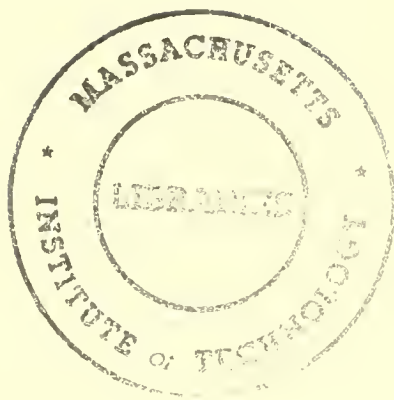


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M.I.T. Working Paper No. 1127-80
OPTIMAL LONG-TERM INVESTMENT WHEN
PRICE DEPENDS ON OUTPUT

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Massachusetts Institute of Technology
Sloan School of Management
May, 1980

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ABSTRACT

In some industries (e.g., mining), major capital investments are necessary to develop resource supplies. Once a commitment has been made, however, low operating costs may make it uneconomic to shut down an old facility before the end of its productive life. Under these circumstances, a new investment will have an impact on total industry supply (and thus on the prevailing price of the product) which will be difficult to reverse in the short run. This "total supply effect" is shown to result in optimal investment criteria which differ substantially from the standard net present value rule. This paper develops optimal investment rules for (a) monopolistic, (b) oligopolistic and (c) competitive industry participants and considers the effect of industry structure on prices and output assuming optimal investment rules are followed.

Dewey



Optimal Long-Term Investment when Price Depends on Output

1. Introduction.

In evaluations of new capital investment projects, the standard financial decision criterion is the net present value rule: a firm seeking to maximize shareholder wealth should accept any and all projects with $NPV > 0$. It is generally recognized that the net present value criterion is not always appropriate: for example, if capital is rationed or if a given project produces significant economies (diseconomies) in other sectors, analysis of the project may have to be expanded to take additional constraints and/or opportunities into account. (see, for example, Chenery [1959]). However, when projects are undertaken in developed economies with large capital markets, the effects of capital rationing and intersectoral economies are assumed to be small. Within a firm, the effect of one project on other projects undertaken by the same firm can (it is generally supposed) be accounted for by adjusting the expected future cash flows attributable to the project. As long as all cash flows are counted properly, the net present value rule will always be consistent with the firm's value maximization objective.

This paper seeks to demonstrate that when firms compete in an imperfect product market, naive application of the NPV rule (i.e., estimation and evaluation of direct cash flows only) is not in general consistent with value maximization. At the same time, an extensive NPV analysis, while correct, may not be computationally feasible. A preferable approach is to modify the acceptance criterion to be "accept Project Z if the direct cash flows have $NPV > z$ " where z is determined via optimization of an investment decision model which is developed in this paper. The value of z (after optimization) is affected by characteristics of industry demand, competition, opportunity, and the longevity of assets.

Given the diversity of the literature on imperfect product markets,

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it is useful, before proceeding, to clarify what type of market this analysis seeks to address. Imperfect markets are frequently defined in opposition to perfectly competitive markets which are characterized in two ways:

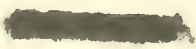
(Perfect Competition)

1. All sellers face a horizontal demand curve for the product, or
2. The number of sellers is very large. (Usually (2) implies that (1) "effectively" holds for all participants).

The imperfect market treated in this paper is one in which the industry total demand curve is downward sloping: the impact of different numbers of market participants on optimal investment decisions will be examined in the course of the analysis.

The basic situation addressed in this paper is as follows: a firm operates in an industry which produces a homogeneous commodity. The price of the commodity is determined by total industry output: as output increases the price will fall. The total supply of output is determined by firms' investment decisions; however, once an investment has been made, the supply thereby created cannot be revoked or eliminated in the short run. The value of an investment is established by the interaction of output price and the costs associated with the individual project.

Uncertainty enters the model in a number of ways. In general, firms are uncertain about (1) the lifetimes remaining to existing productive assets, and (2) the occurrence of and costs associated with future investment opportunities. In the model, these outcomes are governed by exogenous probability distributions: filtered through the optimal investment rule these probabilistic events ultimately determine (1) the output of individual



firms, (2) total industry output, and (3) prices at future points in time.

The basic model may be thought of as applying to a number of different industries. A particular example developed in the paper is that of a mining company, which produces metal for sale at world commodity prices. The opportunities the company faces are new potential mine sites; total output and price are determined by the number of productive mines operating at any time. The question addressed is: what criterion should the firm use to decide whether to develop a new mine site?

The model can also be applied to an industry like chemicals: here technological innovations and/or new refinery configurations can suddenly make new supplies of feedstock available. New technologies or new sources of supply create investment opportunities, which, if taken, will affect the total supply of a chemical product and its price.

In a slightly different vein, a transformation of variables would allow model to encompass innovation in the form of new product development. In this case, the output prices associated with different projects would be essentially independent (reflecting product dissimilarity and differentiation), but the cost of inputs would be an increasing function of total usage. Examples of industries to which this type of economic structure is relevant include (1) food products (those based on a common raw material like chocolate), (2) metals and plastic fabrication, and (3) paper products (the common raw material is pulp).

Previous Literature. The model presented here is a model of investment strategy applicable to particular types of firms and industries. It may be contrasted with a model of a different type of industry developed by Spence [1979]. Spence considered interaction among firms in a growing market (one where demand exceeds

supply providing firms with the opportunity to grow). An essential feature of his model was the firms' exploitation of a transient growth opportunity; investments in the growth phase were key determinants of firms' relative position (value) when the industry reached a static maturity.

This paper considers optimal investment policies in a mature, dynamic industry. Demand for product is essentially fixed and stationary (although it depends on price). On the other hand, output, as well as the relative position and value of firms, varies based on (1) the opportunities which are found and (2) the investment decisions made by the firms. Investments are considered to be not permanent (as in Spence), but transitory (although irreversible in the short run). Finally, consistent with the notion of a company's free access to a frictionless capital market, the model places no limit (as does Spence) on firms' ability to fund profitable investment opportunities; limitations on investment (and deviations from the net present value rule) arise solely because of limitations of the product market and of the cost structure of opportunities.

In addition to Spence's work on investment strategy, there is a small but significant literature which treats the relationship between the rate of return required by stockholders and the marginal (i.e., minimum) rate of return on new investment opportunities in an intertemporal context. Elton and Gruber [1976] demonstrated that the optimal marginal rate of return depends on the evolution of the investment path over time. Aivasian and Callen [1979] considered the effect of market structure on optimal investment and the cost of capital over time. They showed that a monopolist or (Cournot) oligopolist will optimally set the marginal rate of return higher (lower)

than the cost of capital if future investment opportunities are a decreasing (increasing) function of current investment.

This paper extends this previous research in two ways. First, it develops a model of the dynamics of the opportunity set which is based on industry and firm parameters. Second, it explicitly relates investment decisions to the demand curve for output as well as the cost structure of opportunities.

Finally, Leland [1972] made an important contribution to the theory of firm behavior under uncertainty by analyzing a firm facing uncertain demand. Leland considered optimal price and/or quantity setting by a utility-maximizing (risk neutral or risk averse) firm over a single period. The model presented below differs from Leland in that it is dynamic: price and output are fixed and known this period but will fluctuate randomly (as output changes) in future periods. A further difference in this model is that in the short run neither price nor quantity are directly controllable by firms: in the long run, however, firms exercise indirect control over both price and quantity via investment decisions. Finally, it should be noted that the present model assumes firms are value (rather than utility) maximizers: the investment opportunities a firm accepts are presumed to be instantaneously capitalized in a frictionless capital market. For convenience, key variables for the industry are assumed uncorrelated with the rest of the market, thus, in effect, firms behave as risk-neutral decision-makers.

Section 2 describes the formulation of the basic model.

In it, an important assumption is made to simplify the analysis: firms are assumed to use long-term contracts to "lock in a spread" on new investment opportunities. The investments can thus be valued as riskless assets (albeit with uncertain termination dates). Extension of the model to consider the sale of output at spot prices (i.e., without long-term fixed price contracts) is the subject of ongoing research. Section 3 characterizes a firm's optimal investment decision rule. Section 4 presents a numerical example, which is then used to illustrate the effect of concentration on industry value and the price of output. Section 5 presents conclusions and discusses how the model could be implemented for an actual firm.

2. Model Formulation.

Demand. Consider a company which extracts metal (say copper) from mines in different locations. The company sells its output in the world copper market. Prices prevailing in the market at any point in time are determined by the total quantity Q being produced at that time:¹

$$p = p(Q) \tag{1}$$

Total output Q at time t depends on the number of mines open at that time. All mines are assumed to be the same size and indivisible, thus total output may be scaled by the output of one mine:

$$Q = 0, 1, 2, \dots \tag{2}$$

$Q = 0$ implies no mines are open: we assume $p(0)$ is finite. (More generally, $Q = 0$ may be taken to represent a minimum level of output which is always achievable at constant cost ($< p_0$). This minimum level might be sustained in the absence of new mine openings by (1) scrap reprocessing and (2) secondary recovery techniques.²)

We assume that copper is a normal good, thus price declines as industry output increases:

$$p(0) > p(1) > p(2) \dots \tag{3}$$

Objective. The copper company seeks to maximize its value in the capital

market. Equivalently (assuming frictionless capital markets and fully informed investors) it seeks to maximize the present value of its future stream of output.

Investments. The company's investment process is as follows: from time to time company geologists identify a new mine opportunity. We approximate the arrival of new opportunities by means of a Poisson process with frequency λ (average interarrival time $\tau = 1/\lambda$).

A new mine opportunity if undertaken will produce 1 unit of copper per year at a cost c per year. The cost c is affected by factors such as ore richness, location, required capital investment, etc., and is variable across opportunities. The costs related to a specific mine become known at the time the mine opportunity is identified (not before). Characteristic costs for successive mine opportunities are drawn independently from a distribution $f(c)$ which is known to the company.

When a new mine opportunity is identified, the company may choose to open the mine or not. We assume that, at the time a mine is opened, the company may also enter into a long-term fixed-price contractual sale of the output from the mine: it is this price which is given by the price function $p(Q)$. It is assumed that the company, if it decides to open a mine, always sells the associated contract, thus from the company's point of view, the mine represents a riskless stream of returns for as long as it remains open.

We assume that a rejected mine opportunity disappears, i.e., the company cannot recall previously rejected opportunities when economic conditions change. This assumption is made in order to simplify the problem to make it analytically and computationally tractable. The assumption is partly offset by the

steady-state assumption that the opportunity set distribution $f(c)$ is not deteriorating over time. Even though past opportunities cannot be recalled at will, they or their equivalents may be encountered by the company again as it continues to search. Operationally, the situation might be represented as follows: initially, the company doesn't own a mine site but only holds a development option (with a positive exercise price). If the immediate decision is against developing the mine, the option expires unexercised and the site reverts to the general pool of opportunities. Subsequent review of the same site would require a new development option (with a new exercise price); costs associated with the site would thus be changed, making it in effect a new opportunity.

Mines differ in the total extent of their ore bodies, thus with identical extraction rates, will last different lengths of time. The duration of a mine's productivity is uncertain. It is assumed to be determined by a Poisson process having characteristic frequency μ . At any point in time, all open mines have expected productive lives of T years remaining where $T = 1/\mu$.

Cost Structure of Opportunities. We have said that if the company reviews a mine opportunity and accepts it, output for the duration of the mine's natural life will be sold at the contractual price $p(Q)$ (where Q is the number of mines open at the time the opportunity is found). So far, however, we have not characterized the costs of production (except to associate them with a distribution $f(c)$). We shall see below that the breakdown of costs plays an important role in the determination of a company's responses to new developments in the industry (changes in Q).

In general, costs associated with any given production technology may be

classified as: (1) capital costs which are incurred before, and as a condition to production, and (2) operating costs which are contemporaneous with production. Let us denote costs in these categories as k and o , respectively.

To these two traditional categories we add a third: let s denote the shutdown cost associated with closing a mine prematurely. Shutdown costs s can be thought of as the lump sum settlement for all costs which would continue after production was prematurely halted. Shutdown costs include items such as (1) the loss of value on downstream facilities, (2) bonuses and/or pensions paid to terminated employees, (3) remaining debt service, (4) compensatory damages paid on violated contracts, (5) fines or taxes levied as a result of closing.³ Note that if $s < 0$, the mine has a positive net salvage value.

Present Value of an Opportunity. Consider a mine opportunity at the time industry output $Q = i$ ($p(Q) = p_i$). Assume the mine will stay open T years and that shutdown costs are forecast to be zero. By the standard formula, the net present value of the opportunity is:

$$NPV = -k + \int_0^T (p_i - o) e^{-\alpha s} ds \quad (4)$$

where α is an appropriate discount rate. Since the stream $p_i - o$ is by assumption riskless, α should be the rate obtained on risk-free securities of maturity T in the capital markets.

However, we have assumed that the duration of the asset T is

uncertain; expiration is governed by a Poisson process, thus the opportunity's NPV is:

$$\begin{aligned}
 \text{NPV} &= -k + \int_0^{\infty} \int_0^t \left[(p_i - o) e^{-\alpha s} \right] \mu e^{-\mu t} dt \\
 &= -k + \frac{p_i - o}{\mu + \alpha}
 \end{aligned} \tag{5}$$

This NPV is known at the time a mine opportunity arises; in a frictionless capital market the opportunity, if accepted, could be immediately capitalized and sold off for this amount. (Alternatively, the company's value would be immediately increased by the NPV of the opportunity).

Eq. (5) may be rewritten:

$$\text{NPV} = \frac{p_i - o - k(\mu + \alpha)}{\mu + \alpha} \tag{6}$$

We are thus justified in equating c , the flow of costs associated with a mine opportunity, with a linear combination of the operating and capital costs of the mine

$$c \equiv o + k(\mu + \alpha) \tag{7}$$

The density function $f(c)$ naturally reflects the underlying joint distribution of the pair (o, k) .

Cost and Irreversibility. In this section we shall impose three constraints on prices and costs prevailing in the industry.

It will be shown that these constraints limit the economic alternatives open to the firm and thus decrease its flexibility in response to changes in the industry environment. The main purpose of the model will then be to analyze optimal investment decisions when future actions are known to be so constrained in this manner.

The first condition is straightforward and appealing: it is that there exists a finite level of output Q such that :

$$0 < p(Q) < \min c = \min(c_0 + k(\mu + \alpha)). \quad (8)$$

(Min c signifies the lowest value of c to which a positive probability weight is attached.)

Eq. (8) is in part a condition of no free production; it bounds the cost of bringing new output onstream strictly away from zero. Eq. (8) is also a statement about the satiability of demand; it indicates that at some level of production, the price paid for new output will be so low that any new opportunity will have a negative present value. This in turn places an upper bound on total industry output. (This last is an outcome of optimization and not an assumption: it will be proved below. However, the linkage between the two statements should be intuitively quite clear.)

If searching for new opportunities is a costly activity, then (8) is a sufficient condition for there to be an upper bound on production but is actually stronger than necessary. If search is costly (and controllable by

the corporation), search activity will stop when the expected value of encountering an opportunity falls below the search cost.⁴ If search stops at production level $Q = m$ then the probability of the production level reaching $Q = m + 1$ is zero (even if $p(m) > \min c$).

It is important to recall that the production limit is not an exogenous constraint on a firm's ability to undertake new investments, but is the result of the interaction of demand with technological factors determining the supply of new opportunities. Limitations on new investment do not arise because of capital rationing or the scarcity of intangible managerial resources (cf. Spence); but are instead a natural result of the product market's structure.

The second and third conditions imposed on the opportunity set relate to the irreversibility of a mine investment once it is undertaken. Recall that, by assumption, all output from a mine is sold off at a fixed price when the investment is made. The firm is thereby able to "lock in a spread" and thus has no incentive to reduce its own production in response to changes in total industry output. However, the company might in some cases seek to

replace an existing mine with a new and better opportunity or with purchased output. The second and third constraints are designed to exclude such circumstances from consideration (which is not to imply that these circumstances are uninteresting: they are an important area of future research).

Consider the proposed replacement of mine A (currently productive) with mine B (a new opportunity). Replacement of A by B does not change either industry output or the company's revenues; both mines have the same expected lifespan as of the date of the proposal. A decision may thus be made on the basis of cost alone.

The present value cost of keeping mine A open is:

$$\frac{o_A}{\mu + \alpha}$$

The present value cost of opening B is:

$$k_B + \frac{o_B}{\mu + \alpha} .$$

In addition, if mine A is closed prematurely, shutdown costs s_A will be incurred. The option to keep mine A open will be preferred if

$$\frac{o_A}{\mu + \alpha} < s_A + k_B + \frac{o_B}{\mu + \alpha} . \tag{10}$$

(Assume equality is resolved in favor of the status quo.)

From (10) a sufficient condition for the replacement alternative

never to be elected is:

$$\max \left(\frac{o}{\mu + \alpha} \right) - \min \left(\frac{o}{\mu + \alpha} \right) \leq \min (k + s) \quad (11)$$

(11) says that the combined cost of shutting down an old mine and constructing a new one always exceeds the present value of the net difference in operating costs achieved by the change.

Now, let us consider the replacement of mine A with a long-term purchase contract at price $p(Q)$. (It is assumed that a firm can purchase output on the same contractual terms as it sells it.) The firm will choose to keep mine A open as long as

$$\frac{o_A}{\mu + \alpha} \leq s_A + \frac{p(Q)}{\mu + \alpha} \quad (12)$$

A sufficient condition for the purchase alternative never to be chosen is:

$$\max \frac{o}{\mu + \alpha} \leq \min s + \min \frac{p(Q)}{\mu + \alpha} \quad (13)$$

Eq. (13) requires that the purchase price of output plus shutdown cost always exceeds operating cost savings obtained from closing down a facility.⁵

If Eqs. (11) and (13) hold, then an industry's options with respect to new investment opportunities are only two: (1) to reject the opportunity (so that total output remains the same) or (2) accept the opportunity and have output increase by one mine's production. All options which involve the replacement of old (but productive) investments by superior new ones are

excluded by the boundary conditions.

The conditions on o , k , s , and p , imposed by Eqs. (11) and (13) are undoubtedly too strict to be realistic. Sometimes it does pay a firm to shutdown a facility and replace its output by output purchased in the market or by output from a new and more efficient facility. On the other hand, the importance of such alternatives in the determination of optimal investment policy will be directly proportional to the probability of their turning out to be attractive. That is, as long as the probability of the replacement option being chosen is low, excluding such options from the formulation is not likely to have a severe impact on the optimal decision rule.

Eqs. (11) and (13) are interesting in that they indicate which types of technology tend to limit flexibility and impose constraints on investment behavior. Industries most likely to be inflexible in their response to new developments and/or opportunities are those with high capital costs or high shutdown costs relative to the fixed and variable costs of production.

The mechanism whereby high capital cost (capital intensiveness) tends to limit flexibility can be traced as follows: When new opportunities arise or prices change, the capital costs of previous investments are sunk costs. If capital costs are high in general, it then becomes very unlikely that the replacement of old facilities with new (before the end of their productive life) be economically justified. If replacement alternatives are not attractive, that implies that every newly accepted opportunity (a good) necessarily brings about a price decrease with respect to the output of future opportunities (a bad). The analysis below indicates that this inherent dilemma (realizing one opportunity makes the environment for

future opportunities less good) may have serious consequences for firms' investment decisions.

Competition. Before we can completely specify a given firm's investment decision problem, it is necessary to characterize the nature of the competition the firm faces. Given that (by our previous assumptions) companies cannot directly control prices or output, the most important dimension of competition among firms lies in investment and the generation of new opportunities.

We assume that competing firms all engage in basically the same investment process. Let the company whose decision problem we are modeling be designated Firm A, other firms designated B, C, D, etc. By previous assumption Firm A encounters and reviews λ opportunities per year. Let δ be the number of opportunities encountered by all other firms; the arrival rate of new opportunities for the entire industry is then $\lambda + \delta$.

To solve its own optimization problem, Firm A must model the behavior of its competitors with respect to the opportunities they will review. Although in general this would be a difficult task (involving joint optimization of all firms decision problems), two cases of particular interest may be solved directly. Thus, with respect to its competitors, Firm A might assume:

- (1) that all (other) firms apply the standard NPV criterion accepting all opportunities with $NPV > 0$ (equivalently $p_i > c$), or
- (2) (Nash equilibrium) that all other firms apply the same optimization criterion as itself.

Assumption (1) best approximates the case of a large firm competing with many small firms. Assumption (2) is descriptive of behavior in industries

in which several firms of about equal size participate.⁷ To avoid parallel formulations, the functional equations below are developed, under the assumption of symmetric industry structure and behavior. Effects of the alternative assumption will be considered in Section 3 below.

Decision Rule. We shall formulate the firm's decision problem as a dynamic optimization problem in continuous time. We assume the industry is in a steady state (real demand and real costs are not changing over time),⁸ and that the firm's planning horizon is unbounded.

The firm then faces a stationary investment problem. For such problems, it is well known that the optimal decision rule takes the form "accept (the next opportunity) if its reward exceeds z " where the value of z is the outcome of optimization. In the investment decision problem, rewards take the form of value added to shareholder wealth by the acceptance of an opportunity. The standard NPV criterion is then a special case of the general rule for which the hurdle criterion z is zero.

Note that the NPV formula (see Eqs. 5 and 6) permits the hurdle criterion to be expressed in three equivalent ways. These are:

- (1) Accept (an opportunity) if $NPV > z$
- (2) Accept if $p(Q) - c > y$ (Net profit criterion)
- (3) Accept if $c < x$ (Cost criterion)

Because we shall be working with functions of the distribution of c , the

problem is most naturally formulated in terms of c (alternative (3)).

We shall therefore solve for optimal x (x^*); optimal y and z can be obtained from x by substitution into the net profit and present value formulas.

Functional Equations. Let Q , the total output of the copper industry be indexed by i : ($Q = 0, 1, \dots, i, \dots$) and let i be known as the "state of the industry". Consider Firm A when the industry is in state i and there are t years left until the horizon. Define $v_i(t)$ as the value of Firm A at time t assuming it follows an optimal investment policy between t and the horizon. The horizon is called "time 0" and we assume that values of terminating in any state are identically 0:

$$v_i(0) = 0 \quad \forall i \quad (14)$$

At the same time, we assume that Firm A has already realized the values of previous investments through the capital markets; values of its "assets in place" thus are not counted in the forward looking valuation $v_i(t)$.

Now consider Firm A at time $t + dt$ (one instant before time t). In the transitional interval dt , the following events might occur:

- (1) The company encounters a new opportunity, accepts it, and opens a new mine;
- (2) The company encounters a new opportunity and rejects it;
- (3) Another company opens a new mine;
- (4) Some company closes a mine;
- (5) Nothing happens.

Events (1) - (4) are all governed in part by Poisson processes, thus the probability that two events occur simultaneously in the interval dt is negligible.

Expressing $v_i(t + dt)$ in terms of $v_i(t)$ and the parameters and variables previously defined, we have:

$$\begin{aligned}
 v_i(t + dt) = & \text{Max} \{ [1 - \lambda F_i dt - \delta F_{iB} dt - \mu dt] [e^{-\alpha dt} v_i(t)] \\
 & + \lambda F_i dt \left[\frac{p_i F_i - L_i}{F_i (\mu + \alpha)} + e^{-\alpha dt} v_{i+1}(t) \right] \\
 & + \delta F_{iB} dt e^{-\alpha dt} v_{i+1}(t) + \mu dt e^{-\alpha dt} v_{i-1}(t) \} ,
 \end{aligned} \tag{15}$$

where

$p_i \equiv$ the output price prevailing when $Q = i$,

$F_i = F(x_i) = \int_a^{x_i} f(c) dc \equiv$ the probability that c drawn from $f(c)$ is

less than or equal to x_i ; equivalently the probability that a new opportunity will clear the cutoff x_i . (Note: a is the lower bound of c 's distribution. The subscript $B(F_{iB})$ indicates that x_i is determined by other firms in the industry, i.e., the quantity is not under the control of Firm A.)

$$L_i = L(x_i) = \int_a^{x_i} c f(c) dc \equiv \text{the contribution to the total expectation of } c$$

obtained from points lying below (to the left of) the x_i cutoff.

From these definitions $(p_i F_i - L_i) / F_i (\mu + \alpha)$ is the ex ante conditional expectation of the net present value of an opportunity, conditioned on the event that the opportunity satisfies the cutoff x_i and thus is accepted by Firm A.

Substituting the expansion $(1 - \alpha dt \dots)$ for $e^{-\alpha dt}$ in (15) and ignoring terms of dt^2 or higher, subtracting $v_i(t)$ from both sides, dividing through by dt and taking the limit as $dt \rightarrow 0$ gives:

$$\begin{aligned} \frac{dv_i(t)}{dt} = & \text{Max} \left\{ \frac{\lambda}{\mu + \alpha} (p_i F_i - L_i) - \alpha v_i(t) \right. \\ & - (\lambda F_i + \delta F_{iB}) (v_i(t) - v_{i+1}(t)) \\ & \left. + \mu i (v_{i-1}(t) - v_i(t)) \right\} \end{aligned} \quad (16)$$

If the system has reached steady state $dv_i/dt = 0$, and:

$$\begin{aligned} \alpha v_i = & \text{Max} \left\{ \frac{\lambda}{\mu + \alpha} (p_i F_i - L_i) \right. \\ & - (\lambda F_i + \delta F_{iB}) (v_i - v_{i+1}) \\ & \left. + \mu i (v_{i-1} - v_i) \right\} \end{aligned} \quad (17)$$

where $v_i = v_i(\infty)$. Note that if $Q = i = 0$, the final term in (17) disappears.

(17) describes a system of $n + 1$ equations where n is the highest level of output achievable by the industry (i.e., consistent with optimal behavior). Investment policies which maximize value for firms operating in the environment described by (17) are derived in the following section.

3. Optimal Decision Rule

The partial derivative of the maximand in (17) (w.r.t. x_i) is:¹⁰

$$\frac{\lambda}{\mu + \alpha} [p_i f(x_i) - x_i f(x_i)] - \lambda f(x_i)(v_i - v_{i+1})$$

Setting this quantity equal to zero we have that (optimal) x_i^* satisfies:

$$\frac{p_i - x_i^*}{\mu + \alpha} = v_i - v_{i+1} \tag{18}$$

Second-order conditions ensure that this is a maximum.¹¹

Eq. (18) indicates that Firm A should set its cutoff x_i at the point where the net present value of an accepted opportunity just equals the change in the value of A's future opportunities resulting from a higher level of industry total output. The optimal decision rule thus hinges on the relative values of v_i and v_{i+1} .

In Appendix A, it is proved that for the decision problem we have described:

$$v_i - v_{i+1} \geq 0 \quad \forall i \quad (19)$$

and, for $i \leq n$ where n is the minimum level of output such that $p_n < \min c$

$$v_i - v_{i+1} > 0 \quad (20)$$

(subject to $\lambda > 0$ and $p_i \geq p_{i+1}$ for some i - see Proposition 1 below.)

Several interesting propositions follow from this result. We shall state the propositions and their proofs first, then briefly discuss their implications:

Proposition 1. Optimally, for all $i \leq n$ (defined above), $p_i - x_i^* > 0$, unless

(1) $\lambda = 0$ or (2) $p_0 = p_1 = \dots = p_i = \dots$ (no upper bound) in which case $p_i - x_i^* = 0$.

Proof. The statement follows immediately from Eq. (18) and the fact that $v_i - v_{i+1} > 0$, for $i \leq n$. The exceptions arise from the functional equation for $v_i - v_{i+1}$ (see Eq. A-3).

Proposition 2. The standard net present value rule (accept an opportunity if $NPV > 0$) is optimal if and only if (1) Firm A anticipates it will receive no future opportunities or (2) the industry demand curve is horizontal and unbounded.

Proof. As was previously indicated, the standard NPV criterion is equivalent to setting $x_i^* = p_i$. Proposition 2 then follows from the exceptions to Proposition 1.

Proposition 3. n is an upper bound on production.

Proof. Recall that n was defined as the minimum level of output such that $p_n < \min c$. Proposition 1 implies $x_n^* \leq p_n$, thus:

$$x_n^* \leq p_n < \min c \quad (21)$$

Eq. (21) implies that no opportunity can clear the cutoff x_n^* , thus a transition from n to $n + 1$ can never take place. Total industry production is thus bounded by n . Note: in the case where competition is assumed to be following the standard NPV rule, the proposition still holds (because p_n serves as the cutoff for transitions not controlled by Firm A).

Discussion. Propositions 1 and 2 are important because they indicate that in industries where initial capital commitments to new projects are high (relative to the operating cost of existing facilities) and the standard NPV criterion for capital investment is likely to be incorrect; application of the criterion will bias decision-making in the direction of accepting projects which should be rejected.

It is interesting to consider when, according to the model, the standard NPV rule is a valid criterion. This occurs when the firm in question sees itself as having no further growth opportunities in the industry or when industry demand is elastic over an unbounded range. Demand elasticity over a limited range of output is not a sufficient condition for the $NPV > 0$ criterion to be valid. Rather the industry must exhibit something akin to "supply creating its own demand" so that implicitly any level of output can be

absorbed at the given price.

Finally, Proposition 3 may appear intuitively obvious and theoretically uninteresting. It is important, however, because it bounds the size of the problem of estimating an optimal decision rule (optimal cutoff points) for an actual industry. Given Proposition 3, an iterative algorithm to calculate optimal cutoff points for a given demand curve and characteristic opportunity set is feasible. Such an algorithm will be developed in a subsequent paper.

For analytic purposes, however, it is more interesting to consider how investment decisions may be affected by changes in industry demand, competition, and opportunities. To do this, is necessary to reduce the problem to a tractable size. In what follows we shall assume $p_2 < \min c$, thus the maximum number of mines open at any one time is two. (Note: This assumption does not necessarily imply that only two firms operate in the industry. The key to industry participation is the ability to review a stream of opportunities: although at most two firms will actually be producing at any one time, a much larger number might participate in the opportunity search and review process.)

Writing down functional equations for Firm A (after substitution for $v_i - v_{i+1}$ in Eq. (17)) we have a 3-state system:

$$\alpha v_0 = \frac{\lambda}{\mu + \alpha} (p_0 F_0 - L_0) - (\lambda F_0 + \delta F_{0B}) \left(\frac{p_0 - x_0^*}{\mu + \alpha} \right) \quad (21a)$$

$$\alpha v_1 = \frac{\lambda}{\mu + \alpha} (p_1 F_1 - L_1) - (\lambda F_1 + \delta F_{1B}) \left(\frac{p_1 - x_1^*}{\mu + \alpha} \right) + \mu \left(\frac{p_0 - x_0^*}{\mu + \alpha} \right) \quad (21b)$$

$$\alpha v_2 = 2\mu \left(\frac{p_1 - x_1^*}{\mu + \alpha} \right) \quad (21c)$$

Subtracting (21c) from (21b) and (21b) from (21a), substituting for $v_i - v_{i+1}$ and rearranging terms yields:

$$(\alpha + \delta F_{0B} + \mu) (p_0 - x_0^*) - \delta F_{1B} (p_1 - x_1^*) = \lambda(\theta_0 - \theta_1) \quad (22a)$$

$$-\mu(p_0 - x_0^*) + (\alpha + \delta F_{1B} + 2\mu) (p_1 - x_1^*) = \lambda\theta_1 \quad (22b)$$

where $\theta_i \equiv x_i F_i - L_i$. Eq. (22) is a system of two equations in two unknowns (x_0^*, x_1^*) which is linear in $p - x^*$ and θ . Adding (22b) to (22a) simplifies the system:

$$(\alpha + \delta F_{0B}) (p_0 - x_0^*) + (\alpha + 2\mu) (p_1 - x_1^*) = \lambda\theta_0 \quad (23a)$$

$$-\mu(p_0 - x_0^*) + (\alpha + \delta F_{1B} + 2\mu) (p_1 - x_1^*) = \lambda\theta_1 \quad (23b)$$

Solving for $p_0 - x_0^*$ and $p_1 - x_1^*$ by Cramer's Rule, we have:

$$p_0 - x_0^* = \frac{\lambda\theta_0(\alpha + \delta F_{1B} + 2\mu) - \lambda\theta_1(\alpha + 2\mu)}{(\alpha + \delta F_{0B})(\alpha + \delta F_{1B} + 2\mu) + \mu(\alpha + 2\mu)} \quad (24a)$$

$$p_1 - x_1^* = \frac{\lambda\theta_1(\alpha + \delta F_{0B}) + \lambda\mu\theta_0}{(\alpha + \delta F_{0B})(\alpha + \delta F_{1B} + 2\mu) + \mu(\alpha + 2\mu)} \quad (24b)$$

In Eqs. (24a) and (24b) note that x_0^* and x_1^* appear on the left and (because $\theta_1 = \theta(x_1^*)$) right hand sides of each expression. Optimal (x_0^*, x_1^*) are thus implicitly defined by (24a) and (24b).

Interdependencies among x_0^* , $\theta(x_0^*)$, x_1^* , and $\theta(x_1^*)$ make direct analysis of the impact of parameter changes on (24a) and (24b) generally inconclusive. However, looking at the equation system, it is apparent (1) that for large values of λ , the r.h.s. quantities will be large, thus x_0^* , x_1^* necessarily quite small, and (2) that large values of α , δ and μ will tend to make the r.h.s. quantities small, thereby causing x_0^* , x_1^* to approach p_0 and p_1 respectively. Extreme value analysis thus suggests that the optimal cutoff points x_0^* , x_1^* are (1) declining functions of λ and (2) increasing functions of α , δ and μ (subject to the overall requirement that $p_i - x_i > 0$).

λ measures the arrival rate of new opportunities for review by the firm: as λ increases the firm expects to see more opportunities in a given time span. It seems reasonable that as the number of opportunities reviewed per year, goes up, the firm would be more selective in its acceptance criteria: this, in fact, is what is implied by higher $p_0 - x_0^*$ (lower x_0^*).

On the other hand, a high value of μ implies that accepted opportunities don't last very long; any given investment then will not have a very large impact on prices prevailing in the future. In this case, the opportunity loss from accepting an opportunity now is not very great; one then would expect that the acceptance criterion would not be very different from the standard net present value rule.

α represents the discount rate applied to future anticipated cash flows: as α increases the value of money received in the future declines. If α is high, the opportunity loss on accepting a present opportunity will tend to be low; a lower opportunity loss would then favor a laxer acceptance

criterion (lower $p_i - x_i^*$).

Finally δ measures the degree of competition present in the industry: as δ increases the number of opportunities (drawn from the same distribution $f(c)$) reviewed per year by competitors of Firm A goes up. If a competitor makes an investment, Firm A (1) will realize no positive benefit from the investment but (2) will suffer from the price reduction brought about by the increase in total industry output. Turning the situation around, if Firm A makes an investment, it creates a negative externality in the form of a price reduction. The externality is borne only in part by Firm A: other firms also share in the impact because the expected value of their future opportunities has been reduced. Naturally as competition increases, less of the impact of a price reduction bears on Firm A directly: Firm A's willingness to initiate a price reduction by undertaking an investment will then increase (equivalently, $p_i - x_i^*$ will fall).

Although the behavior of $p_i - x_i^*$ appears to be explainable in rational economic terms, it is impossible to assess whether the magnitudes of $p_i - x_i^*$ are significant given "reasonable" values for the parameters p_i , α , λ , μ , and δ and the distribution $f(c)$. In order to gain some idea of how far the optimal decision rate deviates from the $NPV > 0$ criterion a numerical example (based on a three state investment process) is developed in the following section.

4. Numerical Example.

To completely specify the dynamic investment process which Firm A seeks to optimize we need to give values to p_0 , p_1 , λ , μ , δ , α , and to characterize the distribution $f(c)$. For purposes of the example, let us assume:

(1) $p_0 = 2$; $p_1 = 1.5$. Recall that p_0 , p_1 represent the price of 1 years output from 1 mine. In terms of real world mining projects, it would be reasonable to consider units of account for p_0 and p_1 to be hundreds of millions of dollars.

(2) $\lambda = 1$; $\mu = .05$; $\delta = 9$. Firm A encounters and reviews 1 mine opportunity per year on average. A mine investment has an expected life of 20 years. There are 9 other firms in the industry (thus Firm A encounters about 10% of all available opportunities).

(3) $\alpha = .01$; the riskless rate of interest is approximately 1% per annum.

(4) the distribution $f(c)$ is uniform on the interval (1,2). Costs are denominated in the same units of account as prices, thus the cost (including capital cost) of one year's production can range from 100 to 200 million dollars.

(Note: it might be more realistic to suppose that better minds, i.e. those with low cost are scarcer. In this case, the uniform distribution would not be appropriate.

It is used purely for illustrative purposes here.)

From the above assumption, and the definitions of F , L , and θ we have:

$$F(x) = (x - 1) . ,$$

$$L(x) = \int_1^x cf(c)dc = .5(x^2 - 1) ,$$

$$\theta = xF(x) - L(x) = .5 x^2 - x + .5 .$$

Substituting from the assumptions into Eqs. (23a) and (23b) obtains (asterisks have been suppressed):

$$(.01 + 9(x_0 - 1)) (2 - x_0) + (.01 + .1) (1.5 - x_1) - .5x_0^2 + x_0 - .5 = 0 \quad (25a)$$

$$-.05(2 - x_0) + (.01 + 9(x_1 - 1) + .1) (1.5 - x_1) - .5x_1^2 + x_1 - .5 = 0 \quad (25b)$$

Note: we are here assuming the other firms in the industry behave symmetrically to Firm A, thus $F_{0B} = F(x_0)$ and $F_{1B} = F(x_1)$. If we assumed the other firms applied the standard NPV criterion F_{0B} and F_{1B} would simply have been constants (equal to 1 and .5 respectively).

Simplifying the above yields:

$$9.5x_0^2 - 27.99x_0 + 18.48 - .11(1.5 - x_1) = 0 \quad (26a)$$

$$9.5x_1^2 - 23.99x_1 + 13.835 + .05(2 - x_0) = 0 \quad (26b)$$

The reader may verify that $x_0 = 1.9477$ and $x_1 = 1.474$ solve Eqs. (26) (to within an accuracy of .002).

Thus to satisfy the optimal investment criterion when $p = 2$, a project

must earn profits (after annualized capital costs) of something over \$5 million per year. If $p = 1.5$, required net profits after capital charges are about \$2.6 million per year. The profitability criteria may be translated into net present value terms (using the standard formula): to be acceptable a project must have a minimum net present value of \$83 million when $p = 2$ or \$43 million when $p = 1.5$.¹²

These standards imply that approximately 5% of the projects which would be accepted given the standard NPV criterion would be rejected under the optimal decision rule. Although a 5% rejection rate does not appear high, recall that the competitive assumptions that went into the example make it possible for a firm which rejects an opportunity to be shut out of the industry for perhaps 10 years. The effect of strengthening a firm's market power and weakening the degree of competition (by reducing the number of firms in the industry) will be examined next.

Effect of Concentration. To see the impact concentration has on investment decisions when the product market is imperfect, we shall calculate optimal cutoff points and profit margins for (1) a monopoly, (2) a two firm industry, and (3) a four firm industry. In each case, we assume that the total number of opportunities reviewed by all industry participants is the same, that is $\lambda + \delta = \text{constant}$. Given the assumptions previously outlined for the 10 firm industry, this implies:

- | | | | |
|----------|-----------------|---|----------------|
| 1 Firm: | $\lambda = 10$ | ; | $\delta = 0$ |
| 2 Firms: | $\lambda = 5$ | ; | $\delta = 5$ |
| 4 Firms: | $\lambda = 2.5$ | ; | $\delta = 7.5$ |

Equations for each industry structure are obtained by substitution into Eqs. (23a) and (23b) above. Solutions for x_0^* and x_1^* for 1, 2, 4, and 10 firms are given in Table 1:

TABLE 1

Optimal Cutoffs and Required Profit Margins for Differing Concentrations

	x_0^*	$P_0 - x_0^*$	x_1^*	$P_1 - x_1^*$
1 Firm	1.1095	.8905	1.0361	.4639
2 Firms	1.6709	.3291	1.3340	.1660
4 Firms	1.8583	.1417	1.4288	.0712
10 Firms	1.9477	.0523	1.4740	.0260

(\$ hundreds of millions per year)

From Table 1, the effect of concentration on optimal investment decisions is seen to be quite pronounced. Whereas a firm claiming 10% of the industry's opportunities rejects only 5% of those having $NPV > 0$, a monopolist accepts 10% or less of the same opportunity stream. Figure 1 presents a graph of price and maximum cost as a function of the number of industry participants.

[Value of Growth Opportunities] .

[R & D Budgets]

[Average Price]

[Business Cycle]

V. Conclusion

This paper has shown that in cases where (1) the industry demand curve is downward sloping and (2) capital costs are high relative to operating costs (so that the replacement of old capacity before the end of its productive life is uneconomic), the standard $NPV > 0$ criterion applied to direct cash flows from an investment opportunity is not in general optimal. An extensive NPV analysis would be correct but would involve direct estimation of a complicated dynamic program.

We have shown that an alternative to direct estimation of probable future cash flows (conditioned on the proposed investment project) is to modify the NPV rule to be: accept Project Z if its $NPV > z$, where $z > 0$ (equivalently: accept if Net Profit $> y$ or Cost $< x$). Optimal cutoffs z^* (or y^* or x^*) are determined endogenously and depend on characteristics of industry demand (p_i) competition (δ , F_{iB}), the pace and cost structure of new opportunities (λ , $f(c)$) and the longevity of assets (μ). Extreme value analysis as well as the analysis of a numerical example indicated that deviations from the standard NPV rule are most serious when assets are long-lived ($\mu \rightarrow 0$), when the arrival rate of new opportunities (λ) is high, and/or when the industry is highly concentrated.

Operationalizing the Model. Although it incorporates a rather complex dynamic model of industry structure and performance (and can therefore handle a wide range of industries/technologies within the basic specification), the model shows promising signs of being applicable to investment problems actually encountered by firms. In particular, the recursive

structure of the general equation system (17) and the fact that the problem can be bounded (see Proposition 3) indicate that an iterative algorithm for calculating optimal x_i^* for reasonably large n may be feasible.

Thus, the model presented here offers an alternative to scenario planning (which is currently the most popular method of taking risk into account in major investment decisions). Rather than forecasting alternative scenarios and then simulating outcomes, the model specifies a structural relationship between prices and output and dynamic rules governing technology (new opportunities) and competition. This structural approach is particularly useful for evaluating investment decisions in industries where the price of output is difficult to forecast accurately because it depends heavily on the firm's own future actions or those of its competitors.

Extensions. Several extensions of the model would improve its Generality and usefulness in investment evaluations. First, the model may be extended to situations in which spot pricing prevailed (i.e., no futures contracts are available). Second, decisions to replace or shut down old capacity need to be examined in more detail and, if feasible, these options should be incorporated into the model formulation. Third, demand may be a function of other variables besides current output: for example, price may depend on levels of activity in other industries or in the economy as a whole. For both normative and descriptive purposes, it would be interesting to examine the effect of demand linkages (across industries or sectors) on the investment behavior of value maximizing firms operating in imperfect product markets.

FOOTNOTES

(1) Actually the demand curve could be more complicated. For example, p might depend on inventories ($p = p(Q, I)$) or on levels of activity in other industries (for which copper is an input). In this model, we assume that these other factors are not relevant and that price is a deterministic function of output.

(2) The level of output at $Q = 0$ may be thought of as a base level of output to which mine production would be added: defining δ as the base level, actual tonnage produced would be δ in state 0, $\delta + 1$ in state 1 and $\delta + 2$ in state 2. This assumption would be consistent with the existence of a cheap technology (like scrap reprocessing) which is in permanent but limited supply.

Alternatively, output at $Q = 0$ might be made possible by a technology (like secondary recovery) so expensive that any new discovery would cause it to be put on the shelf.

In this case, actual tonnage would be δ in state 0, 1 in state 1 and 2 in state 2.

Obviously, combination systems are possible: the key point for the model is to have $p(Q)$ indexed to whatever total output is if 0, 1, 2, or more mines are open.

(3) It is interesting to note that several New England states have recently considered ordinances providing for penalties to be paid by corporations

seeking to shut down local plants.

- (4) Formally, let q be the per period discretionary cost associated with searching for opportunities, a , the lower bound of c and λ the probability of encountering an opportunity in the next interval. Obviously, the company will stop searching if

$$q > \lambda \int_a^{p_i} \frac{p_i - c}{\mu + \alpha} f(c) dc .$$

Other formulations of search cost structure are of course possible. The impact of search cost on both asset selection and on activities leading to the generation of opportunities is an important area of future research.

- (5) Eq. (13) may be combined with Eq. (8) to give:

$$\max\left(\frac{0}{\mu + \alpha}\right) < \min s + \min\left(\frac{0}{\mu + \alpha}\right) + \min k ,$$

which condition is identical to Eq. (11). Eq. (13) is thus a slightly more restrictive condition than Eq. (11).

- (6) Different assumptions about λ and δ may reflect different aspects of reality. One possible assumption is that $\lambda + \delta = \text{constant}$: in this case, all firms in an industry basically share access to a fixed stream of opportunities. Another possibility would be $\lambda = \text{constant}$:

here each firm would create its own opportunities.

- (7) Size here reflects the firms' share of the total opportunity stream. The observed size of firms (in terms of total output) would of course depend on realized opportunities and could vary over time.
- (8) Alternatively, the demand function and opportunity set could be moving in parallel. In this case, opportunities might be eroding and prices increasing to compensate for the higher cost.)
- (9) Note that

$$\frac{L_i}{F_i} = \frac{\int_a^{x_i} c f(c) dc}{\int_a^{x_i} f(c) dc} = E(c | c \leq x_i)$$

is the conditional expectation of cost given that it satisfies the cutoff criterion x_i . The result then follows from the definition of present value (Eqs. (6) and (7)).

- (10) Note that we assume $F_{iB} = F(x_{iB})$ is not under Firm A's control and thus is not subject to optimization. F_{iB} may be set equal to F_i after the optimal x_i^* have been determined: in this case, Firm A assumes that Firms B, C, etc. are each solving an optimization problem identical to its own and acting on the basis of the resultant decision rules. In

general, even for an asymmetric industry consistent optimal decision rules would exist: these could be solved for by numerical methods.

If we assume F_{1b} controllable by Firm A (necessarily implying collusion or agreement between industry participants), the partial derivative of the maximand becomes:

$$\frac{\lambda}{\mu + \alpha} [p_i f(x_i) - x_i f(x_i)] - (\lambda + \delta) f(x_i) (v_i - v_{i+1})$$

and optimal x_i^* satisfies:

$$\frac{\lambda}{\lambda + \delta} \frac{p_i - x_i^*}{\mu + \alpha} = v_i - v_{i+1} \quad (1)$$

If firms are assumed to be symmetric, the value of Firm A's opportunities (v_i) equals $\frac{\lambda}{\lambda + \delta}$ of the total value of the industry (V_i)

$$v_i = \frac{\lambda}{\lambda + \delta} V_i$$

Substitution for v_i and v_{i+1} in terms of V_i , V_{i+1} in (1) shows Firm A's rule to be identical to that of a monopolist. A symmetric industry can thus achieve joint value maximization without side payments if implicit contracts on investment criteria (the x_i) are generally honored. An asymmetric industry, on the other hand, would have conflicts of interest over the x_i which would have to be reconciled via side payments (or not at all).

(11) The second derivative of the maximand is:

$$\lambda f''(x_i) \left[\frac{p_i - x_i}{\mu + \alpha} - (v_i - v_{i+1}) \right] - \lambda x_i$$

At $x_i = x_i^*$, the term in brackets is zero, and the entire quantity is negative. $x_i = x_i^*$ thus obtains a maximum.

(12) From the net present value formula:

$$p = 2 : \frac{5.3}{\mu + \alpha} = \frac{5.3}{.06} = 83$$

$$p = 1.5 : \frac{2.6}{.06} = 43 .$$

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APPENDIX A

Theorem. $v_i(t + dt) - v_{i+1}(t + dt) \geq 0$, all i , t . For $i \leq n$, where n is the minimum level of output such that $p_n < \min c$, $v_i(t + dt) - v_{i+1}(t + dt) > 0$ all t .

The first part of the proof is by induction. We will show that

(1) if for all i , $v_i(t) - v_{i+1}(t) \geq 0$, then $v_i(t + dt) - v_{i+1}(t + dt) \geq 0$.

(2) $v_i(dt) - v_{i+1}(dt) \geq 0$, $\forall i$.

By definition: $v_i(t + dt) = \max_{x_i} \{ r_{ij} + \sum_j p_{ij} v_j(t) \}$

$$\equiv r_{ij}^* + \sum_j p_{ij}^* v_j(t) .$$

where $r_{ij} \equiv$ the reward earned in a transition from i to j ,

and $p_{ij} \equiv$ the (instantaneous) probability of a transition from i to j .

(r_{ij} , p_{ij} are functions of x_i).

Let us define $x_i^* = \arg v_i(t + dt)$, that is x_{i+1}^* is the optimal value of x when the system starts in state $i + 1$. Define $F_{i+1} = P(x_{i+1}^*)$,

$$L_{i+1} = L(x_{i+1}^*) .$$

Necessarily

$$v_i(t + dt) \geq \hat{v}_i(t + dt) \equiv r_{ij}(x_{i+1}^*) + \sum_j p_{ij}(x_{i+1}^*) v_j(t) . \quad (A-1)$$

The proof consists of showing that

$$\hat{v}_i(t + dt) - v_{i+1}(t + dt) \geq 0 .$$

Then, from (A-1), the first statement in the theorem necessarily follows.

From the definitions of $\hat{v}_i(t + dt)$ and the v_i process (see Eq. (15) of the text):

$$\begin{aligned} \hat{v}_i(t + dt) = & [\lambda F_{i+1} dt] \frac{P_i F_{i+1} - L_{i+1}}{F_{i+1}(\mu + \alpha)} \\ & + e^{-\alpha dt} \{v_i(t) - [\lambda F_{i+1} dt + \delta F_{iB} dt]\} [v_i(t) - v_{i+1}(t)] \\ & + [\mu dt] \{v_{i-1}(t) - v_i(t)\} . \end{aligned} \quad (A-2)$$

Subtracting $v_{i+1}(t + dt)$ from $\hat{v}_i(t + dt)$, we have:

$$\begin{aligned} \hat{v}_i(t + dt) - v_{i+1}(t + dt) = & [\lambda F_{i+1} dt] \frac{P_i - P_{i+1}}{\mu + \alpha} \\ & + e^{-\alpha dt} \{[1 - \lambda F_{i+1} dt - \delta F_{iB} dt - \mu(i-1)dt]\} [v_i(t) - v_{i+1}(t)] \\ & + [\lambda F_{i+1} dt + \delta F_{i+1,B} dt] \{v_{i+1}(t) - v_{i+2}(t)\} \\ & + [\mu dt] \{v_{i-1}(t) - v_i(t)\} \end{aligned} \quad (A-3)$$

The first term in (A-3) is positive given our assumption that $p_i > p_{i+1}$. The whole expression is positive as long as the $(v_i(t) - v_{i+1}(t))$ terms are all positive. This completes the first half of the induction proof.

To see that the second half of the induction proof is true, we consider the process as it approaches the horizon. At $t = 0$, $v_i(0) = v_j(0) = 0$.

Thus, at dt (one instant left to go)

$$v_i(dt) = \max_{x_i} \{ \lambda F(x_i) dt \frac{[p_i F(x_i) - L(x_i)]}{F(x_i)(\mu + \alpha)} \} \quad (A-4)$$

Carrying out the maximization we have:

$$x_i^* = p_i, \quad \forall i. \quad (A-5)$$

By the definitions of F and L :

$$\begin{aligned} F(x_i^*) &= F(p_i) && \text{if } p_i \geq \min c \\ &= 0 && \text{if } p_i < \min c \end{aligned} \quad (A-6)$$

and, similarly:

$$\begin{aligned} L(x_i^*) &= L(p_i) && \text{if } p_i \geq \min c \\ &= 0 && \text{if } p_i < \min c \end{aligned} \quad (A-7)$$

By assumption, there exists n such that $p_n < \min c$.

Substituting from (A-6) and (A-7) into (A-4), we then have

$$\begin{aligned} v_i(dt) &= [\lambda dt] \frac{p_i F(p_i) - L(p_i)}{\mu + \alpha} && i \leq n-1 \\ &= 0 && i > n-1 \end{aligned} \quad (A-8)$$

From the definition of v :

$$\hat{v}_i(dt) = [\lambda dt] \frac{P_i F(p_{i+1}) - L(p_{i+1})}{\mu + \alpha} \quad i \leq n-2$$

$$= 0 \quad i > n-2$$
(A-9)

From (A-8) and (A-9) :

$$\hat{v}_i(dt) - v_{i+1}(dt) = \frac{\lambda F(p_{i+1}) dt [p_i - p_{i+1}]}{\mu + \alpha} > 0 \quad i \leq n-2$$

$$= 0 \quad i > n-2$$
(A-10)

The first statement in the theorem is thus proved.

To see that strict inequality holds given $i \leq n$: note from Eq. (A-3) that a sufficient condition for $\hat{v}_i(t + dt) - v_{i+1}(t + dt)$ to be strictly positive is that $v_{i-1}(t) - v_i(t)$, $v_i(t) - v_{i+1}(t)$ or $v_{i+1}(t) - v_{i+2}(t)$ be strictly greater than 0. From (A-10) and by two-step induction, this condition is satisfied for all $i \leq n$ (as long as λ is positive and $p_i > p_{i+1}$ for some $i, i+1$).

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