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PRELIMINARY NOTES ON A MATHEMATICAL ANALYSIS OF MANPOWER TRAINING PROGRAMS

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Preliminary Notes on
A Mathematical Analysis of Manpower Training Programs

The purpose of this paper is to present specific human ability development models in order to analyze the ability development function of manpower training. Ability development, of course, is one of two main training functions. Socialization is the other. Roughly speaking, an individual is socialized when he not only knows what is expected of him and what he can expect from others, but also when he accepts the roles defined by these expectations. Training, in this paper, is then conceived of as a process which develops the abilities necessary to fulfill such expectations, and the first task will be to inquire into the nature of ability development.

The General Model and Specific Cases

Let $A(t)$ and $C(t)$ represent the amount and maximum possible amount respectively of any specific ability at time $t$. At $t = 0$, let $C(0) = C_0$. Ignoring fatigue, attention span, and motivational effects for the moment, assume that the time rate of change of ability in an environment requiring the exercise of that ability is as given in the equation

$$\frac{dA(t)}{dt} = k \left[ C(t) - A(t) \right]$$

For example, suppose the ability in question is the ability to solve ordinary differential equations. Then in the absence of formal training in differential equations, a specific individual may be characterized by $k > 0$ if he has had previous experience with calculus and $k = 0$ if he has had no previous experience with calculus. Formal training may increase
Equation (1) is a first order linear differential equation as can be seen upon rearrangement to

$$\frac{dA(t)}{dt} + kA(t) = kC(t)$$

It embodies the basic assumption of this paper that the rate of change of ability is directly related to the difference $C(t) - A(t)$, i.e., $dA(t)/dt$ decreases as $C(t) - A(t)$ decreases. The factor $k$ will be given different interpretations in the discussion which follows. For now it can be regarded as a constant of proportionality.

Assuming that $k$ is a constant, the solution to equation (2) is given by

$$A = \frac{1}{u(t)} \left[ k \int u(\xi) C(\xi) \, d\xi + c \right]$$

where $u(t) = \exp \left[ \int k \, dx \right] = e^{kt}$ and $x$ is a dummy variable. In the case at hand

$$A = e^{-kt} \left[ k \int e^{k\xi} C(\xi) \, d\xi + c \right]$$

and it is clear that the solution depends upon whatever additional assumptions are made concerning $C(t)$.

Case I: $C(t) = C_0$

The simplest assumption for $C(t)$ is that it is a constant function of time, i.e., $C(t) = C_0$. Equation (3) then resolves to

\[ A = e^{-kt} (C_0 e^{kt} + c) \]

With the initial condition \( A(0) = A_0, c = A_0 - C_0 \), and

\[ A = C_0 (1 - e^{-kt}) + A_0 e^{-kt} \]

If \( A(0) = 0 \), the simpler result

\[ A = C_0 (1 - e^{-kt}) \]

is obtained.

It is now clear that given the assumption \( C(t) = C_0 \), a standard learning theory assumption,\(^1\) the value of formal training would be difficult to detect unless \( k \) were very small and could be significantly enhanced with formal training. Otherwise, the approach to maximum ability, \( C_0 \), takes approximately the same amount of time with or without formal training.

**Case II: \( C(t) = C_0 \) and \( k = k(t) \)**

On the other hand, suppose \( k \) were very small and subject to the influence of formal training. Then \( k \) would also be a function of \( t \) and equation (2) would become

\[ \frac{dA(t)}{dt} + k(t)A(t) = k(t)C_0 \]

continuing with the assumption that \( C(t) = C_0 \). Now \( A(t) \) depends upon additional assumptions for \( k(t) \) as well as \( C(t) \). A rather innocuous assumption for \( k(t) \) is that it is a monotonically increasing function of a form that approximately satisfies the differential equation

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null
where \( k(t) \) increases with \( t \), it increases at a decreasing rate.\(^1\) Solving for \( k(t) \),

\[
(9) \quad k(t) = \frac{c_1}{c_2} \left( 1 - e^{-c_2 t} \right) + k
\]

where \( k(0) = k \).

The solution for \( u(t) \) is now given by

\[
(10) \quad u(t) = \exp \left[ \int_{t_0}^{t} k(\tau) d\tau \right]
\]

where \( x \) is a dummy variable for \( t \) in equation (9). The integral in equation (10) can be simplified by applying the first mean-value theorem for integrals,\(^2\) and the result is

\[
(11) \quad u(t) = \exp \left[ k(\tau_0) t \right] = e^{k \tau_0}
\]

where \( 0 \leq \tau \leq t \) and \( k_1 = k(\tau) \). A good approximation for \( k_1 \) is obtained by using equation (9). Thus,

\[
(12) \quad k_1 = k(\tau) = \frac{c_1}{c_2} \left( 1 - e^{-c_2 \tau} \right) + k \approx \frac{c_1}{c_2} + k = B
\]

Solving again for \( A(t) \),

\[
(13) \quad A = e^{-Bt} \left[ \int_{t_0}^{t} e^{-B\tau} k(\tau) C_0 d\tau + C \right]
\]

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1. Mathematical tractability, of course, is one reason for choosing this particular functional form.

The integral in equation (13) can be simplified by applying the second mean-value theorem for integrals for positive monotone increasing functions such as $k(t)$. The result is

$$A = \int^{t}_{v_2} B^t \left[ C_0 \cdot k(t) \int^{t}_{v_2} \frac{B^t}{c} dt + c \right]$$

where $0 \leq v_2 \leq t$. Here $B$ is an even better approximation for $k(t)$.

Thus,

$$A = \int^{t}_{v_2} B^t \left[ C_0 \cdot B \int^{t}_{v_2} \frac{B^t}{c} dt + c \right]$$

Equation (15) reduces to

$$A = e^{-Bt} \left( C_0 e^{Bt} + m \right)$$

where the constant $m = c - C_0 e^{Bv_2}$. Again,

$$A = C_0 \left( 1 - e^{-Bt} \right) + A_0 e^{-Bt}$$

when assuming $A(0) = A_0$, or

$$A = C_0 \left( 1 - e^{-Bt} \right)$$

when assuming $A(0) = 0$. The results in equations (6) and (17) are seen to be identical except for the constant $B = (c_1/c_2) + k$.

This last result is particularly useful in that it permits a simple derivation of the total increase in useful ability over a long period of time. The total increase in useful ability is given by

\[1. \quad \text{Ibid., pp. 115-117.}\]
\( R = \int_0^t [C_0 (1 - e^{-at}) - C_0 (1 - e^{-kt})] \, dt \)

This rather simple integral resolves to

\[
R = C_0 \left( \frac{B - k + k e^{-at} - B e^{-kt}}{B k} \right)
\]

which approaches

\[
R = C_0 \left( \frac{B - k}{B k} \right)
\]

for large values of \( t \). Substituting \( B = (c_1/c_2) + k \),

\[
R = C_0 \left( \frac{c_1}{c_1 k + c_2 k^2} \right)
\]

Now, continuing with the assumption that \( k \) is very small, then

\[
c_2 k^2 \ll c_1 k \quad \text{and} \quad
\]

\[
R \approx C_0 \left( \frac{c_1}{c_1 k} \right) = \frac{C_0}{k}
\]

Thus, even under the restrictive assumption that \( C(t) = C_0 \), the intuitive feeling (and oftentimes observable result) that formal manpower training is valuable has been made perfectly clear. The total increase in useful ability, \( R = C_0/k \), must be very large unless \( C_0 \) is very small.

**Case III:** \( \frac{dC}{dt} = p \frac{dA}{dt} \)

The step from the assumption that \( C(t) = C_0 \), a constant, to the alternative assumption that \( C \) varies with time, of course, is a small one. Various functional forms for \( C(t) \), such as those resulting from solutions to \( \frac{dC}{dt} = c_3 e^{-c_4 t} \) and \( \frac{dC}{dt} = c_5 t/(C - C_0) \) where \( c_3, c_4, c_5, \)
and \( C(0) = C_0 \) are constants, can be substituted into equation (2) and the consequences determined. The factor \( k \) may or may not be assumed constant during such an exercise.

Another alternative assumption to \( C(t) = C_0 \) is

\[
\frac{dC}{dt} = p \frac{dA}{dt} = p k [C(t) - A(t)]
\]

and it has more intuitive appeal than arbitrary substitutions for \( C(t) \) in equation (2). In other words, simply assume that the rate of change of the maximum possible amount of any specific ability is directly proportional to the rate of change of ability. A physical analogy to this hypothesized psychological phenomenon would be the process of pouring water into an expansible container such as a balloon. The balloon has an initial capacity \( C_0 \), but its capacity increases under the weight of the water.

Combining equations (1) and (23) with the initial conditions \( A(0) = 0 \) and \( C(0) = C_0 \) leads to the system of first order linear differential equations

\[
\frac{dA}{dt} = kC - kA, \quad A(0) = 0
\]

\[
\frac{dC}{dt} = pkC - pkA, \quad C(0) = C_0
\]

The solution\(^1\) to this system, assuming \( p \) and \( k \) are constants, is given by

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Notice that unless \( p = 1 \), the amount and maximum possible amount of ability do not increase without limit but to \( C_0/(1 - p) \). The distribution of \( p \) and \( k \) is discussed in the following section.

Before reviewing the conclusions of the analyses in these three cases, consider the system (24) written with \( p \) and \( k \) as variables, i.e.,

\[
\begin{align*}
\frac{dA}{dt} &= k(t)(C - A) \\
\frac{dC}{dt} &= p(t) \frac{dA}{dt}
\end{align*}
\]

System (27) is the case where \( p \) and \( k \) are subject to the influence of formal training. Differentiating the first of these equations and substituting the second into the result yields

\[
\frac{d^2A}{dt^2} + k(t)[1 - p(t)] \frac{dA}{dt} + k'(t)A = k'(t)C
\]

Equation (28) can be simplified by relying upon approximations similar to the one posited in equation (9). Thus, assuming that variations in \( p \) and \( k \) are similar under the influence of formal training, the factor \( k(t)[1 - p(t)] \) can be approximated by the product of two constants, \( k^*(1 - p^*) \). This result, not elaborated here, leads to the further approximation that \( k'(t) = 0 \), and equation (28) becomes
Then, using the initial conditions that \( A(0) = 0 \) and \( A'(0) = k^C_0 \), the solution to equation (29) is given by

\[
(30) \quad A = \frac{C_0}{1-\rho^*} \left[ 1 - e^{-k^C(1-\rho^*)t} \right]
\]

The similarity between this result and equation (25) is not surprising given the similarity already obtained between equations (6) and (17).

And, if the effect of training is to bring \( \rho^* \) close to 1.0, its consequence is a rather spectacular increase in the final level of ability, viz.,

\[
(31) \quad \frac{C_0}{1-\rho^*} - \frac{C_0}{1-\rho} = C_0 \left( \frac{\rho^* - \rho}{1-\rho^* - \rho + \rho \rho^*} \right)
\]

Again, this last result can be simplified by noting that if \( \rho^* \approx 1.0 \), then \( \rho \rho^* \approx 0.0 \). Thus,

\[
(32) \quad \frac{C_0}{1-\rho^*} - \frac{C_0}{1-\rho} = C_0 \left( \frac{\rho^* - \rho}{1-\rho^*} \right)
\]

For example, suppose \( \rho = 0.5 \) and \( \rho^* = 0.99 \). Then \( \rho \rho^* = 0.495 \) and \( \rho^* - \rho = -0.005 \). This yields an increase in the final level of ability, over and above what it would have been without formal training, of \( C_0(\rho^* - \rho/1 - \rho^*) = 49C_0 \).
Conclusions

The first conclusion that can be drawn from the preceding analyses is that the training effect is to shift the ability development curve upward and to the left. This is obvious when comparing equation (17) in Case II with equation (6) in Case I and equation (30) with equation (25) in Case III. The Case III upward shift, of course, is more dramatic than the Case II shift. This conclusion may be intuitively obvious, but intuition is not always so reliable in the face of critical analysis. In this respect, a less intuitive conclusion is that the standard learning theory assumptions embodied in Case I make many training programs difficult to justify. Unless $k$, the rate of development, is very small and subject to the influence of training, any manpower training program would appear to be a bad investment.

The development in Case III is an obvious alternative to Case I, and it leads to equation (32) which is the increase in the final level of ability over and above what it would have been without formal training. In the process of obtaining equation (32), however, the constants $p$ and $k$ in equation (25) were not fully discussed. It is not difficult to show that in order to be consistent with equation (1), a reasonable hypothesis is that $0<k<1$. The same hypothesis must hold for the constant $p$ unless one is willing to believe that ability grows without limit or decreases as a result of training. Sometimes ability does decrease as a result of training, e.g., brainwashing may decrease an individual's powers of discrimination. For normal training purposes, however, it would seem reasonable to assume that both $p$ and $k$ lie within the open interval $(0,1)$. 
Thus, prior to training some population of individuals has one
distribution over the unit square, say $S$, described by

$$S = \{(p, k) : 0 < p < 1, 0 < k < 1\}$$

After training this population has another distribution over the same
square.

Returning to equation (32), it is obvious that the most valuable
training programs are those that focus on individuals with low $p$ values
and increase these to $p^*$ values which are close to 1.0. Even the best
training program is worthless if the average value of $p$ is already
close to 1.0. Thus, assuming that ability and performance are
reasonably well correlated, simple measures of increased proficiency
are insufficient for comparing different training methods. The worst
possible training method, for example, will not look too bad when
compared with a good one if both methods are tested on groups that have
high $p$ values.

The development of methods for measuring $p$ and $k$ is then one of
two main problems for further study. Elaborating on the theoretical
foundations established in this paper is the other, and further
elaborations may indicate some reductions in the magnitude of the
measurement problem. Notice that $p$ and $k$ measurements in themselves
are a definition of an individual's intelligence or capacity to learn
in terms of a rate of development, $k$, and a constant of proportionality,
$p$, between $dC/dt$ and $dA/dt$. Lastly, further theoretical developments
should begin to include the motivational, span of attention, and
fatigue factors ignored in this paper as well as the financial
considerations underlying training programs.