Optimum Structure ofCommodity Taxes." The American Economic Review, Vol. 60 No. 3, June 1970, pp 295-301.

9. Feldstein, M.S. "Distributional Equity and the Optimal Structure of Public Prices." The American Economic Review, Vol. 62, No. 1. March 1972, pp 32-36.

10. Feldstein, M.S. "Equity and Efficiency in Public Sector Pricing: The Optimal Two-Part Tariff." The Quarterly Journal of Economics, Vol. 86 No. 2, May 1972, pp 175-187.

11. Feldstein. M.S. "The Pricing of Public Intermediate Goods." Vol. 1, No. 1, April 1972, pp 45-72.

12. Harberger, A. "Three Basic Postulates for Applied Welfare Economics: An Interpretive Essay." The Journal of Economic Literature, Vol. 9 No. 3, Sept. 1971, pp 785-97.

13. Hicks, J.R. Value and Capital. 2nd Ed. Oxford: Clarendon Press, 1946 esp. pp 330-333.

14. Hirshleifer, J. "Peak Loads and Efficient Pricing: Comment." The Quarterly Journal of Economics, Vol. 72 No. 3, Aug. 1958, pp 451-462.

15. Houthakker, H.S. "Electricity Tariffs in Theory and Practice." The Economic Journal, Vol. 61, No. 1, March 1951, pp 1-25.

16. Hotelling, H. "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates." Vol. 40 No. 3, July 1938, pp 577-616.

17. Lerner, A.P. "On Optimal Taxes with an Untaxable Sector." Vol. 60 No. 3, June 1970, pp 284-294.

18. Littlechild, S.C. "A State Preference Approach to Public Utility Pricing and Investment under Risk." The Bell Journal of Economics and Management Science, Vol. 3 No. 1, Spring 1972, pp 340-345.

19. Margling, S.A. Public Investment Criteria. London: Allen and Unwin, 1967.

20. Mishan, E.J. Cost-Benefit Analysis:An Introduction. New York: Praeger, 1971.

21. Mohring, H. "Increasing Returns and Pricing Constraints." The American Economic Review, Vol.60 No. 4. Sept. 70, pp 693-705.

22. Nelson, J.R. ed. Marginal Cost Pricing in Practice. Englewood Cliffs, N.J.: Prentice-Hall, 1964.

23. Pressman I. "A Mathematical Formulation of the Peak Load Pricing Problem." The Bell Journal of Economics and Management Science, Vol. 1 No. 2. Autumn 1970, pp 304-326.

24. Ramsey, F.P. "A Contribution to the Theory of Taxation." The Economic Journal, Vol. 37 No. 1, Mar. 1927, pp 47-61.

25. Salkever, D.S. "Public Utility Pricing and Output Under Risk: Comment." The American Economic Review, Vol. 60. No. 3. June 1970, pp 487-488.

26. Samuelson, P.A. "Comment on Hirshleifer." The American Economic Review, Vol. 54. No. 2. May 1964, pp 93-96.

27. Steiner, P.O. "Peak Loads and Efficient Pricing." The Quarterly Journal of Economics, Vol. 71 No. 4. Nov. 1957, pp. 585-610.

28. Turvey R. "Peak Load Pricing." The Journal of Political Economy, Vol. 76. No. 1, Jan/Feb 1968, pp 101-13.

29. Turvey, R. Optimal Pricing and Investment in Electricity Supply. London: Allen and Unwin, 1968.

30. Turvey, R. "Public Utility Pricing and Output Under Risk: Comment." The American Economic Review, Vol. 60. No. 3. June 1970, pp 485-86.

31. Williamson, O.E. "Peak-Load Pricing and Optimal Capacity under Indivisibility Constraints." The American Economic Review, Vol. 56. No. 4. Sept. 1966, pp 810-827.

32. Vickrey, W. "Comment on Hirshleifer." The American Economic Review, Vol. 54. No. 3. May 1964, pp 89-92.

31. Williamson, O.E., "Peak-load Pricing and Optimal Capacity under Indivisibility Constraints." The American Economic Review, Vol. 56, No. 4, Sept. 1966, pp 810-827.

32. Vickrey, W. "Comment on Hirshleifer," The American Economic Review, Vol. 54, No. 3, May 1964, pp 88-92.