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Abstract

Harrison [5] has introduced a model called a Brownian network, which approximates a multiclass queueing network with dynamic scheduling capability. In the present paper, a particular Brownian network control problem is considered that approximates a mathematically intractable scheduling problem for a two-station multiclass queueing network. A reformulation of the Brownian network control problem is solved. Linear programming is used to reduce the reformulated problem to a singular control problem for a one-dimensional Brownian motion. The objective of the singular control problem is to minimize the long-run expected average holding cost of the controlled process incurred per unit of time, subject to constraints on the long-run expected average amount of control exerted.

Key words. Brownian motion, stochastic control, singular control, queueing networks.

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1. Introduction. This paper is part of a research effort to develop effective scheduling rules for multiclass queueing networks. Harrison [5] has introduced a type of crude but relatively tractable stochastic system model called a Brownian network. A Brownian network approximates a multiclass queueing network with dynamic scheduling capability if the total load imposed on each station in the queueing network is approximately equal to each station’s capacity. Hence, we can approximate a dynamic scheduling problem for a queueing network by a dynamic control problem for a Brownian network.

We consider a control problem for a Brownian network that approximates a particular scheduling problem for a two-station multiclass queueing network. The queueing network scheduling problem, which is described in the next paragraph, has both input control and priority sequencing decisions, and appears to be mathematically intractable. In this paper, the approximating Brownian network control problem, which appears in equations (1.1)-(1.7), is reformulated, and the new formulation is solved exactly. In Wein [10], the solution found in the present paper was interpreted in terms of the original queueing system. This interpretation yielded an effective scheduling rule for the queueing network scheduling problem. A simulation study in Wein [10] showed that the resulting scheduling rule derived from this analysis outperforms conventional input control and priority sequencing rules that appear in the literature.

The queueing network scheduling problem is motivated by a scheduling problem that is encountered in many factories. Consider a queueing network with two single-server stations and K different customer classes. Customers of class \( k = 1, ..., K \) require service
at a specific station $s(k)$ and their service times are independent and identically distributed random variables with finite mean $m_k$ and variance $s_k^2$. Upon completion of service, a class $k$ customer turns into a class $j$ customer with probability $P_{kj}$ and exits the system with probability $1 - \sum_{j=1}^{K} P_{kj}$, independent of all previous history. We assume that the $K \times K$ Markovian switching matrix $P = (P_{kj})$ has spectral radius less than one, so that all customers will eventually exit the system. Because the number of classes is allowed to be arbitrary, this routing structure is almost perfectly general.

The scheduling problem incorporates input and sequencing decisions. We assume there is an endless line of customers who are waiting to gain entry into the network. Each customer in the line has an exogenously specified class designation. These class designations are such that, over the long-run, the proportion of class $k$ customers released into the system is $q_k$, where $\sum_{k=1}^{K} q_k = 1$. The vector $q = (q_k)$ will be referred to as the entering class mix. The input decisions are to choose a non-decreasing process $N = \{N(t), t \geq 0\}$, where $N(t)$ is the cumulative number of customers injected into the system up to time $t$. Thus the input decisions essentially allow full discretion over the timing of the release of customers into the system, but do not allow for the choice of which class of customer to inject.

The sequencing decisions consist of choosing, at each point in time, which class of customer to process at each server in the network. Preemptive resume scheduling is allowed, so that service of a customer may be interrupted at a particular station when a higher priority customer arrives at that station. Due to the rather crude nature of the Brownian approximation that is employed here, the assumptions made regarding preemption do not have an effect on the scheduling policy that emerges from the analysis.

It is assumed that a holding cost $c_k$ is incurred for each unit of time that a class $k$ customer spends in the queueing network. Also, there is a specified lower bound $\bar{\lambda}$ on the long-run average expected throughput rate of the queueing network. The throughput rate of a queueing system is the number of customer departures from the system per unit of
Our queueing network scheduling problem is to choose the input and sequencing decisions so as to minimize the long-run average expected holding costs incurred per unit of time, subject to a lower bound constraint on the long-run average expected throughput rate.

Since this queueing network scheduling problem appears to be mathematically intractable, the best hope for further progress appears to be in the analysis of cruder, more tractable models, such as the Brownian network model. Under conditions of balanced heavy loading, a Brownian network approximates a multiclass queueing network with dynamic scheduling capability. To state these conditions more precisely, we define the two-vector \( \rho = (\rho_i) \) be the relative server utilizations, or traffic intensities, for the two stations. The values of \( \rho_1 \) and \( \rho_2 \) can be computed from the switching matrix \( P \), the vector \( m = (m_k) \) of expected processing times, the entering class mix \( q = (q_k) \) and the specified average throughput rate \( \bar{\lambda} \), as will be shown later in the introduction. The balanced heavy loading conditions assume the existence of a large integer \( n \) such that 
\[
0 \leq \sqrt{n}(1 - \rho_i) \leq 1
\]
for \( i = 1, 2 \). As a canonical example, one may think of \( \rho_1 = \rho_2 = .9 \), in which case \( n = 100 \) satisfies this condition.

Under such conditions, the scheduling problem described above can be approximated by a dynamic control problem for a Brownian network. For a full development of the Brownian network model, see Harrison [5]. Only enough information will be given here to determine the data of the Brownian network in terms of the original queueing network parameters. Also, most of Harrison's notation will be retained for ease of reference. The Brownian network control problem is stated in the next paragraph, but first we describe the set-up that will be adopted. When we say that \( X \) is a \((\mu, \sigma^2)\) Brownian motion, it is assumed there is a given \((\Omega, F, F_t, X, P_x)\), where \((\Omega, F)\) is a measurable space, \( X = X(\omega) \) is a measurable mapping of \( \Omega \) into \( C(R) \), which is the space of continuous functions on the real line \( R \), \( F_t = \sigma(X(s), s \leq t) \) is the filtration generated by \( X \), and \( P_x \) is a family of probability measures on \( \Omega \) such that the process \( \{X(t), t \geq 0\} \) is a Brownian motion with
drift \( \mu \), variance \( \sigma^2 \), and initial state \( x \). Let \( E_x \) be the expectation operator associated with \( P_x \). If \( Y = \{Y(t), t \geq 0\} \) is a process that is \( F_t \)-measurable for all \( t \geq 0 \), then we say that the process \( Y \) is non-anticipating with respect to the Brownian motion \( X \). More generally, we will say that one process \( Y \) is non-anticipating with respect to another process \( X \) when \( Y \) is adapted to the coarsest filtration with respect to which \( X \) is adapted. Also, a stochastic process is said to be RCLL if its sample paths are right continuous and have left limits with probability one.

It follows from Section 9 of Harrison [5] that under the balanced heavy loading assumptions described earlier, the queueing network scheduling problem described earlier can be approximated by the following limiting control problem: choose a pair of RCLL processes \( Y \) and \( \theta \) (\( K \)-dimensional and one-dimensional, respectively) to

\[
\text{minimize } \limsup_{T \to \infty} \frac{1}{T} E_x \left[ \int_0^T \sum_{k=1}^K c_k Z_k(t) dt \right] \tag{1.1}
\]

subject to the constraints: \( Y \) and \( \theta \) are non-anticipating with respect to \( X \),\( (1.2) \)

\[
Z(t) = X(t) + RY(t) - q\theta(t) \quad \text{for all } t \geq 0, \tag{1.3}
\]

\[
U(t) = AY(t) \quad \text{for all } t \geq 0, \tag{1.4}
\]

\( U \) is non-decreasing with \( U(0) = 0 \), \( (1.5) \)

\[
Z(t) \geq 0 \quad \text{for all } t \geq 0, \text{ and} \tag{1.6}
\]

\[
\limsup_{T \to \infty} \frac{1}{T} E[U_i(T)] \leq \gamma_i \quad \text{for } i = 1, 2. \tag{1.7}
\]

The process \( Z \) represents the \( K \)-dimensional scaled queue length process and describes the state of the system. That is, if \( Q \) is the \( K \)-dimensional queue length process for the original queueing system, then the scaled queue length process is defined by

\[
Z(t) = \frac{Q(nt)}{\sqrt{n}}, \quad t \geq 0, \tag{1.8}
\]

where \( n \) is the large integer specified in the balanced heavy loading conditions. Notice that, as in Harrison [5], exactly the same notation that is used for the scaled processes
is used in the approximating Brownian control problem. We retain this notation in order to emphasize the queueing network interpretation of the Brownian network model. The $K$-dimensional process $Y$ represents the scaled centered allocation process, and the one-dimensional process $\theta$ represents the scaled centered input process. These two control processes correspond to the sequencing and input decisions, respectively, in the queueing network scheduling problem. The scaled process $Y = (Y_k)$ is defined by

$$Y_k(t) = \frac{\alpha_k nt - T_k(nt)}{\sqrt{n}}, \ t \geq 0,$$

(1.9)

where $T_k(t)$ is the cumulative amount of time devoted to serving a class $k$ customer in the interval $[0,t]$ ($T_k$ is a decision variable in the original queueing network scheduling problem), and $\alpha_k$, which is defined in equation (1.17), represents the long-run fraction of the server’s active time at station $s(k)$ that must be devoted to class $k$ in order to maintain material balance in the network. The scaled process $\theta$ is defined by

$$\theta(t) = \frac{\bar{\lambda} nt - N(nt)}{\sqrt{n}}, \ t \geq 0,$$

(1.10)

where, as mentioned earlier, $\bar{\lambda}$ is the specified average throughput rate and $N(t)$ is the cumulative number of customers released into the network in $[0,t]$.

The two-dimensional process $U$ represents the scaled cumulative idleness process for the two stations. Thus $U$ is defined by

$$U(t) = \frac{I(nt)}{\sqrt{n}}, \ t \geq 0,$$

(1.11)

where $I(t)$ is the cumulative amount of idleness incurred by the server at station $i$ in the time interval $[0,t]$. (For brevity’s sake, processes such as $Z$, $Y$, $U$ and $\theta$ will often be referred to without the adjective “scaled”.) The $K \times K$ input-output matrix $R = (R_{kj})$ is defined by

$$R_{kj} = m_j^{-1}(\delta_{jk} - P_{jk}),$$

(1.12)
where \( \delta_{jk} \) denotes the Dirac delta function, meaning that \( \delta_{jk} = 1 \) if \( j = k \) and \( \delta_{jk} = 0 \) otherwise. The \( 2 \times K \) resource consumption matrix \( A = (A_{ik}) \) is defined by

\[
A_{ik} = \begin{cases} 
1, & \text{if } i = s(k), \\
0, & \text{otherwise.}
\end{cases}
\]  
Equation (1.13)

The \( K \)-dimensional process \( X \) is a \((\delta, \Sigma)\) Brownian motion, but several definitions are needed before we can describe the \( K \)-dimensional drift vector \( \delta = (\delta_k) \) and the \( K \times K \) covariance matrix \( \Sigma = (\Sigma_{jl}) \). Let \( \lambda = (\lambda_k) \) be defined by

\[
\lambda = q\bar{\lambda},
\]  
Equation (1.14)

so that \( \lambda_k \) represents the average number of class \( k \) customers that must depart from the system per unit of time in order to satisfy the throughput rate constraint.

Since \( P \) was assumed to be transient, it follows that \( R \) is non-singular and there exists a unique non-negative \( K \)-vector \( \beta = (\beta_k) \) satisfying the flow balance equations

\[
\lambda = R\beta.
\]  
Equation (1.15)

Letting \( C(i) \) be the set of all customer classes \( k \) such that \( s(k) = i \), we define the two-vector of traffic intensities \( \rho = (\rho_i) \) by

\[
\rho_i = \sum_{k \in C(i)} \beta_k.
\]  
Equation (1.16)

We now define the \( K \)-vector \( \alpha = (\alpha_k) \) by

\[
\alpha_k = \frac{\beta_k}{\rho_i} \quad \text{for all } k \in C(i).
\]  
Equation (1.17)

Then the drift \( \delta \) and covariance \( \Sigma \) of the Brownian motion \( X \) are

\[
\delta = \sqrt{n}(\lambda - R\alpha)
\]  
Equation (1.18)

and

\[
\Sigma_{jl} = \sum_{k=1}^{K} [\alpha_k m_k^{-1} P_{kj}(\delta_{jl} - P_{kl}) + \alpha_k m_k^{-1} s_k^2 R_{jk} R_{lk}].
\]  
Equation (1.19)
Inequality (1.7), which expresses the throughput rate constraint in terms of the cumulative server idleness process \( U \), is the only relationship in the limiting control problem that does not appear in the Brownian network formulation of [5]. The two-vector \( \gamma = (\gamma_i) \) in (1.7) is defined by

\[
\gamma_i = \sqrt{n}(1 - \rho_i).
\] (1.20)

Inequality (1.7) states that the long-run average fraction of time that server \( i \) is idle is less than or equal to \( 1 - \rho_i \). In Wein [10], it is shown that this inequality holds if and only if the long-run average throughput rate is greater than or equal to \( \bar{\lambda} \).

The remainder of this paper is organized as follows. In Section 2, the Brownian network control problem (1.1)-(1.7) is reformulated so that the state of the system is described by a two-dimensional workload process, rather than a \( K \)-dimensional queue length process. The new formulation will be referred to as the workload formulation, and it is the solution to the workload formulation that is the goal of this paper. The control processes in the workload formulation are \((Z, U, \theta)\), which are \( K \)-, two- and one-dimensional processes, respectively. In Section 3, linear programming is used to find the optimal process \( Z \) in terms of the process \( U \).

In Section 4, the results of Section 3 are employed to reduce the remainder of the workload formulation to a problem involving singular (or instantaneous) control of a one-dimensional Brownian motion. That is, a \((\mu, \sigma^2)\) Brownian motion \( \hat{B} \) is observed by a controller who, at any time, may instantaneously increase or decrease the value of the process \( \hat{B} \). Holding costs are incurred continuously at a rate \( h(\hat{W}(t)) \), where \( \hat{W} \) is the controlled process. No costs are incurred when the controller either increases or decreases the value of the Brownian motion process. However, there are constraints on the long-run expected average amount of control to be exerted in the positive direction and in the negative direction. The objective is to minimize the long-run expected average holding cost incurred per unit of time.

In Sections 5 through 9, we derive the solution to this constrained singular control
problem. The solution is characterized by two constants \( a \) and \( b \), where \( a < b \), and the optimal policy keeps the controlled process \( \tilde{W} \) inside the interval \( [a, b] \) with minimal effort. The corresponding optimal control functionals are the local times at these boundaries. In Section 5, a Lagrangian approach is used to move the two constraints into the objective function. The resulting unconstrained problem has been analyzed by Taksar [9], and his results are used in Section 6 to develop sufficient conditions for optimality of the constrained singular control problem. In Section 7, a candidate policy that satisfies the constraints is derived, and in Section 8, a solution to the optimality equations is found, which includes a pair of Lagrange multipliers. The special case where the drift \( \mu \) of the Brownian motion \( \tilde{B} \) is zero is treated in Section 9, and the solution to the workload formulation is summarized in Section 10.

2. The Workload Formulation

The state of the system in the Brownian network control problem is described by the \( K \)-dimensional queue length process \( Z \), by way of the basic system relationship (1.3). In this section the Brownian network control problem is reformulated so that the state of the system is described by a two-dimensional workload process. In Harrison [5], it is shown that the matrix \( R \) appearing in (1.3) is invertible. Let us define the matrix \( M = (M_{ik}) \) by

\[
M = AR^{-1}.
\]  
(2.1)

This \( 2 \times K \) matrix will be referred to as the workload profile matrix. Define the two-dimensional workload process \( W = (W_t) \) by

\[
W(t) = MZ(t), \ t \geq 0.
\]  
(2.2)

The workload process describes the state of the system in the workload formulation. Also, define the two-dimensional vector \( v = (v_t) \) by

\[
v = Mq.
\]  
(2.3)
It was shown in Wein [10] that

\[
\rho_i = v_i \lambda \quad \text{for } i = 1, 2,
\]

(2.4)

where \( \lambda \) is a positive number that is a parameter of the original queueing network scheduling problem.

Define the two-dimensional Brownian motion \( B = (B_t) \) by

\[
B(t) = MX(t), \quad t \geq 0,
\]

(2.5)

so that \( B \) has drift \( M\delta \) and covariance \( M\Sigma M^T \). It was shown in Wein [10] that the two-dimensional drift vector \( M\delta = -\gamma \), where \( \gamma \) was defined in equation (1.20).

Define the \textit{workload formulation} of the Brownian network control problem as choosing RCLL processes \( Z, U \) and \( \theta \) (\( K \)-, two- and one-dimensional, respectively) so as to minimize

\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T \sum_{k=1}^K c_k Z_k(t) dt \right]
\]

subject to the constraints: \( U \) and \( \theta \) are non-anticipating with respect to \( B \),

(2.7)

\[
U \text{ is non-decreasing with } U(0) = 0,
\]

(2.8)

\[
Z(t) \geq 0 \quad \text{for all } t \geq 0,
\]

(2.9)

\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}[U_i(T)] \leq \gamma_i \quad \text{for } i = 1, 2, \quad \text{and}
\]

(2.10)

\[
MZ(t) = B(t) + U(t) - v\theta(t) \quad \text{for all } t \geq 0.
\]

(2.11)

Let us call a pair of RCLL processes \((Y, \theta)\) a \textit{feasible policy} for the Brownian network control problem if it satisfies equations (1.3)-(1.7) and call a triple of RCLL processes \((Z, U, \theta)\) a feasible policy for the workload formulation if it satisfies equations (2.8)-(2.11).

The following proposition allows us to analyze the workload formulation of the limiting control problem, rather than studying problem (1.1)-(1.7) directly.

\textbf{Proposition 2.1.} Every feasible policy \((Y, \theta)\) for the Brownian network control problem yields a corresponding feasible policy \((Z, U, \theta)\) for the workload formulation and every feasible policy \((Z, U, \theta)\) yields a corresponding feasible policy \((Y, \theta)\).
Proof. Let \((Y, \theta)\) be any feasible policy for the Brownian network control problem, thus yielding an associated triple \((Z, U, \theta)\) from equations (1.3)-(1.4). Premultiplying (1.3) by the workload profile matrix \(M\) and using (1.4), (2.1), (2.3) and (2.5) gives equation (2.11). Since equations (2.8)-(2.10) are satisfied, the triple \((Z, U, \theta)\) is a feasible policy.

Reversing the argument, suppose that \((Z, U, \theta)\) is a feasible policy for the workload formulation. Then constraints (1.5)-(1.7) are satisfied and, using equation (1.3), one can define a corresponding control process \(Y\) by

\[
Y(t) = R^{-1}[Z(t) - X(t) + q\theta(t)], \quad \text{for all } t \geq 0.
\]

(2.12)

The associated queue length and cumulative idleness processes in (1.3) and (1.4) are precisely the processes \(Z\) and \(U\), respectively, that we started with. Thus the pair \((Y, \theta)\) is a feasible policy. □

We now address conditions (1.2) and (2.7) on non-anticipativeness. Since the matrix \(R\) is invertible, if the control process \(Y\) is non-anticipating with respect to the Brownian motion \(X\) in the limiting control problem, then the control process \(U\) is non-anticipating with respect to the Brownian motion \(B\) in the workload formulation. However, notice that in the limiting control problem, \(\theta\) is non-anticipating with respect to \(X\), whereas in the workload formulation, \(\theta\) is non-anticipating with respect to \(B\). By (2.5), it can be seen that more information will be available to the controller of \(\theta\) in the limiting control problem than in the workload formulation. However, as will be seen in Section 4, the solution to the workload formulation remains unchanged whether \(\theta\) is non-anticipating with respect to \(X\) or with respect to \(B\).

The rest of this paper is devoted to finding a solution to the workload formulation (2.6)-(2.11). In Wein [10], the solution to the workload formulation is interpreted in terms of the original queueing network in order to obtain effective scheduling rules for the queueing system.
3. Solving for \( Z \) in Terms of \( T \)

In this section the optimal process \( Z \) in the workload formulation is expressed in terms of the control process \( U \). Suppose we are given a process \( U \) that satisfies (2.7), (2.8) and (2.10). Then the optimal \( Z \) and \( \theta \) processes are found by solving the following linear program at each time \( t \):

\[
\min_{Z(t), \theta(t)} \sum_{k=1}^{K} c_k Z_k(t) \tag{3.1}
\]

subject to

\[
\sum_{k=1}^{K} M_{1k} Z_k(t) + v_1 \theta(t) = B_1(t) + U_1(t) \tag{3.2}
\]

\[
\sum_{k=1}^{K} M_{2k} Z_k(t) + v_2 \theta(t) = B_2(t) + U_2(t) \tag{3.3}
\]

\[
Z_k(t) \geq 0, \text{ for } k = 1, \ldots, K. \tag{3.4}
\]

At each time \( t \), this linear program may have a different set of right hand side values. Since it would be easier to analyze a linear program with a static constraint set, let us define the dual variables \( \pi_1(t) \) and \( \pi_2(t) \) and state the dual linear program to be solved at time \( t \):

\[
\max_{\pi_1(t), \pi_2(t)} [B_1(t) + U_1(t)] \pi_1(t) + [B_2(t) + U_2(t)] \pi_2(t) \tag{3.5}
\]

subject to

\[
M_{1k} \pi_1(t) + M_{2k} \pi_2(t) \leq c_k \text{ for } k = 1, \ldots, K \tag{3.6}
\]

\[
v_1 \pi_1(t) + v_2 \pi_2(t) = 0. \tag{3.7}
\]

Before analyzing the dual LP (3.5)-(3.7), define the one-dimensional workload imbalance process \( \hat{W} \) by

\[
\hat{W}(t) = \rho_2 W_1(t) - \rho_1 W_2(t), \ t \geq 0. \tag{3.8}
\]

At each time \( t \), \( \hat{W}(t) \) measures how imbalanced the workload is between the two stations. When \( W_1(t) \) and \( W_2(t) \) are in the same proportions as the long-run utilizations \( \rho_1 \) and \( \rho_2 \),
respectively, then $\dot{W}(t) = 0$. The process $\dot{W}$ becomes positive (negative, respectively) when the workload in the system becomes imbalanced towards station 1 (station 2, respectively). By (2.2), (2.4), (2.11) and (3.8), the workload imbalance process can also be expressed as

$$\dot{W}(t) = \rho_2 B_1(t) - \rho_1 B_2(t) + \rho_2 U_1(t) - \rho_1 U_2(t), \quad t \geq 0. \quad (3.9)$$

Returning to the dual LP (3.5)-(3.7), use (3.7) to eliminate $\pi_2(t)$ from the problem and use (2.4) to substitute the utilization levels $\rho$ for the vector $v$ to obtain the one-variable LP

$$\max_{\pi_1(t)} \frac{\dot{W}(t)}{\rho_2} \pi_1(t) \quad (3.10)$$

subject to $c_k^{-1}(\rho_2 M_{1k} - \rho_1 M_{2k}) \pi_1(t) \leq \rho_2$ for $k = 1, \ldots, K$. \quad (3.11)

The dual LP (3.10)-(3.11) allows us to express the solution $Z(t)$ of LP (3.1)-(3.4) in terms of $\dot{W}(t)$. Without loss of generality, assume that the classes $k = 1, \ldots, K$ are ordered so that

$$\arg \max_k c_k^{-1}(\rho_2 M_{1k} - \rho_1 M_{2k}) = 1 \quad (3.12)$$

and

$$\arg \min_k c_k^{-1}(\rho_2 M_{1k} - \rho_1 M_{2k}) = 2. \quad (3.13)$$

By complementary slackness, if $\dot{W}(t) > 0$, then the solution $Z(t)$ of the LP (3.1)-(3.4) must satisfy $Z_k(t) = 0$ for all classes $k \neq 1$. Similarly, if $\dot{W}(t) < 0$, the solution must satisfy $Z_k(t) = 0$ for all classes $k \neq 2$. Notice that the solution is $Z_k(t) = 0$ for $k = 1, \ldots, K$ when $\dot{W}(t) = 0$.

When $\dot{W}(t) > 0$, by (2.2) and (3.12),

$$W_2(t) = \frac{M_{21}}{M_{11}} W_1(t). \quad (3.14)$$

Combining this with the definition (3.8) of the workload imbalance process $\dot{W}$, one obtains

$$W_i(t) = \frac{M_{11}}{\rho_2 M_{11} - \rho_1 M_{21}} \dot{W}(t) \quad \text{for} \quad i = 1, 2. \quad (3.15)$$
Therefore, when $\hat{W}(t) \geq 0$, the solution $Z(t)$ to the LP (3.1)-(3.4) is

$$Z_k(t) = \begin{cases} \frac{\hat{W}(t)}{\rho_2 M_{11} - \rho_1 M_{21}}, & \text{if } k = 1, \\ 0, & \text{if } k \neq 1. \end{cases} \tag{3.16}$$

Similarly, when $\hat{W}(t) < 0$, the solution is

$$Z_k(t) = \begin{cases} \frac{\hat{W}(t)}{\rho_2 M_{12} - \rho_1 M_{22}}, & \text{if } k = 2, \\ 0, & \text{if } k \neq 2. \end{cases} \tag{3.17}$$

By the balanced heavy loading conditions on the original queueing system stated in Wein [10], and by conventions (3.12)-(3.13), it follows that

$$\rho_2 M_{11} - \rho_1 M_{21} > 0 \tag{3.18}$$

and

$$\rho_1 M_{22} - \rho_2 M_{12} > 0, \tag{3.19}$$

and therefore that the solution $Z(t) \geq 0$ for all $t \geq 0$. The optimal queue length process $Z$ is constructed from (3.16)-(3.17) for all $t \geq 0$. Notice that this optimal process does not depend on the control process $\theta$ and depends on the control process $U$ only through the workload imbalance process $\hat{W}$, as described in equation (3.9).

4. The Resulting Control Problem

In this section it is shown that by substituting for the optimal queue length process $Z$ using (3.16)-(3.17), the workload formulation (2.6)-(2.11) can be reduced to a problem of finding the optimal two-dimensional cumulative idleness process $U$. Define the one-dimensional Brownian motion $\hat{B}$ by

$$\hat{B}(t) = \rho_2 B_1(t) - \rho_1 B_2(t), \ t \geq 0. \tag{4.1}$$
Recalling that the two-dimensional Brownian motion $B$ has drift $-\gamma$, by equation (1.8) it follows that $\hat{B}$ has drift $\mu$, where

$$\mu = \sqrt{n}(\rho_1 - \rho_2), \quad \text{(4.2)}$$

and has variance $\sigma^2 = \varphi^T M \Sigma M^T \varphi$, where

$$\varphi = \begin{bmatrix} \rho_2 \\ -\rho_1 \end{bmatrix}. \quad \text{(4.3)}$$

Notice that when the system is perfectly balanced (i.e., $\rho_1 = \rho_2$), then $\mu = 0$. This driftless case will be discussed separately in Section 9; until then, assume that $\mu$ is nonzero. Let us also define the processes $R$ and $L$ by

$$R(t) = \rho_2 U_1(t), \quad t \geq 0 \quad \text{(4.4)}$$

and

$$L(t) = \rho_1 U_2(t), \quad t \geq 0 \quad \text{(4.5)}$$

The letters $R$ and $L$ are used to denote cumulative amounts of rightward and leftward movement, respectively, exerted by the controller on the Brownian motion $\hat{B}$. The constraints (2.10) determine an upper bound on the long-run average expected amounts of rightward and leftward controls exerted. Define a policy as a pair of non-decreasing RCLL processes $(R, L)$ such that $R$ and $L$ are non-anticipating with respect to $\hat{B}$, $E_x[R(t)]$ and $E_x[L(t)]$ are finite for each $t \geq 0$ and each initial state $x$, and $R(0) = L(0) = 0$. Associate with the policy $(R, L)$ the controlled process $\hat{B}(t) + R(t) - L(t)$ for all $t \geq 0$. From equations (2.2), (2.11), (4.1) and (4.4)-(4.5), it can be seen that the controlled process is precisely the workload imbalance process defined in (3.8); that is,

$$\hat{W}(t) = \hat{B}(t) + R(t) - L(t) \text{ for all } t \geq 0. \quad \text{(4.6)}$$

Thus, in the resulting control problem, the Brownian motion $\hat{B}$ can be controlled by pushing it to the right (left, respectively), which increases (decreases, respectively) the
workload imbalance of the system. Now define the positive coefficients \( h_1 \) and \( h_2 \) by

\[
h_1 = \frac{c_2}{\rho_1 M_{22} - \rho_2 M_{12}}
\]  

(4.7)

and

\[
h_2 = \frac{c_1}{\rho_2 M_{11} - \rho_1 M_{21}}.
\]  

(4.8)

If the optimal queue length process constructed from (3.16)-(3.17) is used, then one obtains

\[
\sum_{k=1}^{K} c_k Z_k(t) = h(\hat{W}(t)),
\]  

(4.9)

where

\[
h(x) = \begin{cases} 
  -h_1 x, & \text{if } x < 0, \\
  h_2 x, & \text{if } x \geq 0.
\end{cases}
\]  

(4.10)

Thus for each policy \((R, L)\), costs are incurred at a rate \( h(\hat{W}(t)) \). The resulting control problem is to find a policy \((R, L)\) to

\[
\begin{align*}
\text{minimize} & \quad \limsup_{T \to \infty} \frac{1}{T} E_x \left[ \int_0^T h(\hat{W}(t)) dt \right] \\
\text{subject to} & \quad \limsup_{T \to \infty} \frac{1}{T} E_x[R(T)] \leq \frac{\rho_2(1 - \rho_1)\mu}{\rho_1 - \rho_2}, \\
& \quad \limsup_{T \to \infty} \frac{1}{T} E_x[L(T)] \leq \frac{\rho_1(1 - \rho_2)\mu}{\rho_1 - \rho_2}.
\end{align*}
\]  

(4.11) \quad (4.12) \quad (4.13)

If a solution \((R, L)\) to the constrained control problem (4.11)-(4.13) can be found, then the optimal \( \theta \) process is, via equations (3.2) and (4.4),

\[
\theta(t) = v_1^{-1}[B_1(t) + \frac{R(t)}{\rho_2} - \sum_{k=1}^{K} M_{1k} Z_k(t)] \quad \text{for all } t \geq 0,
\]  

(4.14)

where \( Z(t) \) is defined in (3.16)-(3.17). Notice that the optimal \( \theta \) process would be expressed by (4.14) whether \( \theta \) was non-anticipating with respect to the two-dimensional Brownian motion \( B \) or with respect to the \( K \)-dimensional Brownian motion \( X \). The rest of this paper is devoted to finding a solution \((R, L)\) to problem (4.11)-(4.13).
5. The Lagrange Multiplier Method

Instead of analyzing the constrained control problem (4.11)-(4.13) directly, a Lagrangian approach will be used. In particular, let \( r \) and \( l \) be the Lagrange multipliers corresponding to constraints (4.12) and (4.13), respectively. The multipliers \( r \) and \( l \) are interpreted as the per unit costs of exerting the controls \( R \) and \( L \), respectively. Associate with a policy \((R, L)\) the Lagrangian cost function

\[
k(x) = \limsup_{T \to \infty} \frac{1}{T} E_x \left[ \int_0^T h(\dot{W}(t))dt + r(R(T) + lL(T)) \right]. \tag{5.1}
\]

With the aid of the following theorem, the constrained problem (4.11)-(4.13) can be solved by making an appropriate choice of multipliers and then minimizing the Lagrangian cost function. This idea is not at all new in the context of control theory; readers are referred to Bellman [1] and Everett [3] for early applications to dynamic programming problems. The problem of finding a policy \((R, L)\) that minimizes \( k(x) \) will be referred to as the Lagrangian problem.

**Theorem 5.1.** Suppose \( r \) and \( l \) are nonnegative real numbers and suppose \((R^*, L^*)\) is a solution to the Lagrangian problem. Furthermore, suppose

\[
\limsup_{T \to \infty} \frac{1}{T} E_x[R^*(T)] = \frac{\rho_2(1-\rho_1)\mu}{\rho_1 - \rho_2} \tag{5.2}
\]

and

\[
\limsup_{T \to \infty} \frac{1}{T} E_x[L^*(T)] = \frac{\rho_1(1-\rho_2)\mu}{\rho_1 - \rho_2}. \tag{5.3}
\]

Then \((R^*, L^*)\) is a solution to the constrained control problem (4.11)-(4.13).

**Proof.** Let \( \dot{W}^* \) be the controlled process under the optimal policy \((R^*, L^*)\), so that

\[
\dot{W}^*(t) = \dot{B}(t) + R^*(t) - L^*(t) \quad \text{for all} \quad t \geq 0. \tag{5.4}
\]

Then for all policies \((R, L)\)

\[
\limsup_{T \to \infty} \frac{1}{T} E_x \left[ \int_0^T h(\dot{W}^*(t))dt + rR^*(T) + lL^*(T) \right] \leq \ldots
\]

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\[ \limsup_{T \to \infty} \frac{1}{T} E_x \left[ \int_0^T h(\tilde{W}(t)) dt + rR(T) + lL(T) \right]. \] (5.5)

Rearranging terms yields

\[ \limsup_{T \to \infty} \frac{1}{T} E_x \left[ \int_0^T h(\tilde{W}(t)) dt \right] \leq \limsup_{T \to \infty} \frac{1}{T} E_x \left[ \int_0^T h(\tilde{W}(t)) dt \right] \\
+ r \limsup_{T \to \infty} \frac{1}{T} E_x [R(T) - R^*(T)] \\
+ l \limsup_{T \to \infty} \frac{1}{T} E_x [L(T) - L^*(T)]. \] (5.6)

Since (5.6) holds for all policies, it certainly holds for the subset of policies such that

\[ \limsup_{T \to \infty} \frac{1}{T} E_x [R(T)] \leq \frac{\rho_2 (1 - \rho_1) \mu}{\rho_1 - \rho_2} \] (5.7)

and

\[ \limsup_{T \to \infty} \frac{1}{T} E_x [L(T)] \leq \frac{\rho_1 (1 - \rho_2) \mu}{\rho_1 - \rho_2}. \] (5.8)

However, by equations (5.2)-(5.3), the terms

\[ r \limsup_{T \to \infty} \frac{1}{T} E_x [R(T) - R^*(T)] \] (5.9)

and

\[ l \limsup_{T \to \infty} \frac{1}{T} E_x [L(T) - L^*(T)] \] (5.10)

are nonpositive for this subset of policies. Hence the policy \((R^*, L^*)\) is a solution to (4.11)-(4.13).

Notice that, at this point, the Lagrange multipliers \(r\) and \(l\) are not known. In order to invoke Theorem 5.1, we need to find a pair of multipliers \((r, l)\) and a solution \((R, L)\) to the Lagrangian problem that simultaneously satisfy (5.2)-(5.3).

Given \(r\) and \(l\), the Lagrangian problem is a problem of singular (or instantaneous) control of Brownian motion. The name stems from the fact that the state of the controlled process can be instantaneously changed by the controller and, as a result, the optimal
control processes $R$ and $L$ are continuous but singular (i.e., the set of time points at which $R$ and $L$ increase has measure zero). Such problems have been the subject of much study; see, for example, Harrison and Taksar [6], Karatzas [7], Shreve, Lehoczky and Gaver [8], and Taksar [9]. In particular, Taksar [9] has solved the Lagrangian problem defined by (5.1) for a convex holding cost function $h$ that is finite everywhere or infinite on a finite interval. Since the holding cost function $h$ defined by (4.9)-(4.10) satisfies Taksar’s requirement, his results will be used in the next section, where sufficient conditions for optimality are developed for the constrained problem (4.11)-(4.13).

6. The Optimality Equations

The goal of this section is to present sufficient conditions for an optimal solution to the constrained problem (4.11)-(4.13). First, the sufficient conditions for optimality will be stated for the Lagrangian problem with arbitrary nonnegative numbers $r$ and $l$. These latter conditions were derived in Taksar [9]. The sufficient conditions for optimality of problem (4.11)-(4.13) are found by combining Taksar’s results with Theorem 5.1.

Let us start by defining a class of policies that include the optimal policy. Harrison and Taksar define a special class of policies in [6] called control limit policies. This type of policy brings the controlled process $\hat{W}$ within a certain interval $[a, b]$ instantaneously and then keeps it within this interval while exerting a minimum amount of control. The controlled process $\hat{W}$ under such a policy is the Brownian motion $\hat{B}$ reflected at the endpoints of the interval $[a, b]$. A complete analysis of this controlled process is contained in Harrison [4], where it is referred to as a regulated Brownian motion. The control functionals $R$ and $L$ under a control limit policy are the local times of $\hat{W}$ at the points $a$ and $b$, respectively. More specifically, the control limit policy $(R, L)$ on the interval $[a, b]$ is defined by

$$R(t) = \sup_{0 \leq s \leq t} [a - \hat{B}(s) + L(s)]^+$$

(6.1)
and

\[ L(t) = \sup_{0 \leq s \leq t} [\hat{B}(s) + R(s) - b]^+. \] (6.2)

The existence and uniqueness of such functionals was established in Harrison and Taksar [6].

We now present sufficient conditions for optimality for the Lagrangian problem (5.1). Suppose that an optimal solution to the Lagrangian problem is found and let \( g \) be the corresponding value of the Lagrangian cost function \( k(x) \). Thus \( g \) is the minimal average cost per unit time (independent of initial state) and will be referred to as the gain. As is standard with problems using average cost criterion, let \( V(x) \) be the cost incurred under the optimal policy when the initial state of the controlled process \( \hat{W} \) is \( x \) minus the cost incurred under the optimal policy when the initial state of \( \hat{W} \) is 0. It will be assumed that \( V \) has a first and second derivative and \( V \) will be referred to as the potential function.

The following proposition was proved in Taksar [9].

**Proposition 6.1.** Suppose \((g, V(x))\) is a solution to

\[
\begin{align*}
\operatorname{Min} \{V(x) + h(x) - g, r + V'(x), l - V''(x)\} &= 0 \\
V(0) &= 0,
\end{align*}
\] (6.3) (6.4)

and suppose there exists an interval \([a, b]\) such that

\[
\begin{align*}
V(x) + h(x) - g &= 0 \quad \text{for } a \leq x \leq b, \\
V'(x) &= r \quad \text{for } x \leq a, \\
V'(x) &= l \quad \text{for } x \geq b.
\end{align*}
\] (6.5) (6.6) (6.7)

Suppose that for some constants \( N_1 \) and \( N_2 \)

\[
V(x) \leq N_1 + N_2 h(x).
\] (6.8)

Then \( g \) is the minimal average cost for the Lagrangian problem and the control limit policy on the interval \([a, b]\) is an optimal policy.
We are now in a position, using Theorem 5.1 and Proposition 6.1, to state sufficient conditions for optimality of the constrained problem (4.11)-(4.13).

**Theorem 6.2.** Suppose \((g, V(x), r, l, a, b)\) satisfy

\[
\begin{align*}
\text{Min} \{\Gamma V(x) + h(x) - g, r + V'(x), l - V'(x)\} &= 0, \\
V(0) &= 0, \\
\Gamma V(x) + h(x) - g &= 0 \quad \text{for } a \leq x \leq b, \\
V'(x) &= -r \quad \text{for } x \leq a, \\
V'(x) &= l \quad \text{for } x \geq b, \\
\limsup_{T \to \infty} \frac{1}{T} E_x[R(t)] &= \frac{\rho_2(1 - \rho_1)\mu}{\rho_1 - \rho_2}, \quad \text{and} \\
\limsup_{T \to \infty} \frac{1}{T} E_x[L(t)] &= \frac{\rho_1(1 - \rho_2)\mu}{\rho_1 - \rho_2},
\end{align*}
\]

where \(R\) and \(L\) are the control limit policies on the interval \([a, b]\). Suppose \(V \in C^2\) and satisfies condition (6.8). Then the optimal policy \((R, L)\) to the constrained problem (4.11)-(4.13) is the control limit policy on the interval \([a, b]\).

The requirement that the function \(V \in C^2\) is sometimes called the heuristic principle of smooth fit and is often imposed when solving control problems; see, for example, Beneš, Shepp and Witsenhausen [2]. In the next two sections a solution, which is denoted \((g^*, V^*(x), r^*, l^*, a^*, b^*)\), to equations (6.9)-(6.15) will be found.

### 7. A Candidate Policy

The first step towards a solution to equations (6.9)-(6.15) is to find candidate interval endpoints \(a^*\) and \(b^*\). These values lead to a candidate policy \((R^*, L^*)\) defined by

\[
R^*(t) = \sup_{0 \leq s \leq t} [a^* - \hat{B}(s) + L^*(s)]^+
\]
and

\[ L^*(t) = \sup_{0 \leq s \leq t} [\hat{B}(s) + R^*(s) - b^*]^+ \quad (7.2) \]

In order to find these endpoints, the following lemma concerning a regulated Brownian motion will be needed. See Chapter 5 of Harrison [4] for a derivation of this result. Let us first define \( \nu \) by

\[ \nu = \frac{2\mu}{\sigma^2}. \quad (7.3) \]

**Lemma 7.1.** Suppose \( \hat{B} \) is a \((\mu, \sigma^2)\) Brownian motion, \( R \) and \( L \) are as defined in (6.1)-(6.2), and thus \( \hat{W} = \hat{B} + R - L \) is a regulated Brownian motion on the interval \([a, b]\). Then \( \hat{W} \) has a truncated exponential steady state distribution with density function

\[ p(x) = \frac{\nu e^{\nu(x-a)}}{e^{\nu(b-a)} - 1} \text{ for } a \leq x \leq b. \quad (7.4) \]

Furthermore,

\[ \lim_{T \to -\infty} \frac{1}{T} E_x[R(T)] = \frac{\mu}{e^{\nu(b-a)} - 1} \quad (7.5) \]

and

\[ \lim_{T \to -\infty} \frac{1}{T} E_x[L(T)] = \frac{\mu}{1 - e^{-\nu(b-a)}}. \quad (7.6) \]

The candidate interval endpoints will be derived by solving the following problem:

among the class of control limit policies, find the policy \((R, L)\) to

\[
\text{minimize } \limsup_{T \to -\infty} \frac{1}{T} E_x \left[ \int_0^T h(\hat{W}(t)) dt \right] \quad (7.7)
\]

subject to \( \limsup_{T \to -\infty} \frac{1}{T} E_x[R(T)] = \frac{\rho_2(1 - \rho_1)\mu}{\rho_1 - \rho_2}, \quad (7.8) \)

\[
\limsup_{T \to -\infty} \frac{1}{T} E_x[L(T)] = \frac{\rho_1(1 - \rho_2)\mu}{\rho_1 - \rho_2}. \quad (7.9)
\]

That is, we restrict the policy \((R, L)\) to take the form defined in (6.1)-(6.2), and then find the values of \( a \) and \( b \) that solve problem (7.7)-(7.9). Notice that problem (7.7)-(7.9)
is just the constrained problem (4.11)-(4.13) with the inequality constraints (4.12)-(4.13) replaced by conditions (6.14)-(6.15), which are equality constraints. The first step in solving problem (7.7)-(7.9) is to express the constraints directly in terms of the interval endpoints $a$ and $b$.

**Lemma 7.2.** A control limit policy $(R, L)$ on the interval $[a, b]$ satisfies constraints (7.8)-(7.9) if and only if

$$b - a = v^{-1} \ln \left( \frac{\rho_1(1 - \rho_2)}{\rho_2(1 - \rho_1)} \right).$$

(7.10)

**Proof.** The result follows by equating the right side of equations (7.5) and (7.8) and solving for $(b - a)$. The result can also be proved by equating the right side of equations (7.6) and (7.9).

Since the cost function $h$ defined in (4.9)-(4.10) is convex and achieves its minimum at zero, the optimal control limit policy for problem (7.7)-(7.9) will have endpoints $a$ and $b$ that satisfy $a \leq 0 \leq b$. By Lemma 8.2, the search over values of $a$ and $b$ to solve problem (7.7)-(7.9) can be reduced to a search over values of $a$ such that

$$-v^{-1} \ln \left( \frac{\rho_1(1 - \rho_2)}{\rho_2(1 - \rho_1)} \right) \leq a \leq 0.$$

(7.11)

**Proposition 7.3.** The control limit policy with interval endpoints

$$a^* = v^{-1} \ln \left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right)$$

(7.12)

and

$$b^* = v^{-1} \ln \left( \frac{(h_1 + h_2)\rho_1(1 - \rho_2)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right)$$

(7.13)

solves problem (7.7)-(7.9).

**Proof.** By equation (7.4), the objective function (7.7) can be expressed as $\int_a^b h(x)p(x)dx$ for any control limit policy on the interval $[a, b]$. By Lemma 7.2, the optimal value of the endpoint $a$ can be found by minimizing the function $f(a)$ over the interval
Using (7.10) to substitute for \((b - a)\) in (7.4) and using (4.10) to substitute for \(h\) yield

\[
f(a) = \frac{\nu \rho_2 (1 - \rho_1)}{(\rho_1 - \rho_2)} \left( \int_a^0 -h_1 x e^{\nu(x-a)} dx \right.
\]
\[
+ \int_0^{a - \nu^{-1} \ln \left( \frac{\rho_1(1 - \rho_2)}{\rho_2(1 - \rho_1)} \right)} h_2 x e^{\nu(x-a)} dx \right).
\]

Integration by parts yields

\[
f(a) = (\nu(\rho_1 - \rho_2))^{-1} \left( \rho_2 (1 - \rho_1)(h_1 + h_2)e^{-\nu a} 
\right.
\]
\[
+ \nu a (h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)) 
\]
\[
- h_1 \rho_2 (1 - \rho_1) - h_2 \rho_1 (1 - \rho_2) + h_2 \rho_1 (1 - \rho_2) \ln \left( \frac{\rho_1(1 - \rho_2)}{\rho_2(1 - \rho_1)} \right) \Bigg). \tag{7.16}
\]

Setting \(f'(a) = 0\), we obtain

\[
-\nu \rho_2 (1 - \rho_1)(h_1 + h_2)e^{-\nu a} + \nu (h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)) = 0. \tag{7.17}
\]

Solving for the endpoint \(a\) gives equation (7.12) and substituting this value in equation (7.10) yields (7.13). Taking the second derivative of \(f\) with respect to \(a\) yields

\[
f''(a) = \frac{\nu \rho_2 (1 - \rho_1)(h_1 + h_2)e^{-\nu a}}{(\rho_1 - \rho_2)}, \tag{7.18}
\]

which is strictly positive by definitions (4.2) and (7.3), thus completing the proof.

Thus a candidate interval \([a^*, b^*]\) has been found such that the candidate policy \((R^*, L^*)\) defined by equations (7.1)-(7.2) satisfy conditions (6.14)-(6.15). Notice that, as expected, \(a^* = 0\) in the limiting case \(h_2 = 0\), and similarly, \(b^* = 0\) when \(h_1 = 0\). A candidate gain \(g^*\) can now be derived by evaluating the Lagrangian cost function \(k(x)\) defined in (5.1) under the candidate policy.
Corollary 7.4. Under the candidate policy \((R^*, L^*)\), the Lagrangian cost function \(k(x)\) is

\[
g^* = \left(\nu(\rho_1 - \rho_2)\right)^{-1} \left( h_1 \rho_2 (1 - \rho_1) \ln \left( \frac{(h_1 + h_2) \rho_2 (1 - \rho_1)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right) \right. \\
+ h_2 \rho_1 (1 - \rho_2) \ln \left( \frac{(h_1 + h_2) \rho_1 (1 - \rho_2)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right) \\
+ \frac{r \mu \rho_2 (1 - \rho_1)}{(\rho_1 - \rho_2)} + \frac{l \mu \rho_1 (1 - \rho_2)}{(\rho_1 - \rho_2)}.
\]

(7.19)

Proof. By (7.5)-(7.6) and (7.14), the value of the Lagrangian cost function (5.1) under the candidate policy \((R^*, L^*)\) is

\[
g^* = f(a^*) + r \frac{\mu}{e^{\nu(b^* - a^*)} - 1} + l \frac{\mu}{1 - e^{-\nu(b^* - a^*)}}.
\]

(7.20)

Substituting for \((b^* - a^*)\) and \(a^*\) using equations (7.10) and (7.12), respectively, yields expression (7.19).

8. Solution to the Optimality Equations

In this section a solution \((g^*, V^*(x), r^*, l^*, a^*, b^*)\) to equations (6.9)-(6.15) will be presented. In the previous section candidate endpoints \(a^*\) and \(b^*\) were derived and a candidate gain \(g^*\) was found in terms of the Lagrange multipliers \(r\) and \(l\).

We start this section by finding a pair of candidate multipliers \((r^*, l^*)\). By condition (6.12) and the assumption that \(V \in C^2\), we set \(V''(a^*) = 0\) in equation (6.11) to obtain, upon setting \(V'(a^*) = -r\) and using (4.10) and (7.12) to substitute for \(h\) and \(a^*\) respectively,

\[
g = -\mu r - h_1 \nu^{-1} \ln \left( \frac{(h_1 + h_2) \rho_2 (1 - \rho_1)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right).
\]

(8.1)

Similarly, setting \(V'(b^*) = l\) and \(V''(b^*) = 0\) in equation (6.1) yields

\[
g = \mu l + h_2 \nu^{-1} \ln \left( \frac{(h_1 + h_2) \rho_1 (1 - \rho_2)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right).
\]

(8.2)
Equating the right sides of equations (8.1) and (8.2) gives

$$r + l = -\frac{h_1}{\mu v} \ln \left( \frac{(h_1 + h_2)\rho_2 (1 - \rho_1)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right)$$
$$- \frac{h_2}{\mu v} \ln \left( \frac{(h_1 + h_2)\rho_1 (1 - \rho_2)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right).$$

(8.3)

**Lemma 8.1.** Let $r + l$ be as in (8.3). Then $r + l > 0$.

**Proof.** By rearranging terms in (8.3), we see that $r + l > 0$ if and only if

$$\left( \frac{(h_1 + h_2)\rho_2 (1 - \rho_1)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right)^{-h_1} > \left( \frac{(h_1 + h_2)\rho_1 (1 - \rho_2)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right)^{h_2}$$

(8.4)

or, equivalently,

$$\frac{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)}{h_1 + h_2} > (\rho_2 (1 - \rho_1))^{\frac{h_1}{h_1 + h_2}} (\rho_1 (1 - \rho_2))^{\frac{h_2}{h_1 + h_2}}.$$  

(8.5)

Since $\ln(x)$ is an increasing concave function, it follows that $r + l > 0$. ♦

It turns out that there is not a unique pair of multipliers $(r^*, l^*)$ that satisfy the optimality equations (6.9)-(6.15); any nonnegative pair that satisfy equation (8.3) will suffice. To this end, choose any $\lambda^* \in [0, 1]$. The pair of candidate multipliers $(r^*, l^*)$ are defined by

$$r^* = \lambda^* \left( -\frac{h_1}{\mu v} \ln \left( \frac{(h_1 + h_2)\rho_2 (1 - \rho_1)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right) \right)$$
$$- \frac{h_2}{\mu v} \ln \left( \frac{(h_1 + h_2)\rho_1 (1 - \rho_2)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right).$$

(8.6)

and

$$l^* = (1 - \lambda^*) \left( -\frac{h_1}{\mu v} \ln \left( \frac{(h_1 + h_2)\rho_2 (1 - \rho_1)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right) \right)$$
$$- \frac{h_2}{\mu v} \ln \left( \frac{(h_1 + h_2)\rho_1 (1 - \rho_2)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right).$$

(8.7)

With the candidate multipliers $(r^*, l^*)$ in hand, it is now possible to calculate the candidate gain $g^*$ directly in terms of the problem data. This can be done in three ways:
substituting \( r^* \) for \( r \) in equation (8.1), substituting \( l^* \) for \( l \) in equation (8.2), or substituting both \( r^* \) and \( l^* \) in equation (7.19) of Corollary 7.4. Readers may verify that all three ways lead to the result that

\[
g^* = (\lambda^* - 1)h_1\nu^{-1} \ln\left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right) + \lambda^* h_2\nu^{-1} \ln\left( \frac{(h_1 + h_2)\rho_1(1 - \rho_2)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right). \tag{8.8}
\]

To complete the candidate solution of the optimality equations, a candidate potential function \( V^* \) is required. With \( g^*, a^* \) and \( b^* \) now chosen, condition (6.1) becomes

\[
\mu V'(x) + \frac{1}{2} \sigma^2 V''(x) + h(x) - g^* = 0 \quad \text{for} \quad a^* \leq x \leq b^*. \tag{8.9}
\]

The solution to this second order differential equation is given in the following proposition.

**Proposition 8.2.** The solution \( V^{**} \) to (8.9) is

\[
V^{**}(x) = \frac{g^*}{\mu} + C^* e^{-\nu x} - \frac{2}{\sigma^2} e^{-\nu x} \int_{a^*}^{x} h(y) e^{\nu y} dy \quad \text{for} \quad a^* \leq x \leq b^*, \tag{8.10}
\]

where

\[
C^* = \frac{h_1\mu^{-1}\nu^{-1}(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \ln\left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right). \tag{8.11}
\]

**Proof.** The solution \( V^{**} \) to (8.9) is

\[
V^{**}(x) = \frac{g^*}{\mu} + C e^{-\nu x} - \frac{2}{\sigma^2} e^{-\nu x} \int_{a^*}^{x} h(y) e^{\nu y} dy \quad \text{for} \quad a^* \leq x \leq b^*, \tag{8.12}
\]

where \( C \) is an unknown constant. To find \( C \), we evaluate (8.12) at the point \( x = a^* \), set \( V^{**}(a^*) = -r^* \) and substitute for \( a^* \) using (7.12) to get

\[
-r^* = \frac{g^*}{\mu} + C \left( \frac{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)}{(h_1 + h_2)\rho_2(1 - \rho_1)} \right), \tag{8.13}
\]

or, equivalently,

\[
g^* = -\mu r^* - \mu C \left( \frac{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)}{(h_1 + h_2)\rho_2(1 - \rho_1)} \right). \tag{8.14}
\]
Substituting \( g^* \) and \( r^* \) for \( g \) and \( r \) in (8.1) using (8.6) and (8.8) respectively, equating the right sides of equations (8.1) and (8.14), and solving for \( C \) yields the candidate constant \( C^* \) defined in (8.11).

The following three lemmas verify that the boundary conditions \( V'^*(b^*) = l^* \), \( V''*(a^*) = 0 \) and \( V'''*(b^*) = 0 \), respectively, are satisfied.

**Lemma 8.3.** The function \( V'^* \) defined in (8.10)-(8.11) satisfies \( V'^*(b^*) = l^* \).

**Proof.** Evaluating \( V'^* \) in (8.10) at \( x = b^* \) and integrating by parts yield

\[
V'^*(b^*) = \frac{g^*}{\mu} + C^* e^{-\nu b^*} - \frac{2}{\sigma^2} e^{-\nu b^*} \left(-h_1 \nu^{-2} (-1 - (e^\nu a^* (\nu a^* - 1))) + h_2 \nu^{-2} (\nu b^*(\nu b^* - 1) + 1)\right).
\]

Substituting for \( a^* \), \( b^* \) and \( C^* \) using (7.12)-(7.13) and (8.11), we obtain

\[
V'^*(b^*) = \frac{g^*}{\mu} + \left(\frac{h_1 + h_2)\rho_2(1 - \rho_1)}{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right) \frac{h_1}{\mu \nu} \ln \left(\frac{h_1 + h_2)\rho_2(1 - \rho_1)}{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right) + \frac{2h_1}{\sigma^2} \left(\frac{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}{h_1 + h_2)\rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right)
\]

\[
\left(-\nu^{-2} - \nu^{-2} \left(\frac{h_1 + h_2)\rho_2(1 - \rho_1)}{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right) \ln \left(\frac{h_1 + h_2)\rho_2(1 - \rho_1)}{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right) + \nu^{-2} \left(\frac{h_1 + h_2)\rho_2(1 - \rho_1)}{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right) \ln \left(\frac{h_1 + h_2)\rho_2(1 - \rho_1)}{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right) - \nu^{-2} \left(\frac{h_1 + h_2)\rho_2(1 - \rho_1)}{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right) \ln \left(\frac{h_1 + h_2)\rho_2(1 - \rho_1)}{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right)\right).
\]

Cancelling and rearranging terms yield

\[
V'^*(b^*) = \frac{g^*}{\mu} - \frac{h_2}{\mu \nu} \ln \left(\frac{h_1 + h_2)\rho_2(1 - \rho_1)}{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right) - \left(\frac{h_1 + h_2}{\mu \nu}\right)
\]

\[
\left(\frac{h_1 \rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}{h_1 + h_2)\rho_2(1 - \rho_1) + h_2 \rho_1(1 - \rho_2)}\right) + \frac{h_1 \rho_2(1 - \rho_1)}{\mu \nu \rho_1(1 - \rho_2)} + \frac{h_2}{\mu \nu}.
\]

Since the last three terms on the right side of (8.17) sum to zero, using (8.8) we see that \( V'^*(b^*) = l^* \).
Lemma 8.4. \( V' \) satisfies \( V'''(a^*) = 0 \).

Proof. Differentiating (8.10) yields

\[
V'''(x) = -\nu C^* e^{-\nu x} - \frac{2h(x)}{\sigma^2} + \frac{\nu^2}{\mu e^{\nu x}} \int_{a^*}^{x} h(y)e^{\nu y}dy \quad \text{for } a^* \leq x \leq b^*.
\]

Using (4.10) and (7.12) and evaluating equation (8.18) at \( x = a^* \) give

\[
V'''(a^*) = -\nu C^* \left( \frac{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)}{(h_1 + h_2)\rho_2(1-\rho_1)} \right)
\]

\[
- \frac{2}{\sigma^2} \left( \frac{-h_1}{\mu \nu} \ln \left( \frac{(h_1 + h_2)\rho_2(1-\rho_1)}{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)} \right) \right),
\]

which equals zero by (8.11).

Lemma 8.5. \( V' \) satisfies \( V'''(b^*) = 0 \).

Proof. Evaluating (8.18) at \( x = b^* \) and integrating by parts yield

\[
V'''(b^*) = -\nu C^* e^{-\nu b^*} - \frac{2h_2 b^*}{\sigma^2} + \nu^2 \mu^{-1} e^{-\nu b^*} \left( -h_1 \nu^{-2} (-1 - (e^{\nu a^*}(\nu a^* - 1))) + h_2 \nu^{-2}(e^{\nu b^*}(\nu b^* - 1) + 1) \right).
\]

Substituting for \( a^*, b^* \) and \( C^* \) using (7.12)-(7.13) and (8.11), respectively, it follows that

\[
V'''(b^*) = -\nu \left( \frac{(h_1 + h_2)\rho_2(1-\rho_1)}{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)} \right) \frac{h_1}{\mu \nu} \ln \left( \frac{(h_1 + h_2)\rho_2(1-\rho_1)}{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)} \right)
\]

\[
+ \frac{h_1}{\mu} \ln \left( \frac{(h_1 + h_2)\rho_1(1-\rho_2)}{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)} \right)
\]

\[
+ \nu^2 \mu^{-1} \left( \frac{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)}{(h_1 + h_2)\rho_1(1-\rho_2)} \right) \left( h_1 \nu^{-2} + h_1 \nu^{-2} \right)
\]

\[
- \left( \frac{(h_1 + h_2)\rho_2(1-\rho_1)}{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)} \right) \ln \left( \frac{(h_1 + h_2)\rho_2(1-\rho_1)}{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)} \right)
\]

\[
- \left( \frac{h_2 \nu^{-2}(h_1 + h_2)\rho_2(1-\rho_1)}{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)} \right) + \left( \frac{h_2 \nu^{-2}(h_1 + h_2)\rho_1(1-\rho_2)}{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)} \right)
\]

\[
\ln \left( \frac{(h_1 + h_2)\rho_1(1-\rho_2)}{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)} \right) - \left( \frac{h_2 \nu^{-2}(h_1 + h_2)\rho_1(1-\rho_2)}{h_1\rho_2(1-\rho_1) + h_2\rho_1(1-\rho_2)} \right)
\]

\[
+ h_2 \nu^{-2} \right). \tag{8.21}
\]
The \( \ln(\cdot) \) terms cancel out and upon rearranging, we have
\[
V^{*''}(b^*) = \frac{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)}{\mu (h_1 + h_2) \rho_1 (1 - \rho_2)} \left( -\frac{h_1 (h_1 + h_2) \rho_2 (1 - \rho_1)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} \right) \\
- \frac{h_2 (h_1 + h_2) \rho_2 (1 - \rho_1)}{h_1 \rho_2 (1 - \rho_1) + h_2 \rho_1 (1 - \rho_2)} + h_1 + h_2,
\]
which equals zero. 

We now extend the function \( V^{*'} \) derived in Proposition 8.2 beyond the interval \([a^*, b^*]\) in the obvious way suggested by conditions (6.12)-(6.13) and define the candidate potential function \( V^* \) in terms of \( V^{*'} \). Define \( V^* \) by
\[
V^*(x) = \begin{cases} 
\int_0^x V^{*'}(y)dy, & \text{if } x \geq 0, \\
-\int_x^0 V^{*'}(y)dy, & \text{if } x < 0,
\end{cases}
\]
where
\[
V^{*'}(x) = \begin{cases} 
-r^*, & \text{if } x \leq a^*, \\
l^*, & \text{if } x \geq b^*, \\
\frac{\mu}{\kappa} + C^* e^{-\nu x} - \frac{2}{\kappa^2} e^{-\nu x} \int_a^x h(y) e^{\nu y} dy, & \text{if } a^* < x < b^*.
\end{cases}
\]
Notice that \( V^* \in C^2 \), as required, and that \( V^*(0) = 0 \).

The candidate solution \((g^*, V^*(x), r^*, l^*, a^*, b^*)\) is now completely specified, and equations (6.10)-(6.15) of the optimality equations have been verified. To complete the verification of the candidate solution, condition (6.9) needs to be verified. Since (6.11)-(6.13) have already been verified, it suffices to check that
\[
\Gamma V^*(x) + h(x) - g^* \geq 0 \quad \text{for } x \not\in [a^*, b^*]
\]
and
\[
-r^* \leq V^{*'}(x) \leq l^* \quad \text{for } a^* < x < b^*.
\]

Lemma 8.6. Condition (8.25) holds.

Proof. To show (8.25), notice that for \( x < a^* \),
\[
\Gamma V^*(x) + h(x) - g^* = -\mu r^* - h_1 x - g^* \\
> -r^* - h_1 a^* - g^* \\
= 0.
\]
(8.27)
Similarly, for \( x > b^* \),

\[
\Gamma V^*(x) + h(x) - g^* = \mu l^* + h_2 x - g^* \\
> ul^* + h_2 b^* - g^* \\
= 0. \tag{8.28}
\]

**Lemma 8.7.** Condition (8.26) holds.

**Proof.** Since \( V''(a^*) = -r^*, V''(b^*) = l^* \) and \( V'''(a^*) = V'''(b^*) = 0 \), to prove (8.26) it suffices to show that

\[
V'''(x) \geq 0 \quad \text{for} \quad a^* < x < b^*.
\]  

For \( a^* \leq x \leq 0 \), we have, by (8.18) and integration by parts,

\[
V'''(x) = -\frac{h_1}{\mu} \left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right) \ln \left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right) e^{-\nu x} \\
+ \frac{2h_1 x}{\sigma^2} - \frac{h_1 e^{-\nu x}}{\mu} (\nu x e^{\nu x} - e^{\nu x} - \nu a^* e^{\nu a^*} + e^{\nu a^*}). \tag{8.30}
\]

Substituting for \( a^* \) from (7.12) and cancelling and rearranging terms yield

\[
V'''(x) = \frac{h_1}{\mu} \left( 1 - \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} e^{-\nu x} \right) \quad \text{for} \quad a^* \leq x \leq 0 \]  

Letting \( x = 0 \) in (8.31) gives

\[
V'''(0) = \frac{h_1 h_2 (\rho_1 - \rho_2)}{\mu (h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2))}, \tag{8.32}
\]

which is greater than zero by definition (4.2). Also, for \( a < x < 0 \),

\[
V'''(x) = \frac{2h_1 (h_1 + h_2)\rho_2(1 - \rho_1)}{\sigma^2 (h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2))} e^{-\nu x}, \tag{8.33}
\]

which is also greater than zero. Since \( V''(a) = 0 \), \( V'''(0) > 0 \) and \( V'''(x) > 0 \) for \( x \in (a^*, 0) \), it follows that \( V''(x) \geq 0 \) for \( a \leq x \leq 0 \).
Similarly, for $0 \leq x \leq b^*$, we have by (8.18) and integration by parts,

\[
V^{**}(x) = -\frac{h_1}{\mu} \left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \ln\left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right) e^{-\nu x} - \frac{2h_2 x}{\sigma^2} - \frac{h_1 e^{-\nu x}}{\mu} \left( -1 + \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right) \ln\left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right) + \frac{h_2 e^{-\nu x}}{\mu} \left( e^{\nu x}(\nu x - 1) + 1 \right) \right),
\]

which can be rewritten as

\[
V^{**}(x) = -\frac{h_2}{\mu} + \frac{h_2(h_1 + h_2)\rho_1(1 - \rho_2)}{\mu(h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2))} e^{-\nu x} \quad \text{for } 0 \leq x \leq b^*. \tag{8.34}
\]

For $0 < x < b^*$, we can differentiate $V^{**}(x)$ to obtain

\[
V^{***}(x) = -\frac{2h_2(h_1 + h_2)\rho_1(1 - \rho_2)}{\sigma^2(h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2))} e^{-\nu x}, \tag{8.36}
\]

which is less than zero. Since $V^{**}(0) > 0$, $V^{**}(b^*) = 0$ and $V^{***}(x) < 0$ for $x \in (0, b^*)$, we have that $V^{**}(x) \geq 0$ for $0 \leq x \leq b^*$. Thus, (8.29) has been shown, which completes the proof. \[\Box\]

We are now in position to state the solution to problem (4.11)-(4.13).

**Theorem 8.8.** The solution to problem (4.11)-(4.13) is the control limit policy $(R^*, L^*)$ defined in (7.1)-(7.2) on the interval $[a^*, b^*]$, where $a^*$ and $b^*$ are defined in (7.12)-(7.13).

**Proof.** The results from Lemma 7.1 through Lemma 8.7 lead to a solution $(g^*, V^*(x), r^*, l^*, a^*, b^*)$ that satisfies equations (6.9)-(6.15) such that $V \in C^2$. By Theorem 6.2, the result holds if the function $V^*$ satisfies condition (6.8) that $V(x) \leq N_1 + N_2 h(x)$ for some constants $N_1$ and $N_2$. This condition holds by setting

\[
N_1 = \max\{V^*(a^*), V^*(b^*)\} \tag{8.37}
\]

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and
\[ N_2 = \max\left\{ \frac{r^* + l^*}{h_2}, \frac{r^* + l^*}{h_1} \right\}. \] (8.38)

As is usual in control problems dealing with long-run average cost criterion, the optimal policy \((R^*, L^*)\) is not unique. In particular, any policy that brings the controlled Brownian motion process within the interval \([a^*, b^*]\) in a finite expected amount of time and then uses \(a^*\) and \(b^*\) as reflecting barriers will achieve the same average expected cost as the policy \((R^*, L^*)\).

9. The Driftless Case

In this section, the constrained problem (4.11)-(4.13) is solved under the conditions of perfect system balance, where \(\rho_1 = \rho_2\). In this case, the two-dimensional Brownian motion \(\bar{B}\) defined in (4.1) has drift \(\mu = 0\). Defining \(\xi\) by
\[ \xi = \sqrt{n\rho_1(1 - \rho_1)}, \] (9.1)

it is seen that the right side of constraints (4.12) and (4.13) both equal \(\xi\). Taking \(\xi\) as an essential system parameter, problem (4.11)-(4.13) will be solved in terms of it. The procedure used in Sections 4 through 8 is still valid in this case, but the computations are now substantially easier. In particular, the sufficient conditions for optimality are the same as in Theorem 6.2, except that conditions (6.14)-(6.15) are replaced by
\[ \limsup_{T \to \infty} \frac{1}{T} E_x[R(t)] = \limsup_{T \to \infty} \frac{1}{T} E_x[L(t)] = \xi. \] (9.2)

As in Section 7, the first step towards finding a solution to the optimality equations (6.9)-(6.13) and (9.2) is to find candidate interval endpoints \(a^*\) and \(b^*\). See Chapter 5 of Harrison [4] for a proof of the following result, which is the analogue of Lemma 7.1 for the driftless case.
Lemma 9.1. Suppose $\hat{B}$ is a $(0, \sigma^2)$ Brownian motion, $R$ and $L$ are as defined in (6.1)-(6.2), and thus $\hat{W} = \hat{B} + R - L$ is a regulated Brownian motion on the interval $[a, b]$. Then $\hat{W}$ has a uniform steady state distribution on $[a, b]$ and

$$\limsup_{T \to \infty} \frac{1}{T} E_x[R(t)] = \limsup_{T \to \infty} \frac{1}{T} E_x[L(t)] = \frac{\sigma^2}{2(b - a)}.$$  \hfill (9.3)

The candidate interval endpoints are derived by solving the following problem: among the class of control limit policies, find the policy $(R, L)$ to

minimize $\limsup_{T \to \infty} \frac{1}{T} E_x[\int_0^T h(\hat{W}(t))dt]$ \hfill (9.4)

subject to $\limsup_{T \to \infty} \frac{1}{T} E_x[R(t)] = \limsup_{T \to \infty} \frac{1}{T} E_x[L(t)] = \frac{\sigma^2}{2(b - a)}.$ \hfill (9.5)

Lemma 9.2. The control limit policy with interval endpoints

$$a^* = -\frac{h_2}{h_1 + h_2} \frac{\sigma^2}{2\xi},$$ \hfill (9.6)

and

$$b^* = \frac{h_1}{h_1 + h_2} \frac{\sigma^2}{2\xi},$$ \hfill (9.7)

solves problem (9.4)-(9.5).

Proof. Equating the right sides of (9.2) and (9.5) leads to the requirement

$$b - a = \frac{\sigma^2}{2\xi}.$$ \hfill (9.8)

Thus $a^*$ can be found by minimizing $f(a)$ over the interval

$$-\frac{\sigma^2}{2\xi} \leq a \leq 0,$$ \hfill (9.9)

where

$$f(a) = \frac{2\xi}{\sigma^2} \left( \int_a^0 -h_1 x dx + \int_{a + \frac{\sigma^2}{2\xi}} h_2 x dx \right).$$ \hfill (9.10)
Integrating gives
\[ f(a) = \frac{\xi}{\sigma^2} \left( h_1 a^2 + h_2 a^2 + \frac{h_2 \sigma^2 a}{\xi} + \frac{h_2 \sigma^4}{4\xi^2} \right). \] (9.11)

Setting \( f'(a) = 0 \), we have
\[ 2h_1 a + 2h_2 a + \frac{h_2 \sigma^2}{\xi} = 0. \] (9.12)

Solving for \( a \) yields \( a^* \) in (9.6) and using (9.8) to solve for \( b \) yields \( b^* \) in (9.7). These are the candidate interval endpoints, since
\[ f''(a) = \frac{\xi}{\sigma^2} (2h_1 + 2h_2), \] (9.13)

which is positive.

Since, upon cancelling and rearranging terms,
\[ f(a^*) = \frac{\sigma^2 h_1 h_2}{4\xi (h_1 + h_2)}, \] (9.14)

the candidate gain \( g^* \) is
\[ g^* = \frac{\sigma^2 h_1 h_2}{4\xi (h_1 + h_2)} + r\xi + l\xi. \] (9.15)

**Theorem 9.3.** Let \( \lambda^* \in [0, 1] \) and let \( a^* \) and \( b^* \) be defined as in (9.6)-(9.7). Let

\[ r^* = \lambda^* \frac{h_1 h_2 \sigma^2}{h_1 + h_2 4\xi^2}, \] (9.16)

\[ l^* = (1 - \lambda^*) \frac{h_1 h_2 \sigma^2}{h_1 + h_2 4\xi^2}, \] (9.17)

\[ g^* = \frac{h_1 h_2 \sigma^2}{h_1 + h_2 2\xi^2}, \] (9.18)

and

\[ V^*(x) = \begin{cases} \int_0^x V^{*'}(y)dy, & \text{if } x \geq 0, \\ -\int_x^0 V^{*'}(y)dy, & \text{if } x < 0, \end{cases} \] (9.19)

where

\[ V^{*'}(x) = \begin{cases} -r^*, & \text{if } x \leq a^*, \\ l^*, & \text{if } x \geq b^*, \\ \frac{2g^* x}{\sigma^2} - \frac{2}{\sigma^2}\int_{a^*}^x h(y)dy + C_1^*, & \text{if } a^* < x < b^*, \end{cases} \] (9.20)

and

\[ C_1^* = \frac{\sigma^2(-\lambda^* h_1^2 h_2 + (2 - \lambda^*) h_1 h_2^2)}{4\xi^2 (h_1 + h_2)^2}. \] (9.21)
Then \((g^*, V^*(x), r^*, l^*, a^*, b^*)\) solves equations (6.9)-(6.13) and (9.2). Furthermore, \(V \in \mathbb{C}^2\) and satisfies condition (6.8).

Thus, by Theorem 6.2, the solution to problem (4.11)-(4.13) in the driftless case is the control limit policy \((R^*, L^*)\) defined by (7.1)-(7.2) on the interval \([a^*, b^*]\), where \(a^*\) and \(b^*\) are defined in (9.6)-(9.7). The proof of Theorem 9.3 parallels the arguments in Section 8 that lead to Theorem 8.8. However, the computations in this case are substantially simpler and will be omitted. This section is concluded by showing that, as the drift \(\mu \to 0\), the endpoints derived in Section 8 for the nonzero drift case converge to the endpoints derived in the driftless case.

**Proposition 9.4.**

\[
\lim_{\mu \to 0} \frac{\sigma^2}{2\mu} \ln \left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right) = -\frac{h_2}{h_1 + h_2} \frac{\sigma^2}{2\xi} \tag{9.22}
\]

and

\[
\lim_{\mu \to 0} \frac{\sigma^2}{2\mu} \ln \left( \frac{(h_1 + h_2)\rho_1(1 - \rho_2)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right) = \frac{h_1}{h_1 + h_2} \frac{\sigma^2}{2\xi}. \tag{9.23}
\]

**Proof.** Since \(\mu = \sqrt{n}(\rho_1 - \rho_2)\), it follows that

\[
\lim_{\mu \to 0} \frac{\sigma^2}{2\mu} \ln \left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right) = \lim_{\rho_2 \to \rho_1} \frac{\sigma^2}{2\sqrt{n}(\rho_1 - \rho_2)} \ln \left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right). \tag{9.24}
\]

By l'Hôpital's rule, one obtains

\[
\lim_{\rho_2 \to \rho_1} \frac{\sigma^2}{2\sqrt{n}(\rho_1 - \rho_2)} \ln \left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right)
= \lim_{\rho_2 \to \rho_1} \frac{\frac{d}{d\rho_2} \sigma^2 \ln \left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right)}{\frac{d}{d\rho_2} 2\sqrt{n}(\rho_1 - \rho_2)} \tag{9.25}
= \lim_{\rho_2 \to \rho_1} \frac{-\sigma^2 \rho_1 h_2}{2\sqrt{n}(h_1 + h_2)(\rho_1 - \rho_2) + h_2\rho_1(1 - \rho_2)} \tag{9.26}
= \frac{-\sigma^2 h_2}{2\sqrt{n}(h_1 + h_2)\rho_1(1 - \rho_1)}. \tag{9.27}
\]
The definition of $\xi$ in (9.1) completes the proof for the endpoint $a^*$. The proof for $b^*$ is similar and is omitted.

10. Solution to the Workload Formulation: Summary

The purpose of this paper was to obtain a solution $(U, Z, \theta)$ to the workload formulation (2.6)-(2.11) of the limiting control problem. This is a self-contained section that summarizes the solution. The parameters $\rho_i, M_{ik}, v_1, \sigma^2, h_i, \nu$ and $\xi$ appearing in this section are all defined in terms of the primitive problem data. Their definitions can be found in equations (1.5), (2.1), (2.3), (4.3), (4.7)-(4.8), (7.3) and (9.1), respectively.

In the workload formulation, the controller observes a two-dimensional Brownian motion process $B$, from which can be observed the one-dimensional Brownian motion process $\hat{B}$ defined by

$$\hat{B}(t) = \rho_2 B_1(t) - \rho_1 B_2(t), \ t \geq 0. \quad (10.1)$$

If $\rho_1 \neq \rho_2$, then define the interval endpoints $a$ and $b$ by

$$a = \nu^{-1} \ln \left( \frac{(h_1 + h_2)\rho_2(1 - \rho_1)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right) \quad (10.2)$$

and

$$b = \nu^{-1} \ln \left( \frac{(h_1 + h_2)\rho_1(1 - \rho_2)}{h_1\rho_2(1 - \rho_1) + h_2\rho_1(1 - \rho_2)} \right). \quad (10.3)$$

If $\rho_1 = \rho_2$, then let

$$a = -\frac{h_2}{h_1 + h_2} \frac{\sigma^2}{2\xi} \quad (10.4)$$

and

$$b = \frac{h_1}{h_1 + h_2} \frac{\sigma^2}{2\xi}. \quad (10.5)$$

For a particular realization of $\hat{B}$, define the control functionals $(R, L)$ by

$$R(t) = \sup_{0 \leq s \leq t} [a - \hat{B}(s) + L(s)]^+ \quad (10.6)$$
and

\[ L(t) = \sup_{0 \leq s \leq t} [\hat{B}(s) + R(s) - b]^+. \]  \hfill (10.7)

The two-dimensional optimal control process \( U \) is given by

\[ U_1(t) = \frac{R(t)}{\rho_2} \]  \hfill (10.8)

and

\[ U_2(t) = \frac{L(t)}{\rho_1}. \]  \hfill (10.9)

From the functionals \( (R, L) \) in (10.6)-(10.7), next define the process \( \hat{W} \) by

\[ \hat{W}(t) = \hat{B}(t) + R(t) - L(t) \quad \text{for all } t \geq 0. \]  \hfill (10.10)

The \( K \)-dimensional optimal control process \( Z \) is given by

\[ Z_k(t) = \begin{cases} \frac{\hat{W}(t)}{\rho_2 M_{11} - \rho_1 M_{21}}, & \text{if } k = 1 \text{ and } \hat{W}(t) \geq 0, \\ 0, & \text{if } k \neq 1 \text{ and } \hat{W}(t) \geq 0. \end{cases} \]  \hfill (10.11)

and

\[ Z_k(t) = \begin{cases} \frac{\hat{W}(t)}{\rho_2 M_{12} - \rho_1 M_{32}}, & \text{if } k = 2 \text{ and } \hat{W}(t) < 0, \\ 0, & \text{if } k \neq 2 \text{ and } \hat{W}(t) < 0. \end{cases} \]  \hfill (10.12)

Finally, the optimal control process \( \theta \) is given by

\[ \theta(t) = v_1^{-1}[B_1(t) + \frac{R(t)}{\rho_2} - \sum_{k=1}^{K} M_{1k} Z_k(t)], \quad \text{for all } t \geq 0. \]  \hfill (10.13)

Thus the solution \( (U, Z, \theta) \) to the workload formulation (2.6)-(2.11) is given by equations (10.8)-(10.9) and (10.11)-(10.13).

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