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Abstract

In Part II of this study, the basic theory of Part I is applied to the problem of finding the w-center of a polyhedral system $\mathcal{X}$. We present a projective transformation algorithm, analagous but more general than Karmarkar's algorithm, for finding the w-center of $\mathcal{X}$. The algorithm exhibits superlinear convergence. At each iteration, the algorithm either improves the objective function (the weighted logarithmic barrier function) by a fixed amount, or at a linear rate of improvement. This linear rate of improvement increases to unity, and so the algorithm is superlinearly convergent. The algorithm also updates an upper bound on the optimal objective value of the weighted logarithmic barrier function at each iteration. The direction chosen at each iteration is shown to be positively proportional to the projected Newton direction. This has two consequences. On the theoretical side, this broadens a result of Bayer and Lagarias regarding the connection between projective transformation methods and Newton's method. In terms of algorithms it means that our algorithm specializes to Vaidya's algorithm if it is used with a line search, and so we see that Vaidya's algorithm is superlinearly convergent as well. Finally, we show how to use the algorithm to construct well-scaled containing and contained ellipsoids centered at near-optimal solutions to the w-center problem. After a fixed number of iterations, the current iterate of the algorithm can be used as an approximate w-center, and one can easily construct well-scaled containing and contained ellipsoids centered at the current iterate, whose scale factor is of the same order as for the w-center itself.

Keywords: analytic center, w-center, projective transformation, Newton method, ellipsoid, linear program.

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Abbreviated Title: Projective Transformations, Part II.

I. Introduction

Part II of this study uses the projective-centering and the local improvement methodologies developed in Part I [3] , to develop an algorithm for finding the w-center of a polyhedral system, which is the solution $\hat{x}$ to the problem

$$P_w: \text{ maximize } F(x) = \sum_{i=1}^{m} w_i \ln (b_i - A_i x)$$

$$\text{subject to} \quad Ax + s = b \qquad (1.1)$$

$$s > 0$$

$$Mx = g$$

where $w = (w_1, \ldots, w_m)$ are positive weights that are normalized to $\sum_{i=1}^{m} w_i = 1$, and $F(\cdot)$ is a (weighted) logarithmic barrier function. Problem P_w is a generalization of the analytic center problem defined by Sonnevend [11], [12] , for nonuniform positive weights on all constraints. This problem has had numerous applications in mathematical programming, see Renegar [10], Gonzaga [5], and Monteiro and Adler [8], [9], among others.

If there exists a point $x \in \mathcal{X} = \{x \in R^n \mid Ax \leq b, Mx = g\}$ for which $Ax < b$, and if $\mathcal{X}$ is bounded , then P_w will have a unique solution $\hat{x}$, called the w-center of $\mathcal{X}$. To be more precise, we should say that $\hat{x}$ is the w-center of $\mathcal{X}$ (A, b, M, g), since the solution to P_w is dependent on the particular polyhedral representation of $\mathcal{X}$. However, as in Part I of this study, we will refer to $\hat{x}$ as the w-center of $\mathcal{X}$, where it is understood that $\mathcal{X}$ represents a specific intersection of half-space and hyperplanes.

For the case when all weights w_i are identical, there are two other algorithms known to this author that have been developed for the w-center problem. Vaidya [14] has developed an algorithm that constructs the Newton-direction from the current iterate and then performs an inexact line search in this direction. He shows that at each iteration, there is either constant improvement in F(x) or a linear rate of improvement in F(x), and so his algorithm exhibits linear convergence. Censor and Lent [2] present a primal-dual algorithm for finding the center, that is convergent to $\hat{x}$, but not necessarily in any strong sense.

The algorithm developed in this paper is based on the use of projective transformation methods to w-center a given point, as in Section IV of Part I. At each iteration the polyhedron $\mathcal{X} = \{x \in R^n \mid Ax \le b, Mx = g\}$ is projectively transformed to a polyhedron $\mathcal{Z}$ so that the current point $\bar{x}$ is the w-center of $\mathcal{Z}$. A search direction $\bar{d}$ is then determined. This direction is used first to test if P_w is bounded or unbounded. If it is bounded, it will then produce an (updated) upper bound on the value of $F(\hat{x})$. A steplength α is then computed.

The algorithm can be run with a steplength α determined analytically at each iteration, or by the use of a line search. The new point x_{NEW} is then determined by projectively transforming the point $\bar{x} + \alpha \bar{d}$ back from $\mathcal{Z}$ to $\mathcal{X}$. At each iteration, there is either a constant improvement in F(x) , or a linear rate of improvement in F(x). However, because the linear rate of improvement approaches one in the limit, the algorithm exhibits superlinear convergence. The search direction determined at each iteration of the algorithm is positively proportional to the projected Newton direction. If the steplength is chosen at each iteration by a line search, the algorithm specializes to Vaidya's algorithm (with equal weights), and this shows (obliquely) that Vaidya's algorithm exhibits superlinear convergence, verifying a conjecture of Vaidya [15] that his algorithm might exhibit stronger convergence properties.

As was shown in Section II of Part I, at the w-center $\hat{x}$ of $\mathcal{X}$ one can construct an inner ellipsoid E_{IN} and an outer ellipsoid E_{OUT} with property that $E_{IN} \subset \mathcal{X} \subset E_{OUT}$, $\hat{x}$ is the center of E_{IN} and E_{OUT} , and $(E_{OUT} - \hat{x}) = ((1 - \bar{w})/\bar{w})(E_{IN} - \hat{x})$, where $\bar{w} = \min_i \{w_i\}$. This ratio is $(m - 1)$ when all weights are identical. Thus $\hat{x}$ is in a sense a w-balanced point of $\mathcal{X}$. Although the algorithm we present converges to the w-center $\hat{x}$ of $\mathcal{X}$, it may never reach the w-center. However, after a fixed number of iterations it will exhibit a point $\bar{x}$ "close enough" to the w-center, in the following sense: at the point $\bar{x}$, one can easily construct ellipsoids F_{IN} and F_{OUT}, with the property that $F_{IN} \subset \mathcal{X} \subset F_{OUT}$, and $(F_{OUT} - \bar{x}) = (1.75/\bar{w} + 5)\ (F_{IN} - \bar{x})$. When all weights are identical, then this ratio is $(1.75m + 5)$ which is $O(m)$. In general, the order of this ratio is $O(1/\bar{w})$, which is the same as for E_{IN} and E_{OUT} .

The paper is organized as follows: Section II presents the projective transformation algorithm for solving the w-center problem P_w . In Section III, we prove that the optimality tests and unboundedness tests in the algorithm are valid. In Section IV, we prove that the objective value bounds produced by the algorithm at each iteration are valid, and prove that the algorithm is linearly convergent in F(x). In Section V, we show that the algorithm exhibits superlinear convergence. In Section VI we show the relationship between the algorithm and Vaidya's algorithm [14] , by showing that the direction $\bar{d}$ determined at each step is positively proportional to a projected Newton direction. In SectionVII, we show that after a fixed number of iterations, one can easily construct ellipsoids F_{IN} and F_{OUT} about the current iterate $\bar{x}$ with the property that $F_{IN} \subset \mathcal{X} \subset F_{OUT}$, and $(F_{OUT} - \bar{x}) = (1.75/\overline{w} + 5)\ (F_{IN} - \bar{x})$. Section VIII contains closing remarks.

II. A Projective Transformation Algorithm for Finding the w-center of $\mathcal{X}$.

The notation and conventions used here are exactly the same as in Part I of this study [3]. We assume the reader is familiar with these notation and conventions. We also will cite many of the results presented in Part I.

Let the given data (A, b, M, g) define the polyhedral system $\mathcal{X} = \{x \in R^n \mid Ax \le b, Mx = g\}$. Let $w \in R^m$ be a given vector of weights satisfying $w > 0$ and normalized so that $e^T w = 1$, where $e = (1, \ldots, 1)^T$. Our interest lies in solving the w-center problem P_w given in (1.1). We make the following assumptions regarding the data:

(2.1a) The matrix A is $m \times n$ and has rank n.

(2.1)

(2.1b) The matrix M is $k \times n$ and has rank k.

These assumptions are for convenience, and if A or M lacks full rank, then one can either eliminate variables or constraints, or one can replace certain matrix inverse operations with pseudoinverse operations in the analysis.

Exactly in the spirit of Karmarkar's algorithm and consistent with the local improvement algorithm of Section V of Part I, we have the following algorithm for solving P_w. Let the data for the problem be $[w, A, b, M, g, \bar{x}, \varepsilon]$. Here w is the vector of weights satisfying $w > 0$ and $e^T w = 1$, (A, b, M, g) are the data for the polyhedral system $\mathcal{X}$, $\bar{x}$ is the starting point, which must satisfy $\bar{s} = b - A\bar{x} > 0$ and $M\bar{x} = g$, and $\varepsilon > 0$ is the optimality tolerance.

Let $\bar{w} = \min_i \{w_i\}$. Because we will use the quantity $\bar{w}/(1-\bar{w})$ extensively throughout the description of the algorithm and the subsequent analysis, we define the constant $k = \bar{w}/(1-\bar{w})$ for convenience. Also, in the algorithm, F^* is an upper bound on the optimal objective value of P_w.

Algorithm for the W-Center Problem

Step 0 Set $\bar{w} = \min_i \{w_i\}$. Set $k = (\bar{w}/(1-\bar{w}))$. Set $F^* = +\infty$.

Step 1 Set $\bar{s} = b - A\bar{x}$, $y = A^T \bar{S}^{-1} w$.

Step 2 (Projective Transformation of constraints) $\tilde{A} = A - \bar{s}y^T$

Step 3 (Compute direction in $\mathcal{Z}$ space)

Let $\bar{d}$ be the solution to the problem (P3) :

$$\begin{aligned} \underset{d}{\text{maximize}} \quad & -y^T d \\ \text{subject to} \quad & d^T \tilde{A}\, \bar{S}^{-1} W \bar{S}^{-1} \tilde{A} d \le k \\ & Md = 0 \end{aligned}$$

If this problem is unbounded, then stop. P_w is unbounded.

If $y^T \bar{d} = 0$, stop. $\bar{x}$ solves P_w.

Step 4 (Perform Boundedness Tests and Update Upper Bound).

Set $\gamma = (-y^T \bar{d})/k$

If $\gamma \ge 1/k$, stop. Problem P_w is unbounded, and $\bar{d}$ is a ray of $\mathcal{X}$.

If $\gamma < 1$ then set $F^* = \min \{F^*, F(\bar{x}) + \gamma + \gamma^2/(2(1-\gamma))\}$

If $\gamma < .08567$ then set $F^* = \min \{F^*, F(\bar{x}) + .669k\gamma^2\}$

Step 5 (Compute Steplength)

Set $\alpha = 1 - \dfrac{1}{\sqrt{1+2\gamma}}$

Set $z_{NEW} = \bar{x} + \alpha \bar{d}$.

Step 6 (Transform back to original space $\mathcal{X}$)

$$x_{NEW} = \bar{x} + \frac{z_{NEW} - \bar{x}}{1 + y^T(z_{NEW} - \bar{x})}$$

Step 7 (Stopping Criterion) Set $\bar{x} \leftarrow x_{NEW}$. If $F^* - F(\bar{x}) \le \varepsilon$, stop.

Otherwise go to Step 1.

This algorithm has the following straightforward explanation. Note first that P_w is an instance of the canonical optimization problem (5.1) of Part I, namely

$$P_{q,p}: \underset{x,\,s}{\text{minimize}} \; F_{q,p}(x) = \ln(q - p^T x) - \sum_{i=1}^{m} w_i \ln(b_i - A_i x)$$

$$\begin{aligned} \text{subject to:} \qquad Ax + s &= b \\ s &> 0 \\ Mx &= g \\ p^T x &< q \end{aligned} \tag{2.2}$$

where $q = 1$, and $p = (0, \ldots, 0)^T$, and hence $F(x) = -F_{q,p}(x) = -F_{1,0}(x)$. Thus we can proceed with the local improvement algorithm presented in Section V of Part I. Let $\bar{x} \in \text{int}\, \mathcal{X}$ be given, let $\bar{s} = b - A\bar{x}$, and let $y = A^T \bar{S}^{-1} w$. Then we can projectively transform $\mathcal{X}$ to

$$\mathcal{Z} = \{z \in R^n \mid (A - \bar{s}y^T)z \le b - \bar{s}y^T \bar{x},\ Mz = g\}$$

with the function $z = g(x) = \bar{x} + \dfrac{x - \bar{x}}{1 - y^T(x - \bar{x})}$, as in (3.2), (3.3) and (3.4) of Part I.

Problem $P_{q,p} = P_{1,0}$ is transformed to

$$\text{minimize} \quad G(z) = \ln\left([1 - y^T \bar{x}] - [-y^T] z \right) - \sum_{i=1}^{m} w_i \ln t_i$$

$$\begin{aligned} \text{subject to} \quad (A - \bar{s}y^T)z + t &= b - \bar{s}y^T \bar{x} \qquad (2.3) \\ t &> 0 \\ Mz &= g \\ -y^T z &< 1 - y^T \bar{x} \end{aligned}$$

which is equivalent to (2.2) (and, of course (1.1)) according to Lemma 3.2 (ii) of Part I. Note that (2.3) is also an instance of problem (5.1) of Part I with A replaced by $\tilde{A} = A - \bar{s}y^T$, p replaced $-y$, q replaced by $1 - y^T \bar{x}$, etc. Because $\bar{x}$ is the w-center of $\mathcal{Z}$, then the direction $\bar{d}$ given by the solution to (5.3) of Part I (with A and p replaced by $\tilde{A}$ and $-y$) has the property that $\bar{x} + \bar{d}$ maximizes $-y^T z$ over

$z \in E_{IN} = \left\{ z \in R^n \mid Mz = g, (z - \bar{x})^T \tilde{A}^T \bar{S}^{-1} W \bar{S}^{-1} \tilde{A} (z - \bar{x}) \le \bar{w}/(1 - \bar{w}) \right\}$.

Let $\gamma = -y^T \bar{d}(1 - \bar{w})/\bar{w}$ as in (5.6) of Part I (note that $q - p^T \bar{x} = 1$). This is precisely the quantity γ defined in Step 4 of the algorithm. Let $\alpha = 1 - 1/\sqrt{1 + 2\gamma}$, as in Step 5 of the algorithm, and $z_{NEW} = \bar{x} + \alpha \bar{d}$. Then by Corollary 5.1 of Part I, $G(z_{NEW}) = G(\bar{x} + \alpha \bar{d}) \le G(\bar{x}) - (\bar{w}/(1 - \bar{w}))\left(1 + \gamma - \sqrt{1 + 2\gamma}\right)$ if $\alpha = 1 - 1/\sqrt{1 + 2\gamma}$. Projectively transforming back to $\mathcal{X}$ space using the inverse of $g(\cdot)$, namely $x = h(z) = \bar{x} + \dfrac{z - \bar{x}}{1 + y^T(z - \bar{x})}$ (see (3.4) of Part I), we obtain

$x_{NEW} = \bar{x} + \dfrac{z_{NEW} - \bar{x}}{1 + y^T(z_{NEW} - \bar{x})}$ as in Step 6 of the algorithm, and

$$F(x_{NEW}) - F(\bar{x}) = F_{1,0}(\bar{x}) - F_{1,0}(x_{NEW}) \ge (\bar{w}/(1 - \bar{w}))\left(1 + \gamma - \sqrt{1 + 2\gamma}\right), \qquad (2.4)$$

from Lemma 3.2 (ii) of Part I.

In particular, whenever γ is greater than or equal to a given constant, and we will use $\gamma \geq .08567$, then we have

$$F(x_{NEW}) - F(\bar{x}) \geq (\bar{w}/(1-\bar{w}))(.0033) \qquad (2.5)$$

Before proceeding with the analysis and verification of the algorithm, we make the following remarks.

Remark 2.1. Use of a line search. Steps 5 and 6 of the algorithm can be replaced by a line search. Because the projective transformation $g(\cdot)$ ((3.3) of Part I) preserves directions relative to $\bar{x}$, the line search can be performed in the space $\mathcal{X}$ directly. Specifically, one needs to find a value of δ that nearly maximizes $F(\bar{x} + \delta \bar{d})$ over $\delta \geq 0$ and $\bar{x} + \delta \bar{d}$ feasible. Because $F(x)$ is strictly concave over $x \in \mathcal{X}$, then there will be at most one maximizer of $F(\bar{x} + \delta \bar{d})$ over the feasible range of δ. One could start the line search with $\delta = \dfrac{\alpha}{1 + \alpha y^T \bar{d}}$, where $\alpha = 1 - 1/\sqrt{1 + 2\gamma}$ which corresponds to a step of size $\alpha = 1 - 1/\sqrt{1 + 2\gamma}$ in the projectively transformed space $\mathcal{Z}$.

Remark 2.2. Efficient Computation of $\bar{d}$ in Step 3. As in the linear programming algorithm of Part I, one can compute $\bar{d}$ without working with the possible very dense matrices $\tilde{A} = A - \bar{s}y^T$ or $\tilde{Q} = \tilde{A}\,\bar{S}^{-1}W\,\bar{S}^{-1}\,\tilde{A}$. The discussion of this procedure is deferred to Section VI, where we show that $\bar{d}$ is proportional to the projected Newton direction.

III. Optimality Test and Unboundedness Test

In this section, we show that the optimality test of Step 3 of the algorithm is valid, and that the unboundedness tests of Steps 3 and 4 are valid.

Proposition 3.1. If the algorithm terminates in Step 3 because optimization problem (P3) is unbounded, then P_w is unbounded.

Proof: Suppose that at Step 3, that (P3) is unbounded. That means there exists a ray $r \in R^n$ such that $y^T r = -1$, $Mr = 0$, and $\tilde{A}r = 0$. But $\tilde{A}r = 0$ implies $Ar = \bar{s}y^T r = -\bar{s} < 0$. Thus r is a ray of $\mathcal{X}$, and $F(\bar{x} + \theta r) = \sum_{i=1}^{m} w_i \ln(\bar{s}_i (1+\theta)) \rightarrow +\infty$ as $\theta \rightarrow \infty$. ■

Proposition 3.2. If the algorithm terminates at Step 3 with $y^T \bar{d} = 0$, then $\bar{x}$ solves P_w.

Proof: The solution $\bar{d}$ will satisfy the following optimality conditions:

$-y = 2\,\bar{\beta}\tilde{A}^T \bar{S}^{-1} W \bar{S}^{-1} \tilde{A}\,\bar{d} - \bar{\pi}^T M$, where $\bar{\beta} \geq 0$ and $\bar{\beta}(k - \bar{d}^T \tilde{A}^T \bar{S}^{-1} W \bar{S}^{-1} \tilde{A}\,\bar{d}) = 0$. We also obtain $\bar{\beta}\,\bar{d}^T \tilde{A}^T \bar{S}^{-1} W \bar{S}^{-1} \tilde{A}\,\bar{d} = (1/2)(-y^T \bar{d} + \bar{\pi}^T M \bar{d}) = (1/2)(-y^T \bar{d}) = 0$, so that $\bar{\beta} = 0$ or $\tilde{A}\,\bar{d} = 0$. In either case, $y = \bar{\pi}^T M$, i.e. $w^T \bar{S}^{-1} A = \bar{\pi}^T M$ Thus from (2.1) of Part I, $\bar{x}$ solves P_w. ■

Proposition 3.3. If the optimization problem (P3) in Step 3 has a solution with a nonzero optimal value, then that solution is unique.

Proof: If not, then let $r = d^1 - d^2$ where d^1, d^2 solve the optimization problem. Then it is straightforward to show that $\tilde{A}r = 0$ and $Mr = 0$. Because $\tilde{A} = A - \bar{s}y^T$, then $Ar = \bar{s}y^T r$, and $y^T r \neq 0$, for otherwise A would not have rank n. However, $y^T r = y^T d^1 - y^T d^2 = 0$ because d^1 and d^2 are both optimal solutions, which is a contradiction. ■

Proposition 3.4. If Algorithm 3 stops in Step 4, then P_w is unbounded, and $\bar{d}$ is a ray of $\mathcal{X}$.

Proof: From Theorem 4.1 of Part I, $\bar{x}$ is the w-center of $\mathcal{Z}$ as defined in (3.2) of Part I, and $\bar{x} + \bar{d}$ lies in the inner ellipsoid E_{IN} for $\mathcal{Z}$ as given by Theorem 2.1 of Part I. Thus $\bar{x} + \alpha \bar{d} \in \mathcal{Z}$ for all $\alpha \in [-1, 1]$. In particular, let $\alpha = 1/(k\gamma)$. Then if Algorithm 3 stops in Step 4, $\alpha \le 1$ and $\alpha > 0$, whereby $z = \bar{x} + \alpha \bar{d} \in \mathcal{Z}$.

Because $z \in \mathcal{Z}$, $(A - \bar{s}y^T)z \le b - \bar{s}y^T \bar{x}$, so that

$$Az \le \bar{s}y^T z + b - \bar{s}y^T \bar{x} .$$

However, we also have $y^T z = y^T \bar{x} + \alpha y^T \bar{d} = y^T \bar{x} + \dfrac{y^T \bar{d}}{k\gamma} = y^T \bar{x} - 1$. Thus

$$Az \le \bar{s}(y^T \bar{x} - 1) + b - \bar{s}y^T \bar{x} = b - \bar{s} = A\bar{x} .$$

Thus $A(z - \bar{x}) \le 0$. But $z - \bar{x}$ is a positive scalar multiple of $\bar{d}$, so that $A\bar{d} \le 0$. Furthermore $\bar{d} \ne 0$, for otherwise the algorithm would have stopped in Step 3 (see Proposition 3.2). Next, observe that $M\bar{d} = 0$, so that $\bar{d}$ is a ray of $\mathcal{X}$. Because A has full rank, (see (2.1)), $A\bar{d} \le 0$ and $A\bar{d} \ne 0$. Thus $F(\bar{x} + \theta \bar{d}) \to +\infty$ as $\theta \to \infty$ and P_w is unbounded. ■

Section IV. Linear Convergence and Improved Optimal Objective Value Bounds

The purpose of this section is to establish the following three results regarding the algorithm for the w-center problem:

Lemma 4.1. (Optimal Objective Value Bounds) At Step 4 of the algorithm ,

(i) if $\gamma < 1$, then P_w has a unique optimal solution $\hat{x}$, and

$$F(\hat{x}) \leq F(\bar{x}) + \gamma + \frac{\gamma^2}{2(1-\gamma)}.$$

(ii) if $\gamma \leq .08567$, then $F(\hat{x}) \leq F(\bar{x}) + .669\, k\gamma^2$.

Note that Lemma 4.1 validates the upper bounding procedure presented in Step 4 of the algorithm.

Lemma 4.2 (Local Improvement). At Step 6 of the algorithm,

(i) if $\gamma \geq .08567$, $F(x_{NEW}) \geq F(\bar{x}) + (.0033)\, k$.

(ii) if $\gamma \leq .08567$, $F(x_{NEW}) - F(\bar{x}) \geq .4612\, k\gamma^2$.

Lemma 4.3 (Linear Convergence). At each iteration, at least one of the following is true:

(i) $F(x_{NEW}) \geq F(\bar{x}) + (.0033)\, k$.

(ii) $F(\hat{x}) - F(x_{NEW}) \leq .32\,(F(\hat{x}) - F(\bar{x}))$, where $\hat{x}$ is the w-center of $\mathcal{X}$.

In SectionV, we will show a result that is stronger than Lemma 4.3, namely that as the iterates $\bar{x}$ converge to $\hat{x}$, that the constant .32 in Lemma 4.3 (ii) will go to zero, thus establishing superlinear convergence.

Note that Lemma 4.3 is an immediate consequence of Lemma 4.1 and 4.2. Lemma 4.3(i) is a restatement of Lemma 4.2(i). To prove Lemma 4.3(ii), note that if $\gamma \leq .08567$, then from Lemma 4.1(ii) and 4.2(ii),

$$\frac{F(x_{NEW}) - F(\bar{x})}{F(\hat{x}) - F(\bar{x})} \geq \frac{.4612}{.669} \geq .68\,,$$

and so

$$\frac{F(\hat{x}) - F(x_{NEW})}{F(\hat{x}) - F(\bar{x})} = 1 - \frac{F(x_{NEW}) - F(\bar{x})}{F(\hat{x}) - F(\bar{x})} \leq 1 - .68 = .32\,. \blacksquare$$

We thus need to prove Lemmas 4.1 and 4.2. We start by asserting some elementary inequalities.

Proposition 4.1. (Inequalities)

a) $\ln(1 + x) \leq x - \left(\frac{(h - \ln(1 + h))}{h^2}\right)x^2$ whenever $-1 < x \leq h$.

b) $\ln(1 + x) \leq x - .378\,x^2$ whenever $-1 \leq x \leq .5$.

c) $1 + \gamma - \sqrt{1 + 2\gamma} \geq \left[\left(1 + \theta - \sqrt{1 + 2\theta}\right)/\theta^2\right]\gamma^2$ whenever $0 \leq \gamma \leq \theta$.

d) $1 + \gamma - \sqrt{1 + 2\gamma} \geq .4612\,\gamma^2$ whenever $0 \leq \gamma \leq .08567$.

Proof: (a) follows from the fact that $\left[h - \ln(1 + h)\right]/h^2$ is decreasing in h for $h > -1$. (b) follows from (a) by substituting $h = .5$. (c) follows from the fact that $\left(1 + \theta - \sqrt{1 + 2\theta}\right)/\theta^2$ is decreasing in θ for $\theta \geq 0$. (d) follows from (c) by substituting $\theta = .08567$. $\blacksquare$

We now will prove Lemma 4.2, followed by Lemma 4.1(i) and Lemma 4.1(ii).

Proof of Lemma 4.2.: Statement (i) is a restatement of inequality (2.5). According to (2.4), $F(x_{NEW}) - F(\bar{x}) \geq k\left(1 + \gamma - \sqrt{1 + 2\gamma}\right)$. Let $\theta = .08567$. Then according to Proposition 4.1(d), $1 + \gamma - \sqrt{1 + 2\gamma} \geq .4612\,\gamma^2$ for $0 \leq \gamma \leq .08567$, which proves statement (ii). ∎

Proof of Lemma 4.1.(i): Let $\bar{x}$ be the current point, and let $y = A^T \bar{S}^{-1} w$. For any $x \in \mathcal{X}$, let $z = g(x)$ where $g(\cdot)$ is the projective transformation given in (3.3) of Part I, and let $\mathcal{Z}$ be given in (3.2) of Part I. Then P_w is equivalent to the problem (2.3):

$$\text{minimize } G(z) = \ln(1 - y^T\bar{x} + y^T z) - \sum_{i=1}^{m} w_i \ln t_i$$

$$\begin{aligned} \text{subject to} \quad & (A - \bar{s}y^T)z + t = b - \bar{s}y^T\bar{x} \\ & t > 0 \\ & Mz = g \\ & -y^T z < 1 - y^T\bar{x} \end{aligned}$$

and by the remarks following (2.2) and (2.3), and lemma 3.2(ii) of Part I, $F(x) = -F_{1,0}(x) = -G(z)$. It thus suffices to show that if $\gamma < 1$, then $G(z) \geq G(\bar{x}) - \gamma - \gamma^2 / (2(1-\gamma))$. By construction of y, from Theorem 4.1 of Part I we know that $\bar{x}$ is the w-center of $\mathcal{Z}$. Thus for any $z \in \mathcal{Z}$, $-\sum_{i=1}^{m} w_i \ln t_i \geq -\sum_{i=1}^{m} w_i \ln \bar{s}_i = G(\bar{x})$, where t is the slack corresponding to z in $\mathcal{Z}$.

We now must show $\ln(1 - y^T\bar{x} + y^T z) \geq -\gamma - \gamma^2 / (2(1-\gamma))$. To see this, note that because $\bar{x} + \bar{d}$ maximizes $-y^T z$ over $z \in E_{IN}$, then $\bar{x} + \bar{d}/k$ maximizes $-y^T z$ over $z \in E_{OUT}$, and so for any $z \in \mathcal{Z} \subset E_{OUT}$, $-y^T z \leq y^T\bar{x} - y^T\bar{d}/k = -y^T\bar{x} + \gamma$. Thus $\ln(1 - y^T\bar{x} + y^T z) \geq \ln(1-\gamma) \geq -\gamma - \gamma^2 / (2(1-\gamma))$, from Proposition 2.5 of Part I. This proves statement (i) of Lemma 4.1 . ∎

The proof statement (ii) of Lemma 4.1 is very involved, and follows from the following sequence of lemmas:

Lemma 4.4. Let $h > 0$ be a given parameter. Let $\bar{x}$ be the w-center of $\mathcal{X}$, let $\bar{s} = b - A\bar{x}$, and suppose $\hat{x} \in \mathcal{X}$ satisfies

$$(\hat{x} - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A(\hat{x} - \bar{x}) = \beta^2 .$$

Then

$$\sum_{i=1}^{m} w_i \ln (b_i - A_i\hat{x}) - \sum_{i=1}^{m} w_i \ln \bar{s}_i \le \begin{cases} -\dfrac{h - \ln(1+h)}{h^2}\beta^2 & \text{if } \beta \le h\sqrt{k} \\ -\dfrac{h - \ln(1+h)}{h^2} h\sqrt{k}\,\beta & \text{if } \beta \ge h\sqrt{k} \end{cases} .$$

Proof: First we observe that

$$\sum_{i=1}^{m} w_i \ln (b_i - A_i\hat{x}) - \sum_{i=1}^{m} w_i \ln (\bar{s}_i) = \sum_{i=1}^{m} w_i \ln (1 + r_i) \text{ , where}$$

$r = -\bar{S}^{-1}A(\hat{x} - \bar{x})$. Then note that $w^T r = -w^T \bar{S}^{-1}A(\hat{x} - \bar{x}) = \pi^T M(\hat{x} - \bar{x}) = 0$ for some $\pi = R^k$, from (2.1d) of Part I. Also, $r^T W r = \beta^2$. Then

$(r\sqrt{k} / \beta)^T W(r\sqrt{k} / \beta) = k$, and by Proposition 2.2 of Part I, $|r_i| \le \beta / \sqrt{k}$, $i = 1, \ldots, m$.

We now prove the two cases of the Lemma.

Case 1. $(\beta \le h\sqrt{k})$. Then $|r_i| \le h$. From Proposition 4.1(a), $\ln(1 + r_i) \le r_i - \dfrac{(h - \ln(1+h))}{h^2} r_i^2$, $i = 1, \ldots m$. Thus $\sum_{i=1}^{m} w_i \ln (1 + r_i) \le$
$w^T r - \dfrac{(h - \ln(1+h))}{h^2} r^T W r = -\dfrac{(h - \ln(1+h))}{h^2} \beta^2$.

Case 2. $(\beta \geq h\sqrt{k})$. Because $|r_i| \leq \beta\sqrt{k}$, then $r_i\sqrt{k}\; h/\beta \leq h$, and so again by Proposition 4.1(a), $\ln(1 + r_i\sqrt{k}\; h/\beta) \leq r_i\sqrt{k}\; h/\beta - \frac{(h - \ln(1+h))}{h^2} r_i^2\, kh^2/\beta^2$,

$i = 1, \ldots, m$. Thus $\sum_{i=1}^{m} w_i \ln(1 + r_i\sqrt{k}\, h/\beta) \leq -\frac{(h - \ln(1+h))}{h^2} kh^2$, because

$w^T r = 0$ and $r^T W r = \beta^2$. However by the concavity of the log function,

$$(\sqrt{k}\, h/\beta)\sum_{i=1}^{m} w_i \ln(1 + r_i) = (\sqrt{k}\, h/\beta)\sum_{i=1}^{m} w_i \ln(1 + r_i) + (1 - \sqrt{k}\, h/\beta)\sum_{i=1}^{m} \ln(1)$$

$$\leq \sum_{i=1}^{m} w_i \ln((\sqrt{k}\, h/\beta)(1 + r_i) + (1 - \sqrt{k}\, h/\beta)) = \sum_{i=1}^{m} w_i \ln(1 + r_i\sqrt{k}\, h/\beta)$$

$$\leq -\frac{(h - \ln(1+h))}{h^2} kh^2.$$ Thus

$$\sum_{i=1}^{m} w_i \ln(1 + r_i) \leq -\frac{(h - \ln(1+h))}{h^2}\sqrt{k}\, h\beta. \blacksquare$$

Lemma 4.5. Let $\bar{x}$ be the current point in the w-center algorithm, let $\tilde{Q} = \tilde{A}^T \bar{S}^{-1} W \bar{S}^{-1} \tilde{A}$, where $\tilde{A} = A - \bar{s}y^T$ is defined in Step 2, and γ is defined as in Step 4. Suppose $\hat{x}$ is the optimal solution to P_w and $\hat{z} = g(\hat{x})$, where $g(\cdot)$ is the projective transformation given by (3.3) of Part I, and that $(\hat{z} - \bar{x})\tilde{Q}(\hat{z} - \bar{x}) = \beta^2$. If $\gamma < 1$, and $h > 0$ is a given parameter, then

$$F(\hat{x}) - F(\bar{x}) \leq \begin{cases} -\frac{(h - \ln(1+h))}{h^2}\beta^2 + \beta\sqrt{k}\gamma + \beta^2 k\gamma^2/(2(1-\gamma)) & \text{if } \beta \leq h\sqrt{k} \\ -\frac{(h - \ln(1+h))}{h^2} h\sqrt{k}\beta + \beta\sqrt{k}\gamma + \beta^2 k\gamma^2/(2(1-\gamma)) & \text{if } \beta \geq h\sqrt{k} \end{cases}.$$

Proof: According to Lemma 3.2(ii) of Part I, $F(\hat{x}) - F(\bar{x}) = G(\bar{x}) - G(\hat{z})$, where

$$G(z) = \ln(1 - y^T\bar{x} + y^T z) - \sum_{i=1}^{m} w_i \ln(\tilde{b}_i - \tilde{A}_i z)$$

where $\tilde{A}$ is defined in Step 2 of the algorithm, and $\tilde{b} = b - \bar{s}y^T \bar{x}$. Noting that $G(\bar{x}) = -F(\bar{x}) = -\sum_{i=1}^{m} w_i \ln \bar{s}_i$, then

$$G(\bar{x}) - G(\hat{z}) = -\ln(1 + y^T(\hat{z} - \bar{x})) + \sum_{i=1}^{m} w_i \ln(\tilde{b}_i - \tilde{A}_i\hat{z}) - \sum_{i=1}^{m} w_i \ln \bar{s}_i \quad (4.1)$$

However, because $\bar{x}$ is the w-center of $\mathcal{Z}$ (defined in (3.2) of Part I), then by Lemma 4.4,

$$\sum_{i=1}^{m} w_i \ln(\tilde{b}_i - \tilde{A}_i\hat{z}) - \sum_{i=1}^{m} w_i \ln \bar{s}_i \le \begin{cases} -\dfrac{(h - \ln(1+h))}{h^2}\beta^2 & \text{if } \beta \le h\sqrt{k} \\ -\dfrac{(h - \ln(1+h))}{h^2} h\sqrt{k}\beta & \text{if } \beta \ge h\sqrt{k} \end{cases}$$

It thus remains to show that $-\ln(1 + y^T(\hat{z} - \bar{x})) \le \beta\sqrt{k}\gamma + \dfrac{\beta^2 k\gamma^2}{2(1-\gamma)}$.

Let $\hat{d} = (\hat{z} - \bar{x})$. The vector $\bar{d}$ in Step 3 of the algorithm is that vector that maximizes $-y^T(\bar{x} + z)$ over all $z \in E_{IN}$, as discussed in SectionV of Part I, where $E_{IN} = \left\{z \in R^n \mid Mz = g,\ (z - \bar{x})^T \tilde{Q}(z - \bar{x}) \le k\right\}$. Thus $-y^T(\bar{x} + \bar{d}) \ge -y^T(\bar{x} + \hat{d}\sqrt{k}/\beta)$, and so $y^T\hat{d} \ge \beta y^T \bar{d}/\sqrt{k}$. However, because $\gamma = -y^T \bar{d}/k$ then $y^T\hat{d} \ge -\beta\gamma\sqrt{k}$. Also, because $\mathcal{Z} \subset E_{OUT} = \left\{z \in R^n \mid Mz = g,\ (z - \bar{x})^T \tilde{Q}(z - \bar{x}) \le 1/k\right\}$ from Theorem 2.1 of Part I, we must have $\beta^2 \le 1/k$, and so $\beta\sqrt{k} \le 1$. Thus $\gamma\beta\sqrt{k} \le \gamma < 1$. This then implies

$-\ln(1 + y^T\hat{d}) \le -\ln(1 - \beta\gamma\sqrt{k}) \le +\beta\gamma\sqrt{k} + \dfrac{\beta^2\gamma^2 k}{2(1-\gamma)}$, the last inequality being an instance of Proposition 2.5 of Part I, with $\varepsilon = -\gamma\beta\sqrt{k}$ and $\alpha = \gamma$. ■

Lemma 4.6. Under the hypothesis of Lemma 4.5, if

$$\gamma < 1 - \frac{\ln(1+h)}{2h} - 1/2\sqrt{\left[\frac{\ln(1+h)}{h}\right]^2 + 2\left[1 - \frac{\ln(1+h)}{h}\right]} \qquad , \qquad (4.2)$$

then $\beta < h\sqrt{k}$.

Proof: Suppose $\beta \geq h\sqrt{k}$. Then from Lemma 4.5,

$F(\hat{x}) - F(\bar{x}) \leq f(\gamma, \beta)$, where

$$f(\gamma, \beta) = -\frac{(h - \ln(1+h))}{h^2} h\sqrt{k}\beta + \beta\sqrt{k}\gamma + \frac{\beta^2 k \gamma^2}{2(1-\gamma)}$$

Note $f(\gamma, \beta)$ increases in γ for $\beta \geq 0$ and $0 \leq \gamma < 1$. Straightforward calculation reveals that $f(\gamma, \beta) = 0$ if

$$\gamma = \frac{2 - \ln(1+h)/h - \sqrt{(\ln(1+h)/h)^2 + 2\beta\sqrt{k}\,(1 - \ln(1+h)/h)}}{2 - \beta\sqrt{k}} \qquad (4.3)$$

Thus if γ is less than the above quantity, then $f(\gamma, \beta) < 0$, contradicting the optimality of $\hat{x}$. Thus γ must be greater than or equal to the expression in (4.3). Next, borrowing the observation in the proof of Lemma 4.5 that $0 \leq \beta\sqrt{k} \leq 1$, then the expression in (4.3) is greater than or equal to

$$\frac{2 - \ln(1+h)/h - \sqrt{(\ln(1+h)/h)^2 + 2(1 - \ln(1+h)/h)}}{2}$$

which equals $1 - \frac{\ln(1+h)}{2h} - 1/2\sqrt{\left[\frac{\ln(1+h)}{h}\right]^2 + 2\left[1 - \frac{\ln(1+h)}{h}\right]}$.

Thus, if $\beta > h\sqrt{k}$, then $\gamma \geq 1 - \frac{\ln(1+h)}{2h} - 1/2\sqrt{\left[\frac{\ln(1+h)}{h}\right]^2 + 2\left[1 - \frac{\ln(1+h)}{h}\right]}$. ■

Proof of Statement (ii) of Lemma 4.1. : Let us set $h = .5$. Then the expression on the RHS of (4.2) is then greater than .08567. Thus if $\gamma < .08567$, then from Lemma 4.6 and Lemma 4.5, $\beta \le .5\sqrt{k}$, and

$F(\hat{x}) - F(\bar{x}) \le -.3781\,\beta^2 + \beta\sqrt{k}\,\gamma + \beta^2 k\gamma^2 / (2(1-\gamma))$. Let us define

$f(\beta) = -.3781\,\beta^2 + \sqrt{k}\,\gamma\beta + \beta^2 k\gamma^2 / (2(1-\gamma))$. The function $f(\beta)$ is quadratic in β, and because $k \le 1$ and $\gamma < .08567$, then $k\gamma^2 / (2(1-\gamma)) < .3781$, so that $f(\beta)$ is concave. Thus the largest value of $f(\beta)$ is given by $\bar{\beta} = \dfrac{\sqrt{k}\gamma}{.7562 - \dfrac{k\gamma^2}{(1-\gamma)}}$,

with $f(\beta) \le f(\bar{\beta}) = \dfrac{k\gamma^2}{1.5124 - \dfrac{2k\gamma^2}{(1-\gamma)}} \le \dfrac{k\gamma^2}{1.5124 - \dfrac{2\gamma^2}{(1-\gamma)}} \le .669k\gamma^2$ for $\gamma \le .08567$.

This completes the proof of Lemma 4.1 . ■

V. Superlinear Convergence

In the previous section, we showed linear convergence of the algorithm, with a linear convergence rate of .32, by choosing the value $h = .5$ and applying Lemmas 4.4, 4.5, and 4.6. In this section we show that as we choose $h > 0$ and arbitrarily close to zero, then the linear convergence rate goes to zero in the limit, thus showing that the algorithm is superlinearly convergent.

We first present some elementary facts about three particular functions.

Proposition 5.1.

$$\text{Let } f(h) = 1 - \frac{\ln(1+h)}{2h} - 1/2\sqrt{\left[\frac{\ln(1+h)}{h}\right]^2 + 2\left[1 - \frac{\ln(1+h)}{h}\right]} \quad \text{for } h > 0. \tag{5.1}$$

$$\text{Let } j(\theta) = \left[1 + \theta - \sqrt{1 + 2\theta}\,\right]/\theta^2 \qquad \text{for } \theta > 0 \tag{5.2}$$

Let $p(h) = \dfrac{(h - \ln(1 + h))}{h^2}$ for $h > 0$ (5.3)

Then $\lim_{h \to 0} f(h) = 0$, $\lim_{\theta \to 0} j(\theta) = .5$, and $\lim_{h \to 0} p(h) = .5$. ■

We then prove the following three propositions, after which the proof of superlinear convergence easily follows.

Proposition 5.2. For any $h > 0$, at Step 6 of the algorithm, if $\gamma \le f(h)$, then
$F(x_{NEW}) - F(\bar{x}) \ \ge \left[\dfrac{1 + f(h) - \sqrt{1 + 2f(h)}}{(f(h))^2}\right] k\gamma^2$.

Proof: $F(x_{NEW}) - F(\bar{x}) \ge k\left(1 + \gamma - \sqrt{1 + 2\gamma}\right)$ from (2.4). Now substituting $f(h)$ for θ in Proposition 4.1(c), we obtain the desired result. ■

Proposition 5.3. Suppose $h > 0$ and sufficiently small and $f(h)$ and $p(h)$ are defined as in Proposition 5.1. At Step 6 of the algorithm, if $\gamma \le f(h)$, then

$$F(\hat{x}) - F(\bar{x}) \ \le \ \frac{k\gamma^2}{4p(h) - \dfrac{2k(f(h))^2}{(1 - f(h))}} \ ,$$

where $\hat{x}$ is the optimal solution to P_w .

Proof: Because $f(h)$ is just the expression of the RHS of (4.2), we have by Lemma 4.6 and Lemma 4.5 that if $\gamma < f(h)$, then

$$F(\hat{x}) - F(\bar{x}) \ \le \ -p(h)\beta^2 \ + \ \beta\sqrt{k}\,\gamma \ + \ \beta^2 k\gamma^2/(2(1-\gamma)) \ . \qquad (5.4)$$

If h is sufficiently small $p(h)$ is approximately .5 from Proposition 5.1 and $k\gamma^2/(2(1-\gamma)) \le (f(h))^2/(2(1-\gamma)) \le (f(h))^2/(2(1 - f(h))$ is approximately zero. Thus the RHS of (5.4) is quadratic and concave in β. Its maximal value occurs at

$$\bar{\beta} = \frac{\sqrt{k}\,\gamma}{2p(h) - \dfrac{k\gamma^2}{(1-\gamma)}} ,$$

and the maximum value of the RHS in (5.4) is therefore

$$\frac{k\gamma^2}{4p(h) - \dfrac{2k\gamma^2}{(1-\gamma)}} .$$

However $\gamma \le f(h)$, so we have

$$F(\hat{x}) - F(\bar{x}) \le \frac{k\gamma^2}{4p(h) - \dfrac{2k(f(h))^2}{(1-f(h))}} . \quad \blacksquare$$

Proposition 5.4. For $h > 0$ and sufficiently small, if $\hat{x}$ is the optimal solution to P_W, then

i) if $\gamma \ge f(h)$, then $F(x_{NEW}) - F(\bar{x}) \ge \left(1 + f(h) - \sqrt{1 + 2f(h)}\right) k$

ii) if $\gamma \le f(h)$, then

$$\frac{F(\hat{x}) - F(x_{NEW})}{F(\hat{x}) - F(\bar{x})} \le 1 - \left[\frac{1 + f(h) - \sqrt{1 + 2f(h)}}{(f(h))^2}\right]\left[4p(h) - \frac{2k(f(h))^2}{(1 - f(h))}\right] . \quad (5.5$$

Proof: If $\gamma \ge f(h)$, then from (2.4), we know that $F(x_{NEW}) - F(\bar{x}) \ge \left(1 + \gamma - \sqrt{1 + 2\gamma}\right)k$ $\ge \left(1 + f(h) - \sqrt{1 + 2f(h)}\right) k$ since $1 + \theta - \sqrt{1 + 2\theta}$ is an increasing function of $\theta \ge 0$. Statement (ii) follows directly by combining Propositions 5.2 and 5.3. We have that if $\gamma \le f(h)$, then

$$\frac{F(\hat{x}) - F(x_{NEW})}{F(\hat{x}) - F(\bar{x})} = 1 - \frac{F(x_{NEW}) - F(\bar{x})}{F(\hat{x}) - F(\bar{x})} \leq 1 - \left(\frac{k\gamma^2 \left[\frac{1 + f(h) - \sqrt{1 + 2f(h)}}{(f(h))^2} \right]}{k\gamma^2 \left[\frac{1}{4p(h) - \frac{2k(f(h))^2}{(1 - f(h))}} \right]} \right).$$

Cancelling out $k\gamma^2$ and rearranging yields the desired result. ∎

Lemma 5.1. The algorithm for solving P_W exhibits superlinear convergence in F(x).

Proof: It suffices to show that as $h \to 0$, then the RHS of (5.5) goes to zero. As $h \to 0$, $f(h) \to 0$ from Propostion 5.1. Then note that with $\theta = f(h)$, $\theta \to 0$ as $h \to 0$ and $\frac{1 + \theta - \sqrt{1 + 2\theta}}{\theta^2} = j(\theta) \to .5$ as $\theta \to 0$, by Proposition 5.1. The last term of (5.5) is $4p(h) - 2k(f(h))^2/(1 - f(h))$. As $h \to 0$ $p(h) \to .5$ and $f(h) \to 0$ by Proposition 5.1. Thus the entire expression (5.5) approaches $1 - (.5)(4(.5) - 0) = 0$. ∎

VI. Analysis and Computation of the Improving Direction

In this section, we show that the direction $\bar{d}$ of Step 3 of the algorithm is a positively scaled projected Newton direction. As a byproduct of this result, the computation of $\bar{d}$ can be carried out without solving equations involving the matrix $\tilde{Q} = \tilde{A}^T \bar{S}^{-1} W \bar{S}^{-1} \tilde{A}$, which will typically be extremely dense. Vaidya's algorithm for the center problem [14] corresponds to computing the Newton direction and performing an inexact line search. Thus, our algorithm specializes to Vaidya's algorithm when our algorithm is implemented with a line search.

Furthermore, this establishes that Vaidya's algorithm then will exhibit superlinear convergence.

Let $\bar{x}$ be the current iterate of the algorithm, let $\bar{s} = b - A\bar{x}$, and $y = A^T \bar{S}^{-1}w$ and $\tilde{A} = A - \bar{s}y^T$ as in Steps 1 and 2 of the algorithm, and let $Q = A^T \bar{S}^{-1}W \bar{S}^{-1}A$, and $\tilde{Q} = \tilde{A}^T \bar{S}^{-1}W \bar{S}^{-1}\tilde{A}$. From (2.1a), A has full rank, so that Q is nonsingular and positive definite. Let $F(x)$ be the weighted logarithmic barrier function of P_w given in (1.1). Then the gradient of $F(\cdot)$ at $\bar{x}$ is given by $-y$, i.e., $\nabla F(\bar{x}) = -y$, and the Hessian of $F(\cdot)$ at $\bar{x}$ is given by $-Q$, i.e., $\nabla^2 F(\bar{x}) = -Q$.

Thus the projected Newton direction d_N is the optimal solution to

$$\begin{aligned} &\text{maximize } -y^T d - (1/2)\, d^T Q d \\ &\text{subject to } \ Md = 0 \end{aligned}$$

and the Newton direction d_N together with Lagrange multipliers π_N is the unique solution to

$$\begin{aligned} Qd_N - M^T\pi_N &= -y \\ Md_N &= 0 \end{aligned} \tag{6.1}$$

Because Q has rank n and M has rank k, we can write the solution to (6.1) as

$$\begin{aligned} d_N &= -Q^{-1}y + Q^{-1}M^T\pi_N \\ \text{where } \pi_N &= (MQ^{-1}M^T)^{-1}MQ^{-1}y \end{aligned} \tag{6.2}$$

It is our aim to show the following

Lemma 6.1. Let d_N be the Newton direction given by the solution (6.1) or (6.2). Then $1 + y^T d_N \geq 0$, and

(i) if $1 + y^T d_N > 0$, and $d_N \neq 0$, $\bar{d} = d_N \sqrt{k} / \left(\sqrt{d_N \tilde{Q} d_N}\right)$ is the direction of Step 3 of the algorithm.

(ii) If $1 + y^T d_N > 0$, and $d_N = 0$, then $\bar{d} = d_N = 0$ is the direction of Step 3 of the algorithm, and the current iterate $\bar{x}$ solves P_w.

(iii) if $1 + y^T d_N = 0$, then the optimization problem (P3) of Step 3 is unbounded, and P_w is unbounded.

Remark 6.1. Simplified Computation of $\bar{d}$. Lemma 6.1(i) shows that $\bar{d}$ is just a positive scale of the Newton direction d_N. Thus in order to solve for $\bar{d}$, one need not solve a system involving the possibly-very-dense matrix $\tilde{Q}$. Rather one need only solve the equations (6.1) for d_N and then compute $\bar{d} = d_N \sqrt{k} / \sqrt{d_N \tilde{Q} d_N}$.

Remark 6.2. Relation of Algorithm to Vaidya's algorithm. Lemma 6.1(i) shows that $\bar{d}$ is just a positive scale of the Newton direction d_N. Suppose the algorithm is implemented with a line search replacing Steps 5 and 6 , as suggested in Remark 2.1. Then because the projective transformations g(x) and h(z) given by (3.3) and (3.4) of Part I preserve directions from $\bar{x}$, the algorithm direction in the space $\mathcal{X}$ will be d_N. Therefore, when using a line search, the algorithm is just searching in the Newton direction. This is precisely Vaidya's algorithm [14], when all weights w_i are identical. And because the complexity analysis of Sections IV and V carries through with or without a line search, we see that Vaidya's algorithm exhibits superlinear convergence.

Remark 6.3. An Extension of a Theorem of Bayer and Lagarias. In [1], Bayer and Lagarias have shown the following structural equivalence between Karmarkar's algorithm for linear programming and Newton's method: First one can projectively transform the problem of minimizing Karmarkar's potential function over a polyhedron $\mathcal{X}$ to finding the (unbounded) center of an unbounded polyhedron $\mathcal{Z}$, where $\mathcal{Z}$ is the image of $\mathcal{X}$ under a projective transformation that sends the set of optimal solutions to the linear program to the hyperplane at infinity. Then the image of Karmarkar's algorithm (with a line search) in the space $\mathcal{Z}$ corresponds to performing a line search in the Newton direction in the transformed space $\mathcal{Z}$. Lemma 6.1 is in fact a generalization of this result. It states that if one is trying to find the center of any polyhedron $\mathcal{X}$ (bounded or not), then the direction generated at any iteration of the projective transformation method (i.e., the algorithm of Section II) is a positive scale of the Newton direction. Thus, if one determines steplengths by a line search of the objective function, then the projective tranformation method corresponds to Newton's method with a line search.

Another important relationship between directions generated by projective transformation methods and Newton's method can be found in Gill et al. [4].

We now prove Lemma 6.1 by a sequence of three propositions.

Proposition 6.1. If (d_N, π_N) solve (6.1), then $1 + y^T d_N \geq 0$.

Proof: Note that from (6.2), we have

$1 + y^T d_N = 1 - y^T Q^{-1} y + y^T Q^{-1} M^T (M Q^{-1} M^T)^{-1} M Q^{-1} y \geq 1 - y^T Q^{-1} y$.

We thus must show that $y^T Q^{-1} y \leq 1$. Note that $\tilde{Q} = \tilde{A}^T \bar{S}^{-1} W \bar{S}^{-1} \tilde{A}$ is positive semi-definite, and that $\tilde{Q} = Q - yy^T$. Thus $0 \leq y^T Q^{-1} \tilde{Q} Q^{-1} y = y^T Q^{-1} (Q - yy^T) Q^{-1} y = y^T Q^{-1} y (1 - y^T Q^{-1} y)$. Therefore $y^T Q^{-1} y \leq 1$, completing the proof. ■

Proposition 6.2. If (d_N, π_N) solve (6.1) and $1 + y^Td_N = 0$, then the optimization problem (P3) of Step 3 is unbounded, and P_w has no solution.

Proof: From the proof of Proposition 6.1, we see that if $1 + y^Td_N = 0$, then

$$1 - y^TQ^{-1}y = 0 \tag{6.3}$$

and $$y^TQ^{-1}M^T(MQ^{-1}M^T)MQ^{-1}y = 0. \tag{6.4}$$

Let $r = -Q^{-1}y$. From (6.4) we have $r^TM^T(MQ^{-1}M^T)^{-1}Mr = 0$, whereby $Mr = 0$. Also $r^T\tilde{Q}r = r^T(Q - yy^T)r = y^TQ^{-1}y\ (1 - y^TQ^{-1}y) = 0$ from (6.3). Finally, from (6.3) we have $-y^Tr = y^TQ^{-1}y = 1 > 0$. Thus r satisfies $Mr = 0$, $r^T\tilde{Q}r = 0$, and $-y^Tr > 0$, whereby the optimization problem (P3) of Step 3 is unbounded. From Proposition 3.1, P_w is unbounded. ■

Proposition 6.3. If (d_N, π_N) solve (6.1), and $1 + y^Td_N > 0$, then either $d_N = 0$, or $d_N\tilde{Q}d_N > 0$.

Proof: Because $\tilde{Q} = Q - yy^T$,

$$d_N\tilde{Q}d_N = d_N{}^TQd_N - (y^Td_N)^2, \tag{6.5}$$

and from (6.2) we obtain $d_N{}^TQd_N = -\ y^Td_N$. Substituting in (6.5) yields

$d_N\tilde{Q}d_N = -\ y^Td_N - (y^Td_N)^2$. Since $\tilde{Q}$ is positive semi-definite, $d_N\tilde{Q}d_N \geq 0$. If $d_N\tilde{Q}d_N = 0$, then we must have $y^Td_N = -1$, or $y^Td_N = 0$. If $y^Td_N = -1$, then this contradicts the hypothesis that $1 + y^Td_N > 0$. Thus $y^Td_N = 0$, which implies from (6.1) that $d_N = 0$. ■

Proof of Lemma 6.1.: From Proposition 6.1, we have $1 + y^Td_N \geq 0$. Suppose $1 + y^Td_N > 0$ and $d_N \neq 0$. Then $\bar{d} = d_N\sqrt{k} / \sqrt{d_N\tilde{Q}d_N}$, $\bar{\pi} = \pi_N / (1 + y^Td_N)$, and

$\bar{\beta} = \sqrt{d_N \tilde{Q} d_N} \,/\, (2\sqrt{k}\,(1 + y^T d_N))$, are all well-defined (by Proposition 6.3) and satisfy the optimality conditions $\bar{d}\tilde{Q}\,\bar{d} = k$, $M\,\bar{d} = 0$, $-y = 2\,\bar{\beta}\tilde{Q}\,\bar{d} - M^T\,\bar{\pi}$, $\bar{\beta} > 0$, for the optimization problem (P3) of Step 3. Thus $\bar{d}$ is the direction of Step 3 of the algorithm.

Next suppose $1 + y^T d_N > 0$ and that $d_N = 0$. Then $-M^T\,\bar{\pi}_N = -y$, and $\bar{d} = 0$, $\bar{\pi} = \pi_N$ satisfy the optimality conditions for the optimization problem (P3) in Step 3. By Proposition 3.2, the current iterate $\bar{x}$ solves P_w.

Finally, suppose $1 + y^T d_N = 0$. Then from Proposition 6.2, we conclude that the optimization problem (P3) of Step 3 is unbounded and P_w is unbounded. ■

VII. Inner and Outer Ellipsoids at an approximate w-center point $\bar{x}$

One of the special features of the w-center $\hat{x}$ of a polyhedral system $\mathcal{X}$ is the fact that there exist ellipsoids E_{IN} and E_{OUT}, with center at $\hat{x}$, such that $E_{IN} \subset \mathcal{X} \subset E_{OUT}$ and $E_{IN} = (\bar{w}/(1 - \bar{w}))\,E_{OUT}$, see Theorem 2.1 of Part I. Although the iterates of the algorithm of Section II will converge to $\hat{x}$, there may not be finite termination, and in fact the solution $\hat{x}$ may involve irrational data. A natural question is whether one can construct good ellipsoids F_{IN} and F_{OUT} about points near $\hat{x}$, with the property that $F_{IN} \subset \mathcal{X} \subset F_{OUT}$, and $F_{OUT} = c \cdot F_{IN}$, where $c = O(1/\bar{w})$. The main result of this section answers this question in the affirmative:

Theorem 7.1. (Inner and Outer Ellipsoids at an approximate w-center point.) Let $\bar{x}$ be the current iterate of the algorithm, let $\bar{s} = b - A\bar{x}$, and let γ be as defined in Step 4 of the algorithm. Then if $\gamma < .08567$, the ellipsoids

$$F_{IN} = \left\{x \in R^n \mid Mx = g,\ (x - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A (x - \bar{x}) \leq \bar{w}\right\}$$

and

$$F_{OUT} = \left\{x \in R^n \mid Mx = g,\ (x - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A (x - \bar{x}) \leq \left[1.75\sqrt{\frac{1-\bar{w}}{\bar{w}}} + 5\sqrt{\bar{w}}\right]^2\right\}$$

satisfy $F_{IN} \subset \mathcal{X} \subset F_{OUT}$.

Remark 7.1. Note that $(F_{OUT} - \bar{x}) = \left(1.75\sqrt{\frac{1-\bar{w}}{\bar{w}}} + 5\right)(F_{IN} - \bar{x})$. Thus the ratio of the scale of F_{OUT} to F_{IN} is less than $1.75/\bar{w} + 5$. If $w = (1/m)e$, $\bar{w} = 1/m$, and this ratio is less than $1.75m + 5$, which is $O(m)$.

Remark 7.2. The number of iterations of the algorithm needed to produce $\gamma < .08567$ is bounded if P_w is bounded. Let $\hat{x}$ be the optimal solution to P_w . Then if x° is the initial value of x in the algorithm, we must have $\gamma < .08567$ after at most $\left\lceil \frac{(1-\bar{w})}{(\bar{w})}\left(\frac{F(\hat{x}) - F(x^\circ)}{.0033}\right)\right\rceil$ iterations . This follows from Lemma 4.2(i). If F^* is any finite upper bound on the value of $F(\hat{x})$ produced at Step 4 of the algorithm, then $\gamma < .08567$ after at most

$$\left\lceil \frac{(1-\bar{w})}{\bar{w}}\left(\frac{F^* - F(x^\circ)}{.0033}\right)\right\rceil \text{ iterations.}$$

The proof of Theorem 7.1 is a consequence of the following sequence of propositions and lemmas.

Proposition 7.1. Suppose $\bar{x} \in \text{int}\,\mathcal{X}$ is given, and let $\bar{s} = b - A\bar{x}$. Let $\tilde{x} \in R^n$ satisfy $M\tilde{x} = g$ and $(\tilde{x} - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A(\tilde{x} - \bar{x}) \le \delta^2 \bar{w}$, where $\delta < 1$. Then $\tilde{x} \in \text{int}\,\mathcal{X}$, and if $\tilde{s} = b - A\tilde{x}$, $(\bar{x} - \tilde{x})^T A^T \tilde{S}^{-1} W \tilde{S}^{-1} A(\bar{x} - \tilde{x}) \le \delta^2 \bar{w}/(1 - \delta)^2$.

Proof: By supposition, $M\tilde{x} = g$. Let $\tilde{s} = b - A\tilde{x}$. We first must show that $\tilde{s} > 0$, which will imply $\tilde{x} \in \text{int}\,\mathcal{X}$. By supposition above, $(\tilde{s} - \bar{s})\bar{S}^{-1} W \bar{S}^{-1}(\tilde{s} - \bar{s}) \le \delta^2 \bar{w}$, so that $\sum_{i=1}^{m} w_i \left(\frac{\bar{s}_i - \tilde{s}_i}{\bar{s}_i}\right)^2 \le \delta^2 \bar{w}$. Because $\bar{w} \le w_i$, $i = 1, \dots, m$, we have $\left|1 - \frac{\tilde{s}_i}{\bar{s}_i}\right| \le \delta < 1$, so that $\tilde{s}_i > 0$, $i = 1, \dots, m$. This shows that $\tilde{x} \in \text{int}\,\mathcal{X}$.

Furthermore, we obtain $\frac{\tilde{s}_i}{\bar{s}_i} \ge 1 - \delta$, so that $\frac{\bar{s}_i}{\tilde{s}_i} \le \frac{1}{1 - \delta}$.

Next, note that

$$(\bar{x} - \tilde{x})^T A^T \tilde{S}^{-1} W \tilde{S}^{-1} A(\bar{x} - \tilde{x}) = \sum_{i=1}^{m} w_i \left(\frac{\bar{s}_i}{\tilde{s}_i}\right)^2 \left(\frac{\bar{s}_i - \tilde{s}_i}{\bar{s}_i^2}\right)^2 \le$$

$$\left(\frac{1}{1 - \delta}\right)^2 \sum_{i=1}^{m} w_i \left(\frac{\bar{s}_i - \tilde{s}_i}{\bar{s}_i^2}\right)^2 \le \frac{\delta^2}{(1 - \delta)^2}\,\bar{w}\,. \quad \blacksquare$$

Lemma 7.1. Let $\bar{x} \in \text{int}\,\mathcal{X}$ be given, and let $\hat{x}$ be the w-center of $\mathcal{X}$, and suppose that

$$(\hat{x} - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A(\hat{x} - \bar{x}) \le \delta^2 \bar{w} \quad \text{for some } \delta < 1.$$

Let $F_{IN} = \left\{x \in R^n \mid Mx = g,\ (x - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A(x - \bar{x}) \le \bar{w}\right\}$ and

$$F_{OUT} = \left\{x \in R^n \mid Mx = g,\ (x - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A(x - \bar{x}) \le (1+\delta)^2 \left(\sqrt{\frac{1-\bar{w}}{\bar{w}}} + \frac{\delta}{1-\delta}\sqrt{\bar{w}}\right)^2\right\}.$$

Then $F_{IN} \subset \mathcal{X} \subset F_{OUT}$.

Proof of Lemma 7.1.: Let x be an element of F_{IN}. Then by the same argument as in Proposition 7.1, we have $x \in \mathcal{X}$, so that $F_{IN} \subset \mathcal{X}$. Because $\hat{x}$ is the w-center of $\mathcal{X}$, then

$$(x - \hat{x})^T A^T \hat{S}^{-1} W \hat{S}^{-1} A(x - \hat{x}) \le (1 - \bar{w})/\bar{w} \quad \text{for any } x \in \mathcal{X}, \tag{7.1}$$

where $\hat{s} = b - A\hat{x}$, from Theorem 2.1 of Part I.
Also, by Proposition 7.1, with $\tilde{x} = \hat{x}$, we have

$$(\bar{x} - \hat{x})^T A^T \hat{S}^{-1} W \hat{S}^{-1} A(\bar{x} - \hat{x}) \le \delta^2 \bar{w}/(1-\delta)^2 \cdot \tag{7.2}$$

Taking the square-roots of (7.1) and (7.2) and noting the triangle inequality for norms, we have

$$(x - \bar{x})^T A^T \hat{S}^{-1} W \hat{S}^{-1} A(x - \bar{x}) \le \left(\sqrt{\frac{1-\bar{w}}{\bar{w}}} + \frac{\delta\sqrt{\bar{w}}}{1-\delta}\right)^2 \quad \text{for every } x \in \mathcal{X}. \tag{7.3}$$

Next, note that from the hypothesis of the lemma, $\frac{\hat{s}_i}{\bar{s}_i} - 1 \le \delta$, and so $\frac{\hat{s}_i}{\bar{s}_i} \le 1 + \delta$, $i = 1, \ldots, m$. Now let $x \in \mathcal{X}$ be given, and let $s = b - Ax$. Then

$$(x - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A (x - \bar{x}) = \sum_{i=1}^{m} w_i \left(\frac{\hat{s}_i}{\bar{s}_i}\right)^2 \left(\frac{\bar{s}_i - \hat{s}_i}{(\hat{s}_i)^2}\right)^2 \leq$$

$$(1+\delta)^2 (x - \bar{x})^T A^T \hat{S}^{-1} W \hat{S}^{-1} A (x - \bar{x}) \leq (1+\delta)^2 \left(\sqrt{\frac{1-\bar{w}}{\bar{w}}} + \frac{\delta}{1-\delta}\sqrt{\bar{w}}\right)^2 ,$$

completing the proof. ■

Lemma 7.2. If $\bar{x}$ is the current iterate of the algorithm and if $\gamma < .08567$ at Step 4, and if $\hat{x}$ is the w-center of $\mathcal{X}$, then $(\hat{x} - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A(\hat{x} - \bar{x}) \leq .55\ \bar{w}$.

Proof: Let $\hat{z} = g(\hat{x})$ where $g(x)$ is given in (3.3) of Part I. Let $\tilde{Q}$ be as given in Step 2 of the algorithm, and let $\beta^2 = (\hat{z} - \bar{x})^T \tilde{Q} (\hat{z} - \bar{x})$. Then substituting h = .5 in expression (4.2), we obtain from Lemmas 4.5 and 4.6 that if $\gamma < .08567$, then $\beta \leq h\sqrt{k} = .5\sqrt{k}$. Let $\bar{d}$ be the direction defined in Step 3 of the algorithm and let $\hat{d} = \hat{z} - \bar{x}$. Then because $\bar{x} + \bar{d}$ maximizes $-y^T z$ over $z \in E_{IN} = \{z \in R^n \mid Mz = g, (z - \bar{x})^T \tilde{Q} (z - \bar{x}) \leq k\}$ then $-y^T(\bar{x} + (\beta/\sqrt{k})\,\bar{d}) \geq -y^T(\bar{x} \pm \hat{d})$. Thus $\pm y^T \hat{d} \leq (-\beta/\sqrt{k}) y^T \bar{d}$. But $\gamma = -y^T \bar{d}/k$, so $\pm y^T \hat{d} \leq \beta\gamma\sqrt{k}$. Therefore,

$$(y^T \hat{d})^2 \leq \beta^2 \gamma^2 k . \tag{7.4}$$

Next, note that since $\tilde{Q} = Q - yy^T$, where $Q = A^T \bar{S}^{-1} W \bar{S}^{-1} A$, we have $(\hat{z} - \bar{x})^T Q (\hat{z} - \bar{x}) = (\hat{z} - \bar{x})^T \tilde{Q} (\hat{z} - \bar{x}) + \left[(\hat{z} - \bar{x})^T y\right]^2 = \beta^2 + (y^T \hat{d})^2 \leq \beta^2 + \beta^2 k \gamma^2$ from (7.4). Also, $(\hat{x} - \bar{x}) = (\hat{z} - \bar{x}) / (1 + y^T(\hat{z} - \bar{x}))$ from (3.4) of Part I, so that

$$(\hat{x} - \bar{x})^T Q (\hat{x} - \bar{x}) \leq (\beta^2 + \beta^2 k \gamma^2)/(1 + y^T \hat{d})^2 \leq (\beta^2 + \beta^2 k \gamma^2)/(1 - \beta\sqrt{k}\gamma)^2$$

$$\leq \frac{h^2 k + h^2 k^2 \gamma^2}{(1 - hk\gamma)^2} \leq \frac{k(h^2 + h^2\gamma^2)}{(1 - h\gamma)^2} \leq \frac{2\,\bar{w}(h^2 + h^2\gamma^2)}{(1 - h\gamma)^2} \leq .55\ \bar{w} .$$ ■

Proof of Theorem 7.1. : We first show that $F_{IN} \subset \mathcal{X}$. Let $\tilde{x} \in F_{IN}$ and let $\tilde{s} = b - A\tilde{x}$. It suffices to show that $\tilde{s} \geq 0$. Because $\tilde{x} \in F_{IN}$, then $\sum_{i=1}^{m} w_i \left(\frac{\tilde{s}_i}{\bar{s}_i} - 1 \right)^2 \leq \bar{w}$. Therefore, because $w_i \geq \bar{w}$, $i = 1, \ldots, m$,

$1 - \frac{\tilde{s}_i}{\bar{s}_i} \leq 1, i = 1, \ldots, m$, so that $\tilde{s}_i \geq 0$, $i = 1, \ldots, m$. Thus $F_{IN} \subset \mathcal{X}$.

To show that $\mathcal{X} \subset F_{OUT}$, we apply Lemma 7.2, which shows that if $\gamma < .08567$, then the w-center $\hat{x}$ of $\mathcal{X}$ must satisfy $(\hat{x} - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A(\hat{x} - \bar{x}) \leq \delta^2 \bar{w}$, where $\delta = \sqrt{.55}$. Next, applying $\delta = \sqrt{.55}$ in Lemma 7.1, we obtain the conclusion that

$(1+\delta)^2 \left(\sqrt{\frac{1-\bar{w}}{\bar{w}}} + \frac{\delta}{1-\delta}\sqrt{\bar{w}} \right)^2 \leq \left(1.75\sqrt{\frac{1-\bar{w}}{\bar{w}}} + 5\sqrt{\bar{w}} \right)^2$, thus showing that $\mathcal{X} \subset F_{OUT}$. ■

VIII. Concluding Remarks

Alternative Convergence Constants. Lemma 4.3 asserts that at each iterate of the algorithm that we obtain a constant improvement of at least .0033k or a linear convergence to the optimal objective value, with convergence constant .32. The constants .0033 and .32 are derived in Section IV by using the value h= .5 in Lemmas 4.4, 4.5, and 4.6. If instead of choosing h = .5, one chooses h = 2, for example, then by parallelling the methodology in Section 4, one obtains Lemma 4.3 with a constant improvement of at least .0133k or a linear convergence rate with convergence constant .64. The choice of h = .5 was fairly arbitrary. Similar results can be had by choosing a different value of h to obtain different constants for the threshold value of γ and the relative sizes of F_{OUT} and F_{IN} in Theorem 7.1.

Stronger Convergence. Vaidya [14] has shown that his Newton direction algorithm is linearly convergent. Here, we have extended his result and have shown that the algorithm of Section II (and Vaidya's algorithm) exhibit superlinear convergence. A natural question for future study is whether one can show the algorithm here to be quadratically convergent.

The behavior of γ. In Step 3 of the algorithm, the search direction $\bar{d}$ is computed. In Step 4, the parameter γ , as a function of $\bar{x}$ is computed, and we can write $\gamma = \gamma(\bar{x})$. The value of $\gamma(\bar{x})$ is then used to derive upper bounds, a step length, and a guaranteed improvement in the objective value. From Lemma 4.3, it is obvious that $\gamma(\bar{x})$ goes to zero as $\bar{x}$ approaches $\hat{x}$, the optimal solution to P_w. Concerning the behavior of $\gamma(\bar{x})$, it is natural to ask if the level sets of $\gamma(x)$ are convex, if $\gamma(x)$ decreases at each iteration, etc. The author has demonstrated examples where a level set of $\gamma(x)$ is not convex, and where $\gamma(x)$ increases at a particular iteration. Thus $\gamma(x)$ is not as well-behaved as one would hope for.

No Finite Termination. As pointed out in Section VII, the solution $\hat{x}$ to the w-center problem can have irrational components and so the algorithm will not stop after finitely many iterations. Even if the problem P_w is unbounded, the algorithm may never detect unboundedness, and so may not stop after finitely many iterations. This is shown by the example of Section 4 of Bayer and Lagarias [1] . In that example, the iterates of a Newton method with a line search are traced for the w-center problem, where $w = (1/3, 1/3, 1/3)$, and $\mathcal{X} \in \{x \in R^2 \mid x_1 \geq -1,\ x_1 \leq 1,\ x_2 \geq 0\}$, and the starting point is $\bar{x} = (1/3, 2/3)$. The (Newton) direction defined by each iterate is never a ray of $\mathcal{X}$, and so the algorithm will never stop.

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