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Optimal Consumption and Portfolio Policies
When Markets are Incomplete

by

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Abstract

We consider a continuous time version of the consumption portfolio problem under uncertainty, where an agent invests his initial endowment on capital markets to maximize his utility of consumption and possibly final wealth over a fixed horizon. Using the martingale approach, we address the general existence question when markets are dynamically incomplete, i.e., when the dimension of the diffusion process followed by the assets prices is less than the dimension of the Brownian motion that describes the uncertain environment. The sufficient conditions we require are virtually identical to those put forth in the complete markets case and are expressed entirely in terms of the primitives of the model. We show in particular that even when the investor has access to the whole information in the economy, only that which is strictly reflected in the securities price system is relevant to manufacture his best consumption plan.

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1. Introduction

Optimal consumption portfolio policies under dynamic uncertainty and in continuous time have long been studied in the economics literature. The traditional approach originated from the pioneering work of Merton [1969], [1971], and found one of its latest achievements in a recent paper by Karatzas et al. [1986]. A common feature of the contributions in this field is the use of stochastic dynamic programming as a means to derive the optimal policy for an individual. There seem to be two problems with this approach. The first one is its inherent difficulty, especially when non-negativity constraints on consumption are involved. For instance in the last reference given above, the mathematics is quite elaborate, even though the securities price process is assumed to follow a simple geometric Brownian motion. It is therefore uncertain whether many economists will continue to use this technique in most situations of practical interest. Secondly, the very existence of optimal consumption portfolio policies can easily be an issue under general circumstances. One way to solve the existence problem in dynamic programming is to construct a control and to check afterwards— with the verification theorem of control theory—that it is indeed optimal. So the proof of the existence of a solution hinges first on its characterization. To do this unfortunately one has to solve a non-linear partial differential equation, so that it is very difficult to construct a control in general. Another possibility is to invoke directly some existence theorem in control theory, under a compactness assumption of the set of admissible strategies. However, as Cox and Huang [1986] already emphasized, this assumption is unpalatable when financial markets are frictionless. Indeed markets with no transactions costs nor restrictions on short-selling put no constraints on the amount of money one can invest in each security, so that the admissible set is clearly unbounded and thus non-compact. Our conclusion is that stochastic dynamic programming is a very powerful technique but that it is not best suited to the general consumption portfolio problem posed by economists.

An alternative approach has recently been put forward in place of the dynamic programming: notably, Cox and Huang [1986], [1987] and Pliska [1984] in portfolio theory, or Chamberlain [1985] and Huang [1987] in general equilibrium. This approach takes full advantage of some tools developed in martingale probability theory. That a multiperiod securities market is arbitrage free whenever the discounted prices of its securities behave like martingales was first foreshadowed by Harrison and Kreps [1979], then extended by Harrison and Pliska [1981] and Duffie and Huang [1985]. The point is that there must exist some probability measure under which the risky securities earn in average the rate of return on the riskless bond. (This would certainly be true under the original probability beliefs if agents were risk neutral. What the change in probability really does is that it subsumes risk aversion.) However as it stands, this new strand of literature cannot be quite considered on a par with its dynamic programming relative, because it rests on the critical assumption that financial markets are dynamically complete. Roughly speaking, markets are said to be dynamically complete when the number of risky securities is equal to the “dimension” of uncertainty. The goal
of this paper is to fill this gap by extending the martingale approach as an alternative to stochastic dynamic programming, and more specifically to show how an existence proof can be derived when financial markets are dynamically incomplete. Needless to say, we believe that the martingale approach is more appropriate than the dynamic programming one in this application, and it is hoped that this paper will contribute to the idea that the former dominates the latter, both in terms of their respective tractability and of their intuitive appeal.

Before we delve into the arcana of probability theory, we present a brief summary. This paper takes the continuous version of the canonical consumption portfolio problem and examines the behavior of an investor who endeavors to design a best financial strategy to achieve his most preferred consumption plan. His problem is to maximize his expected utility of consumption over time, subject to a budget constraint which equates the agent’s revenue (capital gains on assets, dividends earnings, endowment) and expenses (consumption, net trading on stocks). To be somewhat more explicit about our framework, recall that there is a finite horizon \( T \) in this economy and a single good – taken to be the numéraire – available for consumption; there may be some bequest at the terminal date too, but for expositional reasons we will not bother about it presently. Then a contingent claim or consumption plan is characterized by a distribution of the consumption good across all states of nature and at all times. A strategy is a decision regarding the continuous trading of the existing securities from time 0 to time \( T \). There are \( M \) risky securities, posited to follow a vector diffusion process, and a bond, whose instantaneous rate of return is riskless. Except for his initial endowment, the agent’s income is entirely generated by the capital gains and dividends on his investments in assets. The proceeds are either consumed or reinvested. Finally, the space of marketed bundles consists of all the contingent claims that can be manufactured by withdrawals from the portfolio during the time span \( [0, T] \). Then to say that capital markets are dynamically incomplete just means that the space of marketed bundles is strictly included in the whole consumption space or, if one prefers, that there are distributions of the consumption good over time that the investor might want to pick but which cannot be delivered by an appropriate trading strategy.

We will call \( \mathcal{M} \) the space of marketed bundles and \( X \) the whole consumption space. The first question we have to address is under which circumstances markets happen to be dynamically incomplete, i.e., when \( \mathcal{M} \) does not coincide with \( X \). The typical model considered so far in the literature obtains when the number \( M \) of risky securities is equal to the dimension \( N \) of the underlying Brownian motion which generates uncertainty. It can then be shown – using the martingale representation theorem – that any contingent claim in \( X \) is attainable through some appropriate trading strategy so that markets are dynamically complete and \( \mathcal{M} = X \). This result is no longer true when \( M \) is less than \( N \). In this second case, there are too few securities to “span” the entire consumption space so that \( \mathcal{M} \) becomes strictly included in \( X \).
Not surprisingly, there are consequences in terms of the information structure as well. For convenience the information set (σ-algebra) generated by the securities price system is referred to as the “price information set”. With dynamically complete markets, it turns out that the price information set coincides with the whole information in the economy, so that in particular any process observed at the current time will be a function of the past and present realizations of the securities price process. This is surely a strong requirement, that we can relax by letting markets become incomplete. The relation between $\mathcal{M}$ and the price information set $\mathcal{F}^S$ becomes then a little more subtle ($S$ stands here for the vector of securities prices). Namely, one has the strict inclusions $\mathcal{F}^S \subset \mathcal{M} \subset X$. Because investors have access to more information than that which is contained in the price system, there are bundles in $\mathcal{M}$ which are outside the price information set. Conversely however, it can be shown (though this is not so easy to prove) that all the consumption goods in the price information set are marketed. Hence our first observation has to be that if a particular solution to the investor’s maximization program happens to be price measurable, then it is automatically marketed.

We shall borrow from Cox and Huang [1986] their solution concept with regard to the above maximization program. Their idea is to embed the investor’s problem, which is by assumption restricted to the space $\mathcal{M}$ of marketed bundles, in the whole consumption space, and to bring to bear martingale theory to transform a complex dynamic problem into a static one that has already been studied by mathematicians. Before we do this we find it convenient to normalize all prices in units of the riskless bond. This is without loss of generality, because we assume that the interest rate is positive and bounded from above, so that the price of the bond can neither vanish nor explode. The integrated budget constraint then states the familiar condition that the present discounted value of the stream of consumption is equal to the sum of the initial wealth and of the present discounted value of the financial gains on the risky securities. For any consumption plan $c \in \mathcal{M}$, we can write accordingly

$$
\int_0^T c^*(t) \, dt = W^*(0) + \int_0^T \text{financial gains}.
$$

where $c^*$ denotes the discounted flow of consumption, $W^*(0)$ the initial wealth, and the integral of financial gains the sum of the (discounted) capital gains and dividends accruing to the strategy that manufactures $c$ for the entire time span $[0,T]$.

The discounted price $\pi(c^*)$ of the above consumption stream at time 0 is equal to $W^*(0)$, because in this set-up the initial investment is the only amount required by the strategy that manufactures $c$. The agent’s maximization program can therefore be formulated as

$$
\max_{c \in \mathcal{M}} E \int_0^T u(c(t), t) \, dt \quad \text{s.t.} \quad \pi(c^*) = W^*(0).
$$

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where \( u \) is some utility function and \( \pi \) the linear functional that gives the price of marketed goods at time 0. We now extend this maximization program in the following way

\[
\max_{c \in X} \mathbb{E} \int_0^T u(c(t), t) \, dt \quad \text{s.t.} \quad \phi(c^*) = W^*(0),
\]

where \( X \) stands as before for the whole consumption space and \( \phi \) is a linear functional that extends \( \pi \) over all of \( X \). When markets are incomplete, the above embedding procedure raises two questions:

(i) What prices should one ascribe to a stream of consumption that is not marketed?
(ii) What ensures that a solution to the extended program, if any, can be marketed?

The first question arises from the fact that when \( \mathcal{M} \) is strictly included in \( X \), only the marketed commodities have their price determined by arbitrage. There is an abundance of price functionals \( \phi \) that extend \( \pi \) over all of \( X \), and one could choose a priori any one of them. However, one candidate is of special interest to us: it is the (unique) one which is measurable with respect to the price system, i.e., such that the shadow price of consumption is itself in the price information set. With this particular valuation, it turns out that a solution \( \hat{\pi} \) to the extended maximization program can always be chosen to be price measurable, and thus marketed. To see this, we have to recall a result from option pricing theory which states that the price of any contingent claim can be written as its expectation under some probability. Let then \( Q \) be the probability associated with our choice of the price measurable valuation, and take the conditional expectation of \( \hat{\pi} \) under \( Q \) with respect to the price information set. The new consumption plan is price measurable by construction. In addition, it can be shown that it satisfies the same budget constraint and that it is as least as preferred as \( \hat{\pi} \). But \( \hat{\pi} \) is optimal by assumption, so that it should be clear that the two solutions are in fact indifferent, or even identical when the utility function is strictly concave.

Thus, as far as the optimization problem is concerned, there is nothing more than the securities prices themselves that the agent needs to observe to devise his optimal strategy. Even in a world endowed with a very rich information structure, only that which is contained in the capital markets is relevant to manufacture his optimal consumption plan. The crux of the argument is that an investor can lose nothing in terms of his expected utility by projecting the whole problem onto the information generated by prices. An intuitive account of how this works can tentatively be given as follows. Since there are infinitely many valuations which give the prices of the contingent claims outside \( \mathcal{M} \), the investor may be seen as maximizing his expected utility over the whole consumption space subject to an infinite number of budget constraints. Simply pick the particular valuation which is in the price information set and observe that the corresponding solution \( \hat{\pi} \) is price measurable and thus marketed. Now expected utility can be improved upon only if some of the constraints are slack. But all the valuations agree on \( \mathcal{M} \) so that all constraints are saturated. Hence \( \hat{\pi} \) is indeed a solution to the investor's maximization problem.
The rest of this paper is organized as follows. In section 2, the traditional framework for a dynamic consumption portfolio problem is resumed and simply extended to allow for incomplete markets. What is new about our approach is that we characterize the set of marketed consumption bundles entirely in terms of the original probability beliefs rather than relatively to some arbitrary reference measure. The nice fact about this is that beliefs do provide something more than merely specifying the null sets. The other features of the model are standard. In section 3 we use the martingale representation theorem to show that all price measurable processes are in fact marketed. This is important since otherwise our existence proof would not go through. In section 4, we formally pose the investor’s maximization problem and directly extend the Cox and Huang [1986] results to our economy. The last section concludes with a very simple example carried out for the class of constant relative risk aversion utility functions.
2. The Model

In this section a model of securities markets in continuous time with diffusion price process will be formulated. Before we can define the dynamic maximization problem faced by an investor, it will be useful to characterize the economy in terms of its information structure, the price system of its securities, and the trading strategies its agents can afford.

2.1. The Information Structure

Agents share a common information which is revealed continuously through time—there is no surprise. They have no uncertainty about the present \( t = 0 \) and their uncertainty about the future is gradually resolved over time; at \( t = T \) they learn the true state of nature. We may think of the information structure at an intermediate time \( t \) as the collection of events that can occur at or before time \( t \). The following mathematical model gives these notions a precise meaning.

Taken as a primitive are a probability space \((\Omega, \mathcal{F}, P)\) and a time span \([0, T]\), where \( T \) is some positive real number. Let \((\mathcal{W}(t), \mathcal{F}_t, P)\) be an \( N \)-dimensional Brownian motion defined on it where we take \( \mathcal{F}_t = \sigma[\mathcal{W}(s) : 0 \leq s \leq t] \) to be the \( \sigma \)-algebra generated by \( \mathcal{W} \) up to time \( t \). Furthermore it is assumed that \( \mathcal{F} = \mathcal{F}_T \). We use \( PM \) to denote the \( \sigma \)-field generated by the progressively measurable processes. (A function \( Z : \mathbb{R}^+ \times \Omega \to \mathbb{R}^N \) is said to be \( \mathcal{F}_t \)-progressively measurable if for each \( t \) the restriction of \( Z \) to \([0, t] \times \Omega \) is \( \mathcal{B}_t \times \mathcal{F}_t \)-measurable, where \( \mathcal{B}_t \) is the \( \sigma \)-field of Borel subsets of \([0, t]\).)

All the processes to appear will be progressively measurable. In our framework, the \( \mathcal{F}_t \)-adapted processes (cf. Chung and Williams [1983], p.8) are \( \mathcal{F}_t \)-progressively measurable. The following proposition addresses this rather pedantic measurability question, where the generic \( X \) process is to be interpreted as \( \mathcal{W} \) in our setup.

**Proposition 2.1.** Assume \( X(\cdot, \omega) \) is right continuous for each \( \omega \in \Omega \) and define \( \mathcal{F}_t = \sigma[X(s) : 0 \leq s \leq t] \). If a process is adapted to \( \mathcal{F}_t \), then it is also progressively measurable.

**Proof.** See Stroock and Varadhan (1979), Exercise 1.5.6.

Another important information structure is that generated by the securities price system, whose description is given next.
2.2. The Securities Price System

We consider a securities market with \( M + 1 \) long-lived securities traded, where \( M \leq N \), indexed by \( m = 0, 1, \ldots, M \). The market will be frictionless in that there is no constraint on short selling and no transaction costs (like brokerage fees for instance). Security \( n = 1, 2, \ldots, M \) is risky, pays dividends at rate \( \nu_n(t) \) and sells at time \( t \) for \( S_n(t) \) ex-dividends. Security 0—the bond—is (locally) riskless, pays no dividend and sells at time \( t \) for \( B(t) = \exp\{\int_0^t \rho(s) \, ds\} \). We assume that both \( \nu_n(t) \) and \( \rho(t) \) can be respectively written as \( \nu_n(S(t), t) \) and \( \rho(S(t), t) \) with \( \nu_n(x, t) : \mathbb{R}^M \times [0, T] \to \mathbb{R} \) and \( \rho(x, t) : \mathbb{R}^M \times [0, T] \to \mathbb{R}^+ \) Borel measurable. Finally, we require that \( \rho(x, t) \) be bounded from above. Thus, \( B(t) \) is bounded above and below away from zero. Agents in our economy are assumed to have rational expectations in the sense that they all agree on the law of the price process. More precisely, it is assumed that \( S \in \mathbb{R}^M \) is a right continuous, \( \mathcal{F}_t \)-progressively measurable, \( (a.s., P) \) continuous diffusion process satisfying

\[
S(t) + \int_0^t \nu(S(s), s) \, ds = S(0) + \int_0^t \rho(S(s), s) \, ds + \int_0^t \sigma(S(s), s) \, dW(s) \quad \forall t \in [0, T]. \tag{1} \]

where \( \rho(x, t) : \mathbb{R}^M \times [0, T] \to \mathbb{R}^M \) and \( \sigma(x, t) : \mathbb{R}^M \times [0, T] \to \mathbb{R}^{M \times N} \) are Borel measurable, and \( \sigma(x, t) \) is of full rank for all \( x \) and \( t \). We shall furthermore assume that

\[
|\sigma(t, 0)| \vee |\rho(t, 0)| \leq K \\
|\sigma(x, t) - \sigma(y, t)| \vee |\rho(x, t) - \rho(y, t)| \leq K|x - y|.
\]

for all \( x, y \in \mathbb{R}^M \) and all \( t \in [0, T] \) and some constant \( K \). We have used the following notation: if \( \sigma \) is a matrix, then \( |\sigma| \) denotes the Hilbert-Schmidt norm \( \sqrt{\text{Trace } \sigma \sigma^T} \). These conditions imply that \( S(\cdot, \omega) \) is unique up to a set of \( P \)-measure zero (cf Stroock [1986], p.94).

The sub-\( \sigma \)-algebra generated by prices is denoted by \( \mathcal{F}_t^S \). Since all prices are adapted to \( \mathcal{F}_t \), it follows that this second information set is in general strictly contained in the original one. Let us adopt once and for all the following definition: a process is said to be price measurable if it is progressively measurable (or, equivalently, adapted since \( S \) is right continuous) with respect to \( \mathcal{F}_t^S \). Mathematically, we have that any price measurable variable at time \( t \) can be expressed as a function of the price process evaluated at a countable number of times at or before time \( t \). (cf. Stroock (1979), Exercise 1.5.6). We will show that the best consumption plan for the agent can always be chosen to be price measurable. Thus, his entire strategy will be defined entirely in terms of the current and possibly past values of the price system.

We now use \( G(t) \) to denote the gains process

\[
G(t) = S(t) + \int_0^t \rho(s) \, ds.
\]
and rewrite (1) in units of the 0-th security. Defining \( S^*(t) = S(t)/B(t), \) \( t^*(t) = u(t)/B(t) \) and correspondingly \( G^*(t) = S^*(t) + \int_0^t t^*(s) \, ds. \) Itô's formula implies that

\[
G^*(t) = S^*(0) + \int_0^t \frac{1}{B(s)} [b(S(s), s) - r(S(s), s)S(s)] \, ds + \int_0^t \frac{\sigma(S(s), s)}{B(s)} \, d\mathcal{W}(s)
\]

\[
\equiv S^*(0) + \int_0^t b^*(S(s), B(s), s) \, ds + \int_0^t \sigma^*(S(s), B(s), s) \, d\mathcal{W}(s)
\]

for all \( t \in [0, T]. \) (a.s., \( P). \)

To be a reasonable securities market model in the arbitrage free sense, there must exist a probability measure \( Q \) absolutely continuous with respect to \( P \) such that \( E^P[(dQ/dP)] < \infty \) and that \( (G^*(t), \mathcal{F}_t, Q) \) be a martingale (cf. Harrison and Kreps [1979]). We will say that a probability \( Q \) is an absolutely continuous martingale measure when it satisfies those three conditions. We will elicit momentarily all the absolutely continuous martingale measures under some qualifying assumption on \( \sigma \) and \( b, \) but before that we need two technical lemmas. We use \( S^+(R^N) \) to denote the set of symmetric non-negative definite \( N \times N \) real matrices, \( \text{Hom}(R^N, R^M) \) to denote the set of homomorphisms from \( R^N \) to \( R^M, \) and \( \text{Range}(\sigma) \) to denote the range of \( \sigma \in \text{Hom}(R^N, R^M). \)

**Lemma 2.1.** Let \( a \in S^+(R^N), \) denote by \( \pi_a \) the orthogonal projection of \( R^N \) onto \( \text{Range}(a), \) and let \( a^+ \) be the element of \( S^+(R^N) \) satisfying \( a^+ a = aa^+ = \pi_a. \) Then \( a \mapsto a^+ \) is a measurable function of \( a. \) Next, suppose that \( \sigma \in \text{Hom}(R^N, R^M) \) and that \( a = \sigma^T \sigma. \) Then \( \text{Range}(a) = \text{Range}(\sigma^T) \) and \( \pi_a = \sigma(\sigma^T \sigma)^+ \sigma^T. \)

**Proof.** Stroock [1986], lemma III.2.5.

We introduce the notation \( \dot{\sigma}. \) Intuitively, \( \dot{\sigma} \) is simply the inverse of \( \sigma \) operating on its range.

**Lemma 2.2.** Let \( \sigma \in \text{Hom}(R^N, R^M). \) Then there exists a unique \( \dot{\sigma} \in \text{Hom}(R^M, R^N) \) such that \( \dot{\sigma} \eta \in \text{Range}(\sigma^T) \) and \( \sigma \dot{\sigma} \eta = \pi_a \eta \) for all \( \eta \in R^M. \) Moreover \( \sigma \mapsto \dot{\sigma} \) is a measurable function of \( \sigma. \)

**Proof.** Define \( \dot{\sigma} = (\sigma^T \sigma)^+ \sigma^T \pi_a. \) Then for any \( \eta \in R^N \) there exists some \( \xi \in R^M \) such that \( \pi_a \eta = \sigma \xi. \) Thus

\[
\dot{\sigma} \eta = (\sigma^T \sigma)^+ \sigma^T \sigma \xi = \pi_a \sigma \xi = \pi_a \eta \in \text{Range}(\sigma^T)
\]

by lemma 2.1. In addition,

\[
\sigma \dot{\sigma} \eta = \sigma (\sigma^T \sigma)^+ \sigma^T \pi_a \eta = \pi_a \sigma \pi_a \eta = \pi_a \eta
\]

The uniqueness follows from the fact that for two \( \dot{\sigma}, \dot{\hat{\sigma}} \in \text{Hom}(R^N, R^M) \) and for all \( \eta \in R^N, \)

\[
(\dot{\sigma} - \dot{\hat{\sigma}}) \eta \in (\text{Null} \sigma)^\perp \cap \text{Null} \sigma = 0.
\]

where \( \text{Null} \sigma \) is the null space of \( \sigma. \) The measurability follows also directly from lemma 2.1.
Remark. When $\sigma$ is onto, $\pi_\sigma = I_M$, the $M$-dimensional identity matrix, so that the above relation becomes $\sigma\hat{\eta} = \eta$.

We are now in a position to find all the absolutely continuous martingale measures. We shall see that there are infinitely many such measures, one of which is adapted to the $\sigma$-field $\mathcal{F}_t^S = \sigma[S(s) : 0 \leq s \leq t]$ (i.e., price measurable). All our results will be subject to the following qualifying assumption on $\sigma$ and $b$.

**Assumption 2.1.** Let $\kappa(x,t) = -\dot{\sigma}(x,t)(b(x,t) - r(x,t)x)$. Then $\kappa$ is bounded for all $(x,t) \in \mathbb{R}^M \times [0,T]$.

Note that this assumption is satisfied in particular when $S$ follows a multiplicative geometric Brownian motion as in the model first considered by Merton [1971].

**Proposition 2.2.** Suppose $Q$ is an absolutely continuous martingale measure. Then its Radon-Nikodym derivative $R \equiv \frac{dQ}{dP}$ is equal to $\eta N$ where

$$\eta = \eta(T) = \exp \left\{ \int_0^T \kappa(S(s),s) \cdot dW(s) - 1/2 \int_0^T |\kappa(S(s),s)|^2 ds \right\}$$

and

$$N = N(T) = \exp \left\{ \int_0^T \nu(S(s),s) \cdot dW(s) - 1/2 \int_0^T |\nu(S(s),s)|^2 ds \right\}$$

for some $\nu(t) \in \text{Null } \sigma(t)$ such that $E \int_0^T |\nu(S(s),s)|^2 ds < \infty$.

In this theorem as well as in the sequel, the notation $x \cdot y$ will stand for the inner product $\sum_{i=1}^M x_i y_i$.

**Proof.** $R(t) = E^P( R \mid \mathcal{F}_t)$ is a martingale under $P$ which can be taken to be continuous (a.s., $P$). By the martingale representation theorem (cf. for instance Kunita and Watanabe (1967), Section 4) there exists $\alpha \in PML^2(\lambda \times P; R^N)$ such that

$$R(t) = 1 + \int_0^t \alpha(s) \cdot dW(s),$$

(3)

where $PML^2(\lambda \times P)$ denotes the $\mathcal{F}_t$-progressively measurable processes of $L^2(\lambda \times P)$ and $\lambda$ is the Lebesgue measure. But $(G^*(t), \mathcal{F}_t, Q)$ is a martingale if and only if $(R(t)G^*(t), \mathcal{F}_t, P)$ is a martingale (cf. Dellacherie and Meyer [1982], Lemma VII.48). By Itô's formula then, we have

$$d(R(t)G^*(t)) = R(t) dG^*(t) + G^*(t) dR(t) + dR(t) dG^*(t)$$

$$= R(t) b^*(S(t),t) dt + R(t) \sigma^*(S(t),t) dW(t) + G^*(t) dR(t) + \sigma^*(S(t),t)\alpha(t) dt.$$

Hence by a standard argument it must be that

$$R(t) b^*(S(t),t) + \sigma^*(S(t),t)\alpha(t) = 0.$$

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(cf. Jacod (1979), Proposition 1.43), or equivalently
\[ R(t)(b(t) - r(t)S(t)) + \sigma(t)\alpha(t) = 0, \]
where for brevity \( b(t) \) stands for \( b(S(t), t) \) and similarly for other variables.

To solve the last equation, note that by lemma 2.2 there exists a unique \( \kappa(t) = -(b(t) - r(t)S(t)) \in \text{Range} \sigma^T(t) \) such that \( -(b(t) - r(t)S(t)) = \sigma(t)\kappa(t) \). It then follows that \( \sigma(\alpha(t) - R(t)\kappa(t)) = 0 \) so that the general solution for \( \alpha \) displays the following orthogonal decomposition
\[ \alpha(t) = R(t)\kappa(t) + \nu(t) \]
for some \( \nu(t) \in \text{Null} \sigma(t) \cap \mathcal{ML}^2(\lambda \times P) \). Substituting for \( \alpha \) in (3), we find
\[ dR(t) = \alpha(t) \cdot dW(t) = (R(t)\kappa(t) + \nu(t)) \cdot dW(t). \]
Now let \( \eta(t) \) solve \( d\eta(t) = \eta(t)\kappa(t) \cdot dW(t) \). An application of Itô's formula to \( \log \eta(t) \) shows that \( \eta(t) \) is as in the statement of the theorem with \( T \) replaced by any \( t \) in \([0, T]\). Note that by lemma 2.2 again \( \kappa(S(t), t) \) is progressively measurable and that by the boundedness assumption of \( \kappa \) we have
\[ \int_0^T |\kappa(S(t), t)|^2 dt < \infty; \]
hence the integral defining \( \eta(t) \) is well-defined (cf. Stroock [1986], Section II.3).

Now put \( N(t) = \frac{R(t)}{\eta(t)} \). We have to show that \( \eta(t) \) is strictly positive (a.s., \( \lambda \times P \)). For this it suffices to demonstrate that
\[ \int_0^T \kappa(S(t), t) \cdot dW(t) - 1/2 \int_0^T |\kappa(S(t), t)|^2 dt > -\infty \quad (\text{a.s., } P). \]
But (4) implies
\[ \left| \int_0^T \kappa(S(t), t) \cdot dW(t) \right| < \infty; \]
cf. Lipster and Shiryayev (1977), Theorem 7.1. This, together with (4), ensures that \( \eta(T) > 0 \) (a.s., \( P \)). By Itô's formula again,
\[
\begin{align*}
\frac{dN(t)}{\eta(t)} &= \frac{dR(t)}{\eta(t)} - \frac{R(t)}{\eta^2(t)} d\eta(t) + 1/2 \frac{2R(t)}{\eta^3(t)} (d\eta(t))^2 - \frac{1}{\eta^2(t)} dR(t) d\eta(t) \\
&= \left( N \kappa + \frac{\nu}{\eta} \right) \cdot dW - N \kappa \cdot dW + N |\kappa|^2 dt - \left( \frac{\nu \cdot \kappa}{\eta} + N |\kappa|^2 \right) dt \\
&= \frac{\nu}{\eta} \cdot dW - \frac{\nu \cdot \kappa}{\eta} dt
\end{align*}
\]
Thus we can write
\[ dN(t) = N(t)\nu'(t) \cdot dW(t). \] (5)

since \( \nu \) and \( \kappa \) are orthogonal. The expression for \( N(t) \) can in turn be derived from Itô's formula applied to \( \log N(t) \).

We now proceed to show that \( \eta(t) \) is price measurable.
Proposition 2.3. The process \( \eta(t) \) is price measurable.

Proof. By proposition 2.2,

\[
\log \eta(t) = \int_0^T \kappa(S(s), s) \cdot d\mathcal{W} - 1/2 \int_0^T |\kappa(S(s), s)|^2 \, ds.
\]

Lemma 2.2 shows that the second integral of the right hand side is price measurable. (It can be approximated as the sum of price measurable functions.) Consider the first integral. Since by definition \( \kappa(S(t), t) \in \text{Range} \sigma^\top(t) \),

\[
\kappa(S(t), t) \cdot d\mathcal{W}(t) = \pi_\sigma \kappa(S(t), t) \cdot d\mathcal{W}(t) = \kappa(S(t), t) \cdot \pi_\sigma^\top d\mathcal{W}(t).
\]

But

\[
\pi_\sigma^\top d\mathcal{W}(t) = \sigma^\top (\sigma \sigma^\top)^+ \sigma^\top d\mathcal{W}(t) = \sigma^\top (\sigma \sigma^\top)^+ \left( dS(t) + \iota(S(t), t) \, dt - b(S(t), t) \, dt \right),
\]

and since \( S(t) + \int_0^t \iota(S(s), s) \, ds - \int_0^t b(S(s), s) \, ds \) is certainly price measurable, the result follows.

From now on the probability whose Radon-Nikodym derivative \( \eta \) is price measurable will be denoted by \( Q \). Note that since \( \eta \) is strictly positive, \( (a.s., P) \), \( P \) and \( Q \) have the same null sets. It remains to check that \( \eta \) is \( P \)-square integrable. In fact, a stronger result holds under our assumptions.

Lemma 2.3. \( \eta \) and \( \eta^{-1} \) are in \( L^q(P) \) for any \( q > 0 \) finite.

Proof. Let \( q > 0 \). We have

\[
\eta^q = \exp \left\{ \int_0^T q \kappa \cdot d\mathcal{W} - 1/2 \int_0^T q \kappa^2 \, dt \right\}
\]

\[
= \exp \left\{ \int_0^T q \kappa \cdot d\mathcal{W} - \frac{q^2}{2} \int_0^T \kappa^2 \, dt \right\} \exp \left\{ \frac{q(q-1)}{2} \int_0^T \kappa^2 \, dt \right\},
\]

since \( \kappa \) is bounded (say by \( K \)). \( E_P(\eta^q) \leq \max \left\{ 1, \exp \left( \frac{q(q-1)}{2} T K^2 \right) \right\} \) or \( ||\eta||_{q, P} < \infty \) where \( ||\eta||_{q, P} \) denotes the norm of \( \eta \) in \( L^q(P) \). The same argument works for \( \eta^{-1} \).

We turn now our attention to trading strategies.
2.3. The Trading Strategies

The consumption space for the agents is

\[
PM^2L^2(\lambda \times P) \times L^2(P) \equiv L^2([0, T] \times \Omega, PM, \lambda \times P) \times L^2(\Omega, \mathcal{F}, P).
\]

where \(PM^2L^2(\lambda \times P)\) is the space of consumption rate processes and \(L^2(P)\) the space of final wealth.

A trading strategy is an \(M + 1\)-vector process \((\alpha, \theta) = \{(\alpha(t), \theta_m(t)) : m = 1, \ldots, M\}\), where \(\alpha(t)\) and \(\theta_m(t)\) are the number of shares of the 0-th and the \(m\)-th security, respectively, held at time \(t\). Thus, the value of the strategy \((\alpha, \theta)\) at time \(t\) is that of the portfolio consisting of \(\alpha(t)\) shares of the bond \(B(t)\) and \((\theta_m(t); m = 1, \ldots, M)\) shares of the risky securities \(S(t)\), or

\[
W(t) \equiv \alpha(t)B(t) + \theta(t) \cdot S(t).
\]

For a progressively measurable process \(\rho \in R^N\), we will constantly make use of the following notation:

\[
\left\| \left( \int_0^T |\rho(t)|^2 \, dt \right)^{1/2} \right\|_{\rho, P} = \left\| \left( \int_0^T |\rho(t)|^2 \, dt \right)^{1/2} \right\|_{\rho, R}
\]

where as already mentioned \(\|(-)\|_{\rho, P}\) denotes the \(L^p(P)\)-norm. Accordingly let

\[
\mathcal{L}^2(G, P) = \left\{ \theta \in PM^2L^2(\lambda \times P; R^M) : ||\sigma^T \theta||_{2, P} < \infty \right\}.
\]

Note that \(\mathcal{L}^2(G, P)\) is entirely defined in terms of the probability \(P\).

A trading strategy is said to be admissible if

(i). \(\theta \in \mathcal{L}^2(G, P)\)  

(ii). There exists a consumption plan \((c, W) \in PM^2L^2(\lambda \times P) \times L^2(P)\) such that

\[
\begin{align*}
\alpha(t)B(t) + \theta(t) \cdot S(t) + \int_0^t c(s) \, ds &= \alpha(0)B(0) + \theta(0) \cdot S(0) + \int_0^t \left( \alpha(s) dB(s) + \theta(s) \cdot dG(s) \right) \\
&= \int_0^t \left( \alpha(s) dB(s) + \theta(s) \cdot dG(s) \right)
\end{align*}
\]

for all \(t \in [0, T]\) \((a.s., P)\) and

\[
W = \alpha(T)B(T) + \theta(T) \cdot S(T) \quad (a.s., P)
\]

The consumption plan \((c, W)\) of (8) and (9) is said to be financed by \((\alpha, \theta)\) and the quadruple \((\alpha, \theta, c, W)\) is said to be a self-financing strategy (cf. Cox and Huang [1986], p. 5). Condition (ii) is just the natural integrated budget constraint. First \(\alpha(t)B(t) + \theta(t) \cdot S(t) \equiv W(t)\) is the financial wealth valued at the current price of the existing securities. The integral \(\int_0^t c(s) \, ds\) is the stream of consumption from 0 to \(t\). Of course \(\alpha(0)B(0) + \theta(0) \cdot S(0) \equiv W(0)\) is the initial wealth that the
agent invests at time \( t = 0 \). The quantity \( \int_0^t (\alpha(s) \, dB(s) + \theta(s) \cdot dG(s)) \) can be interpreted as the capital gains of the portfolio \((\alpha, \theta)\) carried from \( 0 \) to \( t \), while \( W = \alpha(T)B(T) + \theta(T) \cdot S(T) = W(T) \) is the final wealth which can be viewed as a bequest. Condition (i) implies that the trading strategy depends on the state only through the information available at that time—an informational constraint—and that \( \int_0^T \theta(s) \cdot dG^*(s) \) is a well-defined stochastic integral, as is shown in the next lemma.

**Lemma 2.4.** Let \( \theta \) satisfy (7). Then

\[
\int_0^T \theta(s) \cdot dG^*(s)
\]

is an \( L^q(Q) \)-martingale for any \( q \in [1, 2) \).

**Proof.** Define \( X(t) = \int_0^t \theta(s) \cdot dG^*(s) \). We first prove that \( X(T) \in L^2(P) \). By (2), \( dG^* = b^* \, dt + \sigma^* \, dW^* = \frac{1}{B}[(b - rS) \, dt + \sigma \, dW] \). The stochastic integral is thus the sum of two terms. Since \( \dot{\theta}(b - rS) = -\kappa \) is bounded, we have \( b - rS = -\sigma \kappa \), and the first term is

\[
\int_0^T \frac{1}{B} \theta \cdot (b - rS) \, dt = - \int_0^T \sigma^* \theta \cdot \kappa \, dt.
\]

By the Cauchy-Schwarz inequality and the fact that \( \kappa \) and \( B \) are bounded, its \( L^2(P) \)-norm is such that

\[
\left\| \int_0^T \sigma^* \cdot \kappa \, dt \right\|_{2,P} \leq K \left\| \int_0^T \sigma^* \theta \cdot \kappa \, dt \right\|_{2,P} \leq K' \left\| \sigma^* \theta \right\|_{2,P} < \infty,
\]

for some constants \( K \) and \( K' \). For the second term

\[
\left\| \int_0^T \frac{1}{B} \sigma^* \theta \cdot dW^* \right\|_{2,P} = \left\| \sigma^* \theta \right\|_{2,P} \leq K \left\| \sigma^* \theta \right\|_{2,P} < \infty.
\]

So \( X(T) \in L^2(P) \). By Hölder’s inequality and lemma 2.3 this implies that \( X(T) \in L^q(Q) \) for any \( q \in (1, 2) \) so that \( \left( \int_0^T \theta(s) \cdot dG^*(s), \mathcal{F}_t, Q \right) \) is an \( L^q(Q) \)-martingale for any \( q \in [1, 2) \).

Let \( \mathcal{H} \) denote the space consisting of the quadruples \((\alpha, \theta, c, W)\). By the linearity of the stochastic integral and the sub-additivity of the norm, we see that \( \mathcal{H} \) is a linear space. It will be convenient to normalize all the prices in units of the 0-th security (the bond). We record a well-known fact about strategies (cf. Cox and Huang [1986], Proposition 3.2).

**Lemma 2.5.** Suppose \((\alpha, \theta, c, W) \in \mathcal{H} \) and let \( W^*(t) = W(t)/B(t) = \alpha(t) + \theta(t) \cdot S^*(t) \) and \( c^*(t) = c(t)/B(t) \) denote the normalized wealth and consumption at time \( t \). Then the value of \((c, W)\) in units of \( B \) at time \( t \) is

\[
E^Q \left[ \int_0^T c^*(s) \, ds + W^* \mid \mathcal{F}_t \right] = W^*(t).
\]
Proof. Let $(\alpha, \theta, c, W) \in \mathcal{H}$. Itô’s formula implies that
\[ dW^*(t) = d(\alpha(t) + \theta(t) \cdot S^*(t)) = d\alpha(t) + d\theta(t) \cdot S^*(t) + \theta(t) \cdot dS^*(t) + d\theta(t) \cdot dS^*(t). \]
But differentiating (8) and dividing through by $B(t)$ one finds
\[ d\alpha(t) + d\theta(t) \cdot S^*(t) + d\theta(t) \cdot (dS^*(t) + S^*(t) r(t) dt) + c^*(t) dt = \theta(t) \cdot r^*(t) dt \]
and since $d\theta(t) \cdot S^*(t) r(t) dt = 0$ we get
\[ dW^*(t) = \theta(t) \cdot (dS^*(t) + r^*(t) dt) - c^*(t) dt = \theta(t) \cdot dG^*(t) - c^*(t) dt. \]
Equivalently,
\[ W^*(t) + \int_0^t c^*(s) ds = W^*(0) + \int_0^t \theta(s) \cdot dG^*(s) \tag{10} \]
for all $t \in [0, T]$. (a.s., $P$). Writing the above equality at $t = T$ and taking conditional expectations under $Q$ with respect to $\mathcal{F}_T$, we get the desired result once we have taken (9) and lemma 2.4 into account.

3. Marketed goods and the price information

In general, a marketed consumption plan $(c, W)$ need not be price measurable. To see this, it suffices to notice that in the definition of a strategy nowhere we assumed that $\theta$ had to be progressively measurable with respect to $\mathcal{F}_t$. This is natural if we insist that agents have access to the whole information in the economy. However, if the wage income flow $y$ is itself price measurable, any price measurable consumption bundle $(c, W)$ can be manufactured by an appropriate trading strategy. This result is essential for the existence proof developed in section 4. In this section we present the tools aimed at showing that the set of marketed bundles is sufficiently large to contain all the price measurable bundles. As a by-product, we will also get that the martingale measure $Q$ is the only one to be price measurable. This certainly restricts the choice of solutions considered in section 4.

Writing (10) at $t = T$, one sees that, for a strategy,
\[ W^* + \int_0^T c^*(t) dt = W_0^* + \int_0^T \theta(t) \cdot dG^*(t), \]
where $W_0^*$ is the initial wealth in units of the 0-th security held by the agent. This prompts the following definition:
\[ \mathcal{M} = \left\{ X(t, \omega) : X = X_0 + \int_0^T \theta(t) \cdot dG^*(t) \quad \text{for some } X_0 \in \mathbb{R} \text{ and } \theta \in L^2(\mathcal{F}, P) \right\}. \]
By lemma 2.4, we know that it is a subspace of $L^2(P)$. We will first show that $\mathcal{M}$ contains all the price measurable functions of $L^2(P)$. The relation between price measurable and marketed consumption goods will then follow easily.
Lemma 3.1. Suppose $X \in L^2(P)$ and let $X_t = E^Q[X \mid \mathcal{F}_t]$ for all $t \in [0, T]$. Then there exists a constant $C$ such that

$$\|X_t\|_{2,r} \leq C \|X\|_{2,r}.$$

Proof. From the Bayes rule (cf. for instance Jacod (1979), Lemma 7.9) and the fact that $\eta(t)$ is adapted

$$X_t = \frac{1}{\eta(t)} E^P[\eta X \mid \mathcal{F}_t] = E^P[\frac{\eta}{\eta(t)} X \mid \mathcal{F}_t].$$

But

$$\frac{\eta}{\eta(t)} = \exp\{\int_t^T \kappa \cdot dW - \frac{1}{2} \int_t^T |\kappa|^2 dt\}$$

so that

$$\left(\frac{\eta}{\eta(t)}\right)^2 = \exp\{\int_t^T 2\kappa \cdot dW - 2 \int_t^T |\kappa|^2 dt\} \exp \int_t^T |\kappa|^2 dt.$$

By exactly the same argument as in lemma 2.3, $E^P\left[(\eta/\eta(t))^2 \mid \mathcal{F}_t\right] \leq e^{(T-t)K^2} \leq e^{TK^2} = C$ for some $K$ majorizing $\kappa$. Thus by the Cauchy-Schwarz inequality

$$|X_t| \leq E^P\left[(\eta/\eta(t))^2 \mid \mathcal{F}_t\right]^{1/2} E^P\left[X^2 \mid \mathcal{F}_t\right]^{1/2} \leq C E^P\left[X^2 \mid \mathcal{F}_t\right]^{1/2},$$

or, taking norms,

$$\|X_t\|_{2,r} \leq C \|X\|_{2,r}.$$

Lemma 3.2. For each $X \in L^2(P)$ there is a unique $X_0 \in R$ and a unique $\rho_X \in PML^2(\lambda \times P; R^N)$ such that

$$X = X_0 + \int_0^T \rho_X(t) \cdot dW^*(t) \quad (a.s., P).$$

In fact $X_0 = E^Q[X]$ and there is a $C' < \infty$ such that

$$\|\rho_X\|_{2,r} \leq C' \|X\|_{2,r}.$$

Proof. Suppose that $X_0 \in R$ and $\rho \in PML^2(\lambda \times P; R^N)$ are such that $X_0 + \int_0^T \rho(t) \cdot dW^*(t) = 0$. By the Cauchy-Schwarz inequality $\|\rho\|_{1, \mathcal{Q}} \leq \|\eta\|_{2,r} \|\rho\|_{2,r}$, so $\left(X_0 + \int_0^T \rho(t) \cdot dW^*(t), \mathcal{F}_t, \mathcal{Q}\right)$ is a continuous martingale and so $X_0 = \int_0^T \rho(t) \cdot dW^*(t) = 0$. Hence $\|\rho\|_{1, \mathcal{Q}} = 0$ and so $\rho = 0$ (a.s., $\lambda \times P$). This proves the uniqueness.

To prove the existence, consider

$$W^*(t) = W(t) - \int_0^t \kappa(S(s), s) ds.$$
By the Cameron-Martin-Girsanov theorem, \( W^*(t) \) is itself a Brownian motion with respect to the law \( \hat{Q} = P \circ W^* \) defined by
\[
\frac{d\hat{Q}}{dP} = \exp \left\{ \int_0^t \kappa(S(s), s) \cdot dW - 1/2 \int_0^t \| \kappa(S(s), s) \|^2 \, ds \right\} = \eta(t)
\]
so \( \hat{Q} = Q \). But \( Q \) is the unique solution to the martingale problem associated with the operator \( L_t = 1/2 \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^N \kappa_i \frac{\partial}{\partial x_i} \) and initiating from zero at \( t = 0 \) (cf. Stroock [1986], Theorem III.4.6). Thus \( Q \) is extremal for \( W^* \).

But if \( X \in L^2(P) \) then by Hölder's inequality and lemma 2.3, \( X \in L^q(Q) \) for any \( q \in (1, 2) \), so if \( X_t = E^Q[X|\mathcal{F}_t] \) we have that \( (X_t, \mathcal{F}_t, Q) \) is an \( L^q(Q) \)-martingale. By the martingale representation theorem (cf. Jacod (1979), Theorem 11.2 and Corollary 11.4),
\[
X_t = E^Q[X] + \int_0^t \rho_X(t) \cdot dW^*(t)
\]
where \( \rho_X \in \text{FMLS}(\lambda \times Q) \). It follows
\[
X_{t+h} - X_t = \int_t^{t+h} \rho_X \cdot dW^* = \int_t^{t+h} \rho_X \cdot dW - \int_t^{t+h} \rho_X \cdot \kappa \, ds
\]
therefore
\[
\left\| \int_t^{t+h} |\rho_X|^2 \, ds \right\|^{1/2}_{2,P} = \left\| \int_t^{t+h} \rho_X \cdot dW \right\|_{2,P}
\]
and
\[
\left\| X_{t+h} - X_t \right\|_{2,P} \leq A\delta^{1/2} \left( \int_t^{t+h} |\rho_X|^2 \, ds \right)^{1/2}_{2,P}.
\]
Choose \( \delta > 0 \) such that \( A\delta^{1/2} < 1/2 \) and assume that \( \|\rho_X\|_{2,P} < \infty \). Then
\[
\left\| \int_t^{t+h} |\rho_X|^2 \, ds \right\|^{1/2}_{2,P} \leq 2 \left\| X_{t+h} - X_t \right\|_{2,P} \leq 4C \|X\|_{2,P}
\]
by lemma 3.1. Finally, choose \( k \geq 1 \) so that \( 1/k < \delta \). Then
\[
\left\| \rho_X \right\|^2_{2,P} = \sum_{i=0}^{k-1} \left\| \int_{i/k}^{(i+1)/k} |\rho_X|^2 \, ds \right\|^2_{2,P} \leq 16kC^2 \|X\|^2_{2,P}
\]
or
\[
\left\| \rho_X \right\|_{2,P} \leq C' \|X\|_{2,P}
\]
where \( C' = 4C\sqrt{k} \).

For the general case, we approximate \( X \) by bounded functions \( X^{(n)} = X1_{[0,n]}(\|X\|) \). From the Lebesgue dominated convergence theorem, \( X^{(n)} \rightarrow X \) in \( L^2(P) \). Now let as before \( X^{(n)} = \)
\[
X_{0}^{(n)} + \int_{0}^{T} \rho_{X}^{(n)} \cdot dW^{*} \quad \text{and} \quad X = X_{0} + \int_{0}^{T} \rho_{X} \cdot dW^{*} \quad \text{be the martingale representations for} \quad X^{(n)} \quad \text{and} \quad X,
\]
respectively. The Cauchy-Schwarz inequality implies that \(|\|\rho_{X}^{(n)}\|_{2, P} \leq \|\rho_{X}^{(n)}\|_{4, Q} \|\eta^{-1}\|_{2, Q}\). Burkholder's inequality in turn implies that \(|\|\rho_{X}^{(n)}\|_{4, Q} \leq B \|X^{(n)} - X_{0}^{(n)}\|_{4, Q}\) for some constant \(B\). Since \(X^{(n)}\) is bounded, it follows that \(|\|\rho_{X}^{(n)}\|_{2, P} < \infty\). We may now apply (12) to get
\[
|\|\rho_{X}^{(n)} - \rho_{X}^{(m)}\|_{2, P} \leq C' \|X^{(n)} - X^{(m)}\|_{2, P} \to 0
\]
when \(n \to \infty\). Thus \(\rho_{X}^{(n)}\) converges in the \(|\| \cdot |\|_{2, P}\) norm to \(\rho_{X}\) and we obtain (12) for any \(X \in L^{2}(P)\) by letting \(n \to \infty\).

The following last lemma gives the relation between \(M\) and the null space of \(\sigma\).

**Lemma 3.3.** Let \(X \in L^{2}(P)\). Then \(X \in M \iff \rho_{X}(t) \perp \text{Null}(\sigma(t)) \quad (a.s., P)\) for each \(t \in [0, T]\).

**Proof.** Say \(X \in M\). Then
\[
X = X_{0} + \int_{0}^{T} \theta(t) \cdot dG^{*}(t) = X_{0} + \int_{0}^{T} \theta(t) \cdot \sigma^{*} dW^{*}(t) = X_{0} + \int_{0}^{T} \sigma^{* T}(t) \theta(t) \cdot dW^{*}(t).
\]
But since \(B\) is bounded away from zero, \(|\|\sigma^{* T} \theta\|_{2, P} \leq K \|\sigma^{* T} \theta\|_{2, P} < \infty\). Thus \(\rho_{X} = \sigma^{* T} \theta \in PML^{2}\) and is orthogonal to the null space of \(\sigma\).

Conversely, suppose \(\rho_{X}(t) \perp \text{Null}(\sigma(t)) \quad (a.s., P)\) for each \(t \in [0, T]\), and let \(\theta(t) = \sigma^{* T} \rho_{X}(t)\). Then \(\sigma^{* T} \theta(t) = \sigma^{* T} \sigma^{* T} \rho_{X}(t) = \pi_{\sigma^{* T} \rho_{X}(t)} = \rho_{X}(t)\) since \(\rho_{X}(t) \in \text{Range} \sigma^{* T}(t)\). Hence
\[
X = X_{0} + \int_{0}^{T} \sigma^{* T}(t) \theta(t) \cdot dW^{*}(t) = X_{0} + \int_{0}^{T} \theta(t) \cdot \sigma^{* T}(t) dW^{*}(t) = X_{0} + \int_{0}^{T} \theta(t) \cdot dG^{*}(t).
\]
Moreover, since \(B\) is bounded
\[
|\|\sigma^{* T} \theta\|_{2, P} \leq K \|\sigma^{* T} \theta\|_{2, P} = K \|\rho_{X}\|_{2, P} \leq Kc' \|X\|_{2, P} < \infty.
\]
for some constant \(K\) by lemma 3.2. Thus \(X \in M\).

Returning to our main concern, we can now state the following result:

**Proposition 3.1.** If \(X \in L^{2}(P)\) is price measurable, then \(X \in M\).

**Proof.** We have seen that \((W^{*}, \mathcal{F}, Q)\) is a Brownian motion. But by (2),
\[
dG^{*}(t) = b^{*} \, dt + \sigma^{*} dW = \sigma^{*} (dW + \kappa \, dt) = \sigma^{*} dW^{*}
\]
so that \(G^{*}\) is a right continuous, \(\mathcal{F}_{t}\)-progressively measurable and \((a.s., P)\) continuous process satisfying
\[
G^{*}(t) = S^{*}(0) + \int_{0}^{t} \sigma^{*}(t) dW^{*}(t) \quad (a.s., P)
\]
Since \( P \) and \( Q \) are equivalent, these statements hold also under \( Q \). But \( B \) is bounded away from zero, hence \( Q \) is the unique solution to the martingale problem associated with the operator \( L_t = \frac{1}{2} \sigma^* \sigma^{*T} \) and initiating from \( S^*(0) \) at \( t = 0 \). Thus \( Q \) is extremal for \( G^* \).

On the other hand, if \( X_t = E^Q[X] \mid \mathcal{F}^S_t \), \( (X_t, \mathcal{F}^S_t, Q) \) is an \( L^q(Q) \)-martingale for any \( q \in (1, 2) \). Hence by Jacod’s martingale representation theorem and the fact that \( X \) is price measurable
\[
X = X_T = X_0 + \int_0^T \theta \cdot dG^*(t) = X_0 + \int_0^T \sigma^{*T} \theta \cdot dW^*(t). \quad (13)
\]
But \( \rho_X = \sigma^{*T} \theta \) is orthogonal to the null space of \( \sigma \), so that by lemma 3.3, \( X \in \mathcal{M} \).

We can now apply proposition 3.1 to the case where the consumption good \((c, W)\) is price measurable. If we define \( X = \int_0^T c^*(s) \, ds + W^* \), we know that \( X \in L^2(P) \) and is price measurable (use sums to approximate the integral defining \( X \)). Therefore we have (13) which we rewrite as
\[
W^* + \int_0^T c^*(t) \, dt = X_0 + \int_0^T \theta(t) \cdot dG^*(t).
\]
Now define \( W^*(t) \) in accordance with lemma 2.5 by
\[
W^*(t) = E^Q \left[ W^* + \int_t^T c^*(s) \, ds \mid \mathcal{F}_t \right].
\]
The choice \( t = T \) shows that \( W = W(T) \) so that (9) is satisfied. On the other hand the choice \( t = 0 \) gives \( W^*(0) = X_0 \). Finally we see that
\[
W^*(t) + \int_0^t c^*(s) \, ds = E^Q \left[ W^* + \int_0^T c^*(s) \, ds \mid \mathcal{F}_t \right] = W^*(0) + \int_0^t \theta(t) \cdot dG^*(t),
\]
so that (10) is also satisfied. It is then clear that if we choose \( \alpha(t) \) so that \( W^*(t) = \alpha(t) + \theta(t) \cdot S^*(t) \), then the corresponding \((\alpha, \theta, c, W) \in \mathcal{Z} \).

We conclude this section by proving a result of independent interest.

**Proposition 3.2.** \( R(t) = \eta(t) \) is the unique price measurable absolutely continuous martingale measure.

**Proof.** Recall from (5) that \( dN_t = N_t \nu(t) \cdot dW^*(t) \) for some \( \nu(t) \in (\text{Null} \sigma)^\perp \), and suppose that \( \nu(t) \neq 0 \). \((a.s., P)\). We have \( \nu(t) \cdot dW^*(t) = \nu(t) \cdot (dW^*(t) - \kappa(t) \, dt) = \nu(t) \cdot dW^*(t) = dN(t) \) since \( \nu \) and \( \kappa \) lie in orthogonal subspaces, so that \( N(t) \) is a martingale under both \( P \) and \( Q \). In particular,
\[
N = 1 + \int_0^T N(t) \rho_N(t) \cdot dW^*(t)
\]
with \( \rho_N(t) = \nu(t) \in \text{Null} (\sigma(t)) \). By lemma 3.3 then, \( N \notin \mathcal{M} \). Therefore, \( N \) is not measurable with respect to the price process and the result follows.
4. Optimization under incomplete markets

In this section we finally address the maximization program faced by an agent. He is endowed with a utility function on consumption rates \( u : R_+ \times [0, T] \to R \cup \{-\infty\} \) and a utility function on final wealth \( V : R_+ \to R \cup \{-\infty\} \). His initial wealth is denoted by \( W(0) \). The problem is to find a strategy \((\alpha, \theta, c, W)\) to maximize his expected utility

\[
\sup_{(\alpha, \theta, c, W) \in \mathcal{H}} E^P \left[ \int_0^T u(c(t), t) \, dt + V(W) \right]
\]

subject to

\[
\alpha(0) B(0) + \theta(0) \cdot S(0) \leq W(0) \tag{14}
\]

\[
c \geq 0 \quad (a.s., \lambda \times P)
\]

\[
W \geq 0 \quad (a.s., P).
\]

To prove that there exists a solution to (14), as in Cox and Huang [1986], we embed this dynamic problem in the following static variational problem

\[
\sup_{(c, W) \in \mathbb{P}ML^2_+(\lambda \times P) \times L^2_+(P)} E^P \left[ \int_0^T u(c(t), t) \, dt + V(W) \right]
\]

subject to

\[
E^Q \left[ \int_0^T c'(t) \, dt + W^* \right] \leq W^*(0). \tag{15}
\]

where \( Q \) is the probability defined in section 2.2.

By lemma 2.5 applied with \( t = 0 \) it is immediate that if a trading strategy satisfies the budget constraint of (14) it satisfies also that of (15). So once we have proved that any solution \((c, W)\) of (15) can be associated with an \((\alpha, \theta, c, W) \in \mathcal{H}\) satisfying the same budget constraint, we are done.

We record the following useful lemma.

**Lemma 4.1.** Let \( X(t) \) be a positive process. Then there exists an \( \mathcal{F}_t^S \)-progressively measurable process \( Y(t) \) such that for every \( \mathcal{F}_t^S \)-stopping time \( \tau \)

\[
E^Q \left[ X, I_{\tau < \infty} \left| \mathcal{F}_t^S \right. \right] = Y, I_{\tau < \infty}.
\]

Moreover,

\[
E^Q \left[ \int_0^T X(t) \, dt \right] = E^Q \left[ \int_0^T Y(t) \, dt \right]
\]

**Proof.** Dellacherie and Meyer [1982], theorems VI.43 and 57, and the fact that all optional processes are necessarily progressively measurable (cf. Chung and Williams [1983], Section 3.4).

For a definition of stopping times, see for instance Chung and Williams [1983], Section 1.7. However, we will restrict ourselves to the case where \( \tau = t \wedge T \), so that (16) becomes simply

\[
E^Q[X(t) \mid \mathcal{F}_t^S] = Y(t)
\]

The process \( Y(t) \) is called the optional projection of \( X(t) \).
Now let \((c, W)\) be a solution of \((15)\), define \(\hat{W} = E^Q[W | \mathcal{F}^S_t]\) and let \(\hat{c}\) be the optional projection of \(c\) as in lemma 4.1. Consider first the budget constraint \((15)\). By lemma 4.1 (choose \(X(t) = c(t)/B(t)\)), we can rewrite it as
\[
E^Q \left[ \int_0^T \hat{c}(t) \, dt + \hat{W}^* \right] \leq W^*(0),
\]
where we have used the fact that the optional projection of \(c^*\) is the discounted value of \(c^*\). But now \((\hat{c}, \hat{W})\) is price measurable so that from section 3, we know that it is marketed and that its initial cost is at most \(W^*(0)\). It remains to check that it is at least as good as the original \((c, W)\).

By Jensen's inequality,
\[
u(\hat{c}(t), t) \geq E^Q[u(c(t), t) | \mathcal{F}^S_t]
\]
and similarly
\[
V(\hat{W}) \geq E^Q[V(W) | \mathcal{F}^S_t].
\]
Thus
\[
E^\mathbb{P} \left[ \int_0^T u(\hat{c}(t), t) \, dt + V(\hat{W}) \right] = E^Q \left[ \int_0^T \frac{1}{\eta(t)} u(c(t), t) \, dt + V(W) \right]
\]
\[
= E^Q \left[ \int_0^T \frac{1}{\eta(t)} u(c(t), t) \, dt + \frac{V(W)}{\eta} \right].
\]
by proposition 2.4 (choose \(X = \frac{1}{\eta} u(\hat{c}(t), t))\),
\[
\geq E^Q \left[ \int_0^T \frac{1}{\eta(t)} E^Q[u(c(t), t) | \mathcal{F}^S_t] \, dt + \frac{1}{\eta} E^Q[V(W) | \mathcal{F}^S_t] \right]
\]
\[
= E^Q \left[ \int_0^T E^Q[u(c(t), t) | \mathcal{F}^S_t] \, dt + E^Q[V(W) | \mathcal{F}^S_t] \right].
\]
since \(\eta\) is \(S\)-measurable (proposition 2.3)
\[
= E^Q \left[ \int_0^T \frac{u(c(t), t)}{\eta(t)} \, dt + \frac{V(W)}{\eta} \right]
\]
by lemma 4.1 again,
\[
= E^\mathbb{P} \left[ \int_0^T u(c(t), t) \, dt + V(W) \right].
\]

Thus, under assumption 2.1, any solution of \((15)\) is matched by a solution of \((14)\), so that the two problems yield in fact equivalent solutions. When the utility function \(u\) is strictly concave in \(c\), the solution is unique so that it will automatically be price measurable. When \(u\) is not strictly concave, one may have of course more than one solution. Consider for instance the consumption good \((\hat{c}, \hat{W})\) where \(\hat{W} = E^\mathbb{P}[W | \sigma[\eta(t) : 0 \leq t \leq T]]\) and similarly for \(\hat{c}\). Since \(\eta(t)\) is price measurable (proposition 2.3), \((\hat{c}, \hat{W})\) is certainly marketed. A verification in all respect similar to the one above shows that \((\hat{c}, \hat{W})\) satisfies the budget constraint and that it is at least as good as
the original \((c, W)\). We may thus have two (here, measurable) solutions yielding the same expected utility and choose the most convenient depending on applications.

The clear advantage of (15) is that it is much easier to solve than (14). General conditions under which there exists a solution to (15) can be found in Cox and Huang [1985], when the commodity is either a consumption rate process or consumption at the final date. When the utility function displays a time-additive structure of the form

\[ u(x, t) = e^{-\beta t} u(x), \]

where there agent’s discount rate \(\beta_t\) is bounded from below, then a sufficient condition for almost all utility functions of practical interest is that the inverse of the shadow price \(\eta(t)\) has a certain finite moment (op. cit., theorems 2.9 and 2.10). Note that this last condition is always satisfied under our own assumptions by lemma 2.3. For the sake of completeness, we report two sets of sufficient conditions for utility functions that are either bounded or unbounded from below.

**Proposition 4.1.** Suppose that \(u : R^+ \mapsto R^+\) is continuous, concave and strictly increasing. Suppose further that there exists \(b \in (0, 1]\) such that for every \(\delta > 0\)

\[ u(x) \leq A_k + B_k x^{1-k+b} \]

asymptotically for some \(A_k \geq 0\) and \(B_k > 0\). Then there exists a solution to (15).

**Proof.** Cox and Huang [1986], Theorem 2.4.

The last proposition includes the utility functions that are of constant relative risk aversion as a special case. For utility functions that are unbounded from below, we have the following.

**Proposition 4.2.** Suppose that \(u : R^+ \setminus \{0\} \mapsto R\) is differentiable, concave and strictly increasing. Suppose further that there exists \(b > 0\) such that for every \(\delta > 0\)

\[ u'(x) \leq K_k x^{-b+k} \]

asymptotically for some \(K_k > 0\). Then there exists a solution to (15).

**Proof.** Cox and Huang [1986], Theorem 2.6
5. An example

We conclude this paper by giving an application to a simple consumption portfolio problem. Since we are primarily interested in existence rather than characterization, we do not seek completeness in this example. We only want to illustrate how easily its solution can be extended to the situation where markets are incomplete. Suppose that consumption occurs only at the final date. The consumption space will then be $L^2(P)$. The interest rate is constant and equal to zero so that $B(t) = 1$. The risky securities gain process follows a multiplicative geometric Brownian motion

$$G(t) = S(t) + \int_0^t \sigma (s) dW(s) + \int_0^t S(s) dS(t) \forall t \in [0, T] \quad (a.s., P),$$

where $b$ is a $M \times 1$ vector of constants, $\sigma$ is an $M \times N$ constant matrix of full rank ($M < N$), and $I_S(t)$ a diagonal matrix whose $i$-th entry is equal to $S_i(t)$, $i = 1, \ldots, M$. The process $\eta(t)$ of proposition 2.2 becomes the lognormal variable

$$\eta(t) = \exp \left\{ -\frac{t}{2} |\sigma b|^2 \right\}. \quad (17)$$

In this case strategies take a particularly simple form. A self-financing strategy is defined by $W(t) = \alpha(t) + \beta(t) \cdot S(t) = W(0) + \int_0^t \beta(t) \cdot dG(t)$ and the space of marketed commodities is

$$\mathcal{M} = \left\{ W \in L^2(P) : W = W(0) + \int_0^T \beta(t) \cdot dG(t) \quad \text{for some} \ W(0) \in \mathcal{R} \text{and} \ \beta \in \mathcal{L}^2(G, P) \right\},$$

where $\mathcal{L}^2(G, P)$ is the set of $\mathcal{F}_t$-progressively measurable $\beta$ such that $|||\sigma^T I_S(t) \beta(t)|||_{2, \mathcal{F}} < \infty$, as before.

We now choose a utility function of constant relative risk aversion:

$$u(x) = \frac{1}{1+\frac{1-\frac{1}{\gamma}}{\gamma}} \quad \text{if} \ x \in (0, \infty) \quad \text{and} \ \gamma > 0;$$

$$= 0 \quad \text{if} \ x = 0 \quad \text{and} \ \gamma > 1;$$

not defined \ \text{if} \ x = 0 \quad \text{and} \ \gamma \in (0, 1).$$

where we have used the convention that $x^0 = \log x$

The problem faced by an investor endowed with one unit of the consumption good at time 0 is

$$\max_{W \in \mathcal{M}} E^P u(W) \quad \text{s.t.} \ W \geq 0 \text{ and } W(0) = 1.$$

Consider the extended maximization program corresponding to (15)

$$\sup_{X \in L^2(P)} \left\{ E^P u(X) : X \geq 0 \quad \text{and} \quad E^Q X = 1 \right\}. \quad (18)$$
where $\frac{dQ}{dt} = \eta$, as before. When $\gamma > 1$ we have
\[
\sup_{X \in L^2(P)} \left\{ E^P X^{1-\frac{1}{\gamma}} : X \geq 0 \text{ and } E^Q X = 1 \right\} = \sup_{X \in L^2(P)} \left\{ E^P |X|^{1-\frac{1}{\gamma}} : E^Q |X| = 1 \right\}.
\]
Now define $Y = X^{1-\frac{1}{\gamma}} \Leftrightarrow X = Y^{\gamma'}$ where $\gamma$ and $\gamma'$ are conjugate ($\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$). Since by the Cauchy-Schwarz inequality $X \in L^1(Q)$, then $Y \in L^{\gamma'}(Q)$ so the above supremum is no greater than
\[
\sup_{Y \in L^{\gamma'}(Q)} \left\{ E^P |Y| : \|Y\|_{\gamma',Q} = 1 \right\} = \sup_{Y \in L^{\gamma'}(Q)} \left\{ E^Q \frac{1}{\gamma'} Y : \|Y\|_{\gamma',Q} = 1 \right\} = \left\| \frac{1}{\gamma'} \right\|_{\gamma',Q}.
\]
One can check that the supremum is achieved for $\hat{Y} = \left( \frac{1}{\gamma'} \right)^{\gamma-1} / \left\| \frac{1}{\gamma'} \right\|_{\gamma',Q}$ which yields
\[
\hat{X} = \left( \frac{1}{\eta} \right)^{\gamma} / \left\| \frac{1}{\gamma} \right\|_{\gamma',Q}.
\] (19)

The case $0 < \gamma < 1$ can be treated in the same way with "inf" replacing "sup", while $\gamma = 1$ can be solved directly.

Since $\hat{X} \in L^2(P)$ by lemma 2.3, $\hat{X}$ is indeed the solution of (18). In addition, $\hat{X}$ is price measurable by proposition 2.3 so that by proposition 3.1, $\hat{W} = \hat{X}$ is marketed. We now elicit the strategy that manufactures $\hat{W}$. Substituting (17) for $\eta$ in (19), we get after some manipulations
\[
\hat{W} = \exp \left\{ \gamma \hat{a} \cdot \hat{W}^*(T) - \frac{\gamma^2 T}{2} |\hat{a} b|^2 \right\},
\]
where as a result of the Cameron-Martin transformation $\hat{W}^*$ is given by $\hat{W}^*(t) = \hat{W}(t) + \hat{a} bt$. Defining
\[
\hat{W}(t) = \exp \left\{ \gamma \hat{a} \cdot \hat{W}(t) - \frac{\gamma^2 T}{2} |\hat{a} b|^2 \right\},
\]
Itô's formula implies that
\[
\hat{W}(t) = E^Q \left[ \hat{W} \mid \mathcal{F}_t \right] = 1 + \gamma \int_0^t \hat{W}'(s) \hat{a} b \cdot d\hat{W}^*(s)
= 1 + \gamma \int_0^t \hat{W}'(s) \sigma^T \hat{a} b \cdot d\hat{W}^*(s)
= 1 + \gamma \int_0^t \hat{W}'(s) I_{\sigma}^{-1}(s) \sigma^T \hat{a} b \cdot dG(s),
\]
since $dG(t) = I_{\sigma} d\hat{W}^*(t)$. Hence by lemma 3.2, $\theta(t) = \gamma I_{\sigma}^{-1}(t) \sigma^T \hat{a} b \hat{W}(t)$. But $\hat{a} b \in \text{Range } \sigma^T$ so that
\[
\hat{a}^T b = (\sigma \sigma^T)^{-1} \sigma \hat{a} b = (\sigma \sigma^T)^{-1} \pi \sigma b = (\sigma \sigma^T)^{-1} b
\]
because $\pi = I_M$ when $\sigma$ is of full rank. We then have
\[
I_{\sigma} \theta(t) = \gamma (\sigma \sigma^T)^{-1} b \hat{W}(t)
\]
so that, as expected, the portfolio shares are constant and independent of the level of wealth.
6. Concluding remarks

In this paper we have extended the martingale approach to the case where markets are dynamically incomplete. We obtained essentially two results. First, it is still possible to duplicate the pay-offs of any contingent claim by using an appropriate trading strategy, provided it is consistent with the information generated by the securities price system. One has, for instance, a constructive technique for hedging the risk associated with holding positions in any option written on stocks. Second, it is also possible to solve a very general version of the intertemporal portfolio problem. Though we did not seek to characterize the optimal individual policies, we provided sufficient conditions for the existence of a solution, which are virtually the same as in the complete markets case. Furthermore, those conditions were entirely defined in terms of the primitives of the model.

The model we have dealt with is a rational expectations one in the sense that its agents all agree on the law of motion satisfied by the securities price process. Under those conditions, we have shown that when markets are incomplete, all actions consistent with the information revealed by prices are necessarily “marketed”, in the sense that they can be implemented by trading the few long-lived securities continuously through time, and this result has provided the clue to the derivation of the optimal strategies.
References
