PROPERTIES OF STORAGE HIERARCHY
SYSTEMS WITH MULTIPLE PAGE SIZES
AND REDUNDANT DATA

Chat-Yu Lam
Stuart E. Madnick

April 10, 1979
Revised
CISR #42
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PREFACE

The Center for Information Systems Research (CISR) is a research center of the M.I.T. Sloan School of Management; it consists of a group of Management Information Systems specialists, including faculty members, full-time research staff, and student research assistants. The Center's general research thrust is to devise better means of designing, generating and maintaining application software, information systems and decision support systems.

Within the context of the research effort sponsored by the National Science Foundation under Grant No. MCS77-20829, CISR proposes to investigate the architecture of the INFOPLEX Data Base Computer which is particularly designed for large-scale information management. INFOPLEX applies the theory of hierarchical decomposition in its design and makes use of multiple microprocessors in its implementation to obtain high performance, high reliability, and large storage capacity. Research issues to be addressed include optimal decomposition of information management functions into a functional hierarchy to be implemented by a hierarchy of microprocessors, and optimal physical decomposition of a data storage hierarchy to support the memory requirements of the information management functions.

In Technical Report No. 1, we discussed the INFOPLEX concept and its research directions. This report focuses on the study of a generalized data storage system for very large databases which can be used to support the memory requirements of INFOPLEX. This data storage system makes use of multiple page sizes in a hierarchy of storage levels and maintains multiple copies of the same information across the storage levels. Important properties of such a data storage system are derived here.
ABSTRACT

The need for high performance, highly reliable storage for very large on-line databases, coupled with rapid advances in storage device technology, has made the study of generalized storage hierarchies an important area of research.

This paper analyzes properties of a data storage hierarchy system specifically designed for handling very large on-line databases. To attain high performance and high reliability, the data storage hierarchy makes use of multiple page sizes in different storage levels and maintains multiple copies of the same information across the storage levels. Such a storage hierarchy system is currently being designed as part of the INFOPLEX database computer project. Previous studies of storage hierarchies have primarily focused on virtual memories for program storage and hierarchies with a single page size across all storage levels and/or a single copy of information in the hierarchy.

In the INFOPLEX design, extensions to the Least Recently Used (LRU) algorithm are used to manage the storage levels. The Read-Through technique is used to initially load a referenced page, of the appropriate size, into all storage levels above the one in which the page is found. Since each storage level is viewed as an extension of the immediate higher level, an overflow page from level 'i' is always placed in level 'i+1'. Important properties of these algorithms are derived. It is shown that, depending upon the types of algorithms used and the relative sizes of the storage levels, it is not always possible to guarantee that the contents of a given storage level 'i' is always a superset of the contents of its immediate higher storage level 'i-1'. The necessary and sufficient conditions for this property to hold are identified and proved. Furthermore, it is possible that increasing the size of intermediate storage levels may actually increase the number of references to lower storage levels, resulting in reduced performance. Conditions necessary to avoid such an anomaly are also identified and proved.

Key Words and Phrases: database computer, very large databases, data storage hierarchy, storage management algorithms, inclusion properties, modelling, performance and reliability analysis.

CR Categories: 4.3 Supervisory Systems, 4.33 Data Base 5.2 Metatheory 6.22 Special-Purpose Computers 6.34 Storage Units
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1. **Introduction**

Two and three-level memory hierarchies have been used in practical computer systems [5, 9, 13]. However, there is relatively little experience with general hierarchical storage systems. Rapid advances in storage technology coupled with the need for high performance, highly reliable on-line databases makes the idea of using a generalized storage hierarchy as the repository for very large shared databases very attractive.

One major area of theoretic study of storage hierarchy systems in the past has been the optimal placement of information in a storage hierarchy system. Three approaches to this problem have been used: (1) Static placement [1, 4, 22] - this approach determines the optimal placement strategy statically, at the initiation of the system; (2) Dynamic placement [7, 16] - this approach attempts to optimally place information in the hierarchy, taking into account the dynamically changing nature of access to information; (3) Information structuring [11, 14] - this approach manipulates the internal structure of information so that information items that are frequently used together are placed adjacent to each other.

Another major area of theoretic study of storage hierarchy systems has been the study of storage management algorithms [2, 3, 8, 10, 17, 21]. Here the study of storage hierarchy and the study of virtual memory systems for program storage have overlapped considerably. This is largely due to the fact that most of the studies
of storage hierarchies in the past have been aimed at providing a virtual memory for program storage. These studies usually do not consider the effects of multiple page sizes across storage levels, nor the problem of providing redundant data across storage levels. These considerations are of great importance for a storage hierarchy designed specifically for very large data bases.

Madnick [15, 18, 19] proposed the design of a generalized storage hierarchy for large data bases that makes use of multiple data redundancy against failure and multiple page sizes in different storage levels for high performance. Such a storage hierarchy system is to be used in the INFOPLEX database computer [12, 20].

Conceptually, the INFOPLEX database computer consists of a functional hierarchy and a physical (storage) hierarchy (See Figure 1).
The functional hierarchy implements all the information management functions of a database manager, such as query language interpretation, security verification, and data path accessing, etc. In INFOPLEX, the functional hierarchy is implemented using multiple microprocessors. Both pipeline and parallel processing are exploited to realize high performance and high reliability. To support the storage requirements of the functional hierarchy, INFOPLEX makes use of a generalized data storage hierarchy system.

In this paper, we extend this work by developing a model of the data storage hierarchy, proposing extensions to the Least Recently Used (LRU) algorithm for managing the storage hierarchy, and deriving important properties of the data storage hierarchy.
2. Model Of A Data Storage Hierarchy

A Data Storage Hierarchy consists of h levels of storage devices, \( M^1, M^2, \ldots, M^h \). The page size of \( M^i \) is \( Q_i \) and the size of \( M^1 \) is \( m_i \) pages each of size \( Q_i \). \( Q_i \) is always an integral multiple of \( Q_{i-1} \), for \( i = 2, 3 \ldots, h \). The unit of information transfer between \( M^i \) and \( M^{i+1} \) is a page, of size \( Q_i \). Figure 2 illustrates this model of the Data Storage Hierarchy.

All references are directed to \( M^1 \). The storage management algorithms automatically transfer information among storage levels. As a result, the Data Storage Hierarchy appears to the reference source as a \( M^1 \) storage device with the size of \( M^h \).

As a result of the storage management algorithms (to be discussed next), multiple copies of the same information may exist in different storage levels.

2.1. Storage Management Algorithms

We shall focus our attentions on the basic algorithms to support the read-through [18] operation. Algorithms to support other operations can be derived from these basic algorithms.

In a read-through, the highest storage level that contains the addressed information broadcasts the information to all upper storage levels, each of which simultaneously extracts the page (of the appropriate size) that contains the information from the broadcast. If the addressed information is found in the highest storage level, the read-through reduces to a simple reference to the addressed information in that level. Figure 3 illustrates the read-through operation.
Common data path

Figure 2. Model of a data storage hierarchy system.

Reference to page $P^1_{ya}$.

Figure 3. Illustration of the READ-THROUGH operation.
Note that in order to load a new page into a storage level an existing page may have to be displaced from that storage level. We refer to this phenomenon as overflow. Hence, the basic reference cycle consists of two sub-cycles, the read-through cycle (RT), and the overflow handling cycle (OH), with RT preceding OH.

For example, Figure 3 illustrates the basic reference cycle to handle a reference to the page \( p^1_{ya} \). During the Read-Through (RT) subcycle, the highest storage level \( M^x \) that contains \( p^1_{ya} \) broadcasts the page containing \( p^1_{ya} \) to all upper storage levels, each of which extracts the page of appropriate size that contains \( p^1_{ya} \) from the broadcast. As result of the Read-Through, there may be overflow from the storage levels. These are handled in the Overflow-Handling (OH) subcycle.

It is necessary to consider overflow handling because it is desirable to have information overflowed from a storage level to be in the immediate lower storage level, which can then be viewed as an extension to the higher storage level.

One strategy of handling overflow to meet this objective is to treat overflows from \( M^i \) as references to \( M^{i+1} \). We refer to algorithms that incorporate this strategy as having dynamic-overflow-placement (DOP).

Another possible overflow handling strategy is to treat an overflow from \( M^i \) as a reference to \( M^{i+1} \) only when the overflow information is not already in \( M^{i+1} \). If the overflow information is already in \( M^{i+1} \), no overflow handling is necessary. We refer to algorithms that incorporate this strategy as having static-overflow-placement (SOP).
Let us consider the algorithms at each storage level for selecting the page to be overflowed. Since the Least Recently Used (LRU) algorithm [6, 21] serves as the basis for most current algorithms, we shall consider natural extensions to LRU for managing the storage levels in the Data Storage Hierarchy system.

Consider the following two strategies for handling the Read-Through Cycle. First, let every storage level above and including the level containing the addressed information be updated according to the LRU strategy. Thus, all storage levels lower than the addressed information do not know about the reference. This class of algorithms is called LOCAL-LRU algorithm. This is illustrated in Figure 4.

The other class of algorithms that we shall consider is called GLOBAL-LRU algorithm. In this case, all storage levels are updated according to the LRU strategy whether or not that level actually participates in the read-through. This is illustrated in Figure 5.

Although the read-through operation leaves supersets of the page $P^j_y$ in all levels, the future handling of each of these pages depends upon the replacement algorithms used and the effects of the overflow handling. We would like to guarantee that the contents of each storage level, $M^i$, is always a superset of its immediately higher level, $M^{i-1}$. This property is called Multi-Level Inclusion (MLI). Conditions to guarantee MLI will be derived in a later section.

It is not difficult to demonstrate situations where handling overflows generates references which produce overflows, which generate yet more references. Hence another important question to resolve is to determine the conditions under which an overflow from $M^i$ is always found to already exist in $M^{i+1}$, i.e., no reference to storage levels lower than $M^{i+1}$ is generated as a result of the overflow. This property is called Multi-Level Overflow Inclusion (MLOI). Conditions to guarantee MLOI will be derived in a later section.

We shall consider these important properties in light of four basic algorithm alternatives based on local or global LRU and static or dynamic overflow. Formal definitions for these algorithms will be provided after the basic model of the Data Storage Hierarchy system is introduced.
Read-through

All these levels are not affected

$M^x$ is the highest level where $P$ is found

Figure 4 LOCAL-LRU

Read-Through

These levels are also updated as if reference to $P$ were made to them

$M^x$ is the highest level where $P$ is found

Figure 5 GLOBAL-LRU
2.2 Basic Model of Data Storage Hierarchy

For the purposes of this paper, the basic model illustrated in Figure 6 is sufficient to model the Data Storage Hierarchy. As far as the Read-Through and Overflow-Handling operations are concerned, this basic model is generalizable to a h-level storage hierarchy system.

$M^r$ can be viewed as a reservoir which contains all the information. $M^i$ is the top level. It has $m_i$ pages each of size $Q_i$. $M^j (j=i+1)$ is the next level. It has $m_j$ pages each of size $nQ_i$ where $n$ is an integer greater than 1.

![Figure 6 Basic model of a data storage hierarchy]

2.3 Formal Definitions of Storage Management Algorithms

Denote a reference string by $r = "r_1, r_2, ..., r_n,"$ where $r_t (1 \leq t \leq n)$ is the page being referenced at the $t$-th reference cycle. Let $S_t^i$ be the stack for $M^i$ at the beginning of the $t$-th reference cycle, ordered according to LRU. That is, $S_t^i = (S_t^i(1), S_t^i(2), ..., S_t^i(K))$, where $S_t^i(1)$ is the most recently referenced page and $S_t^i(K)$ is the least recently referenced page. Note that $K \leq m_i$ ($m_i$ = capacity of $M^i$ in terms of the number of pages). The number of pages in $S_t^i$ is denoted as $|S_t^i|$, hence $|S_t^i| = K$. By convention, $S_t^i = \emptyset$, $|S_t^i| = 0$.
$S^i_t$ is an ordered set. Define $M^i_t$ as the contents of $S^i_t$ without any ordering. Similarly, we can define $S^j_t$ and $M^j_t$ for $M^j$.

Let us denote the pages in $M^j$ by $p^j_1, p^j_2, \ldots$. Each page, $p^j_i$, in $M^j$, consists of an equivalent of $n$ smaller pages, each of size $Q_i = Q_j/n$. Denote this set of pages by $(p^j_i)^j$, i.e., $(p^j_i)^j = \{p^j_{i1}, p^j_{i2}, \ldots, p^j_{in}\}$.

In general, $(M^j_t)^i$ is the set of pages, each of size $Q_i$, obtained by "breaking down" the pages in $M^j$. Formally, $(M^j_t)^i = \bigcup_{k=1}^{x} (S^j_t(k))^i$ where $x = |S^j_t|$. $(p^j_i)^j$ is called the family from the parent page $p^j_y$.

Any pair of pages, $p^i_y$ and $p^i_y$ from $(p^j_i)^j$ are said to be family equivalent, denoted by $p^i_y \equiv p^i_y$. Furthermore, a parent page $p^j_y$ and a page $p^i_y$ (for $1 \leq z \leq n$) from its family are said to be corresponding pages, denoted by $p^i_y \equiv p^j_y$.

$S^i_t$ and $S^j_t$ are said to be in corresponding order, denoted by $S^i_t = S^j_t$, if $S^i_t(k) = S^j_t(k)$ for $k = 1, 2, 3, \ldots, w$, where $w = \min(|S^i_t|, |S^j_t|)$. Intuitively, two stacks are in corresponding order if, for each element of the shorter stack, there is a corresponding page in the other stack at the same stack distance (The stack distance for page $S^i_t(k)$ is defined to be $k$).

$M^i_t$ and $M^j_t$ are said to be correspondingly equivalent, denoted by $M^i_t \equiv M^j_t$ if $|M^i_t| = |M^j_t|$ and for any $k = 1, 2, \ldots, |M^i_t|$ there exists $x$, such that $S^i_t(k) = S^j_t(x)$ and $S^j_t(x) \not\equiv S^i_t(y)$ for all $y \not\equiv k$. Intuitively, the two memories are correspondingly equivalent when each page in one memory corresponds to exactly one page in the other memory.

The reduced stack, $\bar{S}^i_t$, of $S^i_t$ is defined to be $\bar{S}^i_t(k) = S^i_t(j_k)$ for $k = 1, \ldots, |\bar{S}^i_t|$ where $j_k$ is the minimum $j_k$ where $j_k > j_{k-1}$ ($j_0 = 0$) and $\bar{S}^i_t(k) \not\equiv S^i_t(j)$ for $j < j_k$. Intuitively, $\bar{S}^i_t$ is obtained from $S^i_t$ by collecting one page from each family existing in $S^i_t$, such that the page being collected from each family is the page that has the smallest stack distance within the family.
In the following, we define the storage management algorithms.

In each case, assume that the page referenced at time $t$ is $p^i_ya$.

**LRU** $(S^i_t, p^i_ya) = S^i_{t+1}$ is defined as follows:

**Case 1**: $p^i_ya \in S^i_t$, $p^i_ya = S^i_t(k)$:

$$\begin{align*}
S^i_{t+1}(1) &= p^i_ya, \\
S^i_{t+1}(x) &= S^i_t(x-1), \quad 1 < x \leq k
\end{align*}$$

If $|S^i_t| = m_i$ then $p^i_oa = S^i_t(m_i)$ is the overflow, else there is no overflow.

**Case 2**: $p^i_ya \notin S^i_t$:

$$\begin{align*}
S^i_{t+1}(1) &= p^i_ya, \\
S^i_{t+1}(x) &= S^i_t(x-1), \quad 1 < x \leq \min(m_i, |S^i_t|+1)
\end{align*}$$

**LOCAL-LRU-SOP** $(S^i_t, S^j_t, p^i_ya) = (S^i_{t+1}, S^j_{t+1})$ is defined as follows:

**Case 1**: $p^i_ya \in S^i_t$:

$$S^i_{t+1} = \text{LRU}(S^i_t, p^i_ya), \quad S^j_{t+1} = S^j_t$$

**Case 2**: $p^i_ya \in S^i_t$, $p^j \in S^j_t$:

$$\begin{align*}
S^i_t &= \text{LRU}(S^i_t, p^i_ya), \\
S^j_t &= \text{LRU}(S^j_t, p^j),
\end{align*}$$

If there is no overflow from $S^i_t$ then $S^i_{t+1} = S^i_t$ and $S^j_{t+1} = S^j_t$.

If overflow from $S^i_t$ is the page $p^i_oa$

then $(S^i_{t+1}, S^j_{t+1}) = \text{SOP}(S^i_t, S^j_t, p^i_oa)$ defined as:

$$\begin{align*}
S^i_{t+1} &= S^i_t, \\
S^j_{t+1} &= S^j_t, \quad \text{if } p^j \in S^j_t, \text{ then } S^j_{t+1} = S^j_t,
\end{align*}$$

$$\begin{align*}
S^i_{t+1} &= \text{LRU}(S^i_t, p^i_ya), \\
S^j_{t+1} &= \text{LRU}(S^j_t, p^j)
\end{align*}$$

**Case 3**: $p^i_ya \notin S^i_t$ and $p^j \notin S^j_t$:

(handled as in Case 2)

**LOCAL-LRU-DOP** $(S^i_t, S^j_t, p^i_ya) = (S^i_{t+1}, S^j_{t+1})$ is defined as:

**Case 1**: $p^i_ya \in S^i_t$:

$$S^i_{t+1} = \text{LRU}(S^i_t, p^i_ya), \quad S^j_{t+1} = S^j_t$$

**Case 2**: $p^i_ya \notin S^i_t$ and $p^j \in S^j_t$:

$$\begin{align*}
S^i_t &= \text{LRU}(S^i_t, p^i_ya), \\
S^j_t &= \text{LRU}(S^j_t, p^j)
\end{align*}$$

If no overflow from $S^i_t$ then $S^i_{t+1} = S^i_t$, and $S^j_{t+1} = S^j_t$. 
If overflow from $S^j_t$ is $p^j_o$ then
$$(S^i_{t+1}, S^j_{t+1}) = \text{DOP} (S^i_t, S^j_t, p^j_o)$$
which is defined as:
$$S^i_{t+1} = S^i_t,$$
and
$$S^j_{t+1} = \text{LRU} (S^j_t, p^j_o)$$

**Case 3**: $p^i_y \in S^i_t$ and $p^j_y \in S^j_t$:
(handled as in Case 2 above)

$\text{GLOBAL-LRU-SOP} (S^i_t, S^j_t, p^j_o) = (S^i_{t+1}, S^j_{t+1})$ is defined as follows:
$$S^i_{t+1} = \text{LRU} (S^i_t, p^i_y)$$
and
$$S^j_{t+1} = \text{LRU} (S^j_t, p^j_o),$$

If no overflow from $S^i_t$ then $S^i_{t+1} = S^i_t$; and $S^j_{t+1} = S^j_t$.

If overflow from $S^i_t$ is $p^i_o$ then $(S^i_{t+1}, S^j_{t+1}) = \text{SOP} (S^i_t, S^j_t, p^i_o)$

$\text{GLOBAL-LRU-DOP} (S^i_t, S^j_t, p^j_o) = (S^i_{t+1}, S^j_{t+1})$ is defined as:
$$S^i_{t+1} = \text{LRU} (S^i_t, p^i_y)$$
and
$$S^j_{t+1} = \text{LRU} (S^j_t, p^j_o),$$

If no overflow from $S^i_t$ then $S^i_{t+1} = S^i_t$; and $S^j_{t+1} = S^j_t$.

If overflow from $S^i_t$ is $p^i_o$ then $(S^i_{t+1}, S^j_{t+1}) = \text{DOP} (S^i_t, S^j_t, p^i_o)$
3. Properties of Data Storage Hierarchy

One of the properties of a Read-Through operation is that it leaves a "shadow" of the referenced page (i.e., the corresponding pages) in all storage levels. This provides multiple redundancy for the page. Does this multiple redundancy exist at all times? That is, if a page exists in storage level $M^i$, will its corresponding pages always be in all storage levels lower than $M^i$? We refer to this as the Multi-Level Inclusion (MLI) property. As illustrated in Figure 7 for the LOCAL-LRU algorithms and in Figure 8 for the GLOBAL-LRU algorithms, it is not always possible to guarantee that the MLI property holds. For example, after the reference to $P_{31}^i$ in Figure 7(a), the page $P_{11}^i$ exists in $M^i$ but its corresponding page $P_1^j$ is not found in $M_j^i$. In this paper we shall derive the necessary and sufficient conditions for the MLI property to hold at all times.

Another desirable property of the Data Storage Hierarchy is to avoid generating references due to overflows. That is, under what conditions will overflow pages from $M^i$ find their corresponding pages already existing in the storage level $M^{i+1}$? We refer to this as the Multi-Level Overflow Inclusion (MLOI) property. We shall investigate the conditions that make this property true at all times.

Refering to the basic model of a data storage hierarchy in Figure 6, for high performance it is desirable to minimize the number of references to $M^r$ (the reservoir). If we increased the number of pages in $M^i$, or in $M^j$, or in both, we might expect the number of references to $M^r$ to decrease. As illustrated in Figure 9 for the LOCAL-LRU-SOP algorithm, this is not always so, i.e., for the same reference string, the number of references to the reservoir actually increased from 4 to 5.
<table>
<thead>
<tr>
<th>reference to $H^1$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
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<tr>
<td>contents of $H^1$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
</tr>
<tr>
<td>overflow from $H^1$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
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<td>$P_5$</td>
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<tr>
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<td>$P_2$</td>
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<td>$P_4$</td>
<td>$P_5$</td>
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<td>$P_2$</td>
<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
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</tbody>
</table>

(a) LOCAL-LRU-SOP

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<th>reference to $H^1$</th>
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<tr>
<td>reference to $H^2$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
</tr>
</tbody>
</table>

(b) LOCAL-LRU-DOP

Figure 7 Examples of MLI violations for Local-LRU algorithms

<table>
<thead>
<tr>
<th>reference to $H^1$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
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<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
</tr>
<tr>
<td>contents of $H^2$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
</tr>
<tr>
<td>reference to $H^2$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
</tr>
</tbody>
</table>

(a) GLOBAL-LRU-SOP

<table>
<thead>
<tr>
<th>reference to $H^1$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>contents of $H^1$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
</tr>
<tr>
<td>overflow from $H^1$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
</tr>
<tr>
<td>reference to $H^1$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
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<tr>
<td>contents of $H^2$</td>
<td>$P_1$</td>
<td>$P_2$</td>
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<tr>
<td>reference to $H^2$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
</tr>
</tbody>
</table>

(b) GLOBAL-LRU-DOP

Figure 8 Examples of MLI violations for Global-LRU algorithms
after $M^i$ is increased by 1 page in size. We refer to this phenomena as a Multi-Level Paging Anomaly (MLPA). One can easily find situations where MLPA occurs for the other three algorithms. Since occurrence of MLPA reduces performance in spite of the costs of increasing memory sizes, we would like to investigate the conditions to guarantee that MLPA does not exist.

**Figure 9** MLPA for LOCAL-LRU-SOP
3.1 Summary of Properties

The MLI, MLOI, and MLPA properties of the Data Storage Hierarchy have been derived in the form of eight theorems. These theorems are briefly explained and summarized below and formally proven in the following section.

**Multi-Level Inclusion (MLI):** It is shown in Theorem 1 that if the number of pages in $M^j$ is greater than the number of pages in $M^i$ (note $M^j$ pages are larger than those of $M^i$), then it is not possible to guarantee MLI for all reference strings at all times. It turns out that using LOCAL-LRU-SOP, or LOCAL-LRU-DOP, no matter how many pages are in $M^j$ or $M^i$, one can always find a reference string that violates the MLI property (Theorem 2). Using the GLOBAL-LRU algorithms, however, conditions to guarantee MLI exist. For the GLOBAL-LRU-SOP algorithm, a necessary and sufficient condition to guarantee that MLI holds at all times for any reference string is that the number of pages in $M^j$ be greater than the number of pages in $M^i$ (Theorem 3). For the GLOBAL-LRU-DOP algorithm, a necessary and sufficient condition to guarantee MLI is that the number of pages in $M^j$ be greater than or equal to twice the number of pages in $M^i$ (Theorem 4).

**Multi-Level Overflow Inclusion (MLOI):** It is obvious that if MLI cannot be guaranteed then MLOI cannot be guaranteed. Thus, the LOCAL-LRU algorithms cannot guarantee MLOI. For the GLOBAL-LRU-SOP algorithm, a necessary and sufficient condition to guarantee MLOI is the same condition as that to guarantee MLI (Theorem 5). For the GLOBAL-LRU-DOP algorithm, a necessary and sufficient condition to guarantee MLOI is that the number of pages in $M^j$ is strictly greater than twice the number of pages in $M^i$ (Theorem 6). Thus, for the GLOBAL-LRU-DOP algorithm, guaranteeing that MLOI holds will also guarantee that MLI will hold, but not vice versa.
Multi-Level Paging Anomaly (MLPA): We have identified and proved sufficiency conditions to avoid MLPA for the GLOBAL-LRU algorithms. For the GLOBAL-LRU-SOP algorithm, this condition is that the number of pages in $M^j$ must be greater than the number of pages in $M^i$ before and after any increase in the sizes of the levels (Theorem 7). For the GLOBAL-LRU-DOP algorithm, this condition is that the number of pages in $M^j$ must be greater than twice the number of pages in $M^i$ before and after any increase in the sizes of the levels (Theorem 8).

In summary, we have shown that for the LOCAL-LRU algorithms, no choice of sizes for the storage levels can guarantee that a lower storage level always contains all the information in the higher storage levels. For the GLOBAL-LRU algorithms, by choosing appropriate sizes for the storage levels, we can (1) ensure that the above inclusion property holds at all times for all reference strings, (2) guarantee that no extra page references to lower storage levels are generated as a result of handling overflows, and (3) guarantee that increasing the sizes of the storage levels does not increase the number of references to lower storage levels. These results are formally stated as the following eight Theorems. Formal proofs of these Theorems are presented in the following section.
THEOREM 1
Under LOCAL-LRU-SOP, or LOCAL-LRU-DOP, or GLOBAL-LRU-SOP,
or GLOBAL-LRU-DOP, for any \( m_i \geq 2, m_j \leq m_i \) implies \( \exists r, t, (M_t^i)^i \neq M_t^i \)

THEOREM 2
Under LOCAL-LRU-SOP, or LOCAL-LRU-DOP, for any \( m_i \geq 2 \), and any \( m_j \),
\( \exists r, t, (M_t^j)^i \neq M_t^i \)

THEOREM 3
Under GLOBAL-LRU-SOP, for any \( m_i \geq 2 \), \( \forall r, t, (M_t^i)^i \geq M_t^i \) iff \( m_j > m_i \)

THEOREM 4
Under GLOBAL-LRU-DOP, for any \( m_i \geq 2 \), \( \forall r, t, (M_t^j)^i \geq M_t^i \) iff \( m_j \geq 2m_i \)

THEOREM 5
Under GLOBAL-LRU-SOP, for any \( m_i \geq 2 \), \( \forall r, t \), an overflow from \( M_i \)
finds its corresponding page in \( M_j^i \) iff \( m_j > m_i \)

THEOREM 6
Under GLOBAL-LRU-DOP, for any \( m_i \geq 2 \), \( \forall r, t \), an overflow from \( M_i \)
finds its corresponding page in \( M_j^i \) iff \( m_j > 2m_i \)

THEOREM 7
Let \( M_i \) (with \( m_i \) pages), \( M_j^i \) (with \( m_j \) pages) and \( M_i^r \) be System A.
Let \( M_j^i \) (with \( m_j^i \) pages), \( M_j^i^j \) (with \( m_j^j \) pages) and \( M_j^i^r \) be System B.
Let \( m_i^j \geq m_i \) and \( m_j^j \geq m_j \). Under GLOBAL-LRU-SOP, for any \( m_i \geq 2 \),
no MLPA can exist if \( m_j > m_i \) and \( m_j^j > m_i^j \)

THEOREM 8
Let System A and System B be defined as in THEOREM 7.
Let \( m_i^j \geq m_i \) and \( m_j^j > m_j \). Under GLOBAL-LRU-DOP, for any \( m_i \geq 2 \),
no MLPA can exist if \( m_j > 2m_i \) and \( m_j^j > 2m_i^j \)
3.2 Derivation of Properties

THEOREM 1

Under LOCAL-LRU-SOP, or LOCAL-LRU-DOP, or GLOBAL-LRU-SOP,

or GLOBAL-LRU-DOP, for any \( m_i \geq 2, m_j \leq m_i \) implies \( \exists r, t, (M^j_t)^i \not\equiv M^i_t \)

PROOF

Case 1: \( m_j < m_i \)

Consider the reference string \( r = " p_{1a}^i, p_{2a}^i, \ldots, p_{(m_j+1)a}^i " \).

Using any one of the algorithms, the following stacks are obtained at \( t = m_j + 2 \):

\[
S_t^j = (p_{(m_j+1)a}^j, p_{m_j a}^j, \ldots, p_{2a}^j, p_{1a}^j)
\]

\[
S_t^i = (p_{(m_j+1)}^i, p_{m_j}^i, \ldots, p_{3}^j, p_{2}^j)
\]

Thus, \( p_{1a}^i \in M_t^i \) but \( p_{1a}^i \notin (M^j_t)^i \), i.e., \( (M^j_t)^i \not\equiv M^i_t \).

Case 2: \( m_j = m_i = w \)

Consider the reference string \( r = " p_{1a}^i, p_{2a}^i, \ldots, p_{(w+1)a}^i " \)

Using any one of the above algorithms, the following stacks are obtained at \( t = w + 2 \):

\[
S_t^i = (p_{(w+1)a}^i, p_{wa}^i, \ldots, p_{3a}^i, p_{2a}^i)
\]

\[
S_t^j = (p_{1}^j, p_{(w+1)}^j, p_{w}^j, \ldots, p_{4}^j, p_{3}^j)
\]

Thus, \( p_{2a}^i \in M_t^i \) but \( p_{2a}^i \notin (M^j_t)^i \), i.e., \( (M^j_t)^i \not\equiv M^i_t \).

Q.E.D.
THEOREM 2

Under LOCAL-LRU-SOP, or LOCAL-LRU-DOP, for any $m_i \geq 2$, and any $m_j$,

$$\exists r, t, (M^i_t)^j \notin M^i_t.$$

PROOF (For LOCAL-LRU-SOP)

For $m_j \leq m_i$ the result follows directly from THEOREM 1.

For $m_j > m_i$, using the reference string

$$r = "p^i_{za}, p^i_{la}, p^i_{za}, p^j_{2a}, \ldots, p^j_{za}, p^j_{m_j a}",$$

the following stacks will be produced at $t=2m_j+1$:

$$S^i_t = (p^i_{m_j a}, p^i_{za}, p^i_{m_j - 1 a}, \ldots, p^i_{m_j - m_i + 2 a}),$$

$$S^j_t = (p^j_{m_j}, p^j_{m_j - 1}, \ldots, p^j_{2}, p^j_{1})$$

Thus $p^i_{za} \in M^i_t$ but $p^j_{za} \notin (M^j_t)^i$, i.e., $(M^j_t)^i \notin M^i_t$. Q.E.D.

PROOF (For LOCAL-LRU-DOP)

For $m_j \leq m_i$ the result follows directly from THEOREM 1.

For $m_j > m_i$, using the following reference string

$$r = "p^i_{za}, p^i_{la}, p^i_{za}, p^i_{2a}, \ldots, p^i_{za}, p^i_{m_j a}",$$

The following stacks will be produced at $t=2m_j+1$:

$$S^i_t = (p^i_{m_j a}, p^i_{za}, \ldots, p^i_{m_j - m_i + 2 a}), S^j_t = (a_1, a_2, \ldots, a_{m_j})$$

Where for $1 \leq i \leq m_j$, $a_i \in \{p^j_{m_j}, p^j_{m_j - 1}, \ldots, p^j_3, p^j_2, p^j_1\}$

since $p^j_{za}$ is the only overflow from $M^j_t$.

Thus, $p^i_{za} \in M^i_t$ but $p^j_{za} \notin (M^j_t)^i$, i.e., $(M^j_t)^i \notin M^i_t$. Q.E.D.
THEOREM 3

Under GLOBAL-LRU-SOP, for any \( m_i \geq 2 \), \( \forall \ r, t, (M^j_t)^i \supseteq M^i_t \) iff \( m_j > m_i \).

PROOF

This proof has two parts. Part (a) to prove \( \forall \ r, t, (M^j_t)^i \supseteq M^i_t \) iff \( m_j > m_i \), or equivalently, \( m_j \leq m_i \Rightarrow \exists \ r, t, (M^j_t)^i \nsubseteq M^i_t \).

Part (b) to prove \( m_j > m_i \Rightarrow \forall \ r, t, (M^j_t)^i \supseteq M^i_t \).

PROOF of Part (a): \( m_j \leq m_i \Rightarrow \exists \ r, t, (M^j_t)^i \nsubseteq M^i_t \).

This follows directly from THEOREM 1.

Q.E.D.

To prove Part (b), we need the following results.

LEMMA 3.1

\( \forall \ r, t \) such that \( |M^j_t|^i \leq m_i \), if \( m_j = m_i + 1 \), then:

(a) \( (M^j_t)^i \supseteq M^i_t \), and

(b) \( S^j_t \subseteq S^i_t \).

PROOF of LEMMA 3.1

For \( t=2 \) (i.e., after the first reference), (a) and (b) are true.

Suppose (a) and (b) are true for \( t \), such that \( |M^j_t|^i \leq m_i \).

Consider the next reference:

Case 1: It is a reference to \( M^j \):

There is no overflow from \( M^j \), so (a) is still true.

Since Global-LRU is used, (b) is still true.

Case 2: It is a reference to \( M^j \):

There is no overflow from \( M^j \). If no overflow from \( M^i \), the same argument as Case 1 applies. If there is overflow from \( M^i \), the overflow page finds its corresponding page in \( M^j \). Since SOP is used, this overflow can be treated as a "no-op". Thus (a) and (b) are preserved.

Case 3: It is a reference to \( M^r \):

There is no overflow from \( M^j \) since \( |M^j_t|^i \leq m_i \). Thus the same reasoning as in Case 2 applies.

Q.E.D.
LEMMA 3.2

∀ r, t, such that \( |M_t^j| = m_j \), if \( m_j = m_i + 1 \) then
(a) \((M_t^j)^i \supseteq M_t^i\), (b) \( S_t^i \cap S_t^j \), and (c) \((S_t^i(m_j))^i \cap S_t^i = \emptyset\)

Let us denote the conditions (a) (b) and (c) jointly as \( Z(t) \).

PROOF of LEMMA 3.2

Suppose the first time \( S_t^j(m_j) \) is filled is by the \( t^* \)-th reference.
That is, \( S_t^j(m_j) = \emptyset \) for all \( t \leq t^* \) and \( S_t^j(m_j) \neq \emptyset \) for all \( t > t^* \).

From LEMMA 3.1 we know that (a) and (b) are true for all \( t \leq t^* \).

Let \( t_1 = t^* + 1, t_2 = t^* + 2, \ldots \), etc. We shall show, by induction on \( t \), starting at \( t_1 \), that \( Z(t) \) is true. First we show that \( Z(t_1) \) is true as follows:

Case 1: \( M_t^j \subseteq M_t^i \)

\( S_t^i \cap S_t^j \) and \( M_t^j \subseteq M_t^i \) ⇒ \( S_t^j(m_j - 1) \subseteq S_t^j(m_i) \)

As a result of the reference at \( t^* \) (to \( M_r^i \)), \( S_t^j(m_j) = S_t^j(m_j - 1) \) and \( S_t^j(m_i) \) overflows from \( M^i \). This overflow page finds its corresponding page in \( M^j \) because there is no overflow from \( M_j \) and (a).

Since SOP is used, the overflow from \( M_i \) can be treated as a "no-op".

Furthermore, since Global-LRU is used, (b) is true after the \( t^* \)-th reference. (b) and \( |S_t^j(m_j)| > |S_t^j(m_i)| \) ⇒ (a) and (c). Thus \( Z(t_1) \) is true.

Case 2: \( (M_t^j)^i \supseteq M_t^i \) and \( M_t^j \not\subseteq M_t^i \)

\( (M_t^j)^i \supseteq M_t^i \) and \( M_t^j \not\subseteq M_t^i \) ⇒ \( S_t^j(k) \) such that \((S_t^j(k))^i \cap M_t^i = \emptyset\) \( S_t^i \cap S_t^j \) and \((S_t^j(k))^i \cap M_t^i = \emptyset \Rightarrow k > |S_t^i| \) and \((S_t^j(x))^i \cap M_t^i = \emptyset \)

for all \( x \), where \( m_j - 1 \leq x \leq k \). Thus \((S_t^j(m_j - 1))^i \cap S_t^i = \emptyset\)

(i.e., the last page of \( S_t^i \) is not in \( S_t^j \))

\( S_t^i(m_i) \) overflows from \( M_i \). There is no overflow from \( M_j \). Thus the overflow page from \( M_i \) finds its corresponding page in \( M_j \). For the same reasons as in Case 1, (b) is still preserved. (b) and \( |S_t^j(m_j)| > |S_t^j(m_i)| \) ⇒ (a) and (c) are true. Thus, \( Z(t_1) \) is true.
Assume that $Z(t^k)$ is true; to show that $Z(t_{k+1})$ is true, we consider the next reference, at time $t_{k+1}$:

Imagine that the last page of $S^j_{t_k}$ does not exist, i.e., $S^j_{t_k}(m_j) = \emptyset$

If the reference at $t_{k+1}$ is to a page in $M^i_{t_k}$ or $M^j_{t_k}$, then (a) and (b) still hold because Global-LRU is used and because overflow from $M^j$ finds its corresponding page in $M^j$ (See the proof of LEMMA 3.1).

If the reference at $t_{k+1}$ is to a page not in $M^j_{t_k}$, then we can apply the argument as that used in considering the reference at time $t_1$ above to show that $Z(t_{k+1})$ is still true.

Q.E.D.

**LEMMA 3.3**

$\forall r,t, \text{if } m_j = m_i + 1 \text{ then } (a) (M^j_t)^i \supseteq M^i_t \text{ and } (b) (S^j_t(m_j))^i \cap S^i_t = \emptyset$

**PROOF of LEMMA 3.3**

For $t$ such that $|M^j_t| \leq m_j$ (a) follows directly from LEMMA 3.1 and (b) is true because $S^j_t(m_j) = \emptyset$

For $t$ such that $|M^j_t| = m_j$ (a) and (b) follows directly from LEMMA 3.2

Q.E.D.

**LEMMA 3.4**

$\forall r,t, \text{if } m_j > m_i \text{ then } (a) (M^j_t)^i \supseteq M^i_t \text{ and } (b) (S^j_t(m_j))^i \cap S^i_t = \emptyset$

**PROOF of LEMMA 3.4**

Let $m_j = m_i + k$. We shall prove this lemma by induction on $k$.

For $k=1$ (a) and (b) are true from LEMMA 3.3.

Suppose that (a) and (b) are true for $k$.

Consider $m_j = m_i + (k+1)$. That is consider the effects of increasing $M^j$ by 1 page in size:
Since $M^i$ is unchanged, $M^j$ (with $m_i+k+1$ pages) sees the same reference string as $M^j$ (with $m_i+k$ pages). Applying the stack inclusion property (Mattson et al., 70), we have $M^j$ (with $m_i+k+1$ pages) $\supseteq M^j$ (with $m_i+k$ pages). Thus (a) is still true. Suppose $(S^i_t(m_i+k+1))^j \cap S^i_t \neq \emptyset$ then there is a page in $M^j$ that corresponds to this page. But $S^j_t(m_i+k+1)$ is not in $M^j$ (with $m_i+k$ pages). This contradicts the property that $(M^j_t)^i \supseteq M^i_t$. This shows that (b) is still true.

Q.E.D.

**Proof of Part (b):** $m_j > m_i \Rightarrow \forall r,t, (M^j_t)^i \supseteq M^i_t$.

This follows directly from Lemma 3.4. Q.E.D.
THEOREM 4

Under GLOBAL-LRU-DOP, for any $m_j \geq 2$, for all $r, t$, $(M_t^j)^i \not\subseteq M_t^i$ iff $m_j \geq 2m_i$

**PROOF**

This proof has two parts:

Part (a): $m_j < 2m_i \Rightarrow \exists r, t$, $(M_t^j)^i \not\subseteq M_t^i$

Part (b): $m_j \geq 2m_i \Rightarrow \forall r, t$, $(M_t^j)^i \subseteq M_t^i$

**PROOF of Part (a):** $m_j < 2m_i \Rightarrow \exists r, t$, $(M_t^j)^i \not\subseteq M_t^i$

For $m_j \leq m_i$ the result follows from THEOREM 1.

Consider the case for $2m_i > m_j > m_i$:

The reference string $r = p_1^j, p_2^j, p_3^j, \ldots, p_{(2m_i)}^j$

will produce the following stacks:

$S_t^i = (p_1^i, p_2^i, p_3^i, \ldots, p_{(m_i+1)}^i)$,

$S_t^j = (a_1, a_2, a_3, \ldots, a_{m_j})$ where $a_j$'s are picked from $L_1$ and $L_2$ alternatively, starting from $L_1$: $L_1 = (p_1^j, p_2^j, \ldots, p_{m_i}^j)$

and $L_2 = (p_1^j, p_2^j, \ldots, p_{m_i+1}^j)$.

If $m_j$ is even, then $(a_1, a_3, \ldots, a_{m_j-1})$ corresponds to the first $m_j/2$ elements of $L_1$ and $(a_2, a_4, \ldots, a_{m_j})$ corresponds to the first $m_j/2$ elements in $L_2$. We see that $p_{(m_i+1)}^j$ is in $S_t^i$ but its corresponding page is not in $S_t^j$ since $m_j/2 < m_i$.

If $m_j$ is odd, then $(a_1, a_3, \ldots, a_{m_j})$ corresponds to the first $(m_j+1)/2$ elements in $L_1$ and $(a_2, a_4, \ldots, a_{m_j-1})$ corresponds to the first $(m_j-1)/2$ elements in $L_2$. We see that the page $p_{(m_i+1)}^j$ is in $S_t^i$ but its corresponding page is not in $S_t^j$ because max$( (m_j-1)/2 ) = m_i-1$,

thus, $a_{(m_j-1)}^j$ is at most the $(m_i-1)$-th element of $L_2$, $p_{2m_i-(m_i-1)+1}^j = p_{m_i+2}^j$.

In both cases, $(M_t^j)^i \not\subseteq M_t^i$

Q.E.D.
To prove Part (b), we need the following preliminary results.

**LEMMA 4.1**

Under GLOBAL-LRU-DOP, for \( m_i \geq 2, m_j \geq 2m_i \), a page found at stack distance \( k \) in \( M_i \) implies its corresponding page can be found within stack distance \( 2k \) in \( M_j \).

**PROOF of LEMMA 4.1**

We prove by induction on \( t \).

At \( t=1 \), the statement is trivially true. At \( t=2 \) (i.e., after the first reference) \( S_t^i(1) \) and its corresponding page are both at the beginning of the stack, hence the induction statement is still true.

Suppose the induction statement is true at time \( t \), i.e.,
\[
p^i_{za} = S^i_t(k) \Rightarrow p^j_z \text{ can be found within stack distance } 2k \text{ within } S^j_t.
\]

Suppose the next reference is to \( p^i_{wa} \). There are three cases:

**Case 1**: \( p^i_{wa} \in M_t^i \) (\( p^i_{wa} = S^i_t(x) \))

From the induction statement, \( p^j_w \) is found within stack distance \( 2k \) in \( S^j_t \) as illustrated in Figure 10.

---

**Figure 10.**

Consider the page movements in the two stacks as a result of handling the reference to \( p^i_{wa} \):

1. \( p^i_{wa} \) and \( p^j_w \) are both moved to the top of their stack, the induction
(2) Each page in A increases its stack distance by 1, but its corresponding page is in A', each page of which can at most increase its stack distance by 1. Thus the induction statement holds for all pages in A.

(3) None of the pages in B are moved. None of the pages in B' are moved. (See previous diagram) If a page in B has its corresponding page in B', the induction statement is not violated. Suppose a page in B, \( p^i_{ba} = S^i_t(k) \) \((k > x)\), has its corresponding page, \( p^j_{b} = S^j_t(w) \) in A'. Then \( p^j_b \) can at most increase its stack distance by 1. But \( w \leq 2x \) because \( p^j_b \in A' \). Since \( 2k > 2x \), the induction statement is not violated.

**Case 2:** \( p^i_{wa} \notin M^i_t, \ p^j_w \notin M^j_t \)

Each page in \( M^i \) increases its stack distance by 1. Each corresponding page in \( M^j \) can at most increase its stack distance by 2, one due to the reference and one due to an overflow from \( M^i \). Hence if \( p^i_{za} = S^i_t(k), k < m^i \), then \( p^i_{za} = S^i_t+1(k+1) \), and \( p^j_z \) can be found within stack distance \( 2(k+1) \) in \( M^j \) at time \( t+1 \).

**Case 3:** \( p^i_{wa} \notin M^i_t, \ p^j_w \notin M^j_t \)

As a result of the read-through from \( M^i \), each page in \( M^j \) is increased by a stack distance of 1. That is, for \( k < m^i \),

\[
\begin{align*}
& p^i_{za} = S^i_t(k) \Rightarrow p^i_{za} = S^i_t+1(k+1). \\
& \text{Each page in } M^j \text{ can at most increase its stack distance by 2, one due to loading the referenced page and one due to an overflow from } M^i. \text{ Hence, the page } p^j_z \text{ is found within stack distance of } 2k+2 \text{ in } M^j. \text{ Since } \max(2k+2) = 2m^i = m^j, p^j_z \text{ is still in } M^j.
\end{align*}
\]

Q.E.D.
COROLLARY to LEMMA 4.1

\[ m_j > 2m_i \Rightarrow \forall r, t, \ (S_t^j(m_j))^i \cap S_t^i = \emptyset \]

PROOF of COROLLARY

For any \( p^i_z \) in \( S_t^i \), its corresponding page can be found within stack distance \( 2m_i \) in \( S_t^j \), and since pages in \( S_t^j \) are unique, the information in the last page of \( S_t^j \) is not found in \( S_t^i \), i.e., \( (S_t^j(m_j))^i \cap S_t^i = \emptyset \).

PROOF of Part (b) : \( m_j > 2m_i \Rightarrow \forall r, t, \ (M_t^j)^i \supseteq M_t^i \)

This follows directly from LEMMA 4.1.

Q.E.D.
THEOREM 5

Under GLOBAL-LRU-SOP, for any \( m_i \geq 2 \), \( \forall r, t \), an overflow from \( M_i \) finds its corresponding page in \( M_j \) iff \( m_j > m_i \)

COROLLARY

Under GLOBAL-LRU-SOP, for any \( m_i \geq 2 \), \( \forall r, t \), an overflow from \( M_i \) finds its corresponding page in \( M_j \) iff \( \forall r, t \), \( (M_t^j)^i \supseteq M_t^i \)

PROOF

This Proof has two parts as shown below.

**PROOF of Part (a) :** \( m_j > m_i \Rightarrow \forall r, t \), an overflow from \( M_i \) finds its corresponding page in \( M_j \)

From LEMMA 3.4 \( m_j > m_i \Rightarrow \forall r, t \), \( (M_t^j)^i \supseteq M_t^i \) and \( (S_t^j(m_j))^i \cap S_t^i = \emptyset \)
Suppose the overflow from \( M_i \), \( p_{oa}^j \) is caused by a reference to \( M_j \).
Then just before \( p_{oa}^j \) is overflowed, \( p_{oa}^j \) exists in \( M_j \)
After the overflow, \( p_{oa}^j \) finds its corresponding page still existing in \( M_j \)
Suppose the overflow, \( p_{oa}^j \), is caused by a reference to \( M_r \).
Then just before the overflow from \( M_i \), \( p_{oa}^j \) exists in \( M_j \) and \( (S_t^j(m_j))^i \cap S_t^i = \emptyset \)
i.e., the information in the last page of \( M_j \) is not in \( M_i \). This means that the last page of \( M_j \) is not \( p_{oa}^j \), thus, the overflow page \( p_{oa}^j \) finds its corresponding page still in \( M_j \) after an overflow from \( M_j \) occurs.

**PROOF of Part (b) :** \( m_j \leq m_i \Rightarrow \exists r, t \), such that an overflow from \( M_i \) does not find its corresponding page in \( M_j \)

From THEOREM 1, \( m_j \leq m_i \Rightarrow \exists r, t \), \( (M_t^j)^i \nsubseteq M_t^i \), then there exists \( p_{za}^i \in M_t^i \) and \( p_{za}^j \notin M_t^j \). We can find a reference string such that at the time of the overflow of \( p_{za}^i \) from \( M_i \), \( p_{za}^j \) is still not in \( M_j \).
A string of references to \( M_r \) will produce this condition.
Then at the time of overflow of \( p_{za}^i \), it will not find its corresponding page in \( M_j \).

Q.E.D.
THEOREM 6

Under GLOBAL-LRU-DOP, for \( m_1 \geq 2 \), \( \forall r, t \), an overflow from \( M^i \) finds its corresponding page in \( M^j \) iff \( m_j > 2m_i \).

COROLLARY

Under GLOBAL-LRU-DOP, for \( m_1 \geq 2 \), \( \forall r, t \), an overflow from \( M^i \) finds its corresponding page in \( M^j \) implies that \( \forall r, t, (M^i_t) \subseteq M^j_t \).

PROOF

This Proof has two parts as shown below.

PROOF of Part (a): \( m_j > 2m_i \Rightarrow \forall r, t, \) an overflow from \( M^i \) finds its corresponding page it \( M^j \)

THEOREM 4 ensures that \( m_j > 2m_i \Rightarrow \forall r, t, (M^i_t)^j \supseteq M^j_t \) and LEMMA 4.1 ensures that \( (S^j_t(m_j))^j \cap S^i_t = \emptyset \) , we then use the same argument as in Part (a) of THEOREM 5.

PROOF of Part (b): \( m_j \leq 2m_i \Rightarrow \exists r, t, \) such that an overflow from \( M^i \) does not find its corresponding page in \( M^j \)

Case 1: \( m_j < 2m_i \)

\( m_j < 2m_i \Rightarrow \exists r, t, (M^i_t)^j \supseteq M^j_t \) (from the proof of part(a) of THEOREM 4).

We then use the same argument as in Part (b) of THEOREM 5.

Case 2: \( m_j = 2m_i \)

The reference string \( r = " p^i_{1a}, p^i_{2a}, \ldots, p^i_{(2m_i)a}, p^i_{(2m_i+1)a} " \)

will produce the following stacks (at \( t=2m_i+1 \)):

\[ S^i_t = (p^i_{(2m_i)a}, p^i_{(2m_i-1)a}, \ldots, p^i_{(m_i+1)a}) \]

\[ S^j_t = (p^j_{m_i}, p^j_{2m_i}, p^j_{m_i-1}, p^j_{2m_i-1}, \ldots, p^j_{1}, p^j_{m_i+1}) \]

In handling the next reference, to page \( p^i_{(2m_i+1)a} \), the pages \( p^i_{(m_i+1)a} \) and \( p^j_{m_i+1} \) overflow at the same time, hence the overflow page \( p^i_{(m_i+1)a} \) from \( M^i \) does not find its corresponding page in \( M^j \).

Q.E.D.
THEOREM 7

Let $M^i$ (with $m_i$ pages), $M^j$ (with $m_j$ pages) and $M^r$ be System A.
Let $M'^i$ (with $m'_i$ pages), $M'^j$ (with $m'_j$ pages) and $M'^r$ be System B.
Let $m'_i \geq m_i$ and $m'_j \geq m_j$. Under GLOBAL-LRU-SOP, for any $m_i \geq 2$, no MLPA can exist if $m_j > m_i$ and $m'_j > m'_i$.

PROOF

We shall show that $\forall r, t, (M^i_t \cup (M^j_t)^i) \subseteq (M'^i_t \cup (M'^j_t)^i)$
This will ensure that no MLPA can exist.
Since $m'_i \geq m_i$ and LRU is used in $M^i$ and $M'^i$, we can apply the LRU stack inclusion property to obtain $M^i_t \subseteq M'^i_t$.
From THEOREM 5, we know that overflows from $M^i$ or from $M'^i$ always find their corresponding pages in $M^j$ and $M'^j$ respectively. Since SOP is used, these overflows can be treated as "no-ops".
Thus, $M^j$ and $M'^j$ see the same reference string and we can apply the LRU stack inclusion property to obtain $M^j_t \subseteq M'^j_t$ (since $m'_j \geq m_j$ and LRU is used).

$M^i_t \subseteq M'^i_t$ and $M^j_t \subseteq M'^j_t \Rightarrow (M^i_t \cup (M^j_t)^i) \subseteq (M'^i_t \cup (M'^j_t)^i)$

Q.E.D.
THEOREM 8

Let System A and System B be defined as in THEOREM 7.
Let \( m_j' \geq m_j \) and \( m_j \leq m_i' \). Under GLOBAL-LRU-DOP, for any \( m_i \geq 2 \), no MLPA can exist if \( m_j > 2m_i \) and \( m_j' > 2m_i' \).

PROOF

We need the following preliminary results for this proof.

LEMMA 8.1

Let \( S_t^j \) be partitioned into two disjoint stacks, \( W_t \) and \( V_t \) defined as follows: \( W_t(k) = S_t^j(j_k) \) for \( k=1, \ldots, |W_t| \) where \( j_0 = 0 \), and \( j_k \) is the minimum \( j_k > j_{k-1} \) such that \( \exists p_{za}^i \in S_t^i \) and \( p_{za}^i \notin S_t^i(j_k) \).
\( V_t(k) = S_t^j(j_k) \) for \( k=1, \ldots, |V_t| \) where \( j_0 = 0 \), and \( j_k \) is the minimum \( j_k > j_{k-1} \) such that \( \forall p_{za}^i \in S_t^i \), \( p_{za}^i \notin S_t^i(j_k) \). (Intuitively, \( W_t \) is the stack obtained from \( S_t^j \) by collecting those pages that have their corresponding pages in \( M_t^i \) such that the order of these pages in \( S_t^j \) is preserved. \( V_t \) is what is left of \( S_t^j \) after \( W_t \) is formed.)
Then, \( \forall r,t, (a) W_t \subseteq S_t^i \) and \( (b) V_t \subseteq O_t \) where \( O_t \) is the set of pages corresponding to all the pages that ever overflowed from \( M_t^i \), up to time \( t \).

PROOF of LEMMA 8.1

From THEOREM 4, \( m_j > 2m_i \Rightarrow \forall r,t, (M_t^j)^i \geq M_t^i \). Thus, for each page in \( M_t^j \), its corresponding page is in \( M_t^j \). This set of pages in \( M_t^j \) is exactly \( W_t \), and \( W_t \subseteq S_t^i \) by definition. Since the conditions for \( V_t \) and \( W_t \) are mutually exclusive and collectively exhaustive, the other pages in \( M_t^j \) that are not in \( W_t \) are by definition in \( V_t \).
Since a page in \( V_t \) does not have a corresponding page in \( M_t^j \), its corresponding page must have once been in \( M_t^i \) because of Read-Through, and later overflowed from \( M_t^i \). Thus a page in \( V_t \) is a page in \( O_t \).

Q.E.D.
LEMMA 8.2

Any overflow page from $M^j_t$ is a page in $V_t$

PROOF of LEMMA 8.2

From THEOREM 4, $m_j > 2m_i \Rightarrow \forall r, t, (M^j_t)^i \supseteq M^i_t$

From THEOREM 6, $m_j > 2m_i \Rightarrow \forall r, t$, an overflow from $M^j$ always finds its corresponding page in $M^j$

An overflow from $M^j_t$ is caused by a reference to $M^r$. An overflow from $M^j_t$ also implies that there is an overflow from $M^i_t$.

Suppose the overflow page from $M^j_t$ is $P^j_o$. Also suppose $P^j_o \in W_t$, i.e., $P^j_o \in V_t$. We shall show that this leads to a contradiction.

The overflow page from $M^i_t$ is either $P^i_o$ or $P^i_ya$ (yf o).

If $P^i_o \supseteq P^j_o$ is overflowed from $M^j_t$, THEOREM 6 is violated since $P^i_o$ and $P^j_o$ overflow at the same time so $P^i_o$ will not find its corresponding page in $M^j$.

If $P^i_ya \supseteq P^j_o$ is overflowed from $M^j_t$, THEOREM 4 is violated since after the overflow handling, there exists a page $P^i_{ob} \subset P^j_o$ in $M^i$ (since $P^j_o \in W_t$) but $P^j_o$ is no longer in $M^j$.

Q.E.D.

LEMMA 8.3

If there is no overflow from either $M^j$ or $M^i$ then $\forall r, t$, $V_t$ and $V^j_t$ have the same reverse ordering.

Two stacks $S^i$ and $S^j$ are in the same reverse ordering, $S^i \equiv S^j$, if $rS^i(k) = rS^j(k)$ for $1 \leq k \leq \min(|S^i|, |S^j|)$, where $rS$ denotes the stack obtained from $S$ by reversing its ordering. By convention, $S^i \equiv S^j$ if $S^i = \emptyset$ or $S^j = \emptyset$.
**PROOF of LEMMA 8.3**

To facilitate the proof, we introduce the following definitions.

1) The ordered parent stack, \((S^i)^j\), of the stack \(S^i\) is the stack of parent pages corresponding to, and in the same ordering as, the pages in the reduced stack, \(\bar{S}^i\), of \(S^i\). Formally, \((S^i)^j \equiv \bar{S}^i\) and \((S^i)^j \equiv \bar{S}^i\).

2) Define a new binary operator, concatenation \(\mid\mid\), between two stacks, \(S^1\) and \(S^2\), to produce a new stack, \(S\), as follows:

\[ S = S^1 \mid\mid S^2, \text{ where } S(k) = \begin{cases} S^1(k) & \text{for } k = 1, 2, \ldots, |S^1| \\ S^2(k) & \text{for } k = |S^1| + 1, \ldots, |S^1| + |S^2| \end{cases} \]

3) Define a new binary operator, ordered difference \(\circ\), between a stack \(S^1\) and a set \(T\), to produce a new stack, \(S\), as follows:

\[ S = S^1 \circ T, \text{ where } S(k) = S^1(j_k) \text{ for } k = 1, 2, \ldots, (|S^1| - |S^1 \cap T|), \]

where \(j_0 = 0, j_k = \text{minimum } j_k > j_{k-1} \text{ such that } S^1(j_k) \cap T = \emptyset\).

Intuitively, \(S\) is obtained from \(S^1\) by taking away those elements of \(S^1\) which are also in \(T\).

Figure 11 illustrates the LRU ordering of all Level i pages ever referenced up to time \(t\). Since there is no overflow from either \(M^j\) or \(M'^j\), the length of this LRU stack is less than or equal to \(\min(m_j, m'_j)\).

![Figure 11](image)

By the definition of \(V_t^i\), \(V_t^i = (Y_t^j) \circ (S_t^i)^j\).

But \((S_t^i)^j \mid\mid (X_t^j) \circ (S_t^i)^j\),

hence \(V_t^i = (Y_t^j) \circ ((S_t^i)^j \mid\mid (X_t^j) \circ (S_t^i)^j) = (Y_t^j) \circ ((S_t^i)^j \cup (X_t^j)^j)\).

Similarly, by the definition of \(V_t\), \(V_t = (Z_t^j) \circ (S_t^i)^j\).

But \((Z_t^j) = (X_t^j) \mid\mid ((Y_t^j) \circ (X_t^j))^j\),

hence \(V_t = ((X_t^j) \circ (S_t^i)^j) \mid\mid (((Y_t^j) \circ (X_t^j))^j) \circ (S_t^i)^j)\)

= \(((X_t^j) \circ (S_t^i)^j) \mid\mid (Y_t^j) \circ ((S_t^i)^j \cup (X_t^j)^j)) = ((X_t^j) \circ (S_t^i)^j) \mid\mid V_t^i\)

Thus, the two stacks are in the same reverse ordering. Q.E.D.
LEMMA 8.4

\forall r, t, (a) M^j_t \leq M^\hat{j}_t, (b) V_t and V'_t are either in the same reverse ordering or the last element of V'_t is not an element of V_t.

PROOF of LEMMA 8.4

(a) and (b) are true for any time before there is any overflow from either M^j or M'^j. (a) is true because any page ever referenced is in Level j, so a page found in M^j is also found in M'^j. (b) is true because of the result from LEMMA 8.3.

Assume that (a) and (b) is true for t. Consider the next reference at t+1.

Suppose this reference does not produce any overflow from either M^j or M'^j, then (a) still holds because M^j_t \supseteq M^\hat{j}_t and M'^i_t \supseteq M^i_t (See THEOREM 7).
(b) still holds because overflows from M^j and M'^j are taken from the end of stacks V_t and V'_t respectively, and since there is no overflow from Level j, (b)'s validity is not disturbed.

Suppose this reference does produce overflow(s) from Level j.

Case 1: overflow from M^j, no overflow from M'^j:

This cannot happen since overflow from M'^j implies reference to M'^i which in turn implies overflow from M^j also.

Case 2: overflow from M'^j, no overflow from M'^j:

Suppose the last element in V'_t is not an element of V_t. Then starting from the end of V'_t, if we eliminate those elements not in V_t, the two stacks will be in the same reverse ordering. This follows from LEMMA 8.3 and is illustrated in Figure 12.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12.png}
\caption{Figure 12.}
\end{figure}
Thus we see that overflow from $M^j$, i.e., overflowing the last page of $V_t$, will not violate (a) since this page is still in $V_t$. (b) is still preserved since the last page in $V_t$ is still not in $V_t$.

Suppose $V_t$ and $V_t$ are in the same reverse ordering. Then overflowing the last page of $V_t$ does not violate (a) and results in the last page of $V_t$ not in $V_t$.

**Case 3:** overflow from $M^j$ and overflow from $M^j$:

- Suppose the last element in $V_t$ is not in $V_t$. Refering to the diagram in Case 2, we see the result of overflowing the last element of $V_t$ and the last element of $V_t$ does not violate (a) and still preserves the condition that the last element of $V_t$ is not in $V_t$.

- Suppose $V_t$ and $V_t$ are in the same reverse ordering. Then overflowing the last elements of $V_t$ and $V_t$ leaves $V_t$ and $V_t$ still in the same reverse ordering. (a) is not violated since the same page is overflowed from $M^j$ and $M^j$.

Q.E.D.

**Proof of Theorem 8**

$M^i \supseteq M^i$ for the same reasons as those used in Theorem 7.

From Lemma 8.4 $M^j \supseteq M^j$.

Hence, $(M^i_t \cup (M^j)^i_t) \subseteq (M^i_t \cup (M^j)^i_t)$

Q.E.D.
4. Conclusions

We have developed a model of a data storage hierarchy system specifically designed for very large databases. This data storage hierarchy makes use of different page sizes across storage levels and maintains multiple copies of the same information in the hierarchy.

Four algorithms obtained from natural extensions to the LRU algorithm are studied in detail and key properties of these algorithms that affect performance and reliability of the data storage hierarchy are derived.

It is found that for the LOCAL-LRU algorithms, no choice of sizes for the storage levels can guarantee that a lower storage level always contains all the information in the higher storage levels. For the GLOBAL-LRU algorithms, by choosing appropriate sizes for the storage levels, we can (1) ensure the above inclusion property to hold at all times, (2) guarantee that no extra page references to lower storage levels are generated as a result of handling overflows, and (3) guarantee that no multi-level paging anomaly can exist.

Several areas of further study emerge from this investigation. These include the study of store-behind algorithms [18] and the study of extensions to other known storage management algorithms. We hope that this study motivates further work in the area of generalized data storage hierarchy systems for very large databases.
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References


