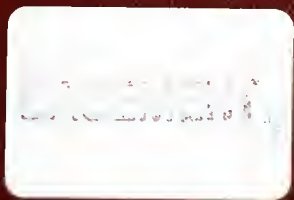


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for the Euclidean Traveling Salesman Problem**

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# On the Spacefilling Curve Heuristic for the Euclidean Traveling Salesman Problem

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## Abstract

Bartholdi and Platzman [3] proposed the spacefilling curve heuristic for the Euclidean Traveling Salesman Problem and proved that their heuristic returns a tour within an  $O(\lg n)$  factor of optimal length. They conjectured that the worst-case ratio is in fact  $O(1)$ . In this note we exhibit a counterexample showing the  $O(\lg n)$  upper bound is tight.

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# 1 Introduction

Bartholdi and Platzman [3] proposed a heuristic for the Euclidean Traveling Salesman Problem (ETSP) based on a spacefilling curve. Their curve  $\phi$  is a uniformly continuous map from the unit interval to the unit square. For our purposes we define  $\phi$  as a total linear ordering  $<_\phi$  on the points of the unit square  $[0, 1]^2$ . This ordering may be defined by a recursive procedure; for any two distinct points  $(x, y)$  and  $(x', y')$  the procedure will eventually decide which point comes first in the order.

In the unit square, if  $x - y \leq 0 < x' - y'$ , then  $(x, y) <_\phi (x', y')$ . In figure 1(a), this means all points from the lower right triangle precede points from the upper left triangle. Otherwise, both points fall in the same triangle, and we apply a recursive ordering on points in such a triangle. First rotate and enlarge the triangle into standard position with vertices at  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 1)$  (for this transformation to be uniquely defined, we have to orient the triangles; in our figures this orientation is denoted by an arrow). In the following we refer to this triangle as the ‘unit triangle’. If  $x + y \leq 1 < x' + y'$  in the unit triangle, then  $(x, y) <_\phi (x', y')$ . In figure 1(b) this means all points from the lower left subtriangle precede points from the upper right subtriangle. Otherwise, both points fall in the same subtriangle. Now this (oriented) subtriangle is similar to the original unit triangle, so we may recurse. After  $t$  iterations of taking subtriangles, the subtriangle we are considering has a hypotenuse of length  $2^{(1-t)/2}$  (in the scale of the original unit square), hence the process is guaranteed to halt after  $O(\log(1/d))$  iterations, where  $d$  is the Euclidean distance between the two input points  $(x, y)$  and  $(x', y')$ .

Now we describe the heuristic of [3]: given a set  $S$  of  $n$  points in the unit square, visit the points in the order  $\phi$  defined above, and finish the tour by returning to the first point. As remarked in [4], this heuristic is very fast. Given point  $(x, y)$  input as a pair of  $k$ -bit fractions, a  $2k$ -bit sorting key  $t$  may be computed in  $O(k)$  bit operations ( $t$  is really an inverse  $\phi^{-1}(x, y)$  where  $\phi$  is defined as a continuous map from the unit interval onto the unit square). Given  $n$  such points, all the keys may be computed and then sorted (by radix-sort [1]) in  $O(kn)$  bit operations, i.e. in time linear in the input size.

Let  $L^\phi(S)$  be the length of the tour produced by this heuristic, and let  $L^*(S)$  be the length of the optimal tour. In [5] they proved that  $L^\phi(S)/L^*(S) = O(\lg n)$ . They further conjecture that the worst case ratio is in fact  $O(1)$ . We refute this conjecture by exhibiting a simple set  $S_n$  of  $n$  points with  $L^\phi(S_n)/L^*(S_n) = \Theta(\lg n)$ .

## 2 A $\Theta(\lg n)$ Example

Consider  $n$  points uniformly spaced along the line from  $(1/3, 1/3)$  to  $(1, 1/3)$ . Precisely, define the set of points

$$S_n = \left\{ (x_i, y_i) = \left( \frac{1}{3} + \frac{2}{3} \cdot \frac{2i-1}{2n}, \frac{1}{3} \right) \text{ for } 1 \leq i \leq n \right\}. \quad (1)$$

This definition of  $S_n$  is convenient for our proof, but in fact any reasonably uniform distribution on the line  $y = 1/3$  would suffice. The optimal tour on  $S_n$  visits the points in the order of increasing  $i$ , and has length  $L^*(S_n) < 4/3$ . It now suffices to show:

**Theorem 2.1** *For  $n = 2^k$ , traversing  $S_n$  in the order of curve  $\phi$  produces a tour of length  $L^\phi(S_n) > 2k/9$ .*

**Proof:** The basic idea is to recursively decompose the order that the spacefilling curve visits  $S_n$ , and observe that at each level there are significantly long ‘jumps’.

Let  $a_k$  be the length of the path produced by traversing  $S_n$  in the order of curve  $\phi$  (note  $a_k < L^\phi(S_n)$  since  $L^\phi(S_n)$  counts the extra edge used to close the path into a cycle). Let  $b_k$  be the length of the path traversed by the heuristic on the following similar set of  $n = 2^k$  points:

$$S'_n = \left\{ (x_i, y_i) = \left( \frac{2}{3} + \frac{1}{3} \cdot \frac{2i-1}{2n}, \frac{2}{3} \right) \text{ for } 1 \leq i \leq n \right\}. \quad (2)$$

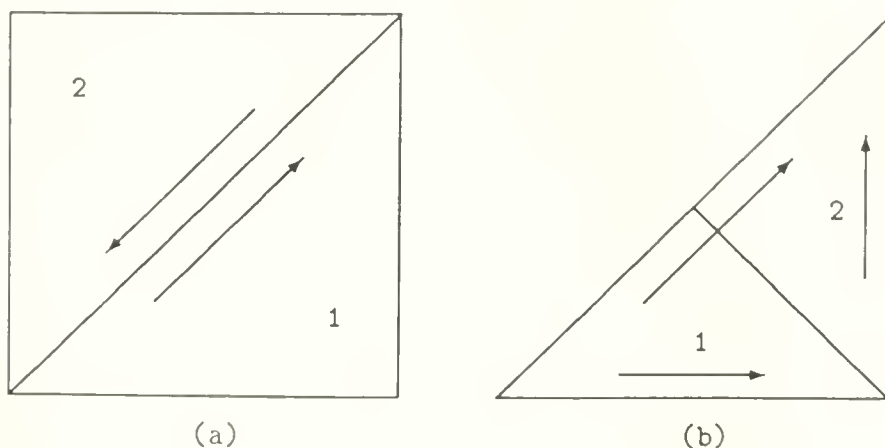


Figure 1: In both the square (a) and the triangle (b), points of subtriangle 1 precede points of subtriangle 2.

By the recursive definition of  $\phi$ , the points of  $S'_n$  are visited in the same order as those in  $S_n$ ; hence  $b_k = a_k/2$ .

We now derive a recursion for  $a_k$ . In figure 2, let  $T$  denote the entire unit triangle, and let  $T_1, T_2, T_3, T_4$  denote the subtriangles in the order visited by  $\phi$ . The points of  $S_n$  lie in subtriangles  $T_1, T_2$ , and  $T_3$ . Hence  $a_k$  is the sum of the path lengths within each subtriangle, together with the lengths of the jumps between consecutive subtriangles.

The  $n/4$  points of  $S_n \cap T_1$  form a half-scale image of  $S'_{n/4}$ , contributing  $b_{k-2}/2$  to  $a_k$ . Similarly the  $n/4$  points of  $S_n \cap T_2$  form a *reversed* half-scale image of  $S'_{n/4}$ . Since  $\phi$  is reversible (i.e. 'reversing the arrow' of a triangle exactly reverses the order of points within that triangle), the length of their path is the same as the path in  $T_1$ , contributing another term of  $b_{k-2}/2$  to  $a_k$ . Finally the  $n/2$  points of  $S_n \cap T_3$  form a half-scale reversed image of  $S_{n/2}$ , contributing  $a_{k-1}/2$  to  $a_k$ . Hence

$$a_k = b_{k-2} + a_{k-1}/2 + j_{12} + j_{23} = (a_{k-2} + a_{k-1})/2 + j_{12} + j_{23}$$

where  $j_{12}$  is the length of the jump from the last point in  $T_1$  to the first point in  $T_2$ , and  $j_{23}$  is the length of the jump from the last point in  $T_2$  to the first point in  $T_3$ . To estimate these jump lengths we need to know the first and last points of  $S_n$  visited in each subtriangle.

**Lemma 2.2** *For  $n = 2^k$ ,  $k \geq 1$ , the first point in  $S_n$  (and by similarity  $S'_n$ ) under order  $\phi$  is  $(x_1, y_1)$ , and the last point is  $(x_p, y_p)$  where  $p = (n/2) + 1$ .*

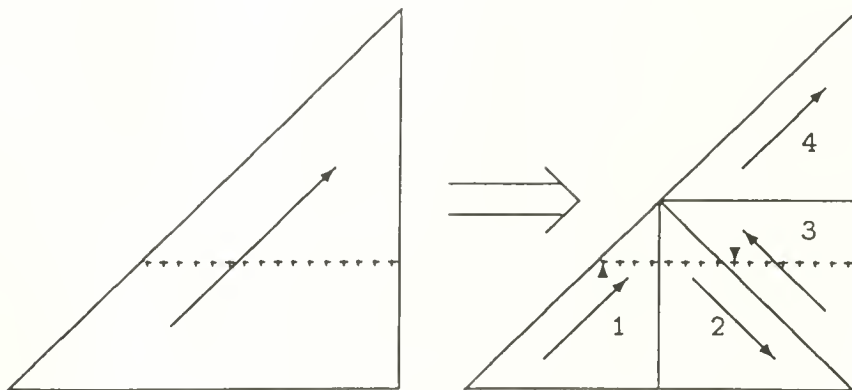


Figure 2: The set  $S_{16}$  in the unit triangle  $T$  decomposes into a copy of  $S'_4$  in  $T_1$ , a reversed copy of  $S'_4$  in  $T_2$ , and a reversed copy of  $S_8$  in  $T_3$ . Curve  $\phi$  visits the leftmost point first and the ninth point last (marked by arrowheads).

**Proof:** Let the points of  $S_n$  be indexed as in the definition (1). By inspection for  $k \leq 2$ . For  $k \geq 3$ , we again use figure 2. The first point of  $S_n$  in  $T$  is the first point in  $T_1$ . The points in  $S_n \cap T_1$  are  $\{(x_i, y_i), 1 \leq i \leq n/4\}$ , with identical indices. They form a half-scale image of  $S'_{n/4}$ , and inductively the first point among them is  $(x_1, y_1)$ .

The last point of  $S_n$  in  $T$  is the last point in  $T_3$ . The points in  $S_n \cap T_3$  are a half-scale reversed image of  $S_{n/2}$ , label them  $\{(x'_i, y'_i) = (x_{(n/2)+i}, y_{(n/2)+i}), 1 \leq i \leq n/2\}$ . Since the order is reversed, the last point visited in  $T_3$  corresponds to the *first* point in  $S_{n/2}$ . Inductively the first point visited in  $S_{n/2}$  would be  $(x'_1, y'_1)$ ; this is point  $(x_{(n/2)+1}, y_{(n/2)+1})$  of  $S_n$ .  $\square$

Now we find the jumps distances  $j_{12}$  and  $j_{23}$ . For  $k \geq 3$ , we may apply the lemma to the find the first and last points of  $S_n$  in  $T_1$ ,  $T_2$ , and  $T_3$ . The last point in  $T_1$  has index  $i = n/8 + 1$ , and the first point in  $T_2$  (since it is reversed) has index  $3n/8$ . Similarly the last point in  $T_2$  has index  $i = n/2$ , and the first point in  $T_3$  has index  $3n/4 + 1$ . Hence  $j_{12} = (n/4 - 1)(2/3n) = 1/6 - 2/3n$ ,  $j_{23} = (n/4 + 1)(2/3n) = 1/6 + 2/3n$ , so for  $k \geq 3$  we have the simple recurrence:

$$a_k = \frac{1}{2}(a_{k-1} + a_{k-2}) + \frac{1}{3}.$$

From the base cases  $a_1 = 1/3$ ,  $a_2 = 2/3$  follows that  $a_k = 2k/9 + (4/27)(-1/2)^k + 5/27$ . Finally we have  $L^\phi(S_n) = a_k + 1/3 > 2k/9$  as claimed.  $\square$

### 3 Application to the PTSP

Bertsimas [2] applies the spacefilling curve heuristic to the Euclidean Probabalistic Traveling Salesman Problem which is defined as follows: we are given a set  $S$  of  $n$  points and a probability  $p_i$  for each point  $(x_i, y_i) \in S$ . A random instance  $X \subset S$  is then generated by including each point  $(x_i, y_i)$  in  $X$  independently with probability  $p_i$ . For a tour  $\tau$  on  $S$ , let  $L^\tau(X)$  be the length of the tour generated by visiting the points of  $X$  in the order followed in  $\tau$ . Suppose tour  $\sigma$  minimizes  $E[L^\sigma(X)]$ ; the goal is to choose tour  $\tau$  to get  $E[L^\tau(X)]$  close to  $E[L^\sigma(X)]$ . Let  $L^\phi(X)$  be the length when  $\tau$  was chosen by the spacefilling heuristic. Then by a similar analysis as in section 2 we may show that in the worst case,  $E[L^\phi(X)]/E[L^\sigma(X)] = \Theta(\lg n)$ .

## 4 Concluding Remarks

Similar examples hold for other curves in the unit square, in particular the Hilbert curve (figure 3(a)) and ‘zig-zag’ curve (figure 3(b)) mentioned in [4]. For the Hilbert curve, take  $n$  points along the diagonal line  $x + y = 2/3$ . For the zig-zag curve, take points uniformly along the horizontal line  $y = 1/2$ . Numerical experiments with the curve  $\phi$  strongly suggest that  $n$  points uniformly spaced on a ‘random’ line across the unit square have expected tour length  $E[L^\phi] = \Theta(\lg n)$ .

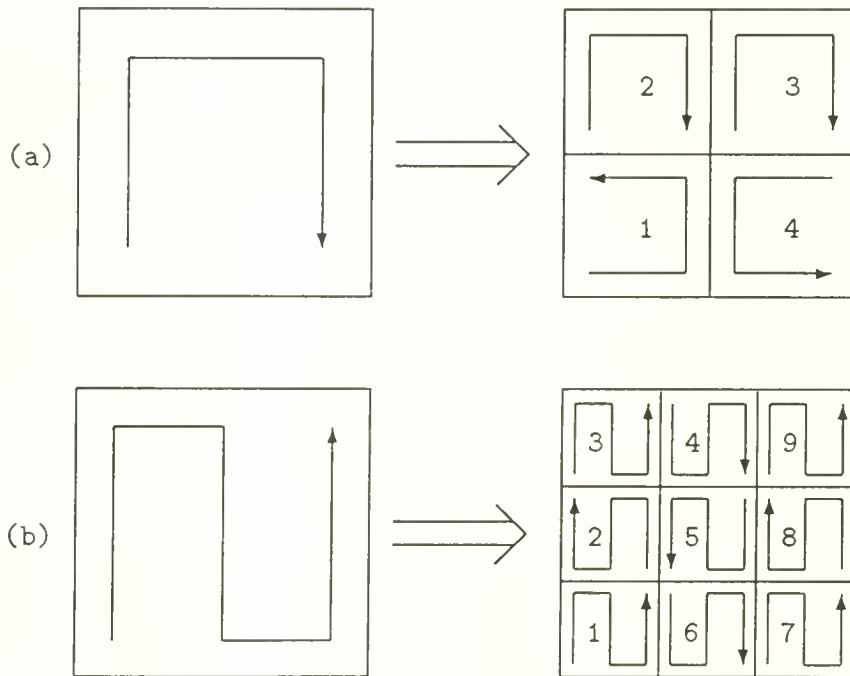


Figure 3: Recursively define the Hilbert curve (a) and the zig-zag curve (b) as several copies of themselves. The numbers indicate the relative order of the subcurves.

We conjecture that this is true for ‘all’ spacefilling curves. More concretely, we conjecture that for any ordering of the vertices of the  $n \times n$  mesh, there is some subset of vertices such that the length of their induced tour is an  $\Omega(\lg n)$  factor off from optimal.

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