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This report is essentially the same as a doctoral thesis in the Department of Electrical Engineering, M.I.T.

Abstract

The magnetron is described in terms of its properties as a feedback oscillator, to show that nonlinear circuit theory may be applied in discussing mode interactions. The interaction of modes in a nonlinear feedback oscillator with two resonances is considered, and it is shown that when large-amplitude oscillation associated with one mode of resonance is present, this oscillation tends to suppress oscillation associated with the other mode. These theoretical observations are supported directly by measurement of the loading effect of an oscillating mode in a magnetron upon small-amplitude externally-supplied oscillations in another mode. They are also supported indirectly by observation of the performance of several different types of magnetrons.
I. Introduction

The Problem

One of the most perplexing problems in magnetron work has been the "moding" problem. This problem arises because of the multiplicity of resonances in the anode structure, and because it is possible for the electron stream to support oscillation in any one of several modes of resonance. Also associated with the moding problem is the fact that under certain circumstances, the oscillating mode may fail altogether, and this event may or may not be followed by the starting of another mode of oscillation. Causes for mode failure are often obscure, especially when such failure is accompanied by the starting of oscillation in another mode.

In order to achieve proper operation of a magnetron, it is necessary to establish stable large-amplitude oscillation in only one of these modes. Some of the most widespread applications of magnetrons require microsecond-pulse operation. In magnetrons designed for such operation, it is necessary to establish oscillation in the desired mode positively and quickly, and to maintain it stably for the duration of the pulse. In magnetrons designed for c-w operation, quick starting is usually not a requirement, but stability is as important as in pulse magnetrons. Thus, the problems to be discussed here are of two kinds. First, the mode selection problem involves the establishing of oscillation in the desired mode positively and quickly in pulse magnetrons. Second, the mode stability problem involves keeping the established oscillation stable, in either pulse or c-w magnetrons. Either of these two problems may be concerned with the avoidance of oscillation in unwanted modes.

The significance of the mode stability problem is emphasized by the fact that one of the limits on high power in magnetrons is the maximum power for which the desired mode of oscillation is stable.

The general form of the solution of the wave equation for the electric field in the interaction space, assumed to be an infinitely long cylinder, is: \( E \exp(jp\theta) Z_p(kr) \), where \( Z_p \) indicates a linear combination of Bessel and Neumann function of the \( p \)th order, and \( k \) is the propagation constant such that \( k = 2\pi/\lambda \), where \( \lambda \) is the free-space wavelength at the frequency considered. To match the solution to the boundary conditions at the inside periphery of the interaction space, Hartree (ref. 16) resolved the r-f electric field into Fourier components in space, and these components are called Hartree harmonics. In an anode structure where all \( N \) cavities are alike, the values of \( p \) for which r-f field components are actually present in the \( n \)th mode are given by \( p = n + AN \), where \( A = 0, \pm 1, \pm 2, \) etc. If the cavities alternate between two sizes, as in the rising-sun anode structure, \( p = n + N/2 \).

The field configurations described above lead to \( N \) possible modes. The actual pattern around the anode is described by
\[
\sum_{A=-\infty}^{\infty} E_p \exp j(\omega t - \frac{2\pi}{p} \theta) + \sum_{A=-\infty}^{\infty} E_p' \exp j(\omega t + \frac{2\pi}{p} \theta).
\]

Thus there are two sets of waves, traveling in opposite directions. Only one such configuration, and therefore only one mode, can exist for \( n = N/2 \) and for \( n = 0 \). However, the solutions of the wave equation for other values of \( n \) in a perfectly symmetrical anode structure are not unique because the phase and amplitude relationships between the two sets of traveling waves are not determined. The presence of a coupling to the external circuit at one cavity removes the degeneracy of the solution (ref. 3, p. 215), and electric fields of the form

\[
E \cos \frac{2\pi m}{N}
\]

and

\[
E \sin \frac{2\pi m}{N}
\]

are possible across the mouth of the \( m \)th cavity, where \( m = 0 \) at the output cavity, providing the loading is small. Hence there are two solutions for each integral value of \( n \) between \( n = 0 \) and \( n = N/2 \). If \( n \) is greater than \( N/2 \), say \( (N/2) +1 \), it is easy to show that a traveling wave in the opposite direction, with \( n = (N/2) -1 \), has the same field configuration as the original traveling wave, and therefore \( n = (N/2) +1 \) does not represent an additional mode.

It is significant that a resonance characterized by an r-f electric field of the form:

\[
E \sin \frac{2\pi m}{n},
\]

where \( m = 0 \) at the output cavity, is very lightly loaded. This condition follows from the fact that there is a very weak r-f electric field at the mouth of the output cavity, so very little energy is carried out through the output circuit.

Oscillation in a given mode takes place when the rotating electron stream couples to a traveling wave corresponding to one of the Hartree harmonics of one of the resonant modes described. Under the proper conditions, a regenerative action takes place in which the r-f electric field tends to bunch electrons in synchronism with itself, and in such phase that the electrons give up energy to it. Operating modes are designated by \( \gamma/n/N \), where \( \gamma \) represents the particular Hartree harmonic to which the electrons are coupled; \( n \) and \( N \) have been defined. It is convenient to visualize the electron configuration as being in the form of \( \gamma \) spokes, as shown in Fig. 1. The \( \pi \)-mode in an eight-cavity magnetron, shown in the figure, is designated 4/4/8.

Mode Selection

If an electron is to fall into synchronism with a rotating r-f wave in a cylindrical magnetron, it is necessary that its velocity in absence of the r-f field be somewhere near the velocity of the rotating wave. This is particularly true during the build-up transient, when the r-f electric field which is available to act on the electron, and thus maintain synchronism, is small. According to Slater (ref. 4, p. 107), in order
that an electron in a circular orbit at radius $r_o$ should have a velocity $v$, the following conditions must be met

$$E = v(B - \frac{m}{e} \frac{v}{r_o})$$

($E$ = electric field intensity, $B$ = magnetic flux density, $m$ = mass, and $e$ = charge of the electron; units are rationalized mks). From this he estimates the d-c potential difference between anode and cathode for synchronism between electrons and r-f traveling waves. Hartree (ref. 16) has developed an expression for the minimum d-c anode voltage for an electron to reach the anode in the presence of an infinitesimal r-f rotating wave. Hartree's expression for voltage, like Slater's, is a function of the velocity of the r-f rotating wave, of anode and cathode diameters, and of magnetic flux density. Hartree's criterion provides better agreement with observed magnetron operation than Slater's, although they are qualitatively similar in most respects.

In Fig. 2, the Hartree voltages for three modes in a typical eight-resonator magnetron are shown as functions of magnetic field. Also shown is the d-c cut-off curve, at which it is possible for electrons leaving the cathode with zero velocity to reach the anode in the presence of constant fields (ref. 3, p. 177).

Hartree (ref. 16, also ref. 1, p. 340) was apparently the first to point out the possible rotating wave components associated with any given mode in a multisegment resonant anode structure, and that it is possible for the electron stream to become coupled, under different conditions, to different components corresponding to the same mode of resonance.

Bunemann (ref. 1, Ch. 6, p. 253) uses another approach to establish a necessary condition for oscillation. He assumed first the existence of the Brillouin steady state, which represents a solution to the electron motion problem. In such a solution, the radial current is zero, and the electrons are all moving in concentric circles. This is
also referred to as a single-stream solution of the electron-motion problem. (Twiss (ref. 31) has shown that a single-stream solution is impossible in the presence of a Maxwellian distribution of emission velocities, and that a double-stream solution is always possible under such circumstances. The single-stream solution may be approached as a limit as emission velocities approach zero. The question of how the electrons in a magnetron could ever fall into the single-stream solution was also raised.)

Under certain conditions of voltage and magnetic field, a perturbation of the single-stream steady state corresponding to an r-f rotating wave may cause the space charge to become unstable, and spokes (cf. Fig. 1) will build up, thus initiating oscillation. When such a voltage, corresponding to regeneration in a particular mode, is reached, it is called the instability voltage for that mode.

Instability voltages have been computed (ref. 22), and it is found that for values of $\gamma$ of four or less, the instability voltages are comparable with threshold voltages and represent plausible standards for minimum starting voltages in terms of observed starting voltages. For larger values of $\gamma$, the instability voltages become further removed from both the threshold voltages and observed starting voltages. It is necessary to conclude that the instability voltage criterion is, in general, not applicable to actual magnetrons. On the other hand, threshold voltages are never in very great disagreement with observed starting voltages.

Fletcher and Rieke (ref. 14; ref. 1, Ch. 8) have pointed out the importance of pulse modulator characteristics, especially the rate of rise of the voltage pulse applied to the magnetron, and the pulser's output impedance. They have shown how a pulse may rise so rapidly that it passes through the range in which the desired mode can start before oscillation in that mode can build up appreciably, and into a region where an undesired mode, or no mode at all, can start. If oscillation in the desired mode could have built up more quickly, or if the rise in voltage had been slower, oscillation would have caused direct current to flow, and the flow of current would have held the voltage down to a value where oscillation in the desired mode could have persisted. On the other hand, it is pointed out that if the open-circuit voltage of the pulser falls within the range in which the desired mode can start, or if the rate of rise of the pulse can be reduced, it should be possible to eliminate misfiring (that is, failure to start in the desired mode).

The problem of competition between modes, which occurs when the anode voltage of the magnetron is rapidly raised from zero to a value at which oscillation in either of two modes could be supported, is discussed by Rieke (ref. 1, Ch. 8). The process of one mode gaining ascendancy over the other and eventually suppressing it is described from a nonlinear circuit point of view. In particular, he points out that when the amplitudes of oscillation in both modes are so large that nonlinear effects are important, it is necessary that the rate of build-up in either mode be affected more by the amplitude of oscillations in the other mode than by its own amplitude. Such a condition is necessary in order that mode selection be definite on any particular pulse. This condition does not preclude uncertainty of selection as between successive pulses of the same
amplitude.

Here, Rieke has described the assumption that the rate of build-up of one mode is affected more by the amplitude of the other mode than by its own amplitude, as being open to question. It is evident that it must be valid for all, or at least nearly all, of the observed cases, because of the definiteness of mode selection on any particular pulse. In a later section, theoretical reasons tending to confirm Rieke's assumption will be advanced.

Mode Stability and Mode Changes

Mode selection has been discussed as a transient problem; here, mode stability, a steady-state problem, will be considered. The only kind of magnetron in which a strictly steady-state analysis is applicable is the c-w magnetron. In many stability problems in pulse magnetrons, the time required for appreciable changes in operating conditions is very long in terms of r-f cycles, and the steady-state analysis is entirely acceptable. In many other pulse magnetrons, a borderline condition between a mode-stability problem and a mode-change problem is encountered. This borderline situation is considered, although some of the fundamental principles previously mentioned apply.

It should be emphasized that starting criteria may not necessarily be expected to apply for a given mode, once large-amplitude oscillation has been established in another mode. For example, the Hartree starting criterion, i.e. the threshold voltage, is no longer as applicable as it was before oscillation started. This criterion specifies the anode potential at which an electron can just reach the anode in the presence of an infinitesimal r-f wave rotating with a given velocity, and in the presence of a given magnetic field, for any particular magnetron, assuming zero emission velocity. In the presence of large-amplitude oscillation in one mode, another r-f rotating wave of small amplitude can have little effect upon whether an electron reaches the anode. It is significant that the threshold voltage represents a necessary but not sufficient condition for oscillation.

Many of those who have worked with magnetrons have accepted starting criteria, which were intended to apply only to the given mode in the absence of others, as being equally applicable to the given mode in the presence of large-amplitude oscillation in another mode. Some of these ideas often seemed to be confirmed when magnetron design changes were made. An example of such an idea, which became fairly widespread, at least tacitly, is that raising the starting voltage of the next higher-voltage mode above the π-mode will necessarily increase the maximum voltage at which the π-mode can be operated stably. There are two possible reasons for reaching this conclusion. First, the use of straps increased the upper limit of input current and input voltage for which a magnetron was stable in the π-mode. The straps also increased the resonant frequency for other modes as compared with the π-mode, and therefore increased their respective starting voltages. Hence, the voltage of the π-mode could be raised further above the threshold value without reaching the starting
voltage for another mode. Some were led to the conclusion that the fundamental improvement in stability resulted from the separation of threshold voltages. The presently accepted explanation for the principal cause of the improvement is that the separation of the resonant frequencies prevents contamination of the π-mode r-f pattern with components of other modes, as discussed previously (ref. 1, Ch. 4). Another cause of confusion was the application of the expression "moding," both to improper starting and to instability of the desired mode. It is reasonable to expect that starting in the wrong mode becomes more difficult as its range of starting voltages becomes further removed from the starting range for the desired mode.

Another concept of mode stability based entirely on other possible modes has been advanced by Copley and Willshaw (ref. 22; ref. 8, p. 1000). They first assume that both the threshold and the instability voltage criteria must be met before oscillation takes place. After oscillation is established, the applied voltage may be increased indefinitely with corresponding increase in power, until the instability voltage for another mode is reached. At this point oscillation in the original mode ceases, but oscillation in the second mode will not start unless (or until) the threshold voltage for the second mode has been reached. Thus it is possible that there will be a range of anode voltages in which no oscillation will take place. Calculations showed that if the cathode diameter were increased, the values of threshold voltage decreased while values of instability voltage increased. Thus, the voltage range between the threshold voltage of the desired mode (in all cases under discussion by these authors, the π-mode) and the instability voltage of the next higher-voltage mode can be increased; since, according to the theory, the maximum input current is approximately proportional to this voltage range, to maximize this voltage is to maximize input power. If the increase in cathode diameter does not reduce efficiency too much, such a procedure should be expected to lead to greater output power.

The application of this mode-change criterion to actual magnetrons produced agreement with theory in some respects, at least qualitatively. The most significant bit of agreement was observed when the cathode diameters were increased. This change actually led to considerable increases in output, as well as input, power.

The latter theory of mode change is based on several assumptions which are very much open to question. Doubt as to the possible existence of a Brillouin steady state has been mentioned. Another doubtful point is whether the calculated instability voltages could have any significance when large-amplitude oscillation is already present. Dunsmuir (ref. 9) has raised a much more fundamental question: whether a lower-voltage mode, whose conditions for oscillation have already been met in terms of threshold and instability voltages, should necessarily give way to a higher-voltage mode as soon as the latter's conditions for oscillation are met.

Experimental evidence refutes this mode change criterion, and any other general criterion based solely, or even primarily, on the unwanted modes. Some of the experimental work reported on in a subsequent section shows clearly a set of mode changes.
depending primarily on conditions in the originally oscillating mode, instead of the mode into which the change took place; in another test, \( \pi \)-mode oscillation is maintained stably as the applied voltage is raised past both threshold and instability voltages for at least one other mode, and it stops only when d-c cutoff is reached.

The magnetron improvements which were accomplished by application of the instability-voltage criterion can be explained otherwise. An increase in cathode radius leads to more stable operation because of the increased intensity of the r-f electric field available near the cathode for bunching; the intensity of the latter field for the \( p \)-th order Hartree harmonic is proportional to \( r^p \), where \( r \) is one of the coordinates of the cylindrical system. This effect will be discussed further.

The maximum-current limitation in magnetrons had also been encountered at the Bell Telephone Laboratories (ref. 18, ref. 1, p. 378). The problem was to design a magnetron in the 1220-Mc/sec to 1350-Mc/sec range (L-band) for high-power operation. The maximum current limitation in other L-band magnetrons had been recognized as being related to the rate of rise of the applied voltage pulse. It was further recognized that the failure to operate in the \( \pi \)-mode was independent of the presence of other modes.

This problem involves some of the aspects of both mode skip or misfiring and the kind of stability problem met with in c-w magnetrons and in those with slowly rising pulses. A quantity, critical current \( I_c \), was defined as the input current at which there was first observed a narrowing of the current pulse at the leading edge. It was found that as the rate of voltage rise became less, \( I_c \) increased. A maximum value was approached which could not be exceeded by any further decrease in the rate of voltage rise.

A systematic experimental study was made to determine the effect of the variation of design parameters upon \( I_c \). As in the magnetrons reported on by Copley and Willshaw, it was found that stability of the \( \pi \)-mode was increased by increasing the ratio of cathode radius to anode radius; it was also found that lighter loading increased \( I_c \). Unfortunately, each of these changes which might be made to increase \( I_c \) also would decrease electronic efficiency. The idea of building a magnetron which was very much dependent on a slow rise of applied voltage was rejected, because this would limit the versatility of the magnetron more than was considered desirable. As a result of the above considerations a magnetron was designed with a much larger ratio of cathode radius to anode radius than previous eight-segment magnetrons.

Problems arising from high rate of voltage rise as compared with the rate of build-up are more acute in lower frequency magnetrons, such as L-band, as compared with S-band and higher frequencies. In magnetrons which are equivalent otherwise, but have different frequencies, the build-up rates per cycle should be the same.

An explanation for this type of combined misfiring and stability problem apparently depends on the fact that the building up of oscillation causes current to flow from the pulser, and thus reduces the applied voltage, and also that greater r-f amplitude increases the voltage range for which electrons can be kept in synchronism with the
r-f traveling wave. The faster r-f build-up can take place in relation to the rate of rise of the applied voltage, the greater the chance of stability. There is a continuous transition from the mode-skip transient problem to the steady-state stability problem.

Some experiments in magnetron design were made at the M.I.T. Radiation Laboratory for the purpose of making a mode less stable (ref. 15). The test magnetrons all had K-7 (2J32) anodes, with anode diameter of 13.6 mm, and cathode diameters of 4, 5, 6, and 8 mm were used. The problem was to discourage oscillation in the 5/3/8 mode which has a slightly lower starting voltage than the π-mode (4/4/8). Most efficient operation was found using the 4-mm cathode, but the cathode melted at moderate powers. Operation with 5-mm cathodes (for which the tube was originally designed) was satisfactory, while operation with 6-mm or 8-mm cathodes led to large areas on the performance chart where the unwanted mode interfered with operation in the π-mode. Thus, increasing cathode diameter has made the unwanted lower voltage mode more stable, to the extent that it interferes with π-mode operation.

Litton Industries (ref. 21) of San Carlos, California, has altered the design of a tunable c-w magnetron for the purpose of increasing the maximum current for which the π-mode is stable. The approach used here was to increase the perveance (E. E. Dept. M.I.T.: Applied Electronics, p. 169, Wiley, 1943) of the magnetron considered as a simple d-c diode. The perveance was increased 20 percent in one magnetron by increasing the cathode diameter by 10 percent. The maximum input current for π-mode operation was increased 75 percent or more, depending on tuning, with little change in efficiency. It is also interesting to observe that the increase in $I_o^*$ brought about by the increase in the cathode radius was only 18 percent. A further small increase in perveance was made by decreasing the anode radius, and this change of dimension led to almost no change in $I_o$. The latter change led to a decrease in the upper current limit in the π-mode at the maximum-frequency setting of the tuner, but to an increase by a factor of about four times at the minimum frequency.

The failure of a mode of oscillation as a result of the failure of the bunching mechanism to maintain synchronism between rotating electrons and the r-f field was implied by Slater in 1941 (ref. 4). It has been discussed by this writer (ref. 19) and investigated by the General Electric Research Laboratories (ref. 20). The failure of oscillation is interpreted as being due to the tendency of a high d-c electric field to make the electrons rotate at a rate higher than that of the r-f traveling wave. These phenomena will be discussed in detail in the following section.

An attempt has been made recently by Welch and others at the University of Michigan to establish more definite mode stability criteria, and an estimate has been made of the space-charge limited current in an oscillating c-w magnetron (ref. 35). This does not necessarily lead to an adequate criterion for the maximum current associated with stable π-mode oscillation, because space-charge limited current is a function of anode voltage, and there is yet no adequate theory which leads to a maximum voltage for stability.

$I_o$ is one of the magnetron characteristic scale factors. See Ref. 1, Ch. 10.
II. The Magnetron as a Feedback Oscillator

Energy Conversion in Oscillators

To make it possible to describe properly the characteristics of any particular mode of oscillation, the magnetron will be considered briefly here in terms of its properties as a feedback oscillator.

In electronic oscillators in general, there is an active electronic system, which converts d-c energy into a-c energy, and there is a passive system which is frequency-sensitive and which controls electron flow or electron motion in the active system in such a way that the active system can produce a net output great enough to supply energy for losses and load. In the cylindrical magnetron, electrons tend to travel in more or less circular paths; if these electrons can be caused to rotate in synchronism with an r-f traveling wave, and to be bunched in the proper phase to give up energy to the wave, then energy conversion in the electron stream of the magnetron can take place.

In usual magnetron operation, the initial bunching results from one mechanism, and bunching is maintained by another. These have been described in some detail by Fisk, Hagstrum, and Hartman (ref. 3, pp. 189-196), and the fundamentals are reviewed briefly below.

Upon leaving the cathode with zero velocity, in the presence of nontime-varying fields (radial electric, axial magnetic field), the electron would start toward the anode, but its path would be bent by the magnetic field and it would return toward the cathode, reaching the cathode surface with zero velocity. If there is, in addition, an r-f wave rotating in the same direction as the electron, then the electron may receive from or give up energy to the r-f field. If it absorbs energy, it will be driven into the cathode with a finite velocity, and will be taken out of the system. If it gives up energy to the system it will return to a point of zero radial velocity in the space between cathode and anode. Thus it has had work done on it by the d-c electric field, and has done work on the r-f field. It is now in a position to stay in the system, and under favorable conditions, it may make more such loops toward the anode, giving up energy to the r-f field each time. Ideally when such an electron finally reaches the anode, it will have converted most of the energy put into it by the d-c field into r-f energy. The electrons emitted in such a phase as to extract energy from the r-f field will be referred to as unfavorable, and those emitted so as to add energy to the r-f field will be called favorable electrons.

In the reference mentioned above (ref. 3, pp. 189-196), the phase-focusing mechanism is also described. Under normal conditions, this mechanism will apply principally to favorable electrons, with the others removed from the system as they are driven back into the cathode. When an electron leads the position in the r-f traveling wave at which the electron would give up the maximum amount of energy to the tangential component of the r-f field, it is also acted upon by the radial component of the r-f field. The latter component is directed in such a way that, in combination with the
axial magnetic field, the electron is accelerated toward the position at which it would give up maximum energy to the r-f field. On the other hand, a lagging electron is subjected to the action of the opposite r-f radial field, and is likewise accelerated toward the position where it would give up maximum energy to the tangential r-f field.

Build-Up Process

Small-amplitude phenomena in magnetrons are much less well understood, qualitatively, than are large-amplitude phenomena. For this reason a completely satisfactory explanation of the build-up process is not yet possible. An attempt will be made here to supply a description of r-f build-up that is plausible from a qualitative point of view.

In the presence of only one r-f rotating wave, and with zero emission velocity, the rejection of unfavorable electrons should be complete as soon as any coherent r-f wave is present. In the presence of a Maxwellian distribution of emission velocities, it is to be expected that the rejection process will be incomplete for small r-f amplitude, and will increase in effectiveness with increasing r-f amplitude. During early stages of build-up, synchronism of electrons with the r-f wave will not necessarily be maintained all the way to the anode. In the first place, the average velocity toward the anode is small, being proportional to the r-f amplitude; in the second place, the effect of the r-f phase-focusing field may be expected to be small, in general, as compared with whatever tendency the d-c field, in combination with the magnetic field, may have toward making the electrons rotate at a velocity other than in synchronism with the r-f wave. If the velocity of electrons is even approximately equal to that of the r-f wave, the bunched electrons, starting in the right phase (the unfavorable electrons having been rejected), should stay in the right phase long enough to lead to a net contribution of energy by electrons to the r-f wave.

As r-f amplitude increases further, the phase-focusing mechanism becomes increasingly effective in maintaining synchronism of the electron stream. The amplitude of r-f voltage necessary for effective phase focusing depends on the difference between the rotational velocity of the r-f wave and that of the electrons in the absence of the r-f field.

The final stage occurs when both of these bunching mechanisms are complete, and the electron stream is, to a first approximation, a constant-current generator. Then the r-f voltage $V_{rf}$ approaches a limit $V_{\text{max}}$ as follows

$$V_{rf} = V_{\text{max}} \left( 1 - e^{-at} \right).$$

Some r-f build-up measurements were made by Fletcher and Lee (ref. 32; ref. 1, pp. 373-376) at the M.I.T. Insulation Research Laboratory. These results are plotted in Figs. 3 and 4. Figure 4 shows that in the first stage, the build-up process is exponential, and not very sensitive to anode voltage, once the process has started. (The later starting when anode voltages are lower may be due to the greater delay in
reaching threshold voltage on the anode when peak amplitude is less, or to less noise in the electron stream.) This stage in Fig. 4 may well be associated with the first stage, when the electron rejection process is becoming more effective. Above 200 watts r-f power output, the rate of build-up is more dependent on anode voltage. This may reasonably be associated with the second stage, when rate of build-up is increasingly dependent on the effectiveness of the phase-focusing. For still higher levels of output power, the curves take on the form $V_{\text{rf}} = V_{\text{max}}(1 - e^{-at})$, indicating that bunching is practically complete.

While the above explanation appears reasonable, it is also evident that there is much room for further investigation on this subject.

"Strength" of Modes

Reference has been made to competition between modes, especially during build-up, and to the ability of a mode of oscillation to persist at high current and high power output. The strength of a mode may be considered a measure of its ability to persist either against possible competition from other modes, or against the destructive effect of excess anode voltage, which would
tend to accelerate electrons to a velocity greater than that of the r-f wave. Evidently
the principal factor determining the strength of a mode is simply the effectiveness of
feedback, as might be expected for any feedback oscillator. (No attempt will be made
to take into account the fact that the relative strengths of two modes might not neces-
sarily be the same in the transient case as in steady-state conditions.)

The effectiveness of feedback may be expressed in terms of the loop transmission
in a feedback amplifier. In such an amplifier, gain = \( \mu / (1 - \mu \beta) \), where \( \mu \) = gain without
feedback, and \( \beta \) represents the proportion of output signal fed back to the input. In an
oscillator, for steady-state operation, \( \mu \beta = 1 \). When \( \mu \beta \) is greater than unity, oscilla-
tions are building up. In usual feedback oscillators, \( \beta \) depends on the passive circuit
and is constant. This may be taken to be the case for magnetrons. Then \( \mu \) is dependent
on amplitude of oscillation, and on loading of the resonant circuit. The feedback ratio,
\( \beta \), depends on the geometry of the anode and of the space between cathode and anode. It
is quite important that bunching be effective near the cathode, but it is in this region
that the r-f electric field is weakest, since the magnitude of the r-f radial electric field
is approximately proportional to \( r^p \), for the \( p^{th} \) order Hartree harmonic. Hence, \( \beta \) is
quite dependent upon \( r_c / r_a \), the ratio of cathode radius to anode radius. The dependence
of stability on this ratio was pointed out by Slater in 1941 (ref. 4); he estimated optimum
values of \( r_c / r_a \) as a function of \( N \) for \( \pi \)-mode operation, seeking a good compromise
between efficiency and stability.

For a high degree of stability, it is desirable that the feedback signal be large, so
that oscillations will be as insensitive as possible to disturbances. This leads to a high
value of \( \mu \beta \) for small signals, resulting in a relatively fast build-up. The effect of such
a condition in terms of mode interactions will be discussed in more detail.

If only one Hartree component is considered in analyzing the magnetron as a feed-
back oscillator, the analogy between the magnetron and a conventional feedback oscil-
lator is quite straightforward. The feedback mechanisms in a magnetron oscillator
with one r-f rotating wave have been discussed. It has been assumed that the traveling
waves corresponding to other Hartree harmonics pass by the electrons so quickly as to
have no net effect. The validity of this assumption is supported by the agreement
between calculated and measured threshold voltages, since threshold voltages are cal-
culated using such an assumption.

Under some circumstances other Hartree harmonics cannot be safely ignored in
considering the feedback process. This is particularly true when the component to
which the electron stream is coupled is of higher order than the lowest order compon-
ent present. It has been pointed out that the intensity of the radial r-f field component
of a \( p^{th} \) order Hartree harmonic is approximately proportional to \( r^p \). Thus, r-f field
components which are of comparable magnitude near the anode may differ greatly near
the cathode, with the advantage to the lowest-order component. An extreme example
is illustrated in Fig. 5 by the detected r-f radial field pattern picked up by a rotating
probe, displayed on a cathode-ray screen, as a function of angular position of the probe.
The mode of resonance corresponds to $n = 8$ in an 18-segment rising-sun anode, but near the cathode the $p = 1$ component predominates in spite of the fact that at the anode the magnitude of the $p = 8$ component is larger. (It has been mentioned that possible orders of Hartree harmonics of the $n^{th}$ mode are given by $p = n + A(N/2)$ in rising-sun anodes, where $A$ is a positive or negative integer or zero.)

The result is that the rejection of unfavorable electrons by the desired component is seriously disrupted by the presence of a much larger component of the same mode. The $8/8/18$ mode has never been observed in magnetrons having anode structure similar to the one which produced the field pattern of Fig. 5, nor has the lower-voltage $10/8/18$ mode, corresponding to a higher-order harmonic of the same resonance, been observed. It is the latter mode which would be more likely to interfere with $\pi$-mode operation, because its theoretical threshold voltage is only slightly less than that of the $\pi$-mode.

A factor which must not be neglected in considering the strength of a mode of oscillation is the loading of the system, including both power to the load and circuit losses. In general, lighter loading leads to greater r-f amplitudes, and greater r-f amplitude leads to a system less easily disturbed by anything outside that mode of oscillation.

Mode Failure in Absence of Other Modes

Failure of oscillation in magnetrons, as anode voltage and current are raised, may, in general, be considered the result of failure of the feedback mechanisms. The fundamental causes of such failure appear to be first, the inability of the r-f field to maintain synchronism between itself and the electrons; second, competition from other modes of oscillation; and third, the flow of direct current to the anode when d-c cutoff is reached, where current can flow to the anode in absence of r-f field. There are secondary factors which contribute to the first two primary causes of failure, but only the three named here seem fundamental.

Here only the first cause will be considered. The second will be discussed in detail in subsequent sections. The third is rarely encountered, and it is evident that nothing can be done about it, except to increase magnetic field, and thus increase the value of d-c cutoff voltage.

In the first case the r-f radial field is no longer strong enough to keep electrons in synchronism with itself, and the feedback mechanism becomes seriously impaired. As the anode voltage is increased in an oscillating magnetron, the tendency to pull electrons

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*The magnetrons referred to here are the MF-series, which have been wavelength scaled to 10.7 cm from the 3.16 cm AX-9 (Columbia Radiation Laboratory). The MF-series are under development at the Research Laboratory of Electronics, M.I.T.*
ahead of synchronous velocity also increases. The electrons now tend to lead the point in the rotating wave at which maximum energy would be given up, and their effectiveness becomes diminished. Eventually the combination of excessive acceleration of the d-c field, combined with the decreased effectiveness of the electrons in building up the r-f field, leads to failure of oscillations.

The increase in anode voltage also leads to increased amplitude of oscillation and therefore the phase-focusing action may increase at a rate comparable with or greater than the increasing tendency of the d-c field to make electrons exceed synchronous velocity. Another effect of the increased r-f amplitude is that the transit time of an electron between cathode and anode is shortened. This has the effect of decreasing the electronic efficiency (detrimental to stability), but it also decreases the time during which a favorable electron can get out of phase.

Another requirement placed upon the phase-focusing mechanism is to prevent dispersion of electron spokes, which would otherwise take place as a result of the mutual repulsion of electrons. Little information is available on how charge density within the spokes varies with changes in operating conditions. If, at higher power levels, the density is higher, the debunching effect due to mutual repulsion is stronger. This effect will increase the tendency toward instability. However, at higher power levels, greater debunching should be opposed by greater r-f field intensity.

A secondary cause of failure is inadequate emission. In order to get an increase of r-f amplitude necessary to maintain stability in the face of increased d-c voltage, an increase in current is necessary. If the cathode cannot supply the necessary current, the collapse of oscillation then results from deficient phase focusing. The cathode also produces more subtle effects, since the distribution of electric fields may depend on whether emission is space-charge limited, and since the emission velocities may also affect the distribution of electric fields. The relative importance of the various effects produced by the cathode is not clear. The most obvious of the cathode effects, that is, the presence of sufficient emission, is evidently inadequate to explain all observed phenomena, as has been pointed out by Dench, of Raytheon, and reported by Welch of the University of Michigan (ref. 35, pp. 37-44). For low values of temperature-limited emission (from an oxide cathode), the maximum anode current for stable π-mode oscillation increased with increasing temperature; for higher values of emission, a maximum was reached, after which there was a slight decrease in maximum current with increasing temperature.

The fundamental reasons for instability, described above, remain the same in principle when there are significant nonuniformities in the magnetron in an axial direction. It is pointed out by Feldmeier (ref. 1, Ch. 13) that uniformity of magnetic field is necessary because of the large variation in current which can result from small variations in magnetic field. Rieke and Fletcher (ref. 17; ref. 1, p. 380) suggested a change in the pole piece design of the 2J39 magnetron which led to a more uniform magnetic field, and a great increase in stability was achieved. (Experimental work on the 2J39 will be
described in a later section.) It seems reasonable to attribute the poor stability of magnetrons with nonuniform magnetic fields to the fact that when one portion of the anode is drawing a large current, another portion (where the magnetic field is stronger) may be drawing little or none; and when the anode voltage is raised to give the latter portion of the anode a chance to draw a reasonable amount of current, the electrons in the portion of the interaction space where magnetic field is weak may well find themselves out of synchronism, and therefore not contributing their share of energy to the r-f wave. If the velocity of these unsynchronized electrons in the weak magnetic field is near that corresponding to another mode, it is not inconceivable that competition will take place, with the second attempting to build up and suppress the first mode.

Axial nonuniformities in the r-f field are also possible. They are most prevalent in strapped anodes (ref. 18, p. 17) and in closed-end rising-sun anodes (ref. 1, Ch. 3, p. 110). The above reasoning suggests that these nonuniformities affect both the stability and efficiency adversely. For example, it is to be expected that synchronism between r-f field and electrons will be lost in a region of weak r-f field much more readily than where the field is strong. The extent of these effects has apparently not been studied, and there seems to be no evidence to indicate that it is very serious, especially in strapped magnetrons.

There has been one kind of axial nonuniformity which has been used to improve operation. It has been found that a slight enlargement of the cathode for a small distance at each end helped to prevent misfiring without adversely affecting efficiency or stability to any great extent (ref. 1, Ch. 8, p. 379).

We have indicated that the two principal factors which are most important to stable operation are axial uniformity (especially magnetic) and the effectiveness of the r-f feedback. The latter depends principally on the intensity of the r-f electric field and on its freedom from disturbances by other modes and by components of the desired mode other than the one to which the electrons are coupled. Disturbance by another mode may take place when both the wanted and unwanted modes are near each other in frequency, and this may be prevented by adequate separation of mode frequencies. Effective bunching of electrons by an r-f electric field tends to suppress any tendency of the electron stream to supply energy to other modes, and this will be discussed in detail. Unwanted Hartree harmonics of the desired mode ordinarily do not cause trouble in π-mode oscillation, with the exception of zero-order component interference in rising-sun magnetrons (ref. 1, Ch. 3, pp. 98-100). The latter type of interference can be effectively eliminated by proper choice of magnetic field.

There are two principal means by which the r-f feedback can be made more intense. The first is to lighten loading. Thus, a given amount of r-f power leads to larger r-f electric fields, but such a change also leads to lower efficiency, both electronic and in the r-f circuit. The second means for increasing the r-f feedback is to alter the geometry of the system, especially by increasing \( r_c/r_a \) in order to increase the r-f intensity near the cathode. (The latter change was discussed in Sec. I.) This change also
leads to lower electronic efficiency. Thus, in these two instances, it is necessary to sacrifice efficiency in order to gain stability, and magnetron design becomes a compromise.

III. Mode Interactions: Nonlinear Circuit Theory

Nonlinear Triode Oscillator

The fundamental theory on which this section is based was first studied by van der Pol about thirty years ago (ref. 23 and 24). The earliest theory was for a simple nonlinear triode oscillator; later the theory was extended to cover nonlinear triode oscillators with two degrees of freedom, that is, two modes in the resonant circuit.

For the triode oscillator operating with a simple resonant circuit, as in Fig. 6, it was assumed that the relationship between instantaneous voltage $v$ across the resonant circuit, and the instantaneous current $i$ through the triode, may be represented by $i = \psi(v)$. The differential equation expressing the performance of a circuit with such a triode operating with an RLC parallel resonant circuit (cf. Fig. 6) is

$$C \frac{d^2v}{dt^2} + \frac{1}{R} \left[ \frac{v}{R} + \psi(v) \right] + \frac{v}{L} = 0. \quad (1)$$

This is one form of what has become known as van der Pol's equation.

The analysis can be advanced further by representing $\psi(v)$ as a polynomial containing first and third powers of $v$. It is shown by van der Pol that even powers in the expression for $\psi(v)$ have a negligible effect if the resonant system is high-Q. If $\psi(v)$ is of the form

$$\psi(v) = av - bv^3 \quad (2)$$

then Eq. 1 can be reduced to the following form, normalized in terms of final magnitude of the oscillations (ref. 24, p. 1052)

$$v - a(1 - v^2)v + \omega^2 v = 0 \quad \epsilon = \frac{a}{\omega} << 1. \quad (3)$$

When there is a resonant circuit with two possible modes of resonance, as in Fig. 7, there are two simultaneous equations (ref. 24, p. 1053) derived in the same manner as

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Fig. 6
Elementary feedback oscillator.

Fig. 7
Triode oscillator with two degrees of freedom (ref. 23).
In Eqs. 4a and 4b, the coupling coefficient between the two resonant branches of the circuit is \( k \), where \( k^2 = k_1 k_2 \). In finding an approximate solution to these simultaneous equations, two possible frequencies of oscillation were found, and designated \( \omega_I \) and \( \omega_{II} \). Therefore, the system may be described as having two possible modes of oscillation.

It will be shown how this kind of analysis can be applied to a magnetron with two possible modes of oscillation.

Magnetron Oscillator: One Mode

When conditions in a magnetron are such that oscillation in only one mode is possible, Eq. 1 may be used to represent the build-up of oscillations. The means of feedback discussed in Sec. II now replace the grid circuit of Fig. 6. If it is assumed as a rough approximation that Eq. 2 can be applied to the nonlinear characteristics of a magnetron, then Eq. 3 replaces Eq. 1 and an approximate solution can be obtained.

Assume the following solution of Eq. 3 for \( v \)

\[
v = A \cos(\omega t - \phi) = A \cos u.
\]  

In Eq. 5, \( A \) and \( \phi \) may be functions of time. Differentiating

\[
\dot{v} = \dot{A} \cos u - A \dot{u} \sin u
\]

and

\[
\ddot{v} = \ddot{A} \cos u - 2 \dot{A} \dot{u} \sin u - A \ddot{u} \cos u + A \dot{u} \sin u.
\]

Since \( u = \omega t - \phi \), then \( \dot{u} = \omega - (d\phi/dt) \), and \( u = - (d^2\phi/dt^2) \).

In order to find \( \dot{v}v^2 \), which appears in Eq. 3, the quantity \( v^3 \) may be found and then differentiated, since \( 1/3 (d/dt)v^3 = \dot{v}v^2 \)

\[
\dot{v}v^2 = \frac{1}{3} \frac{d}{dt} v^3 = A^3 \dot{u} \left( - \frac{1}{4} \sin u - \frac{1}{4} \sin 3u \right) + A^2 A \left( \frac{3}{4} \cos u + \frac{1}{4} \cos 3u \right).
\]  

If the resonant circuit is high-Q, any component of current at frequency 3\( \omega \) (which is approximately 3\( \omega t \)) will give rise to a very small voltage, and will therefore be neglected. Any \( v^2 \) terms, which might appear in \( \psi(v) = i \), lead only to a component of current at frequency 2\( \omega \) and a d-c component, neither of which should generate any appreciable voltage across the resonant circuit.

Substituting Eqs. 5, 6, 7, and 8 into Eq. 3, an equation is obtained containing terms in \( \sin u \) and in \( \cos u \). If the terms containing \( \sin u \) are equated, and the result divided by \( \sin u \)

\[
-2 \dot{A} \dot{u} + A \ddot{u} + a A \dot{u} - \frac{1}{4} a A^3 \dot{u} = 0.
\]  

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If terms containing $\cos u$ are equated, and divided by $\cos u$

$$
\ddot{A} - A\dot{u}^2 - a\dot{A} + \frac{3}{4} aA^2 \dot{A} + \omega^2 A = 0. \tag{10}
$$

In a high-$Q$ system, such as a magnetron, the first and third terms of Eq. 3 are very large as compared with the second. Therefore, the quantity $u$ is very nearly equal to $\omega$, and $d\phi/dt$ is much less than $\omega$. To simplify the solution of Eq. 9 for $\dot{A}$, $u (= d^2\phi/dt^2)$ will be neglected, and then Eq. 9 can be divided by $u$

$$
\dot{A} = \frac{1}{2} aA \left( 1 - \frac{1}{4} A^2 \right). \tag{11}
$$

It is possible to integrate Eq. 11 and find an explicit solution for $\dot{a}t$ in terms of $A$

$$
\dot{a}t = \ln \frac{A^2}{4 - A^2} + c. \tag{12}
$$

Building-up of oscillations must proceed from some finite magnitude of voltage. Let $A = A_0$ when $t = 0$. Then Eq. 12 becomes

$$
\dot{a}t = \ln \frac{A^2(4 - A^2)}{A_0^2(4 - A_0^2)}. \tag{13}
$$

The solution to the build-up equation given by Eq. 13 is shown in Fig. 8, in which $A$ is plotted as a function of $\dot{a}t$.

For oscillation to build up and become stable, the signs in Eq. 2 must be as they

---

**Fig. 8**

Approximate solution of van der Pol's equation, where the solution is assumed to be of the form, $A \cos(\omega t - \phi)$, and $\omega >> dA/dt$. The actual form of this equation is expressed by Eq. 3.

**Fig. 9**

(a) Equivalent circuit for magnetron oscillator with two modes in resonant circuit; (b) Simplified equivalent circuit for magnetron oscillator with two modes in resonant circuit.
are given there, i.e. \( a \) and \( b \) must be positive. If \( a \) is not positive, the solution to Eq. 3 does not correspond to the possible build-up of small-amplitude oscillations. If \( b \) is not positive, there will be no limit to the magnitude of oscillations.

Magnetron with Two Modes of Operation

The equivalent circuit for a resonant cavity with more than one mode of resonance, and with each resonant mode coupled, to a greater or less degree, to a single load, is shown in Fig. 9a. The circuit which will be considered in investigating the interaction of two modes in a magnetron is shown in Fig. 9b. It is assumed that the magnetron is a high-Q system, and that resonances are well separated in frequency, so that the impedance of one of the parallel-resonant circuits is negligible at the resonant frequency of the other. Therefore, any interaction in the passive circuit may be neglected, and the simpler circuit of Fig. 9b will be studied.

First, we shall assume that the voltage developed across the circuit in Fig. 9b may be represented by

\[
v = V_1 e^{\sigma_1 t} \cos \omega_1 t + V_2 e^{\sigma_2 t} \cos \omega_2 t
\]

\[
v = \Re \left( V_1 e^{\lambda_1 t} + V_2 e^{\lambda_2 t} \right)
\]

\[
\lambda_1 = \sigma_1 + j\omega_1
\]

\[
\lambda_2 = \sigma_2 + j\omega_2
\]

The phase of each frequency component has not been generalized (i.e. the voltage has not been expressed in the form \( V_1 \cos(\omega_1 t + \phi) \)), because the solution which will be found here will not depend on the phase relationship, provided \( \omega_1 / \omega_2 \) can not be represented by a rational fraction.

Unless indicated otherwise, in all the equations which follow involving complex quantities, it will be implied that it is the real part which is significant. Therefore Eq. 15 may be rewritten simply

\[
v = V_1 e^{\lambda_1 t} + V_2 e^{\lambda_2 t}
\]

Likewise

\[
i = I_1 e^{\lambda_1 t} + I_2 e^{\lambda_2 t}
\]

It is assumed further that \( \lambda_1 \) is approximately equal to \( j(1/\sqrt{C_1 L_1}) \), and that \( \lambda_2 \) is approximately equal to \( j(1/\sqrt{C_2 L_2}) \). Now let \( Z_1 \) represent the impedance across the \( L_1-C_1-R_1 \) part of the circuit, and \( Z_2 \) the impedance across the \( L_2-C_2-R_2 \) part. Then \( V_1 \approx I_1 Z_1 \) and \( V_2 \approx I_2 Z_2 \). These expressions result from the fact that \( Z_1(\lambda_1) \) is large but \( Z_1(\lambda_2) \) is small, and \( Z_2(\lambda_1) \) is large while \( Z_2(\lambda_2) \) is small. Then

\[
I_1 = V_1 \left( \frac{\lambda_1 C_1}{L_1} + \frac{1}{R_1} + \frac{1}{\lambda_1 L_1} \right)
\]
\[ I_2 = V_2 \left( \lambda_2 C_2 + \frac{1}{R_2} + \frac{1}{\lambda_2 L_2} \right). \]  

(19)

In the equivalent circuit, the electron stream is represented by a single current source, and the voltage fed back is proportional to \((V_1 + V_2)\). The current is therefore a nonlinear function of the sum of the two voltages.

Now let \(i = av - bv^3\). In analyzing the effect of the cubic term, the exponential representation is no longer valid; from Eq. 21

\[
v^3 = V_1^3e^{\frac{3}{4} \sigma_1^1 t} \cos \omega_1 t + 3V_1^2V_2^e e^{\frac{1}{2} \sigma_1^1 t} e^{\frac{1}{4} \sigma_2^2 t} \cos \omega_1 t \cos \omega_2 t \\
+ 3V_1 V_2^2 e^{\frac{1}{2} \sigma_1^1 t} e^{\frac{1}{4} \sigma_2^2 t} \cos \omega_1 t \cos \omega_2 t \\
+ \frac{3}{4}V_2^3 e^{\frac{1}{2} \sigma_2^2 t} \cos \omega_2 t. \]  

(20)

It will be assumed that all current components other than those at frequencies \(\omega_1\) and \(\omega_2\) will lead to negligible voltage components across the resonant circuits. The products, \(\cos \omega_1 t \cos \omega_2 t\) and \(\cos \omega_1 t \cos \omega_2 t\), lead to frequency components of \((2\omega_1 \pm \omega_2)\) and \((2\omega_1 \pm \omega_2)\), respectively. \(v^3\) will be approximated by

\[
v^3 = \frac{3}{4}V_1^3 e^{\frac{3}{4} \sigma_1^1 t} \cos \omega_1 t + \frac{3}{2}V_1^2V_2^e e^{\frac{1}{2} \sigma_1^1 t} e^{\frac{1}{4} \sigma_2^2 t} \cos \omega_1 t \cos \omega_2 t \\
+ \frac{3}{4}V_2^3 e^{\frac{1}{2} \sigma_2^2 t} \cos \omega_2 t. \]  

(23)

Now if \(i = I_1 e^{\frac{\lambda_1^1 t}{V_1}} + I_2 e^{\frac{\lambda_2^1 t}{V_2}}\) then

\[
I_1 = aV_1 - \frac{3}{4}bV_1 \left( V_1^2 e^{\frac{2\sigma_1^1 t}{V_1}} + 2V_2^2 e^{\frac{2\sigma_2^2 t}{V_2}} \right) \]  

(24)

\[
I_2 = aV_2 - \frac{3}{4}bV_2 \left( V_2^2 e^{\frac{2\sigma_2^2 t}{V_2}} + 2V_1^2 e^{\frac{2\sigma_1^1 t}{V_1}} \right). \]  

(25)

Now, current in the active part of the circuit must equal current in the passive circuit; Eq. 24 must be equated to Eq. 18, and Eq. 25 to Eq. 19. But first, Eq. 18 can be
simplified
\[ \lambda_1 C_1 + \frac{1}{\lambda_1 L_1} = \frac{C_1}{\lambda_1} \left( \lambda_1^2 + \frac{1}{L_1 C_1} \right). \] (26)

If \( \omega_1 = \left(1/\sqrt{L_1 C_1}\right) \)
then
\[ \lambda_1 C_1 + \frac{1}{\lambda_1 L_1} = \frac{C_1}{\lambda_1} \left( \lambda_1^2 + \omega_1^2 \right). \] (27)

Since \( \omega_1 \) is much greater than \( \sigma_1 \), then
\[ \lambda_1 C_1 + \frac{1}{\lambda_1 L_1} \approx \frac{2 \sigma_1}{\omega_1} + j \frac{\sigma_1}{\omega_1} C_1. \] (28)

It is the real part which is of greatest interest here. The small imaginary part of
Eq. 28, which will be neglected, would lead to only a very small change in frequency of
the kind expressed in the previous section by \( \frac{d\phi}{dt} \).

Therefore, equating Eq. 24 to Eq. 18, and dividing by \( V_1 \)
\[ 2C_1 \sigma_1 + \frac{1}{R_1} = a - \frac{3}{4} b \left( V_1 e^{\sigma_1 t} + 2V_2 e^{2\sigma_2 t} \right). \] (29)

Now let
\[ a_1 = a - \frac{1}{R_1}, \quad A_1 = V_1 e^{\sigma_1 t}, \quad \text{and} \quad A_2 = V_2 e^{\sigma_2 t}. \]

In other words, at any time \( t \), \( A_1 \) and \( A_2 \) represent the respective magnitudes of voltage
developed at frequencies \( \omega_1 \) and \( \omega_2 \). Then
\[ \sigma_1 = \frac{a_1}{2C_1} - \frac{3b}{8C_1} \left( A_1^2 + 2A_2^2 \right). \] (30)

A new parameter, \( A II^2 = 4a_1/3b \) can be introduced. \( A I \) is the magnitude that \( A_1 \) will
approach in steady state, \((dA_1/dt) = (dA_2/dt) = 0, \) if \( A_2 = 0 \). Therefore
\[ \sigma_1 = \frac{3b}{8C_1} \left( A_1^2 - A_1^2 - 2A_2^2 \right). \] (31)

By symmetry
\[ \sigma_2 = \frac{3b}{8C_2} \left( A II^2 - A_2^2 - 2A_1^2 \right) \] (32)

where \( A II^2 = 4a_2/3b \).

It is seen from Eqs. 31 and 32 that the magnitude of oscillations at each frequency
affects the rate of build-up at the other frequency, i.e. oscillation in one mode reduces
the rate of build-up in other modes. It is significant that, according to this analysis,
the rate of build-up in one mode is affected less by the magnitude of its own oscillation
than by the magnitude of oscillation in the other mode. This condition is one which
Rieke assumed (ref. 1, Ch. 8), at the same time stating that it was open to question.

It is also significant to point out that, although van der Pol did not use quite the same method of solution as given above, his approximate solution for the circuit of Fig. 7 led to the same results as in Eqs. 31 and 32. van der Pol did not assume that coupling between the two resonant sections was small, nor that their resonant frequencies were separated by any large amount.

The results in Eqs. 31 and 32 may be plotted, as in Fig. 10. The coordinate axes correspond to $A_1^2$ and $A_2^2$, respectively. Lines of constant $\sigma_1$ and $\sigma_2$ are drawn, and these are shown in order to separate the plane into regions according to which modes are building up or decaying. Region I, which includes portions of both axes, corresponds to $\sigma_1 > 0$ and $\sigma_2 > 0$. Therefore, if the state of oscillation is such that the magnitudes in each mode, when plotted in the plane, line in region I, then both modes are building-up. In region II, $A_1^2$ is increasing, $A_2^2$ decreasing; in region III, $A_1^2$ is decreasing, $A_2^2$ increasing; and in region IV, both are decreasing.

van der Pol (refs. 23 and 24) has examined the results for "possible" solutions and for stable solutions among the possible solutions. Possible solutions were defined as those for which $dA_1/dt = dA_2/dt = 0$. There are four, as shown by the encircled Arabic numerals in Fig. 10a. Of these, he finds that only two are stable. Solution 1, where $A_1 = A_2 = 0$, is unstable because both $\sigma_1$ and $\sigma_2$ are greater than zero, and any small disturbance, such as shot noise, will start the building-up of both modes. Solution 2 represents the condition where $\sigma_1 = \sigma_2 = 0$, and where $A_1^2 = \frac{1}{3}(2A_II^2 - A_1^2)$, and $A_2^2 = \frac{1}{3}(2A_1^2 - A_II^2)$. The instability here is more subtle than in solution 1. It may be shown that if the respective values of $A_1^2$ and $A_2^2$ are altered slightly so as to get into region II, the build-up of $A_1^2$ will continue, together with decay of $A_2^2$, and the operating point will move away from point 2. Any disturbance into region III will also lead to the moving away from point 2 by the point representing actual operating conditions. Any excursion of the operating point into region I or region IV will in general be followed by movement of the point into regions II or III, rather than back to point 2, and conditions will proceed away from that point.

Points 3 and 4 are similar to each other in character. For example, point 3 represents a condition where $A_1^2 = A_1^2$, $\sigma_1 = 0$, $A_2^2 = 0$, and $\sigma_2 < 0$. $A_1$ is both stationary in
magnitude (because $\sigma_1 = 0$) and stable (because any disturbance of $A_1^2$ from this position will change conditions so as to cause it to return). At point 3, $A_2^2$ is also both stationary and stable, because $dA_2^2/dt = 0$, and $\sigma_2 < 0$. Thus any oscillation which might appear corresponding to frequency $\omega_2$ would be quickly damped out.

For similar reasons, at point 4, $A_1^2$ and $A_2^2$ are both stationary and stable, with $A_1^2 = 0$ and $\sigma_1 < 0$ here.

Another possible set of conditions may give rise to a set of build-up characteristics, which can be represented by Fig. 10b. Here, solution 2 and region III of Fig. 10a are absent. But what is much more important is that at point 4, $\sigma_1$ is still positive. Therefore, even if this point could be reached, with $A_2^2$ at a stationary value, the value of $A_1^2 = 0$ is unstable, and any disturbance would cause it to build up, and the operating point would proceed into a region in which $A_2^2$ must decay. Therefore, only point 3 is both stationary and stable. This situation is characterized by the fact that $1/2 A_1^2 > A_1^2$. A comparable situation, where the only stable oscillation which can take place corresponds to $A_2^2 = A_2^2$, is given by $1/2 A_2^2 > A_1^2$.

The expression "oscillation hysteresis" was used by van der Pol to express what happened as relative conditions for the two modes were changed continuously, during which time oscillation was maintained. Suppose oscillation were started under conditions shown by Fig. 10b. Then if the system is changed so that conditions of Fig. 10a prevail, the oscillating mode will still correspond to $A_1^2$. Only when $1/2 A_1^2$ becomes greater than $A_1^2$ will $A_2$ build up and suppress $A_1^2$. Now beginning from the latter condition, with $A_2^2$ present and $A_1^2 = 0$, let the conditions be gradually changed in the opposite direction. This time the mode change point does not correspond to $1/2 A_2^2 = A_1^2$, as before, but instead to $A_2^2 = 1/2 A_1^2$. It was this effect that was called oscillation hysteresis. It may be described by saying that the existing mode tends to persist.

The applicability of such an analysis to the magnetron should be examined. The feedback mechanisms of a magnetron were discussed in Sec. II. The effects are easily shown, both experimentally and theoretically, to be nonlinear. Although the r-f fields corresponding to various modes of oscillation may be superimposed in the passive circuit, they must act together on the same nonlinear electron system, and thus interaction must be present. A difference between magnetrons and conventional triodes is that essentially, the triode performance is representable by a single lumped element, whereas feedback in the magnetron is distributed throughout the electron interaction space, for which the transit time is of the order of cycles instead of a small fraction of a cycle; however, the summation of effects in the magnetron can lead to a system representable by some mathematical expression comparable to that for a triode. In a magnetron, and in a triode circuit as well, it is not necessarily true that the feedback ratio is the same for both resonances (as it was taken to be here); if it is not the same, this can be remedied in the equivalent circuit by using a new equivalent circuit where, for example, $\sqrt{L_1/C_1}$ is changed, without changing the resonant frequency, which is determined by $\sqrt{L_1/C_1}$. By this means, the original current output at the original
frequency will lead to a different voltage fed back. It is also not necessarily true in a magnetron that the nonlinearities will be similar for the two modes, i.e. that \( a \) and \( b \) (in the active part of the circuit, \( i = av - bv^3 \) was assumed) will bear the same ratio to each other; but this ratio was effectively destroyed in Eq. 30, where \( a_1 \) enters the picture instead of \( a \), because in general, \( a_1 = a - (1/R_1) \) is not equal to \( a_2 = a - (1/R_2) \).

The parts of the above discussion which seem most open to question are the representation of the nonlinear portion of the system, first by a nonlinear circuit element in which the output current is always in phase with the control voltage, and second by the very elementary form of the nonlinear expression. The author has not undertaken the analysis of a nonlinear system in which the phase of the fed-back signal varies. The analysis of nonlinear circuits with somewhat different characteristics than those discussed here will be taken up in the next section. It is also true that the nonlinear characteristics of any electron device depend upon external conditions: applied voltages, magnetic field (for a magnetron), etc. But, in general, the external conditions may be affected by the state of oscillation. Therefore, the interaction between external effects and active circuit characteristics would have to be taken into account before a complete picture of the problem of nonlinear systems could be obtained.

Nonlinear Oscillators with More General Nonlinear Characteristics: One Mode

One of the most significant features of the nonlinear theory developed so far has been the changing of an instantaneous function of voltage into another function which expresses the equivalent voltage magnitudes in the sinusoidal case, assuming that only the fundamental frequency is of importance. For example, the simple nonlinear function of voltage, \( i = av - bv^3 \), expressed earlier, leads to

\[
\mathfrak{f} = aA - \frac{3}{4}bA^3
\]  

(cf. Eq. 24). The method for deriving this type of function from instantaneous relationships has already been done for polynomials. It consists simply of expanding the function of the sinusoid into all of its frequency components by means of well-known trigonometric identities, and discarding all but the fundamental, as in Eq. 23. The procedure is therefore relatively simple in the power series case.

A procedure for determining the fundamental component of more general functions of sinusoids can also be derived. Again suppose \( i = \psi(v) \) (cf. Eq. 2). If the new equation, analogous to Eq. 33, is expressed by \( \mathfrak{f} = F(A) \), then

\[
F(A) = \frac{1}{\pi} \int_0^{2\pi} \psi(A \cos \omega t) \cos \omega t \, d(\omega t).
\]  

This equation has been derived by finding the fundamental Fourier component of \( \psi(A \cos \omega t) \) which is in phase with voltage, and thus in such a phase as to contribute power to the load. If there is any time lag between the output and the effect upon the
active system produced by the output, there may be an out-of-phase component at the fundamental frequency. Such a component may be found from

$$G(A) = \frac{1}{\pi} \int_{0}^{2\pi} \psi(A \cos \omega t) \sin \omega t \, d(\omega t).$$  \hspace{1cm} (35)$$

The primary effect of the latter component is to shift the operating frequency; any effect upon the rate of build-up will be secondary.

The results obtained by using Eq. 34 may be compared with Eq. 33 by letting $$\psi(v) = av - bv^3,$$ as before (cf. Eq. 2). Then

$$F(A) = \frac{1}{\pi} \int_{0}^{2\pi} (a \cos^2 \omega t - bA^2 \cos^4 \omega t) \, d(\omega t).$$ \hspace{1cm} (36)$$

When the integral is evaluated

$$F(A) = aA - \frac{3}{4} bA^3$$ \hspace{1cm} (37)

(cf. Eq. 33).

There are some disadvantages in using the expression of Eq. 2 for $$\psi(v)$$. Neither in a triode nor in a magnetron is there any effect which accompanies large r-f voltages which would lead to $$\psi(v)$$ being proportional to $$(-v^3)$$ for very large r-f amplitude. Instead, the principal effect met with in a triode (or pentode or tetrode) is that either saturation or the low value of instantaneous plate voltage limits current on the positive swing of the grid, and current is cut off on the negative swing of the grid. Keeping in mind that when $$\psi(v) = v^{2n}$$, where n is any integer, $$F(A) = 0$$ (cf. Eq. 34), an odd function for $$\psi(v)$$ will be assumed such that

$$i = \psi(v) = kv, \quad \text{if} \quad -B < kv < B$$

$$= B \quad \text{if} \quad kv > B$$

$$= -B \quad \text{if} \quad kv < -B.$$ \hspace{1cm} (38)$$

In a magnetron, the character of the nonlinearity may be estimated by considering the electrons as being bunched in one region in respect to the r-f traveling wave, and completely absent elsewhere. It will be assumed that after a certain point, any greater intensity in the r-f electric field will not produce a greater r-f current by more effective bunching or greater total circulating charge. Thus, magnetron characteristics may also be investigated by using $$\psi(v)$$ as in Eq. 38.

When $$\psi(v)$$, defined by Eq. 38, is substituted into Eq. 34, numerical integration leads to a build-up in r-f amplitude of the form shown in Fig. 11. In this solution of the build-up equation, the small-signal loop gain of the equivalent feedback oscillator was taken as five, and $$A$$ was taken as unity. The position of $$\tau = 0$$ (where $$\tau = t/2RC$$) was arbitrarily selected to correspond to $$kA = B$$ (see Eq. 38). To the right of $$\tau = 0$, 

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the build-up curve gradually departs from the exponential form, and eventually approaches a maximum value as $\tau$ approaches infinity. This whole pattern is in agreement with the description of build-up of a magnetron made in Sec. II. Such agreement is to be expected, because fundamentally, the assumptions are similar. In Sec. II, it was assumed that the relationship between electric field and bunching was linear for small amplitudes, after which it approached a point where further increase in electric field intensity leads to little or no increase in effectiveness in bunching. It is also significant to observe that the r-f build-up shown in Fig. 11 is much more like the observed build-up in magnetrons than that shown in Fig. 8.

Nonlinear Oscillators with More General Nonlinear Characteristics: Two Modes

The application of the above principles to the two-mode problem is considerably more complicated. Although $\psi(v)$ may be expressed in the same way as before, the expression for current is much more difficult to reach. Current will be expressed here by $F_1(A_1, A_2) + F_2(A_1, A_2)$; $F_1$ corresponds to current at frequency $\omega_1$, and $F_2$ corresponds to current at frequency $\omega_2$. Now let $v = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t$; let $\omega_1$ and $\omega_2$ be integrally related, so

$$A_1 \cos \omega_1 t + A_2 \cos \omega_2 t = A_1 \cos \omega_1 (t+T) + A_2 \cos \omega_2 (t+T)$$

for all values of $t$. The period of time $T$ will be taken to be the shortest time during which both voltages execute an integral number of cycles.

To find the component of current which supplies power at frequency $\omega_1$, the same procedure as that which led to Eq. 34 may be followed

$$F_1(A_1, A_2) = \frac{1}{n} \int_0^{2\pi/n} \psi(v) \cos \omega_1 t \, d(\omega_1 t). \quad (40)$$

In Eq. 40 it has been assumed that the period $T$ corresponded to $n$ cycles of $\sin \omega_1 t$. It is possible to rewrite Eq. 40 in the form

$$F_1(A_1, A_2) = \frac{1}{n} \sum_{p=0}^{n-1} \int_{2\pi p}^{2\pi(p+1)} \psi(A_1 \cos \omega_1 t + A_2 \cos \omega_2 t) \cos \omega_1 t \, d(\omega_1 t). \quad (41)$$

Each phase of $\omega_1 t$ is reached $n$ times during the period, and for each phase there are $n$ different phases of $\omega_2 t$. The totality of phases of $\omega_2 t$ for each phase position of
\( \omega_1 t \) leads to \( n \) equally-spaced phase positions of \( \omega_2 t \). Therefore

\[
F_1(A_1, A_2) = \frac{1}{n\pi} \sum_{p=0}^{n-1} \int_0^{2\pi} \psi\left[A_1 \cos \omega_1 t + A_2 \cos \left(\omega_2 t + \frac{2\pi p}{n}\right)\right] \cos \omega_1 t \, d(\omega_1 t). \tag{42}
\]

As \( n \) becomes very large the summation may become an integral; therefore, when the conditions of Eq. 39 are not met in any finite length of time, Eq. 42 may be replaced by

\[
F_1 = \frac{1}{\pi} \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \psi(A_1 \cos \omega_1 t + A_2 \cos \omega_2 t) \cos \omega_1 t \, d(\omega_1 t) \, d(\omega_2 t). \tag{43}
\]

In Eq. 43, it is considered that time \( t \) is not the independent variable in the double integration, but rather that \( \omega_1 t \) and \( \omega_2 t \) are independent variables. In using Eq. 43 to express conditions during build-up, it is necessary that value of \( |\omega_1 - \omega_2| \) be much greater than either \( \sigma_1 \) or \( \sigma_2 \). It is necessary that the phase difference between the two components change rapidly as compared with the rate of build-up in order for the averaging effect implicit in the derivation to be reasonably well approximated. The effect which must be avoided is that which occurs when the two frequencies are nearly equal. When the voltages are nearly in phase, the output power over one cycle is nearly proportional to \( (A_1 + A_2)^2 \), rather than to \( A_1^2 + A_2^2 \); the excess power averages out over a large number of cycles.

As a simple example of the application of Eq. 43, let \( \psi(v) = av - bv^3 \), as before.

Then

\[
F_1(A_1, A_2) = aA_1 - \frac{3}{4} bA_1^3 - \frac{3}{2} bA_1 A_2^2. \tag{44}
\]

This will lead to the same results as were derived in the section on the magnetron with two modes of oscillation.

It is of interest to examine mode interactions when the characteristics of the active circuit simply level off with large signal voltage, instead of decreasing in proportion to the cube of voltage as in Eq. 2. The characteristics expressed by Eq. 38 do not lead to an expression for \( F_1(A_1, A_2) \) which is easily found from Eq. 43, because \( \psi(v) \) is an implicit function in both frequency components. Thus when \( \psi(v) \) changes from \( kv \) to \( A \), the values of \( \omega_1 t \) and of \( \omega_2 t \) at this point can be determined only by knowing the value of the function.

A reasonably good substitute for Eq. 38 is achieved by the following

\[
\psi(v) = a \tanh b v. \tag{45}
\]

This function has in common with Eq. 38 the properties of approaching \(-1\) for large negative values of \( v \); of approaching \(+1\) for large positive values of \( v \); and of having a slope of \(+a\) for small \( v \). The integral in Eq. 43 would be easy to evaluate if \( \psi(v) \) could be expanded in a Taylor series about \( v = 0 \); but such an expansion is convergent only when
| bv | (π/2). Therefore a numerical integration is again required. Approximate integration of Eq. 43, using ψ(v) as in Eq. 45, has been carried out. When F_1 is divided by A_1, the result is an electronic conductance, a function of A_1 and A_2, and the values of this conductance, g_1, are plotted in Fig. 12. Values of g_2 (= F_2/A_2) can be found from Fig. 12 by interchanging A_1 and A_2. The rates of build-up of A_1 and A_2 may be expressed in terms of σ_1 and σ_2 as in the following expressions:

\[ 2C_1σ_1 A_1 = F_1(A_1, A_2) - \frac{A_1}{R_1} \]  
\[ σ_1 = \frac{1}{2R_1C_1} \left[ \frac{R_1}{A_1} F_1(A_1, A_2)^{-1} \right]. \]

Using the results shown in Fig. 12, it is possible to make a plot of regions of build-up and decay of A_1 and A_2 in the same manner as Fig. 10. These results are shown in Fig. 13a for the case where 1/R_1 = 1/R_2 = 0.5a/b. Thus, the condition, σ_1 = 0 is satisfied for bA_2 = 0, bA_1 = 2.3; for bA_2 = 0.5, bA_1 = 2.25, etc. This is exactly the same type of results as shown in Fig. 10a. For purposes of comparison, (bA_1)^2 and (bA_2)^2 are used as coordinates. Like the case illustrated by Fig. 10a, stable oscillation in either mode is possible. The condition where σ_1 = σ_2 = 0, and both A_1 and A_2 are finite, is unstable, as in Fig. 10a. There is no difference in principle in any respect; only the particular shapes of the σ = 0 curves are changed.

Another important aspect in which the solution expressed by Eqs. 31 and 32 and the solution illustrated by Fig. 12 are similar is that in each case, the rate of build-up in one mode is affected less by the magnitude of its own oscillation than by the magnitude of oscillation in the other mode. For example, when
Results presented in Fig. 12 plotted to show areas of build-up and decay of each mode for the case where $1/R_1 = 1/R_2 = 0.5a/b$. (a) Either mode may oscillate stably (see Fig. 10a). (b) The only stable oscillation possible is that for which $(b_1)^2 = 5.3$, if $\sigma_1 = 0$ is shown by the solid line. Either mode may oscillate stably when $\sigma_1 = 0$ is shown by the dashed line (see Fig. 10b).

Application of Nonlinear Theory to Magnetrons

The most important result which comes from applying this type of nonlinear circuit theory to magnetrons is that large-amplitude oscillation in one mode has a strong tendency toward discouraging oscillation in other modes. It is particularly significant that the rate of build-up of one mode is affected more by the amplitude of oscillation in the other mode than by its own amplitude. The latter condition is necessary if the intersection of $\sigma_1 = 0$ and $\sigma_2 = 0$ in diagrams such as Fig. 10 and Fig. 13 (also cf. Fig. 8.47, ref. 1, Ch. 8, p. 383) is to be unstable. If this intersection were stable, steady-state oscillation in the two modes simultaneously would be possible, and this is contrary to what is normally observed.

The results presented here also show that after one mode of oscillation has been

\* According to Dwight: Table of Integrals and Other Mathematical Data, Macmillan, 1947, $\tanh x = x - 1/3x^3 + 2/15x^5 \ldots \ldots$, $x^2 < \pi^2/4$. 

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established, it becomes much more difficult for a second mode to build up. This statement is in disagreement with all mode change theories in which conditions in the second mode, rather than in the originally oscillating mode, are considered to be the principal factors in a mode change. Instead, this type of nonlinear theory suggests that the most important factor in giving another mode a chance to build up is the weakening of the first mode. However, if the first one is not a very "strong" mode to start with, conditions may be not far removed from those in Fig. 13b. Such a condition is illustrated by assuming that originally \( \sigma_1 \) is represented by the broken line in Fig. 13b and that oscillation is taking place at frequency \( \omega_2 \), with \( A_2 \) finite and \( A_1 = 0 \). Then let a slight change in conditions shift the line representing \( \sigma_1 = 0 \) from the position of the broken line to that of the solid line. Now \( A_1 \) must build up, and \( A_2 \) will be suppressed. It is under such circumstances, then, that a mode change can take place primarily as a result of conditions in the mode which builds up and suppresses the first one. The most significant fact brought out is that if two modes have nearly equal strength, a condition comparable to that in Fig. 13a arises, and no small change associated with the non-oscillating mode will cause it to build up and suppress the oscillating mode.

IV. Mode Interactions: Study of Electron Motion

The study of one aspect of mode interactions will be taken up from a more fundamental point of view than was used in the preceding section. Some drastically simplifying assumptions will be made concerning the nature of electron motion in the magnetron, so it is not practical to try to apply the results quantitatively to actual magnetrons, but it is possible to gain some useful qualitative information from this type of study.

For simplification, the cylindrical magnetron will be temporarily abandoned, and a linear magnetron, such as would result if the cylindrical magnetron were developed into an infinitely long structure with a cross section as shown in Fig. 14, and with a magnetic field perpendicular to the paper, will be considered. The condition which will be assumed is that there is a large-amplitude steady-state r-f traveling wave present, and that the electron stream in that part of the interaction space under consideration here consists of tightly bunched electron spokes. A single electron in one of these spokes will be considered. This electron is assumed to be moving with a constant velocity in such phase as to contribute the maximum possible energy to the above r-f traveling wave. The effect of this electron upon another r-f traveling wave of small amplitude is to be investigated.

If we assume that the effect of space charge upon the form of the r-f traveling wave is small, the resulting electric fields for the large-amplitude traveling wave are described by

\[
E_x = \xi_x \cosh k_x x \sin k_y (y - v_1 t) \tag{48}
\]

\[
E_y = \frac{k_x}{k_y} \xi_y \sinh k_x x \cos k_y (y - v_1 t) \tag{49}
\]
In Eqs. 48 and 49, $k_x$ and $k_y$ are the propagation constants in the $x$- and $y$-directions, respectively, such that between $x = 0$ and $x = X$, $k_x^2 - k_y^2 = \omega^2/c^2$, where $c$ is the free-space velocity of light; $v_1$ is the velocity of the traveling wave, and depends primarily on the cavity resonators, which appear as slots in Fig. 14.

The effect of the r-f magnetic field upon electron motion will be neglected in comparison with the externally applied constant magnetic field, $B$. Therefore, if a constant d-c field $E_0$ between cathode and anode is included, the equations of electron motion are

\begin{align*}
\dot{y} &= \frac{e}{m}(E_y + Bx) \\
\dot{x} &= \frac{e}{m}(E_x + E_0 - By).
\end{align*}

In Eqs. 50 and 51, $\dot{x}$ and $\dot{y}$ represent first and second derivatives, respectively, of $x$ with respect to time, and $\ddot{y}$ and $\ddot{y}$ represent similar derivatives of $y$. Here, $e/m$ is the ratio of charge to mass of the electron. Signs have been chosen so that a positive acceleration corresponds to work done on the electron.

Since the large-amplitude traveling wave has a velocity equal to $v_1$, a new coordinate system may be chosen such that all motion may be expressed in relation to this traveling wave. Therefore, a new coordinate $u$ may be introduced such that $u = y - v_1t$. At the same time, the magnetic field may be expressed in terms of the cyclotron frequency (ref. 2, p. 327) so that $\omega_c = eB/m$. Hence, Eqs. 50 and 51 may be replaced by

\begin{align*}
\ddot{u} &= \frac{e}{m}(E_y + \omega_c \dot{x}) \\
\dot{x} &= \frac{e}{m}(E_x + E_0) - \omega_c(\dot{u} + v_1).
\end{align*}

The change in coordinates is also applied to Eqs. 48 and 49, giving

\begin{align*}
E_x &= \xi_x \cosh k_x x \sin k_y u \\
E_y &= \xi_y \sinh k_x x \cos k_y u.
\end{align*}

If the electron under consideration travels in synchronism with the r-f traveling wave, its velocity in the $y$-direction is $v_1$. If the electron is to give up the maximum possible energy to the tangential component of the r-f electric field (i.e. $E_y$), it is necessary either that $k_y u = (1 + 2n)\pi$, where $n$ is zero or an integer, or for greater convenience, $\xi_x$ may be considered negative, and the position of the electron may then
be taken as \( u = 0 \).

In the analysis of electron motion which follows, electron motion in the \( y \)-direction will be considered relatively small, and therefore variations in electric field as a function of \( x \) will be neglected. Keeping in mind that \( \dot{E}_x \) is negative, Eqs. 54 and 55 become

\[
\begin{align*}
E_x &= -A_x \sin k_y u \\
E_y &= -A_y \cos k_y u.
\end{align*}
\] (56)

(57)

Here, \( A_x \) and \( A_y \) will be treated as positive and constant.

A kind of steady-state electron motion may now be described by setting \( \ddot{x} = 0 \), and \( \dot{y} = \ddot{u} = 0 \). Then, Eqs. 52 and 53 are reduced to

\[
\begin{align*}
\frac{e}{m} A_y \cos k_y u &= \omega_c \dot{x} \\
\frac{e}{m} A_x \sin k_y u &= \frac{e}{m} E_o - \omega_c (\dot{u} + v_1).
\end{align*}
\] (58)

(59)

For synchronism, \( \dot{u} = 0 \). If the electron is to contribute maximum energy to the tangential electric field, \( u = 0 \). These conditions change Eqs. 58 and 59 to

\[
\begin{align*}
\frac{e}{m} A_y &= \omega_c \dot{x} \\
0 &= \frac{e}{m} E_o - \omega_c v_1.
\end{align*}
\] (60)

(61)

The requirement placed upon \( E_o \) by Eq. 61, in terms of \( \omega_c \) and \( v_1 \), cannot, in practice, be met exactly, but the picture is not changed greatly by small changes in \( E_o \). The same kind of steady state may be achieved by a small shift in the position of the electron. For small \( u \), \( \cos k_y u = 1 \), and \( \sin k_y u = k_y u \). Therefore, as a result of small changes in \( E_o \), Eq. 60 is unchanged, and Eq. 61 may be replaced by

\[
\begin{align*}
\frac{e}{m} A_x k_y u &= \frac{e}{m} E_o - \omega_c v_1.
\end{align*}
\] (62)

The stability of the kind of steady state expressed by Eqs. 60 and 62 may be examined by means of the general solution to Eqs. 52 and 53, taking \( E_y = -A_y \), and \( E_x = -A_x k_y u \), as above. Then Eqs. 52 and 53 may be combined to give

\[
\ddot{u} = \omega_c \frac{e}{m} (-A_x k_y u + E_o) - \omega_c^2 (\dot{u} + v_1)
\] (63)

or

\[
\dddot{u} + \omega_c^2 \ddot{u} + Ku = \omega_c \frac{e}{m} E_o - \omega_c^2 v_1.
\] (64)

A constant \( K \) has been introduced to replace \( \omega_c (e/m)A_x k_y \).

To investigate the interaction of the large-amplitude oscillations and an additional small-amplitude r-f traveling wave, a new wave will be introduced into the system. A pair of r-f field equations will be written, which are analogous to Eqs. 56 and 57, to describe the new r-f wave.
When Eqs. 56, 57, 65, and 66 are compared, \( A_x \) is much greater than \( B_x \), and \( A_y \) is much greater than \( B_y \). The simplifying assumptions involving Eqs. 60 and 61, which led to the steady-state approximation to electron motion, will again be adopted, because orbital calculations show that, in principle, this is a correct view of electron motion. Electron motion of this sort will be subjected to perturbation by the r-f wave described by Eqs. 65 and 66. Since it is the perturbations which will be of greatest interest, an inaccurate assumption as to the unperturbed electron motion will be less harmful than it might be if results were based more directly on that assumption. The equations of electron motion now become

\[
\ddot{u} = \frac{e}{m} \left[ -A_y \cos k_y u + B_y \cos k'_y (u + v_3 t) \right] + \omega_c \dot{u} \quad (67)
\]

\[
\ddot{x} = \frac{e}{m} \left[ -A_x \sin k_y u + B_x \sin k'_y (u + v_3 t) \right] - \omega_c \dot{u} \quad (68)
\]

where \( v_3 = v_1 - v_2 \). When Eqs. 67 and 68 are solved simultaneously

\[
\ddot{u} = \frac{e}{m} \left[ k_y \dot{u} A_y \sin k_y u - k'_y (\dot{u} + v_3) B_y \sin k'_y (u + v_3 t) \right] = \frac{e}{m} \left[ -A_x \sin k_y u + B_x \sin k'_y (u + v_3 t) \right] - \omega_c \dot{u}^2. \quad (69)
\]

As before, it will be assumed that deviations of electrons from \( u = 0 \) are small. This assumption implies that \( \sin k_y u = k_y u \), and \( \cos k_y u = 1 \), to a reasonable degree of approximation, as in Eq. 62. Second and higher powers of \( u \) will be neglected, along with terms which include \( uu \), etc. \( B_x \) and \( B_y \) are small, so terms including products involving \( B_x \) or \( B_y \) in combination with \( u \) will be neglected. Other expressions which are found in these equations are \( \sin k'_y (u + v_3 t) \) and \( \cos k'_y (u + v_3 t) \). These may be approximated in the following manner

\[
\sin k'_y (u + v_3 t) = \sin k'_y u \cos k'_y v_3 t + \cos k'_y u \sin k'_y v_3 t
\]

\[
\approx k'_y u \cos k'_y v_3 t + \sin k'_y v_3 t \quad (70)
\]

\[
\cos k'_y (u + v_3 t) = \cos k'_y u \cos k'_y v_3 t - \sin k'_y u \sin k'_y v_3 t
\]

\[
\approx \cos k'_y v_3 t - k'_y u \sin k'_y v_3 t. \quad (71)
\]

Using the above approximations, and letting \( k'_y v_3 = h \), Eq. 69 may be rewritten

\[
\ddot{u} + \omega_c^2 \dot{u} = \frac{e}{m} \left[ -h B_y \sin ht \right] + \omega_c \frac{e}{m} \left[ -A_x \dot{u} + B_y \sin ht \right] \quad (72)
\]

or
\[ u + \omega_c^2 u + Ku = \frac{e}{m} (-hB_y + \omega_c B_x) \sin ht. \]  

(73)

In Eq. 71, \( K \) has been substituted for \( \omega_c (e/m) A_k x y \'), as in Eq. 64. Therefore, the left-hand side of Eq. 73 will be recognized as being identical with the left-hand side of Eq. 64.

In practical magnetrons, \( \omega_c \) is of the same order of magnitude as the operating frequency. It may be somewhat higher, but is rarely much less. The value of \( k_y' \omega_2 \) represents the angular frequency (i.e. \( 2\pi f \)) of the small-amplitude mode of oscillation. Since \( v_1 \) and \( v_2 \) must not be very different, in order that the two modes may compete under any one set of conditions, \( k_y' (v_1 - v_2) \), or \( h \), may be expected to be much smaller than \( \omega_c \). When the velocity of propagation of the wave (i.e. \( v_2 \)) is much less than that of a wave in free space, \( k_x' \) is nearly equal to \( k_y' \). The quantity \( B_x \) equal to \( e \cosh k_x' x \), is almost always larger than \( B_y \), which is equal to \( (k_x'/k_y') \cosh k_x' x \), if \( k_x' \) is nearly equal to \( k_y' \). For small values of \( k_x' \), \( B_x \) is much larger than \( B_y \). Since \( \omega_c \) is much greater than \( h \), and \( B_x \) is greater than \( B_y \), it must also be true that \( \omega_c B_x \) is much greater than \( hB_y \). Thus, the quantity in parentheses, on the right-hand side of Eq. 73, is positive, and a positive constant \( D \) will be introduced such that Eq. 73 becomes

\[ u + \omega_c^2 u + Ku = D \sin ht. \]  

(74)

Equation 74 will be examined for a particular solution in the presence of the small-amplitude r-f wave. A solution of the form, \( u = a \sin ht + \beta \cos ht \), will be assumed, and substituted into Eq. 74. If the solution is to be valid for all values of time, it must be possible for the coefficients of \( \sin ht \) and of \( \cos ht \) to be equated independently. Such a procedure leads to the following equation, when coefficients of \( \sin ht \) are equated

\[ h(h^2 - \omega_c^2) \beta + Ka = D. \]  

(75)

In a similar manner, coefficients of \( \cos ht \) may be equated

\[ -h(h^2 - \omega_c^2) a + K\beta = 0. \]  

(76)

When Eqs. 75 and 76 are solved simultaneously

\[ a = \frac{KD}{h^2(\omega_c^2 - k_x^2)^2 + K^2} \]  

(77)

\[ \beta = \frac{-hD}{h^2(\omega_c^2 - k_x^2)^2 + K^2}. \]  

(78)

Since \( \omega_c^2 \) is greater than \( h^2 \) and all of the other quantities involved are positive, then

*From the wave equation, \( k_x^2 - k_y^2 = \omega^2/c^2 \). In a practical magnetron, \( k_x' \gg \omega/c \) and \( k_y' \gg \omega/c \).
\( a \) is positive, while \( \beta \) depends on the sign of \( h \). The work \( W \) done on the electron by the tangential electric field, is found by \( \int e E_y dy \), where \( E \) is the entire electric field in the \( y \)-direction. Here we are interested only in \( E'_y \), which is equal to \( B_y \cos k'_y(u + v_3t) \).

The increment of distance is given by \( dy = du + v_1 dt \), where \( du = (ha \cos ht - h\beta \sin ht) dt \). When \( \int eE'_y dy \) is found for the whole cycle of \( ht \), the total work done on the electron by the \( E'_y \) component of the electric field is

\[
W = e \int_0^{2\pi/h} \left[ B_y \cos k'_y(u + v_3t) \right] \left[ ha \cos ht - h\beta \sin ht + v_1 \right] dt. \tag{79}
\]

In Eq. 71 it was shown that \( \cos k'_y(u + v_3t) \) is approximately equal to \( \cos ht - k' u \sin ht \). Since \( a \) and \( \beta \) are small terms, having been generated by a perturbation, any terms which would include the product of \( u \) with \( a \) or \( \beta \) would be very small. Therefore, the only term in the product which leads to any net value of \( W \) is that containing \( \cos^2 ht \). The work done on the electron by the electric field during one cycle is

\[
W = eB_y \int_0^{2\pi/h} ha \cos^2 ht \, dt
= \pi e B_y a
= \pi e B_y \frac{h^2}{2 (h^2 - h^2) + K^2}.
\tag{80}
\]

The most important result here is that the perturbations of electron motion produced by the small-amplitude wave cause the electron to absorb energy from it. It is significant to observe that for small values of \( A_x \) (proportional to \( K \) by definition), the amount of energy absorbed is proportional to \( A_x \), provided of course that bunching is more or less complete. For greater values of \( A_x \), the rate of energy absorption by an electron may decrease as a result of an increase in \( A_x \) if the denominator, proportional to a constant plus \( K^2 \), increases proportionately more rapidly than the numerator, which is proportional to \( K \).

As a consequence, the small-amplitude r-f traveling wave does not interact with the single electron, described here, in such a manner as to lead to regeneration, but instead, it must supply energy to the electron. This discussion of mode interactions leads to a somewhat similar conclusion, as was reached in Sec. III. This conclusion is that large-amplitude oscillation in one mode has a strong tendency toward discouraging oscillation in other modes.

The analysis of electron motion carried out in this section suffers from some rather drastic approximations, which were necessary in order to reach any conclusions, but there appears to be little reason to believe that the results obtained here are not correct in principle. The calculation was carried out for one electron at one radial position.
The integration led to an average effect for electrons at this radius, and a further summation of electrons at all radii should lead to an effect for the entire electron stream; this effect must remain the same in principle as for the electrons at one radius only.

V. Mode Stability and Mode Interaction Experiments

In this section, experimental data will be discussed which pertain to various mode problems. An experiment will be described which is intended to measure as directly as possible some of the mode interaction phenomena described in Secs. III and IV. Other experiments which will be discussed provide information which supports the preceding theory in terms of both mode stability and fundamental causes of mode changes.

Mode Interactions

The effect of large-amplitude oscillation in one mode upon resonances corresponding to other modes has been discussed theoretically from two different points of view in Secs. III and IV. The experiment described here was carried out in order to measure this type of effect as directly as possible. The object was to determine the loaded Q of a resonance corresponding to one mode when the magnetron was actually oscillating in another mode.

A block diagram of equipment used in this experiment is shown in Fig. 15. The magnetron (718EY) oscillated in the \( \pi \)-mode \((4/4/8)\), and the characteristics of the \( n = 3 \) resonance were measured by conventional "cold-test" methods (ref. 2, pp. 89-95), based on standing-wave measurements.

A signal, which would be tuned through a wide enough range to study the \( n = 3 \) resonance under a wide variety of conditions, was supplied by the QK-61 magnetron through a directional coupler. Standing-wave measurements were made by means of a slotted
line, which picked up a signal which was fed to the spectrum analyzer through the klystron amplifier. The klystron and the spectrum analyzer were both tuned to the frequency of the QK-61, thus eliminating the π-mode signal so completely that it could not be observed on the spectrum analyzer. Thus it was possible to measure characteristics of the n = 3 resonance while the π-mode was oscillating.

Since the 718EY magnetron was oscillating only during the applied 5-μsec pulse, the standing wave measurements were made only within this period. The signal supplied by the QK-61 was therefore pulsed with a duration of 3.5 μsec, and the latter pulse was synchronized so that it began after the 5-μsec pulse had begun, and ended before the 5-μsec pulse ended. It was also found desirable to pulse the klystron together with the QK-61, because it was observed that the 718EY output contained considerable energy at the n = 3 frequency during the build-up transient and at the end of the pulse. Therefore, the only signal reaching the spectrum analyzer was that which occurred during the interval in which the small-amplitude signal was supplied to the system.

The first series of tests of this kind was performed with constant loading of the π-mode (i.e. magnetron coupled to r-f line according to specifications, and with the line terminated by a matched load). Magnetic field was held constant (1220 gauss), and four different values of input current were used. The resulting four sets of measurements of standing-wave ratio as a function of wavelength are plotted in Fig. 16. Oscillation was taking place in the π-mode only for the two highest values of anode current. In the first test, there was no power supplied to the 718EY, and in the second test, with low anode current, no coherent oscillation could be observed.

From the data in Fig. 16, the loaded Q of the n = 3 resonance was computed using

![Fig. 16](image)

Standing-wave measurements of 718EY magnetron: n = 3 mode of resonance. (1) No anode power. (2) Peak anode current = 1.1 amp, no oscillation. (3) Peak anode current = 6.8 amp, π-mode oscillation taking place. (4) Peak anode current = 9.9 amp, π-mode oscillation taking place. Magnetic field = 1220 gauss for each case.
the method described by Slater (ref. 2, pp. 89-95). It should be noted that only in the first, and narrowest, curve, did the position of the standing-wave minimum shift by half of a wavelength as the wavelength passed through resonance. The results of these calculations are given in Table V-1. Although the external Q is normally only a measure of coupling between the resonant system and the load, considerable variation is shown here, in spite of the fact that no change in the coupling was made. The change in the loading of the n = 3 resonance, according to theory, should appear only as a change in the unloaded Q, as determined from external measurements.

Table V-1

<table>
<thead>
<tr>
<th>Peak Anode Current</th>
<th>External Q</th>
<th>Unloaded Q (internal Q)</th>
<th>Loaded Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>344</td>
<td>1640</td>
<td>285</td>
</tr>
<tr>
<td>1.1</td>
<td>600</td>
<td>112</td>
<td>95</td>
</tr>
<tr>
<td>6.8(a)</td>
<td>420</td>
<td>108</td>
<td>86</td>
</tr>
<tr>
<td>9.9(a)</td>
<td>420(b)</td>
<td>45</td>
<td>40</td>
</tr>
</tbody>
</table>

(a) π-mode oscillation taking place

(b) External Q could not be obtained from graph. It was assumed to be the same as for the preceding value of anode current.

Table V-2

<table>
<thead>
<tr>
<th>Peak Anode Current (amp)</th>
<th>External Q</th>
<th>Unloaded Q (internal Q)</th>
<th>Loaded Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>506</td>
<td>1680</td>
<td>388</td>
</tr>
<tr>
<td>1.1</td>
<td>600</td>
<td>87</td>
<td>76</td>
</tr>
<tr>
<td>7.9(a)</td>
<td>325</td>
<td>55</td>
<td>47</td>
</tr>
<tr>
<td>16</td>
<td>600</td>
<td>158</td>
<td>128</td>
</tr>
</tbody>
</table>

(a) π-mode oscillation taking place.

These results show that as intensity of oscillation increases and anode current is also increasing, loading of the n = 3 resonance by the electron stream increases. Although the calculations for Q when peak anode current is 1.1 amp appear to lead to a loaded Q nearly equal to that found from the calculations for 6.8 amp, the 1.1-amp curve in Fig. 16 appears narrower, leading to the appearance of much higher Q.

These results show definitely that the nonoscillating mode is loaded by the electron...
stream, and that for larger values of anode current, corresponding to stronger oscillation, the effect is greater. It is apparent in the 1.1-amp case that such a loading effect is present, even when coherent $\pi$-mode oscillation is not. Therefore, these data are insufficient to evaluate the effect of the actual $\pi$-mode oscillation upon loading the non-oscillating mode, apart from effects resulting merely from the flow of anode current.

In Fig. 17, data are plotted for a similar set of tests (cf. Fig. 16), using another 718EY magnetron. Calculated results are given in Table V-2.

It can be seen from Fig. 17 and from Table V-2 that when the applied voltage is so high that oscillation stops, the loading of the $n = 3$ mode by the electron stream is much less than when oscillation is taking place. This observation supports the conclusion, reached in Secs. III and IV from two different theoretical points of view, that large-amplitude oscillation in one mode tends to suppress oscillation in other modes.

The next test, performed with the second of the two magnetrons described above (which had provided the data in Fig. 17 and Table V-2), was the investigation of the effect produced by unloading the $\pi$-mode upon the loading of the $n = 3$ mode, during $\pi$-mode oscillation. Therefore, under circumstances which were similar otherwise, r-f voltage in the $\pi$-mode should be greater in the more lightly loaded case.

The output coupling of the 718EY magnetron is easily altered by removing the section of the center conductor of the output coaxial line nearest the magnetron, and replacing it by another. The original section of the center conductor had a portion one-quarter of the $\pi$-mode wavelength long with a larger diameter than elsewhere, which acted as an impedance transformer. The section with which it was replaced had a constant diameter throughout its length. As a result of this change, the loaded $Q$ of the
The data found in the tests described above, for the two different conditions of r-f loading, are plotted in Fig. 18, and calculated results are given in Table V-3.

Table V-3
Measurements of $n = 3$ Resonance in 718EY Magnetron: $\pi$-Mode Loading is Varied

<p>| $\pi$-Mode | $\pi$-Mode |</p>
<table>
<thead>
<tr>
<th>Normal</th>
<th>Lightly Loaded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normally Loaded</td>
<td>Lightly Loaded</td>
</tr>
<tr>
<td>External Q</td>
<td>195</td>
</tr>
<tr>
<td>Unloaded Q</td>
<td>700</td>
</tr>
<tr>
<td>Loaded Q</td>
<td>152</td>
</tr>
</tbody>
</table>

1. Cold test of $\pi$-mode

2. Cold test of $n = 3$ mode

3. Cold test of $n = 3$ mode during $\pi$-mode operation

4. R-f voltage developed in $\pi$-mode

When the $\pi$-mode is normally loaded, the loaded Q of the $n = 3$ mode changes from 388, with no power applied, to 47, when 7 kv is applied, with the peak anode current at 7.9 amp, and $\pi$-mode oscillation taking place. When the $\pi$-mode is lightly loaded, the loaded Q of the $n = 3$ mode changes from 258, with no power applied, to 53, when 7 kv is applied, with peak anode current 7.4 amp, and the $\pi$-mode r-f voltage more than twice as much as in the normally loaded case. Therefore, the higher value of r-f voltage has less effect upon the other mode than has the lower value of r-f voltage. This observation is inconsistent with the results obtained from the nonlinear oscillator theory in Sec. III, in which greater r-f voltage in one mode should always have greater adverse effect on other modes. This observation is not inconsistent with the results of the electron-motion analysis of Sec. IV, in which the adverse effect of one mode upon other modes may decrease with increasing r-f voltage, when the latter voltage becomes large enough.
Standing-wave measurements of $n = 3$ resonance in 718-EY magnetron, with $\pi$-mode oscillations taking place. (1) Normal output coupling. (2) Output coupling altered to lighten $\pi$-mode loading.

It is interesting to compare electronic conductances as a function of the magnitude of r-f voltage in the $\pi$-mode. In all of the results in Table V-4, the anode voltage is 7.0 kv and the magnetic field is 1220 gauss, as before. The electronic conductance in the $\pi$-mode for no r-f voltage was estimated from the shape of the detected r-f envelope observed during the early part of build-up. Means of calculating electronic conductance are described by Rieke (ref. 1, Chs. 7 and 8).

Table V-4

Electronic Conductances in an Oscillating Magnetron

<table>
<thead>
<tr>
<th>$\pi$-Mode (n=4) Voltage</th>
<th>$\pi$-Mode (n=4) Electronic Conductance</th>
<th>n=3 Electronic Conductance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{G_{e, 4}}{\omega_4 C_4}$</td>
<td>$\frac{G_{e, 3}}{\omega_3 C_3}$</td>
</tr>
<tr>
<td>0</td>
<td>-0.013</td>
<td>0 (?</td>
</tr>
<tr>
<td>$832/\sqrt{\omega_4 C_4}$</td>
<td>-0.0066</td>
<td>+0.019</td>
</tr>
<tr>
<td>$1760/\sqrt{\omega_4 C_4}$</td>
<td>-0.0022</td>
<td>+0.015</td>
</tr>
</tbody>
</table>

In Table V-4, negative conductance is regarded as supplying power, and positive as absorbing power. The quantities $\omega_4$ and $C_4$ represent angular frequency and equivalent capacity, respectively, for the $\pi$-mode (n=4), and $\omega_3$ and $C_3$ represent similar quantities for the $n = 3$ mode. It is of interest to observe that the positive electronic conductance in the $n = 3$ mode is, in the cases for which it was measured, much larger than the

-41-
magnitude of negative conductance in the π-mode, except for very small π-mode r-f voltages.

The results of these tests are, to some extent, doubtful as to accuracy, because it was not possible to plot very accurate curves of standing wave ratio. Apparently the observed values of standing wave ratio were subject to considerable random error, and the calculated values for loaded Q, based on the same resonance curve, showed considerable inconsistency. The variations of external Q were particularly disconcerting. When electronic conditions were changed, the loading of a resonant mode should be expected to occur entirely within the magnetron. Yet, there appeared to be substantial variations of external Q, even though no change had been made in the r-f output circuit. These effects are thought to be an experimental error not of a fundamental nature. Because the worst inconsistencies in external Q appear where the internal (or unloaded) Q has a much lower value, it is the internal Q which affects the loaded Q most under these circumstances, and inconsistencies in external Q are considered to be much less important than values of internal Q.

In spite of the apparent inconsistencies in data, the effects observed are so large that it is difficult to conceive how any reasonably large errors in measurements could obscure the correct results in principle.

Observations of Mode Changes

Mode Changes: Low-Power Rising-Sun Magnetron

The first experiment to be described here involves a specially constructed 18-vane low power magnetron of the rising-sun type. This magnetron was originally intended for c-w operation, but this was prevented because of the fact that under c-w conditions,
cathode heating was excessive. Instead, operation with long pulses was carried on, with pulse durations of 10 to 20 μsec, or more.

All the tests described here were performed with a magnetic field of 1410 gauss. With the r-f load matched to the output coaxial transmission line, the voltage-current relationships were as shown in Fig. 19. Under these conditions, three different modes were observed, labeled A, B and C. These modes are described in Table V-5. The identification of each mode was made by means of a rotating probe. The values of n, for modes B and C especially, were not perfectly clear, but the values specified in the table seem to be consistent with mode-spectrum theory for rising-sun magnetrons (ref. 3, p. 229). If values of n are known, the values of γ are quite unambiguous, because the starting voltages, functions of γ and frequency, were observed.

Table V-5

<table>
<thead>
<tr>
<th>Mode Identification in Low-Power Rising-Sun Magnetron</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identification of Mode</td>
</tr>
<tr>
<td>Mode A 9/9/18 (π-mode)</td>
</tr>
<tr>
<td>Mode B 6/3/18</td>
</tr>
<tr>
<td>Mode C 7/2/18</td>
</tr>
</tbody>
</table>

When a slowly rising voltage pulse (rise time, 20 μsec) was applied to this magnetron, oscillation was observed successively in modes A, B and C, and during the fall of voltage, oscillation shifted from C back to A; these changes are shown in Fig. 20.

The test described here was to determine whether the change from one mode to the next depended primarily upon conditions in the original mode, conditions in the next mode, or a combination of both. To reach any satisfactory conclusions on this question, it was necessary to alter conditions in one mode without affecting others. The condition which was altered for each mode was the external loading. This was accomplished separately for each mode by means of an absorption-type wavemeter, as shown in Fig. 21. The loading for each mode is changed by a small but appreciable amount as the wavemeter is tuned to that mode; but tuning to the frequency of one particular mode has virtually no effect on other modes.

When the wavemeter was tuned to
mode A, the boundary between A and B was changed from the position shown by the solid line in Fig. 20 to that shown by the dashed line. No effect upon the boundary between C and A was produced. Tuning the wavemeter to the frequency of mode B affected the ending of that mode in a manner shown by the dashed line, but did not affect its starting. Tuning the wavemeter to the frequency of mode C showed the same kind of results. These results indicate that the conditions for shifting from any one of these modes to another are determined primarily by the characteristics of the mode initially oscillating, and are not affected by the mode which is found immediately after the initial mode has become unstable, no matter whether the voltage shift which gave rise to the mode change is upward or downward.

These results are confirmed by synchroscope observation of the anode current, as shown in Fig. 22. Each of the mode change boundaries is accompanied by a vertical spike, indicating a sharp reduction of current during the transition. Since current in an oscillating magnetron can flow only as a consequence of oscillation, the reduction in current evidently corresponds to cessation of oscillation in the original mode, followed by the building-up of oscillation in the subsequent mode. It should be pointed out that the signal on the synchroscope (Fig. 22) has passed through a video amplifier, and therefore, the limited frequency response (about 10 Mc/sec) may have reduced the sharpness of the spike which corresponds to the reduction of current.

Mode Changes: 2J54 Magnetron

The 2J54 magnetron has an eight-cavity strapped anode with hole-and-slot construction, fundamentally similar to the 2J32 (ref. 1, pp. 751-756). It differs from the 2J32 slightly in wavelength, and also because it is tunable. The characteristic which is of interest here is that it sometimes oscillates in the 5/3/8 mode at a slightly lower anode voltage than that for which it begins to oscillate in the π-mode. For certain values of magnetic field and a reflectionless output line, oscillation was observed to begin in the 5/3/8 mode, and to persist even after the applied voltage had reached a value at which the π-mode could build up. The r-f envelope observed under these circumstances is shown in Fig. 23. The step observed in the leading edge was found to correspond to the n = 3 resonance, and the value of applied voltage identified it as the 5/3/8 mode.
To determine the nature of the transition from the 5/3/8 mode to the \( \pi \)-mode, the output coaxial line was terminated in a movable mismatch. The mismatch consisted of a movable short circuit, with a high-power attenuator (about 10 dB) between the short circuit and the magnetron. Thus, the wave reflected by the short circuit represents about 0.01 in power and 0.1 in voltage of the wave traveling toward the load, when this measurement is made between the magnetron and the attenuator. Thus, a considerable change in magnetron loading could be made by the position of the reflection.

It was observed that as the short circuit was shifted in position, the time of the transition between modes also changed in relation to the start of the pulse. The change in the relative time of the transition could be made to be a relatively large part of the pulse duration if the value of applied voltage was only slightly greater than that at which the transition takes place. A shift of the position of the short circuit was made over a distance great enough so that the relative position of the mode transition moved from one extreme to the other and back again. The distance over which the short circuit had to be shifted, to accomplish the change from one extreme and back to the same extreme, was measured several times; this measurement was repeated with the relative time of the transition starting from and returning to the other extreme value. The average values of the distance over which the short circuit had to be moved in order to restore conditions to their original state were found to be 4.64 cm and 4.65 cm for the two sets of measurements. Since moving a reflection by half of a wavelength produces no net change in impedance as measured at the generator, the above values for the distance over which the reflection was moved should be helpful in indicating which of the two modes controlled the transition. The wavelength for the \( \pi \)-mode is 9.45 cm, and that for \( n = 3 \) is 8.16 cm. The above experimental values correspond more nearly to a half wavelength of the \( \pi \)-mode.

It is therefore apparent that mode competition takes place during this transition, and that conditions in the second mode to oscillate, the \( \pi \)-mode, affect the transition more than do conditions in the first. The mode-competition interpretation of this mode change is further supported by the fact that no decay of the 5/3/8 mode preliminary to starting of the \( \pi \)-mode can be observed in the synchroscope trace of the r-f envelope, shown in Fig. 23. In making this observation, it is significant that a broadband diode detector (100-Mc/sec bandwidth) was used, without a video amplifier between detector and deflecting plates.

Mode Stability: Suppression of Unwanted Modes in the 2J39 Magnetron

An attempt was made to measure mode changes of the type described in the above
subsections, using a 2J39 magnetron (ref. 1, pp. 747-751). This attempt was thwarted by the absence of any oscillating modes other than the π-mode. Stable oscillation was observed, for normal magnetron loading, over a range of anode voltages extending from somewhere below the theoretical π-mode threshold voltage all the way to d-c cutoff, and for a wide range of values of magnetic field. Thus, none of the usual mode change or mode stability problems were encountered at all. The range over which π-mode stability was observed is shown in Fig. 24.

When the d-c cut-off voltage was approached, and oscillation began to fall off sharply, anode current continued to increase smoothly and with no suggestion of a discontinuity. Theoretical values of anode current which should flow at d-c cutoff were computed (ref. 5) for two different values of magnetic field. At 1750 gauss, the theoretical d-c current at cutoff is 21.2 amp; the maximum current at which stable π-mode oscillation was observed was 19.6 amp. At 1950 gauss, theoretical direct current is 25.8 amp; maximum current for stable π-mode oscillation was 25.2 amp. Therefore, the theoretical anode current at d-c cutoff was approached very closely before π-mode oscillation ceased.

Another important point shown by Fig. 24 is that stable π-mode oscillation was observed well above the theoretical starting voltage for the 3/3/8 mode. This refutes, if any further refutation is necessary, the mode change theories which have been based primarily on the notion that a lower voltage mode cannot exist when the anode voltage

![Fig. 24](image)

Limits of π-mode oscillation in 2J39 magnetron.
reaches the threshold value for the next higher-voltage mode. It is also significant that π-mode operation is observed here at voltage levels in excess of the instability voltage for the 3/3/8 mode (refs. 22 and 8). (The instability voltage was taken up in Sec. I.)

The history of the development of the 2J39 magnetron is interesting in the light of the results presented here. Early models of the 2J39 experienced π-mode instability at low values of anode current. The problem was studied by Fletcher and Rieke (ref. 17) who recommended changes in the magnet pole pieces which would make the magnetic field in the interaction space more uniform. The model of the 2J39 tested here was one in which these changes had been made.

Summary

In the preceding subsections, various types of mode change phenomena have been discussed. In all of these, the most fundamental observation expressed by the theoretical work in Secs. III and IV is borne out: that large amplitude oscillation in one mode tends to suppress other modes. In the low-power rising-sun magnetron, it was observed that after oscillation in one mode had been established, other modes of oscillation could not start until the one had collapsed, and in each case this collapse was apparently independent of other modes. In the 2J39, suppression of other modes by the π-mode was so effective that the other modes were not observed. Of the magnetrons considered here, only in the 2J54 was the build-up of another mode in competition with the original mode observed. The latter case is by no means in contradiction with the theoretical arguments of Secs. III and IV. The maximum amplitude achieved by oscillations in the 5/3/8 mode, the initially oscillating mode, was very small as compared with π-mode oscillations (see Fig. 23). For reasons explained in Sec. II, it should be expected that the electron bunching in the 5/3/8 mode would be less effective than in the π-mode, due to the presence of a third-order component, as well as the fifth-order component to which the electrons are coupled. Therefore, the suppression of the π-mode by the 5/3/8 mode might well be expected to be relatively ineffective. Furthermore, it was perfectly clear that the presence of the 5/3/8 mode caused an appreciable delay in the starting of the π-mode.

VI. Correlation of Theory and Experiment: Summary and Conclusions

Mode Competition During Build-Up

The subject of mode competition during the build-up transient has been analyzed rather thoroughly by Rieke (ref. 1, Ch. 8). It was stated in Sec. III that the validity of his analysis depends upon the assumption that when nonlinear effects become important, the dependence of the rate of build-up in one mode upon r-f amplitude in the other mode is greater than upon its own amplitude. Rieke stated that this assumption is open to question. Evidently he questioned the assumption only because he had no theoretical justification for it. It is apparent that he realized that it must be true, if mode selection
for any one starting transient is to be definite. (This is not to say that different modes cannot be selected on successive pulses, even though the applied pulses are identical.) Otherwise simultaneous stable oscillation in two modes is possible, a condition rarely, if ever, met with in practice.

From a theoretical point of view, Rieke's assumption is supported by the nonlinear oscillator theory discussed in Sec. III. Confirmation of the assumption is most obvious in Eqs. 31 and 32 of that section, and these equations are based on the instantaneous voltage-current relationship, \( i = ae - be^3 \). It is less obvious, but equally true, that the conditions assumed in Sec. III, which led to the results shown graphically in Fig. 13, also cannot lead to simultaneous stable oscillation in two modes. It is not entirely impossible that some other r-f voltage-current characteristic could lead to circumstances in which two modes of oscillation could be stable simultaneously, but the most plausible types of these relationships do not lead to such results. (See ref. 1, Ch. 8.) Thus Rieke's assumption, according to him, "open to question," can be supported both theoretically and by observation of actual magnetron performance.

Therefore, magnetron build-up, when conditions allow either of two modes, proceeds, presumably, from random noise, which tends to excite both modes. Build-up in each mode is exponential in form, and more or less independent of the other, until nonlinear effects begin to become important. Then competition begins to take place, and the advantage lies more and more with the mode which has greater r-f field intensity for electron bunching. It is r-f field intensity near the cathode, in the region in which rejection of unfavorable electrons takes place, which is most important. Then it becomes necessary that the magnitude of oscillation in the mode which has the greater bunching effect should reduce the rate of build-up in the other mode more than the magnitude of oscillation in the other mode should affect its own rate of build-up. This condition causes the mode with more effective bunching to become progressively stronger than the other, and eventually to suppress it altogether.

Mode Stability

The mode stability problem can be summarized more quickly than the mode selection problem. The results which have been obtained show that if the anode voltage applied to a magnetron is raised slowly, as compared with the rate of build-up of the mode or modes under consideration, until the starting voltage of another mode has been reached, the advantage lies with the mode which started first, and unless the first mode is relatively weak, the second will not build up. Among all the magnetrons discussed in Sec. V, there was none in which the \( \pi \)-mode failed as a result of competition with another mode, and only one other mode in one magnetron which failed as a result of competition. In the other cases, any shift which took place from one mode to another was primarily the result of collapse of the first mode, independent of others, rather than the result of competition from the second.

The outstanding example of \( \pi \)-mode stability was the 2J39, in which the upper limit
on input voltage, and consequently on input current, was d-c cutoff. From the mode interaction point of view, it is significant to point out that π-mode oscillation continued at a value of anode voltage far above the theoretical threshold voltage of the next higher-voltage mode.

Magnetron Design Considerations

The ultimate object of studying the fundamental principles of mode interactions in magnetrons is to improve magnetron design, and methods of magnetron design. A clearer understanding of fundamentals not only makes better magnetrons possible, but makes the process of reaching a good design more direct and less expensive.

The fundamental requirements involving magnetron modes in pulsed magnetrons are, first, the establishment of large amplitude oscillation in the desired mode (nearly always the π-mode), and second, the maintenance of stable oscillation in that mode for the duration of the applied pulse. In c-w magnetrons, the maintenance of stability is as important as in pulsed magnetrons.

R-f Feedback

In promoting quick starting, in order to avoid misfiring, and in promoting greater steady-state stability, in order to increase the maximum power obtainable from a magnetron, the r-f feedback system is very important. It can be altered by changing the loading of the magnetron, or by changing the magnetron structure itself. Of the possible modifications of the structure, the one which has the most effect is changing the ratio of cathode radius to anode radius. The reason for the great effect which this change leads to is the rate at which the intensity of the r-f electric field falls off as the cathode is approached, as discussed in Sec. II. Therefore, the rate of build-up, and to a smaller degree, the steady-state stability, are very sensitive to changes in cathode diameter, with a larger cathode radius leading to faster build-up and more stability. The larger cathode leads incidentally to greater preoscillation noise. Since oscillations must build up from noise, more noise means a greater initial amplitude of oscillation, and the length of time required for building-up is less. Such an effect has been reported by Forsberg (Private communication from P. W. Forsberg to F. F. Rieke; reported by Rieke in ref. 1, Ch. 8, p. 379).

Many magnetron designers have improved stability and reduced misfiring by increasing the ratio of cathode radius to anode radius. This was done by the General Electric Company, Ltd., in England, where it was associated with instability voltages (see Sec. II, also refs. 22 and 8). It was done by the Bell Telephone Laboratories (ref. 18), who associated with an increase in $I_o$, the characteristic current for a given mode in any particular magnetron; this characteristic current was derived from d-c magnetron considerations by Allis, and was applied to the oscillating magnetron by Slater, in absence of any other current which could be clearly defined from theoretical considerations (ref. 5). Finally, it has been done by the Litton Industries, where it
was associated with the pervenance of a simple diode (ref. 21). The sensitivity of misfiring and stability performance of actual magnetrons to this kind of design change is not adequately explained by any of these criteria.

There may be two adverse effects of increasing the cathode radius with respect to the anode radius. The first is the increased stability and the faster build-up of lower voltage modes, which may delay or prevent the starting of the desired mode. The second is a reduction in electronic efficiency.

Mode Interactions

The first and best-known requirement involving interactions between modes is to keep oscillation in one mode from exciting r-f field components of any other mode. This requires adequate separation of other modes from the desired modes in frequency; the most usual means of separation is the use of strapped anodes or of rising-sun anodes. What constitutes adequate separation has never been clearly set forth. Certainly it must be at least equal to several bandwidths of both the desired mode and of any potentially interfering mode. It may also be desirable to separate modes to the extent that the resonant frequency of a mode whose threshold voltage might be lower than that of the desired π-mode instead becomes higher than that of the π-mode; therefore the unwanted mode has no chance to build up ahead of the π-mode and interfere with π-mode starting. For example, in the 2J54 magnetron (see Sec. V) oscillation in the 5/3/8 mode was found to interfere with the starting of the π-mode; one possible method of removing this interference is to increase the frequency of the n = 3 resonance so that the 5/3/8 mode would have a higher starting voltage than the π-mode.

The topic which has been of chief concern here has been the interaction of modes through the electron stream. These results point to the conclusion that once an oscillating mode, for which electron bunching is complete, or almost complete, has been established, it is extremely unlikely and perhaps impossible for another mode to build up in its presence, and suppress it. Even when a mode of oscillation is relatively weak, such as the 5/3/8 mode in the 2J54 magnetron, it still can have a considerable adverse effect on a substantially stronger mode. Unless the initially oscillating mode is quite weak, oscillation will shift to another mode primarily as a result of the failure of the initial mode, substantially apart from any effect by the subsequently oscillating mode.

There are two respects in which these conclusions may be applied to magnetron design: first, the increase of maximum power in the desired mode; second, the suppression of unwanted lower voltage modes.

The suppression of unwanted lower voltage modes has been definitely demonstrated to be of importance. A strongly oscillating unwanted mode can prevent the starting of the π-mode until the unwanted mode collapses as a result of its own properties; a less strongly oscillating mode can interfere seriously with π-mode starting, as in the 2J54.

The unwanted modes can be discouraged by systematically making them less stable by interfering with the feedback mechanism (see subsection on r-f feedback); it is
regrettable that decreasing the cathode diameter for this purpose also affects the \( \pi \)-mode stability adversely. Their r-f field pattern can be distorted, as by strap breaks, which are presumed to affect the \( \pi \)-mode pattern less than that of other modes. However, the effectiveness of strap breaks has never been clearly established. The unwanted modes can also be removed from the region of lower voltage modes to the region of modes with higher voltages than the \( \pi \)-mode, as described above. When the starting voltage of the unwanted mode is higher than that of the \( \pi \)-mode, it can be completely suppressed by a strongly oscillating \( \pi \)-mode; this time, the tendency of the originally oscillating mode to persist against modes which might oscillate under the existing conditions of magnetic field and anode voltage if the original mode were not there works against the unwanted mode.

However, lower voltage modes are usually not a serious problem. The low voltage mode in the 2J54 disappeared when a higher magnetic field was used. A similar mode, although theoretically possible in the 2J39 magnetron, was never observed. Rising-sun magnetrons are usually free from interference by modes of lower starting voltage than the \( \pi \)-mode.

Turning our attention now to the application of these principles to extending the maximum power limit of \( \pi \)-mode operation in the face of possible competition from unwanted modes, we find that the presence of other possible modes at the desired operating voltage does not necessarily prevent satisfactory oscillation in the \( \pi \)-mode at this voltage. Instead, we find that it is possible to establish \( \pi \)-mode oscillation in a voltage range suitable for \( \pi \)-mode build-up, after which the voltage may continue upward indefinitely until some inherent instability of the \( \pi \)-mode itself sets in. This type of limitation is not well understood from a quantitative point of view, but methods of extending it have been discussed above. It may be recalled here that in a properly loaded 2J39 magnetron, the limitation on \( \pi \)-mode stability coincided with d-c cutoff. On the basis of competition from other modes, there appears to be no fundamental limitation on magnetron power. There appears to be no basis yet for predicting the maximum power available from a magnetron from \( \pi \)-mode failure considerations, other than that imposed by d-c cutoff.

Magnetic Field Uniformity

A very important factor in stability is the uniformity of the magnetic field. This problem is usually not serious in unpackaged magnetrons, where the magnet is entirely exterior to the magnetron; only a small portion of the magnet gap, both in terms of diameter and in terms of length, is included in the interaction space. This is not the case for packaged magnetrons. In these magnetrons, the magnet gap has been shortened and decreased in diameter by bringing the magnet poles through the end plates and as close to the interaction space as possible. Under these circumstances, it is more difficult to make the magnetic field uniform.

Two examples of improvements in magnetic field uniformity will be discussed here.
The 2J39 was changed from a magnetron with unsatisfactory stability characteristics to one with unusually good stability by redesigning the pole pieces to produce a more uniform magnetic field (ref. 17). A relatively uniform magnetic field in the 4J50 magnetron was made possible by means of cathode and shields made of permendur, a ferromagnetic alloy with a very high Curie temperature (ref. 3, pp. 329-330).

Reasons for the adverse effects of nonuniform magnetic fields have been discussed in Sec. II. Their importance is emphasized by the great improvement in 2J39 operation by means of small changes in the pole-piece dimensions.

Cathode Performance

The need for adequate cathode emission has long been recognized as fundamental to good magnetron oscillation. It has been shown here that good starting and stability characteristics which result from good cathode performance characteristics may be associated with the properties of a magnetron considered as a feedback amplifier. In a magnetron, the cathode performance is analogous to cathode performance in a triode feedback oscillator. Good performance in the triode oscillator or in the magnetron is dependent on the loop transmission (defined in Sec. II); the loop transmission may be substandard if cathode performance is poor. In the magnetron it is especially important that the equivalent loop transmission be high for high values of anode voltage, for under these conditions, the tendency for electrons to get out of step with the r-f wave is strong.

Summary

The principal idea which has come from this research is that large-amplitude oscillation in one mode tends quite strongly to suppress oscillation in other modes. This supports the mode-competition theory advanced by Rieke (ref. 1, Ch. 8, pp. 380-387). On the other hand, it contradicts all mode stability and mode change criteria based primarily on the effects of other modes upon a strongly oscillating mode. For example, it is not necessarily true that when the necessary conditions for oscillation in the next higher voltage mode are met, the originally oscillating mode will give way to the mode with the higher starting voltage; on the contrary, if the first mode is one which oscillates strongly, it is much more likely that the first one will persist, and the second will arise only after conditions are such that the first mode will collapse, primarily independent of conditions in the second mode.

It is therefore true that no fundamental limitation upon high power in magnetrons exists as a result of the possible presence of unwanted modes, provided oscillation in the desired mode can be firmly established. No fundamental limitation on high power in magnetrons could be found which is associated with mode instability, except when d-c cutoff was reached; whether or not another limit exists is still open to question.

Therefore, the fundamental requirement both for quick starting, in order to avoid misfiring, and for stability, with or without the possible presence of other modes, is
to establish and maintain effective electron bunching throughout the entire anode length.

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