WORKING PAPER

ALFRED P. SLOAN SCHOOL OF MANAGEMENT

THE RELIABILITY OF SUBJECTIVE PROBABILITIES OBTAINED THROUGH DECOMPOSITION[1]

By

Don N. Kleinmuntz
H.V. Ravinder
James S. Dyer

WP #1847-86

December 1986

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139

THE RELIABILITY OF SUBJECTIVE PROBABILITIES OBTAINED THROUGH DECOMPOSITION[1]

By

Don N. Kleinmuntz
H.V. Ravinder
James S. Dyer

WP #1847-86 December 1986

[1]Please address correspondence to: Don N. Kleinmuntz, Sloan School of Management, E52-568, Massachusetts Institute of Technology, 50 Memorial Drive, Cambridge, MA 02139.

The Reliability of Subjective Probabilities Obtained Through Decomposition[1]

H. V. Ravinder
Anderson School of Management
University of New Mexico
Albuquerque, New Mexico 87131

Don N. Kleinmuntz
Sloan School of Management
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

James S. Dyer
Department of Management
University of Texas at Austin
Austin, Texas 78712

Revised, November 1986

[1]Please address correspondence to: Don N. Kleinmuntz, Sloan School of Management, E52-568, Massachusetts Institute of Technology, 50 Memorial Drive, Cambridge MA 02139

Abstract

The use of decomposition as a procedure for improving the consistency of subjective probability encoding is discussed. Using a psychometric measurement model, an expression is developed that describes the random error associated with decomposition estimates as a function of characteristics of the component assessments. Decomposition is compared to direct assessment in terms of the percent change in measurement error that can be attributed to the use of decomposition. The results are derived for the case of independent elicitation errors and for the more general case of dependent errors. Potential benefits of decomposition are specified and recommendations made on how to utilize decomposition as an approach for error control.

1 Introduction

A critical issue in decision making is the elicitation and encoding of uncertainty in the form of subjective probabilities. There are numerous reviews and tutorials that describe and compare a variety of methods for eliciting probabilities (e.g., Hogarth, 1975; Huber, 1974; Lichtenstein, Fischhoff, & Phillips, 1977, 1982; Lindley, 1986; Lindley, Tversky, & Brown, 1979; Spetzler & Staël von Holstein, 1975; Wallsten & Budescu, 1983). A basic problem discussed throughout this literature has been the potential for encoded probabilities to contain serious inaccuracies and inconsistencies. For instance, Wallsten & Budescu (1983) note that:

> there is, in addition, a growing literature suggesting that human judgment is frequently malleable, inconsistent, and subject to bias (e.g., Fischhoff, Slovic, & Lichtenstein, 1980; Hogarth, 1975; Slovic, Fischhoff, & Lichtenstein, 1977; Tversky & Kahneman, 1973; 1974). Note, however, that the fact that people can be shown to be irrational when left to their own devices does not imply that they will necessarily be irrational in all cases. In particular, a careful decision or risk analyst should be able to structure a situation so that bias and inconsistency are minimized and the encoded probabilities closely represent the judge's carefully considered opinion. (p. 155)

This paper is concerned with how an analyst might structure the encoding situation in order to reduce inconsistency. Specifically, we provide a framework for evaluating the effectiveness of elicitations that rely upon *decomposition.* The evaluation uses a psychometric measurement perspective to provide a comparative analysis of the reliability of probability assessments based upon decomposition versus direct elicitation. Addressing this question leads to a better understanding of decomposition processes for probability encoding and has the potential to improve both the theory and practice of decision analysis.

The remainder of the paper is organized as follows: Section 2 describes the decomposition approach. Section 3 describes the psychometric model that is used to evaluate the elicitations. Section 4 utilizes that model to derive results on the relative reliability of decomposition and direct assessment. Section 5 extends those results while relaxing certain restrictive assumptions of the original model. Finally, section 6 discusses the implications of these results for the practice of probability encoding.

2 Decomposition

Decomposition is often regarded as a useful technique for reducing the complexity of difficult judgment problems: A large, messy problem is divided into a set of smaller and presumably easier judgments. These judgments are then combined using an aggregation rule based upon statistical or mathematical considerations. An obvious advantage of this approach is the reduction of information processing demands, since the judge can isolate and concentrate on selected portions of the problem in turn. Furthermore, the laborious task of information combination can then be handled by a formal model, perhaps implemented on a computer.

This paper examines the application of decomposition to the elicitation of subjective probabilities. Consider the following examples:

Example 1. The market price of oil is an important variable in many economic planning models. An analyst might ask an oil industry expert to assess a probability distribution for the price of oil five years in the future. The expert could consider a variety of factors that might influence the future price of the commodity, including the political situation in the Middle East, the market attractiveness of alternate energy sources, the level of oil exploration activities, and the projected economic health of various national economies.

Example 2. A physician might need to predict the survival time of a patient who has recently undergone an experimental treatment (e.g., an artificial heart implant). The physician's assessment could be based upon a number of related considerations: How advanced was the patient's disease? Was the procedure correctly implemented? Is the patient susceptible to particular complications? What is the patient's emotional and physical strength?

Example 3. A venture capitalist might be considering investing in a firm developing an untried technology. In estimating the probability of the venture's success, key factors include: the professional qualifications of the research and development team, the experience and competence of the management team, the size of the potential market for the eventual product, the general state of the economy, and other factors that might influence the attractiveness of competing investment opportunities.

In each of these examples, the target probabilities are heavily context dependent: It is difficult to think of the target event without reference to other events that may be related either causally or statistically. Holistic or *direct* elicitation of the probability may be problematic since keeping track of these other events vastly increases the complexity of the assessment. This could lead to an ambiguous or inconsistent consideration of these background events, leading in turn to encoding errors.

A decomposition approach explicitly captures the relationship between the target event and the background events in the form of *conditional* probabilities of the target event. For instance, different distributions for the price of oil might be assessed conditional upon the outcome of the Iran-Iraq war. Similarly, distributions could be assessed conditional upon the occurrence of scenarios consisting of multiple events (e.g., the Iran-Iraq war remains stalemated, no significant new oil fields are discovered, and U.S. demand for oil remains level).[1]

Once individual conditional distributions are elicited, the law of total proba-

[1] While the decomposition approach can be readily extended to the continuous case, the present discussion is limited to discrete events, both for the target and background events.

bility provides a convenient method for aggregation. Suppose the probability of the target event, denoted event A, is assessed conditional upon each of a set of background events, denoted $B_1, \ldots, B_n$. Each of the events B_i could either be a single event or be a scenario, composed of the intersection of multiple events. The only requirement is that B_1 through B_n form a mutually exclusive and exhaustive partition of the relevant event space. If this condition is satisfied, then $\sum_{i=1}^{n} \Pr(B_i) = 1$ and the probability of the target event is as follows:

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A|B_i) \Pr(B_i). \tag{1}$$

Clearly, elicitation by decomposition requires greater effort than direct assessment of uncertainty. For *each* background event, both a conditional and marginal probability estimate must be obtained. Knowing whether this additional effort is warranted requires a greater understanding of the decomposition process. This paper provides a first step in this direction by establishing the impact of the decomposition procedure on the reliability of the assessed probabilities.

3 The Psychometric Model

In their recent paper, Wallsten and Budescu (1983) describe a psychometric framework for evaluating the accuracy of probability encoding. Drawing from the literature on psychological measurement and scaling, they propose the following model: Each encoded probability is considered to be a random variable, x, that is composed of a fixed true measure, t, and a variable error, e:

$$x = t + e. \tag{2}$$

The following basic assumptions are made about equation 2: (a) The expected value of the error is zero: $\mathrm{E}(e) = 0$. (b) The true and error values are uncorrelated: $\mathrm{Corr}(t, e) = 0$. (c) For any pair of independent measurements the error scores are uncorrelated: $\mathrm{Corr}(e_i, e_j) = 0$, where $i \neq j$. (d) The true score of one measurement is uncorrelated with the error in another: $\mathrm{Corr}(t_i, e_j) = 0$, where $i \neq j$.

Wallsten and Budescu also note that care should be taken in interpreting the sense in which the t scores are "true" measures of probability:

> It would be inappropriate to think of a person's opinion about a set of events as existing within that person in a precise, fixed fashion just waiting to be measured (Fischhoff, Slovic, & Lichtenstein, 1980). Rather, an individual's opinion is more-or-less vaguely formulated, and upon being asked to evaluate the probability of an outcome, a person will search his or her memory for relevant knowledge, combine it with the information at hand, and (presumably) provide the best judgment possible. ... If the same situation were replicated a large number of times, and if the person had no memory of his or her previous judgments, the encoded probabilities, x, would give rise to a distribution for that particular individual. A convenient way to define the true score, t, is as the expected value of this hypothetical distribution that would be obtained across a series of statistically independent judgments by a given individual. (p. 153)

Further, the variance of the repeated assessments can be represented by the variance of the error scores, σ_e^2, termed the *error variance*. Its square root, σ_e, is the *standard error of measurement*. Using this framework, maximizing the reliability of probability encodings is acheived by minimizing this error variance.

When working with this model, it is important to note and maintain the distinction between *random* and *systematic* sources of error. In particular, an analysis of reliability only considers the former. For instance, when comparing two different measurement techniques, one might observe consistent, predictable differences in the true scores depending upon which technique is utilized. An examination of reliability does not address this potentially important source of error. On the other hand, if random errors were large enough, accurate measurement would not be possible with either technique. Thus, controlling random error can be thought of as a necessary though not sufficient condition for accurate elicitation (for further details, see Wallsten & Budescu, 1983, pp. 152–154, and the references cited there).

4 Independent Component Elicitations

In order to compare decomposition to direct assessment of probabilities, first, notation is defined that applies the measurement model to the two assessment techniques. Second, the measurement error associated with decomposition is derived. Finally, the two techniques are compared in terms of the percent change in measurement error that can be attributed to the use of decomposition.

4.1 Notation and Assumptions

The decomposition and direct estimates of the probability of the target event, A, are denoted a and a' respectively. The probabilities of the target event conditional upon each of the background events are denoted $\Pr(A|B_i) = c_i$, while the marginal probabilities of the background events are denoted $\Pr(B_i) = b_i$, so that equation 1 can be rewritten as:

$$a = \sum_{i=1}^{n} c_i b_i. \tag{3}$$

In order to maintain the assumptions of the measurement model, it is important to note the marginal probabilities of the background events are not completely independent: $\sum_{i=1}^{n} b_i = 1$. Independence is preserved by treating the first $n-1$ marginal probabilities as independent and expressing b_n in terms of the others, $b_n = 1 - \sum_{i=1}^{n-1} b_i$. This allows equation 3 to be rewritten as

$$a = \sum_{i=1}^{n-1} c_i b_i + c_n \left(1 - \sum_{j=1}^{n-1} b_j\right), \tag{4}$$

which is equivalent to:

$$a = c_n + \sum_{i=1}^{n-1} (c_i - c_n)\, b_i. \tag{5}$$

The measurement model can now be directly applied to both the direct estimate and the components of the decomposition estimate. For the direct estimate,

$$a' = \alpha' + \delta', \tag{6}$$

where α' is the true score and δ' is the random error, with its standard error of measurement (SEM) denoted by σ_0. Similarly, for the marginal probabilities of the background events, where $i = 1 \ldots n$,

$$b_i = \beta_i + \delta_i \tag{7}$$

with each SEM denoted σ_i. The true score for b_n is denoted $\beta_n = 1 - \sum_{i=1}^{n-1} \beta_i$, while its error term is denoted $\delta_n = -\sum_{i=1}^{n-1} \delta_i$. Finally, for the conditional probabilities, where $i = 1 \ldots n$,

$$c_i = \gamma_i + \epsilon_i \tag{8}$$

with each SEM denoted τ_i.

Applying assumptions (b), (c), and (d) of the measurement model implies that the various error terms (δ_i and ϵ_i) are uncorrelated, with one exception: The correlation between δ_n and $\sum_{i=1}^{n-1} \delta_i$ is -1.0. These admittedly restrictive assumptions, which make the results easier to interpret, are relaxed in section 5.

4.2 The Error Variance of Decomposition Estimates

This notation and the assumptions described above can be used to derive, in turn, the expected value and variance of the decomposition assessment. The expected value of the decomposition estimate, $\mathrm{E}(a)$, can be readily obtained from equation 5:

$$\mathrm{E}(a) = \mathrm{E}(c_n) + \sum_{i=1}^{n-1} \mathrm{E}\left((c_i - c_n)\, b_i\right). \tag{9}$$

Using the independence assumptions, it follows that:

$$\mathrm{E}(a) = \gamma_n + \sum_{i=1}^{n-1} (\gamma_i - \gamma_n)\,\beta_i. \tag{10}$$

One consequence of assumption (a) of section 3 is that the estimates of the various probabilities are unbiased. If this is the case, then the expected value of the decomposition estimate, given in equation 10, should be equal to the expected value of the direct assessment, which, from equation 6, is α'. If there are *systematic* sources of error in either assessment process, then the two expected values would not normally be equal.

The error variance of the decomposition estimate is denoted by $\mathrm{Var}(a) = \sigma_d^2$. Using equation 4, this variance can be written as a linear combination of three components:

$$\begin{aligned}\sigma_d^2 &= \mathrm{Var}\left(\sum_{i=1}^{n-1} c_i b_i\right) + \mathrm{Var}\left(c_n\left(1 - \sum_{j=1}^{n-1} b_j\right)\right)\\ &\quad + 2\mathrm{Cov}\left(\sum_{i=1}^{n-1} c_i b_i, c_n\left(1 - \sum_{j=1}^{n-1} b_j\right)\right).\end{aligned} \tag{11}$$

In the appendix, this expression is shown to be equivalent to:

$$\begin{aligned}\sigma_d^2 &= \sum_{i=1}^{n-1} \sigma_i^2 (\gamma_i - \gamma_n)^2 + \sum_{i=1}^{n} \beta_i^2 \tau_i^2\\ &\quad + \sum_{i=1}^{n-1} \sigma_i^2 \tau_i^2 + \tau_n^2 \sum_{i=1}^{n-1} \sigma_i^2.\end{aligned} \tag{12}$$

A useful special case occurs when the various probabilities (conditional and background) are assumed to have the same SEM: $\sigma_i = \sigma$ for $i = 1 \ldots n-1$ and $\tau_i = \tau$ for $i = 1 \ldots n$. In this case, equation 12 simplifies to:

$$\sigma_d^2 = \sigma^2 \sum_{i=1}^{n-1} (\gamma_i - \gamma_n)^2 + \tau^2 \sum_{i=1}^{n} \beta_i^2 + 2(n-1)\sigma^2\tau^2. \tag{13}$$

Equation 12 is a central result of this paper, describing the random error associated with decomposition as a function of the true scores and error variances of its components. Several observations can be made from a direct examination of equation 12: First, the measurement error associated with decomposition, σ_d, is proportional to the measurement errors of the component probabilities, σ_i and τ_i. Not surprisingly, the less error there is in the conditional and background probability estimates, the better the decomposition estimate.

Second, the decomposition error is also related to the true scores of the components, although the relationship is not as straightforward. The decomposition error expressed as a function of β_i is minimized when the expression $\sum_{i=1}^{n} \beta_i^2 \tau_i^2$ is minimized with respect to the values β_i. By differentiating this expression with respect to each of the β_i, and recalling the constraint that $\sum_{i=1}^{n} \beta_i = 1$, it can be shown that error variance is minimized when $\beta_i = \beta_n(\tau_n^2/\tau_i^2)$ for each $i = 1 \ldots n-1$. One implication of this result is that under the special case when the errors associated with the conditional probabilities are uniform, $\tau_i = \tau$ for all $i = 1 \ldots n$, then the ideal situation occurs when the true values of the marginal probabilities are equal ($\beta_i = 1/n$).

A similar result can be derived for the true scores of the conditional probabilities, γ_i: Decomposition error is minimized when $\sum_{i=1}^{n-1} \sigma_i^2(\gamma_i - \gamma_n)^2$ is minimized. Clearly, this occurs when the differences $(\gamma_i - \gamma_n)$ approach zero, which implies $\gamma_i = \gamma_n$ for all values of $i = 1 \ldots n-1$. In other words, if one can control which conditioning events to use, it would be desirable to use events all having the same conditional probabilities. [2]

It is also possible to examine decomposition error as a function of the number of conditioning events: Using equation 13 (which assumes equal variances) with uniform values for the true scores of both the conditionals ($\gamma_i = \gamma_n$ for $i = 1 \ldots n-1$) and the marginals ($\beta_i = 1/n$ for $i = 1 \ldots n$) produces the following

[2] If all the conditional probabilities were known to be equal, then decomposition would not really be needed. The extent to which the true values of the conditionals can be controlled by either the assessor or the analyst is not clear.

expression for decomposition error:[3]

$$\sigma_d^2 = \frac{\tau^2}{n} + 2(n-1)\sigma^2\tau^2. \tag{14}$$

Taking the first derivative of the right side of equation 14 with respect to n,

$$\frac{\partial}{\partial n}\sigma_d^2 = -\left(\frac{\tau}{n}\right)^2 + 2\sigma^2\tau^2.$$

Setting this result equal to 0 and solving for n produces the following expression for n^*, the value of n that minimizes variance:

$$n^* = \frac{1}{\sqrt{2}\sigma}. \tag{15}$$

Thus, the optimal number of conditioning events appears to be a function of the amount of error in the marginal probabilities, since n^* increases as σ decreases. Further insight into this result is obtained by examining figure 1, which uses equation 14 to plot decomposition error as a function of n for different values of σ and τ. Decomposition error drops rapidly as n increases, but quickly levels off and begins to increase. Thus, it seems inadvisable to use either too few or too many conditioning events. However, the exact number of events does not seem to be critical since, in the neighborhood of the minimum, the function is relatively flat, particularly for smaller values of σ.

Place figure 1 about here

One way to interpret the error of the decomposition estimate is to conceptualize decomposition as a weighted average of n independently estimated conditional probabilities. The ideal situation occurs when these independently estimated quantities have identical expected values. This corresponds to drawing a sample of n from a single distribution. Hence, the observation that the standard error of the estimate decreases as n increases is a reflection of the law of large numbers. The fact that error actually begins to rise (albeit slowly)

[3] An analysis with less restrictive assumptions produces similar results.

for large n is an indication that the weights used in the weighted average (the marginals) contribute error as well.

Furthermore, the results derived above regarding the ideal distribution of marginal probabilities can be thought of as an attempt to specify an optimal weighting scheme. When the conditional probabilities are all equally prone to error, then equal weights are best. However, if the conditional probabilities have differing degrees of error associated with them, it is desirable to shift weight away from conditional probabilities that are less precisely assessed. Also, if *all* the conditionals are assessed without error, $\tau_i = 0$, the true scores of the marginals become irrelevant, and the specifics of the weighting scheme do not matter.

Viewing the decomposition estimate as a weighted average also provides some insight into the standard error's sensitivity to the true scores of the conditional probabilities. There are two aspects of the conditionals that contribute to the total error associated with decomposition: (1) The error associated with the assessment of each conditional probability, τ_i, and (2) the extent to which the true scores disagree. Even if the conditionals were assessed without error, $\tau_i = 0$, there would still be a contribution to the total error, unless all the conditionals agreed. Also, there is an interaction between this source of error and the errors associated with the weights: If the marginals are assessed without error, $\sigma_i = 0$, then the true scores of the conditionals are irrelevant.

4.3 Comparing Decomposition to Direct Elicitation

What are the relative sizes of the errors associated with direct elicitation and decomposition? This question can be addressed through a direct comparison of the SEM of direct elicitation, σ_0, to the SEM of decomposition, σ_d, as derived above. To simplify the calculations, consider the special case of uniform SEMs for both background and conditional probabilities, $\sigma_i = \sigma$ and $\tau_i = \tau$. Furthermore, compare the two techniques while varying the SEMs of the component probabilities as a function of the SEM of direct elicitation: The SEM of the background probabilities is $\sigma = k_b\sigma_0$, while the SEM of the conditionals is $\tau = k_c\sigma_0$, where k_b and k_c are positive constants. Using these assumptions, it follows from equation 13 that:

$$\begin{aligned} \sigma_d^2 &= (k_b\sigma_0)^2 \sum_{i=1}^{n-1}(\gamma_i - \gamma_n)^2 + (k_c\sigma_0)^2 \sum_{i=1}^{n} \beta_i^2 \\ &\quad + 2(n-1)(k_b k_c)^2 \sigma_0^4. \end{aligned} \tag{16}$$

The relative size of the measurement errors is expressed as the percent change due to decomposition, denoted:

$$\pi = \frac{\sigma_d - \sigma_0}{\sigma_0} = \frac{\sigma_d}{\sigma_0} - 1. \tag{17}$$

Using equation 16, π can be expressed as:

$$\pi = \left\{ k_b^2 \sum_{i=1}^{n-1}(\gamma_i - \gamma_n)^2 + k_c^2 \sum_{i=1}^{n} \beta_i^2 + 2(n-1)k_b^2 k_c^2 \sigma_0^2 \right\}^{1/2} - 1. \tag{18}$$

A value of $\pi = 0$ indicates identical levels of measurement error for both techniques. Negative values indicate an improvement due to decomposition, while positive values indicate that decomposition is inferior.

It is particularly informative to derive upper and lower bounds for π. The upper bound can be derived by noting that:

$$\sum_{i=1}^{n-1}(\gamma_i - \gamma_n)^2 \leq n-1$$
$$\sum_{i=1}^{n}\beta_i^2 \leq 1$$
$$\sigma_0^2 \leq \frac{1}{12}.$$

The last constraint follows from the fact that the variance of a random variable defined over a closed interval must itself be constrained. The particular constraint chosen here is based upon the variance of a random variable uniformly distributed on the interval [0, 1]. Using equation 18, the upper bound is:

$$\pi \leq \left\{k_b^2(n-1) + k_c^2 + k_b^2 k_c^2\left(\frac{n-1}{6}\right)\right\}^{1/2} - 1. \tag{19}$$

Similarly, the lower bound can be derived from the following constraints:

$$\sum_{i=1}^{n-1}(\gamma_i - \gamma_n)^2 \geq 0$$
$$\sum_{i=1}^{n}\beta_i^2 \geq \frac{1}{n}$$
$$\sigma_0^2 \geq 0.$$

This results in the expression:

$$\pi \geq \frac{k_c}{\sqrt{n}} - 1. \tag{20}$$

Figures 2, 3, and 4 plot the upper and lower bounds of π against the number of conditioning events, n, for different levels of k_b and k_c.

Place figures 2, 3, & 4 about here

Figure 2 shows how π can vary when the component probabilities have the same amount of error as direct elicitation, $k_b = k_c = 1.0$. The potential improvement is fairly large, with a 30% or more reduction in error under the best conditions, and the potential improvement becoming larger as the number of conditioning events increases. On the other hand, the worst case analysis indicates that decomposition can actually result in a substantial *increase* in measurement error.

What accounts for this wide variation in the effectiveness of decomposition? Inspection of equation 18 indicates that the values of the true scores for both the conditional and background probabilities are critical. Even modest deviations from the optimal configuration of either set, as described above, causes rapid deterioration in performance. Consider the following numerical example, for the case of $n = 4$ conditioning events: Suppose that the SEM of direct elicitation is moderately high ($\sigma_0 = 0.1$).[4] Assuming the ideal case, with uniform values of the background probabilities, $\beta_i = 0.25$ for $i = 1 \ldots 4$, and the values of the conditionals also equal, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4$, then the result, from equation 18, is a substantial error reduction due to decomposition, $\pi = -0.44$. However, if while holding everything else constant, the values of the background probabilities are changed so that $\beta_1 = 0.7$ while $\beta_2 = \beta_3 = \beta_4 = 0.1$, then the improvement is cut almost in half, $\pi = -0.24$. The results are even more sensitive to the values of the conditional probabilities: If everything else is held constant except that the value of γ_4 is changed by 0.45, then the improvement due to decomposition almost disappears, $\pi = -0.04$.

Figure 3 illustrates the effect of increasing the values of k_b and k_c, in this case assuming that the errors associated with the components are 50% larger than the direct assessment ($k_b = k_c = 1.5$). Clearly, the potential gains from decomposition are reduced and, for $n = 2$, error reduction is *impossible* under even the best circumstances. As k_b and k_c increase, error reduction becomes infeasible for larger and larger values of n.

The converse case is illustrated in figure 4, where the errors associated with the components are 50% smaller than for direct assessment ($k_b = k_c = 0.5$). In this case, the opportunities for error reductions are greater and, for small

[4] The results of this example are not sensitive to the precise value of direct elicitation error.

n, improvements occur for any values of the true scores. Thus, as k_b and k_c decrease, error reduction becomes easier and less dependent on the choice of conditioning events.

5 Dependent Component Elicitations

There are two assumptions in the previous analysis that warrant further discussion. These are the assumption of independence for conditional probability elicitations, and the assumption of partial independence of marginal probability elicitations. These are now discussed in turn.

In practice, errors in the elicitation of conditional probabilities could well be correlated with each other and with the error in the direct elicitation of $\Pr(A)$. For instance, suppose an assessor uses an *anchoring-and-adjustment* process to estimate each $\Pr(A|B_i)$, where the assessor starts with ('anchors on') the direct elicitation of $\Pr(A)$ and adjusts it according to knowledge about the strength of the relationship between events A and B_i. The various conditional probabilities all share the same anchor, but each one has a unique adjustment. Since *each* conditional probability is influenced by the error in the shared anchor, this process can be expected to produce positive correlations among the errors in the conditional probabilities (for a proof, see Ravinder, 1986, chapter 5).

As already noted, the marginal probabilities can not be completely independent, since coherence requires that they sum to 1. This problem was initially handled by assuming that the first $n-1$ marginals were independent while the last one, b_n, was completely determined by those independent probabilities. A more plausible assumption might be that all n marginals be assessed simultaneously, with the errors interrelated in a more complex fashion. However, if coherence is maintained, then any single probability b_i can be determined from the other $n-1$ values (i.e., the correlation between b_i and $\sum_{i\neq j}^{n} b_j$ is -1.0). This in turn implies that any two marginals b_i and b_j have negatively correlated errors.

In this section, our analysis is extended to cover these two cases. In particular, a generalization of equation 12 is derived that allows for correlated errors among the elicited probabilities. Following this, results comparable to those presented above are derived for the case of uniform negative correlations among the marginals and uniform positive correlations among the conditionals.

5.1 The Error Variance with Dependent Components

The notation and assumptions of the previous section are maintained with the following modifications: (a) The errors in the marginal elicitations need not be independent, but rather, have a correlation denoted by $\mathrm{Corr}(\delta_i, \delta_j) = \phi_{ij}$. Note that $-1 \leq \phi_{ij} \leq 1$ for $i \neq j$, while $\phi_{ii} = 1$. (b) Similarly, the errors in the conditional elicitations may be correlated, denoted by $\mathrm{Corr}(\epsilon_i, \epsilon_j) = \rho_{ij}$. Also note that $-1 \leq \rho_{ij} \leq 1$ for $i \neq j$, while $\rho_{ii} = 1$. (c) Finally, independence *between* the marginals and conditionals is maintained: $\mathrm{Corr}(\delta_i, \epsilon_j) = 0$ for all i and j.

Using equation 3 as the starting point, the variance of the decomposition estimate can be written as the sum of a set of covariances:

$$\sigma_d^2 = \mathrm{Var}\left(\sum_{i=1}^{n} c_i b_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(c_i b_i, c_j b_j). \qquad (21)$$

An expression for $\mathrm{Cov}(c_i b_j, c_j b_j)$ can be readily obtained because the c_i and b_i terms are independent:

$$\begin{aligned}\mathrm{Cov}(c_i b_i, c_j b_j) &= \mathrm{E}(c_i c_j b_i b_j) - \mathrm{E}(c_i b_i)\mathrm{E}(c_j b_j) \\ &= \mathrm{E}(c_i c_j)\mathrm{E}(b_i b_j) - (\gamma_i \beta_i)(\gamma_j \beta_j).\end{aligned}$$

Noting that

$$\mathrm{E}(c_i c_j) = \gamma_i \gamma_j + \mathrm{Cov}(c_i, c_j),$$

$$\mathrm{E}(b_i b_j) = \beta_i \beta_j + \mathrm{Cov}(b_i, b_j),$$

$$\mathrm{Cov}(c_i, c_j) = \tau_i \tau_j \rho_{ij},$$

and that

$$\mathrm{Cov}(b_i, b_j) = \sigma_i \sigma_j \phi_{ij},$$

it then follows that:

$$\begin{aligned}
\mathrm{Cov}(c_i b_i, c_j b_j) &= (\sigma_i \sigma_j + \tau_i \tau_j \rho_{ij})(\beta_i \beta_j + \sigma_i \sigma_j \phi_{ij}) - (\gamma_i \beta_i)(\gamma_j \beta_j) \\
&= (\gamma_i \sigma_i)(\gamma_j \sigma_j)\phi_{ij} + (\beta_i \tau_i)(\beta_j \tau_j)\rho_{ij} + (\sigma_i \tau_i)(\sigma_j \tau_j)\phi_{ij}\rho_{ij}.
\end{aligned}$$

Using the right-hand-side of this last expression in equation 21 produces a general formula for the decomposition error:

$$\begin{aligned}
\sigma_d^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}(\gamma_i \sigma_i)(\gamma_j \sigma_j)\phi_{ij} \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_i \tau_i)(\beta_j \tau_j)\rho_{ij} \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}(\sigma_i \tau_i)(\sigma_j \tau_j)\phi_{ij}\rho_{ij}. \qquad (22)
\end{aligned}$$

While this expression is quite general, only certain combinations of values of σ_i and ϕ_{ij} will produce coherent probabilities (i.e., marginal probabilities that sum to 1). In order to satisfy the coherence requirement, it is necessary to impose restrictions on the variance-covariance matrix for the marginal assessments. Specifically, the restriction is that for each $i = 1, \ldots, n$, the correlation between b_i and $\sum_{j \neq i}^{n} b_j$ is -1. Any variance-covariance matrix that satisfies these n simultaneous equations will be coherent.

As an example, consider the assumptions of the previous section: The first $n-1$ marginals were assumed to be independent, $\phi_{ij} = 0$ for all $i \neq j$, where $i \leq n-1$ and $j \leq n-1$. In order to maintain the coherence constraint, it then follows that $\phi_{in} = -(\sigma_i/\sigma_n)$ for $i = 1, \ldots, n-1$, where $\sigma_n^2 = \sum_{i=1}^{n-1} \sigma_i^2$. Furthermore, by letting $\rho_{ij} = 0$ for $i \neq j$, and substituting these values into equation 22, it is easy to derive equation 12 as a special case.

5.2 Uniform Negative Correlation of Marginals

Although the previous assumptions about the marginals satisfy the coherence requirement, many other assumptions also satisfy that requirement. A plausible alternative is to let the correlation among the marginals be uniform and negative, $\phi_{ij} = \phi < 0$ for $i \neq j$. For simplicity, assume uniform error variances for the marginal probabilities, $\sigma_i^2 = \sigma^2$ and for the conditional probabilities, $\tau_i^2 = \tau^2$, although the results could readily be extended to the general case.

Since coherence implies that the correlation between b_i and $\sum_{j\neq i}^n b_j$ is -1, it follows that:

$$\frac{\mathrm{Cov}\left(b_i, \sum_{j\neq i}^n b_j\right)}{\sigma\left\{\mathrm{Var}\left(\sum_{j\neq i}^n b_j\right)\right\}^{1/2}} = -1$$

Since

$$\mathrm{Cov}(b_i, \sum_{j\neq i}^n b_j) = (n-1)\phi\sigma^2$$

and

$$\mathrm{Var}(\sum_{j\neq i}^n b_j) = (n-1)\sigma^2 + (n-1)(n-2)\phi\sigma^2,$$

solving for ϕ indicates that the coherence condition is satisfied when $\phi = -1/(n-1)$. Furthermore, substituting this value into equation 22 produces the following expression for the error variance of decomposition with uniform negative correlation of errors:

$$\begin{aligned} \sigma_d^2 &= \sigma^2 \sum_{i=1}^n \gamma_i^2 - \frac{2\sigma^2}{n-1} \sum_{i=1}^n \sum_{j>i}^n \gamma_i \gamma_j \\ &\quad + \tau^2 \sum_{i=1}^n \beta_i^2 + n\sigma^2\tau^2. \end{aligned} \tag{23}$$

Examination of this equation leads to a number of interesting conclusions. In the first place, decomposition error no longer depends upon the specific choice of event b_n, as it did in the previous section. On the other hand, a number of other implications of equation 23 are essentially the same as those derived from equation 13 in the previous section. For instance, error variance is still minimized when $\beta_i = 1/n$. Furthermore, error variance is minimized when the quantity

$$\sum_{i=1}^{n} \gamma_i^2 - \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j>i}^{n} \gamma_i \gamma_j$$

is minimized. This occurs, as in section 4, when $\gamma_i = \gamma_j$ for $i \neq j$. Finally, as the variances τ^2 and σ^2 approach 0, the error variance of decomposition decreases linearly. Thus, the introduction of uniform negative correlation of errors in the marginals does not substantially alter the conclusions derived from these results.

Several of the results from the previous section are somewhat modified. For instance, consider the derivation of n^*, the number of conditioning events that minimizes error. Letting $\beta_i = 1/n$ and $\gamma_i = \gamma$, equation 23 produces the following expression for the error variance of decomposition:

$$\sigma_d^2 = \frac{\tau^2}{n} + n\sigma^2\tau^2 \tag{24}$$

Taking the first derivative of the right-hand-side of this equation with respect to n,

$$\frac{\partial}{\partial n}\sigma_d^2 = -\left(\frac{\tau}{n}\right)^2 + \sigma^2\tau^2.$$

Setting this result equal to 0 and solving for n produces the following expression for n^*:

$$n^* = \frac{1}{\sigma}. \tag{25}$$

This value is somewhat larger than the value derived in section 4, particularly when σ is relatively small.

A direct comparison of equation 24 with equation 14 is also informative. In particular, the error variance derived from equation 24 is smaller than the value computed in the previous section, with the difference equal to $(2-n)\sigma^2\tau^2$. The results in section 4 seem to have overstated the error variance, particularly for large values of n.

Finally, the bounds on π, the percent change due to decomposition, need to be reevaluated. Using the definitions from section 4.3 and equation 23:

$$\pi = \left\{ k_b^2 \sum_{i=1}^{n} \gamma_i^2 - \frac{2}{n-1} k_b^2 \sum_{i=1}^{n} \sum_{j>i}^{n} \gamma_i \gamma_j + k_c^2 \sum_{i=1}^{n} \beta_i^2 + n k_b^2 k_c^2 \sigma_0^2 \right\}^{1/2} - 1. \tag{26}$$

The upper bound for π can be derived by noting that:

$$\sum_{i=1}^{n} \gamma_i^2 - \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j>i}^{n} \gamma_i \gamma_j \leq \frac{n^2}{4(n-1)}$$

$$\sum_{i=1}^{n} \beta_i^2 \leq 1$$

$$\sigma_0^2 \leq \frac{1}{12}.$$

Using these inequalities in equation 26 results in a revised upper bound:

$$\pi \leq \left\{ k_b^2 \left(\frac{n^2}{4(n-1)} \right) + k_c^2 + k_b^2 k_c^2 \left(\frac{n}{12} \right) \right\}^{1/2} - 1. \tag{27}$$

The expression for the lower bound is unchanged from equation 20. Comparison of equation 27 to equation 19 indicates that the revised upper bound for π are significantly lower than the original. This is shown in figure 5, which plots the original and revised upper bound for $k_b = k_c = 1$.

Figure 5 about here

In summary, the revised assumption of uniform negative correlation among marginals has no serious implications for the analysis. If anything, the results presented in section 4 were conservative, understating the potential error reduction benefits of decomposition.

5.3 Positively Correlated Conditionals

The other extension of the previous analysis is to allow the errors in the conditional probabilities to be positively correlated. Again, for simplicity, assume that $\rho_{ij} = \rho > 0$ for all $i \neq j$. As before, assume that $\phi_{ij} = \phi = -1/(n-1)$, $\sigma_i = \sigma$, and $\tau_i = \tau$. From equation 22, it follows that:

$$
\begin{aligned}
\sigma_d^2 &= \sigma^2 \sum_{i=1}^{n} \gamma_i^2 - \frac{2\sigma^2}{n-1} \sum_{i=1}^{n} \sum_{j>i}^{n} \gamma_i \gamma_j \\
&\quad + \tau^2 \sum_{i=1}^{n} \beta_i^2 + n\sigma^2\tau^2 \\
&\quad + 2\rho\tau^2 \sum_{i=1}^{n} \sum_{j>i}^{n} \left(\beta_i \beta_j - \frac{\sigma^2}{n-1} \right). \qquad (28)
\end{aligned}
$$

Equation 28 is quite similar to the previous expressions for decomposition error. The impact of positive values of ρ is linear. When $\beta_i = 1/n$, the first derivative of equation 28 with respect to ρ is:

$$
\frac{\partial}{\partial \rho} \sigma_d^2 = \tau^2 (1 - \frac{1}{n} - n\sigma^2).
$$

This expression is nonnegative when $\sigma^2 \leq (n-1)/n^2$. Recalling that the variance is itself constrained ($\sigma^2 \leq 1/12$), implies that this condition is satisfied for all $n \leq 10$. Thus, for ten or less conditioning events, positive correlation of conditionals increases the error variance of decomposition. For larger numbers of conditioning events, positive correlation increases decomposition error when σ^2 is small, while positive correlation actually decreases decomposition error for large values of σ^2.

The impact of correlated conditionals can be further demonstrated from a revision of the bounds for π. Using equation 28, the expression for percent change due to decomposition is:

$$
\begin{aligned}
\pi = \\
&\left\{ k_b^2 \sum_{i=1}^{n} \gamma_i^2 - \frac{2}{n-1} k_b^2 \sum_{i=1}^{n} \sum_{j>i}^{n} \gamma_i \gamma_j + k_c^2 \sum_{i=1}^{n} \beta_i^2 \right. \\
&\left. + (1-\rho) n k_b^2 k_c^2 \sigma_0^2 + 2\rho k_c^2 \sum_{i=1}^{n} \sum_{j>i}^{n} (\beta_i \beta_j) \right\}^{1/2} - 1. \qquad (29)
\end{aligned}
$$

Positive values of ρ can cause dramatic increases in the lower bound for π.

For instance, when $k_b = k_c = 1$, it can be shown from equation 29 that:

$$\pi \geq \left\{ \frac{1}{n} + \rho \left(1 - \frac{1}{n} \right) \right\}^{1/2} - 1. \tag{30}$$

Thus, large values of ρ reduce the potential for error reduction (i.e., the lower bound increases), particularly when the number of conditioning events is large.

On the other hand, positive values of ρ can actually improve the upper bound for π. Again, for $k_b = k_c = 1$, it can be shown from equation 29 that:

$$\pi \leq \left\{ \frac{n^2}{4(n-1)} + 1 + (1-\rho)\frac{n}{12} \right\}^{1/2} - 1. \tag{31}$$

As ρ approaches 1, the right-hand-side of this expression decreases. Thus, positive correlation of conditionals brings the upper and lower bounds for π closer together.

In summary, then, positively correlated conditionals may, under certain circumstances, seriously under cut the error reduction potential of decomposition. If one adopts the view, introduced in section 4, that the decomposition estimate is a weighted average of the conditional probabilities, then this result seems reasonable. Positive correlation implies that the quantities being averaged are to some extent redundant, reducing the benefit to be derived from the averaging process. The exception occurs when the errors in the weights (σ^2) are large. In this case, a large number of redundant conditionals may help offset the error in the weights.

6 Discussion

Our results provide some concrete recommendations for analysts interested in structuring probability encoding tasks in order to minimize inconsistency:

1. With the proper choice of conditioning events, decomposition can reduce random errors associated with probability encodings.

2. As the number of events increases, error reduction will only occur up to a point, and the effort required to assess additional probabilities will, on the margin, not be warranted. On the other hand, excessive concern with the number of events is probably not warranted, since the amount of error is generally not sensitive to small deviations from the optimal number of conditioning events.

3. The marginal probabilities of the conditioning events should be selected to be equal, unless there are known differences in the precision of the conditional probabilities. In that case, adjustments to the relative sizes of the marginals are desirable.

4. The conditional probabilities should ideally be as close as possible. Of course, if all the conditional probabilities are *known* to be equal, then we would not really need to bother with decomposition. The extent to which the assessor or the analyst has control over the range of values of conditional probabilities is not at all clear, so this recommendation may be difficult to implement.

5. The errors associated with the assessment of the component probabilities should be as small as possible. In particular, selection of conditioning events whose marginals are precisely known is desirable.

6. The analyst should try to verify and maintain the independence of the component assessments. Specifically, dependencies among the conditional probabilities could easily undermine the value of decomposition.

Unfortunately, in practice it may be quite difficult to follow all of these recommendations simultaneously. For instance, a class of conditioning events that satisfies several of the conditions above are equiprobable events like the toss of a coin or the roll of a die. Not only are the marginal probabilities uniform, but they can be assessed without error. For instance, one might estimate the probability of a target event conditional upon the toss of a fair coin resulting in heads, and then assess the probability conditional upon the coin-toss resulting in tails. Unfortunately, a single individual will probably not be able to assess these two conditional probabilities without introducing dependencies, and the advantages of averaging may be dissipated. Alternately, the assessments of conditional probabilities could be obtained from different assessors. This essentially transforms the decomposition process into the problem of aggregating the opinions of n independent judges. Of course, even introducing multiple assessors will not guarantee the removal of dependencies (for further discussion, see Clemen & Winkler, 1985).

As with any analysis of this type, the utility of the results depends upon the credibility of the assumptions. For instance, the calculation of error bounds depend upon a *ceteris paribus* assumption—that the error of the component elicitations remains constant as the number of conditioning events increase. In point of fact, as the number of conditioning events increases, the assessor's work load will increase, potentially increasing fatigue and decreasing the consistency of the component elicitations. Fortunately, this simply reinforces the conclusion that overly large values of n are undesirable.

Another assumption, equal error variances, introduced in equation 13 and used for the derivation of error bounds, is in large part a convenience rather than a necessity. Similar results could be derived using equation 12 for any arbitrary set of error variances. What, then, are the circumstances where equal error variances are a reasonable assumption? An important factor could be the size of the expected (true) values. Consider the case where the error distribution is assumed to be continuous and unimodal over the interval $(0, 1)$. As the true values approach either 0 or 1, the error variance will be constrained, and will approach zero. (Larger variances are only possible if the distribution has more than one mode or is discontinuous.) Thus, for events whose true probabilities vary widely (e.g., from 0.5 down to 0.0001), equal error variances seem less plausible than for events whose probabilities are roughly equal. If the probabilities utilized in the decomposition do cover an extreme range, the analyses should be repeated with more appropriate variance estimates.

This research describes a very general framework for analyzing the nature of random errors and evaluating strategies for controlling those errors. To the extent that some of the assumptions seem arbitrary or unfounded, empirical research may be appropriate. In particular, measurement of the variances and covariances of random assessment errors in a variety of assessment domains may prove quite informative. Observation of actual assessors using decomposition may also shed light on important sources of systematic errors that are not addressed by the present analytical framework. For instance, probabilities derived from different decompositions could well differ in a systematic fashion. To reiterate a point made above, control of random error is a necessary though not sufficient condition for accurate assessment. It is informative to note that systematic errors have proven to be quite difficult to control (e.g., Fischhoff, 1982). On the other hand, a decomposition approach is quite simple from a conceptual perspective, is relatively easy to implement, and seems to have real potential for reduction of an important source of error.

Appendix

This appendix demonstrates that equation 11 is equivalent to equation 12. For the first component of the right-hand side of equation 11, because of independence assumptions, it follows that $\mathrm{Var}(\sum_{i=1}^{n-1} c_i b_i) = \sum_{i=1}^{n-1} \mathrm{Var}(c_i b_i)$. The independence conditions also imply that

$$\mathrm{Var}(c_i b_i) = \gamma_i^2 \sigma_i^2 + \beta_i^2 \tau_i^2 + \sigma_i^2 \tau_i^2.$$

Therefore,

$$\mathrm{Var}\left(\sum_{i=1}^{n-1} c_i b_i\right) = \sum_{i=1}^{n-1} \gamma_i^2 \sigma_i^2 + \sum_{i=1}^{n-1} \beta_i^2 \tau_i^2 + \sum_{i=1}^{n-1} \sigma_i^2 \tau_i^2. \tag{32}$$

The second component can be expanded as follows:

$$\mathrm{Var}\left(c_n\left(1-\sum_{j=1}^{n-1} b_j\right)\right) =$$
$$\gamma_n^2 \mathrm{Var}\left(1-\sum_{j=1}^{n-1} b_j\right) + \left(1-\sum_{j=1}^{n-1} \beta_j\right)^2 \tau_n^2 + \mathrm{Var}\left(1-\sum_{j=1}^{n-1} b_j\right) \tau_n^2.$$

Since $\mathrm{Var}(1-\sum_{j=1}^{n-1} b_j) = \sum_{j=1}^{n-1} \mathrm{Var}(b_j)$ and since $\mathrm{Var}(b_j) = \sigma_j^2$, it follows directly that:

$$\mathrm{Var}\left(c_n\left(1-\sum_{j=1}^{n-1} b_j\right)\right) =$$
$$\gamma_n^2 \sum_{j=1}^{n-1} \sigma_j^2 + \left(1-\sum_{j=1}^{n-1} \beta_j\right)^2 \tau_n^2 + \tau_n^2 \sum_{j=1}^{n-1} \sigma_j^2. \tag{33}$$

The third term from the right-hand side of equation 11 (excluding the constant) can be rewritten as follows:

$$\mathrm{Cov}\left(\sum_{i=1}^{n-1} c_i b_i, c_n\left(1-\sum_{j=1}^{n-1} b_j\right)\right) =$$

$$\sum_{i=1}^{n-1}\mathrm{Cov}\left(c_i b_i, c_n\right) - \sum_{i=1}^{n-1}\mathrm{Cov}\left(c_i b_i, c_n \sum_{j=1}^{n-1} b_j\right).$$

This expression simplifies considerably, since $\mathrm{Cov}(c_i b_i, c_n) = 0$ for all values of $i \le n-1$, and $\mathrm{Cov}(c_i b_i, c_n \sum_{j=1}^{n-1} b_j) = 0$ except when $i = j$. Since

$$\mathrm{Cov}(c_i b_i, c_n b_i) = \mathrm{E}(c_n c_i b_i^2) - \gamma_n \gamma_i \beta_i^2,$$

and since

$$\mathrm{E}(b_i^2) = \beta_i^2 + \sigma_i^2,$$

therefore:

$$\mathrm{Cov}\left(\sum_{i=1}^{n-1} c_i b_i, c_n\left(1-\sum_{j=1}^{n-1} b_j\right)\right) =$$

$$-\gamma_n \sum_{i=1}^{n-1}\gamma_i(\beta_i^2+\sigma_i^2) + \gamma_n \sum_{i=1}^{n-1}\gamma_i\beta_i^2,$$

which is equivalent to:

$$\mathrm{Cov}\left(\sum_{i=1}^{n-1} c_i b_i, c_n\left(1-\sum_{j=1}^{n-1} b_j\right)\right) = -\gamma_n \sum_{i=1}^{n-1}\gamma_i\sigma_i^2. \tag{34}$$

Substituting the right-hand-sides of equations 32, 33, and 34 into equation 11 and noting that

$$\sum_{i=1}^{n-1}\gamma_i^2\sigma_i^2 + \gamma_n^2\sum_{i=1}^{n-1}\sigma_i^2 - 2\gamma_n\sum_{i=1}^{n-1}\gamma_i\sigma_i^2 = \sum_{i=1}^{n-1}\sigma_i^2(\gamma_i-\gamma_n)^2$$

and that

$$\sum_{i=1}^{n-1}\beta_i^2\tau_i^2 + \left(1-\sum_{i=1}^{n-1}\beta_i\right)^2\tau_n^2 = \sum_{i=1}^{n}\beta_i^2\tau_i^2$$

produces equation 12.

Acknowledgements

Initial drafts of this paper were written while the first two authors were affiliated with the Department of Management, University of Texas at Austin. This research was supported by the Office of Naval Research, under contract N00014-85-K-0732. Helpful comments from the Departmental Editor and two referees are gratefully acknowledged.

References

1. Clemen, R. T. & Winkler, R. L. "Limits for the Precision and Value of Information from Dependent Sources." *Operations Research*, Vol. 33 (1985), 427–442.

2. Fischhoff, B. "Debiasing," in D. Kahneman, P. Slovic, & A. Tversky (Eds.), *Judgment under Uncertainty: Heuristics and Biases*, Cambridge University Press, New York, 1982.

3. Fischhoff, B., Slovic, P., & Lichtenstein, S., "Knowing What You Want: Measuring Labile Values," in T. S. Wallsten (Ed.), *Cognitive Processes in Choice and Decision Behavior*, Erlbaum, Hillsdale, N.J., 1980.

4. Hogarth, R. M., "Cognitive Processes and the Assessment of Subjective Probability Distributions," *Journal of the American Statistical Association*, Vol. 70 (1975), 271–289.

5. Huber, G. P., "Methods for Quantifying Subjective Probabilities and Multiattribute Utilities," *Decision Sciences*, Vol. 5 (1974), 430–458.

6. Lichtenstein, S., Fischhoff, B., & Phillips, L., "Calibration of Probabilities: The State of the Art," in H. Jungermann & G. de Zeeuw (Eds.), *Decision Making and Change in Human Affairs*, Reidel, Dordrecht, Netherlands, 1977.

7. Lichtenstein, S., Fischhoff, B., & Phillips, L., "Calibration of Probabilities: The State of the Art to 1980," in D. Kahneman, P. Slovic, & A. Tversky (Eds.), *Judgment under Uncertainty: Heuristics and Biases*, Cambridge University Press, New York, 1982.

8. Lindley, D. V. "The Reconciliation of Decision Analyses." *Operations Research*, Vol. 34 (1986), 289–295.

9. Lindley, D. V., Tversky, A., & Brown, R. V. "On the Reconciliation of Probability Assessments (with discussion)." *Journal of the Royal Statistical Society A*, Vol. 142 (1979), 146–180.

10. Ravinder, H. V. *The Reliability of Subjective Probability Estimates Obtained Through Decomposition.* Unpublished Doctoral Dissertation, Department of Management, University of Texas at Austin, 1986.

11. Slovic, P., Fischhoff, B., & Lichtenstein, S., "Behavioral Decision Theory," *Annual Review of Psychology*, Vol. 28 (1977), 1–39.

12. Spetzler, C. S., & Staël von Holstein, C.-A. S., "Probability Encoding in Decision Analysis," *Management Science*, Vol. 22 (1975), 340–358.

13. Tversky, A., & Kahneman, D., "Availability: A Heuristic for Judging Frequency and Probability," *Cognitive Psychology*, Vol. 5 (1973), 207–232.

14. Tversky, A., & Kahneman, D., "Judgment under Uncertainty: Heuristics and Biases," *Science*, Vol. 185 (1974), 1124–1131.

15. Wallsten, T. S., & Budescu, D. V., "Encoding Subjective Probabilities: A Psychological and Psychometric Review," *Management Science*, Vol. 29 (1983), 151–173.

Figure Captions

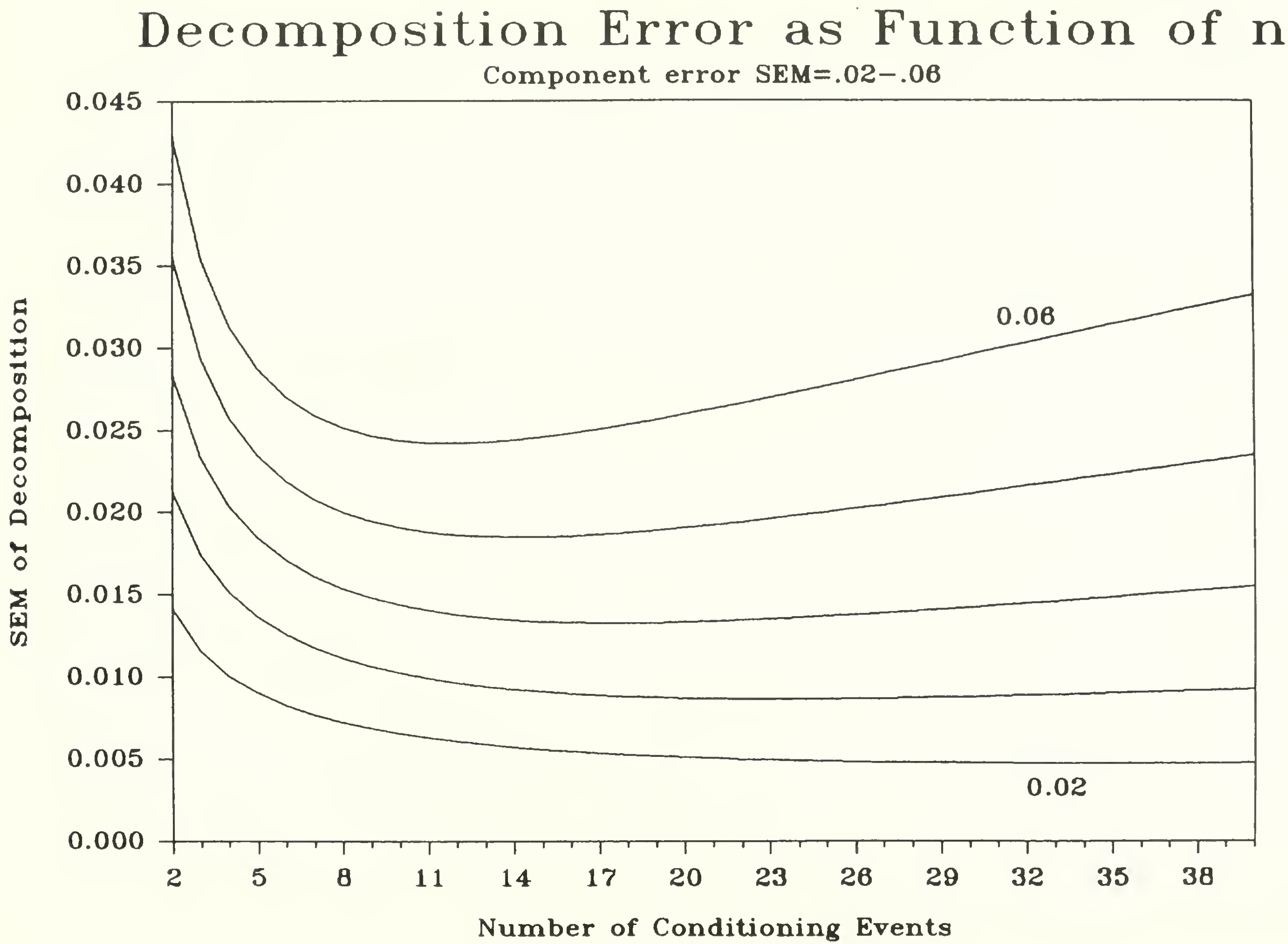
Figure 1: Error as a Function of Number of Conditioning Events

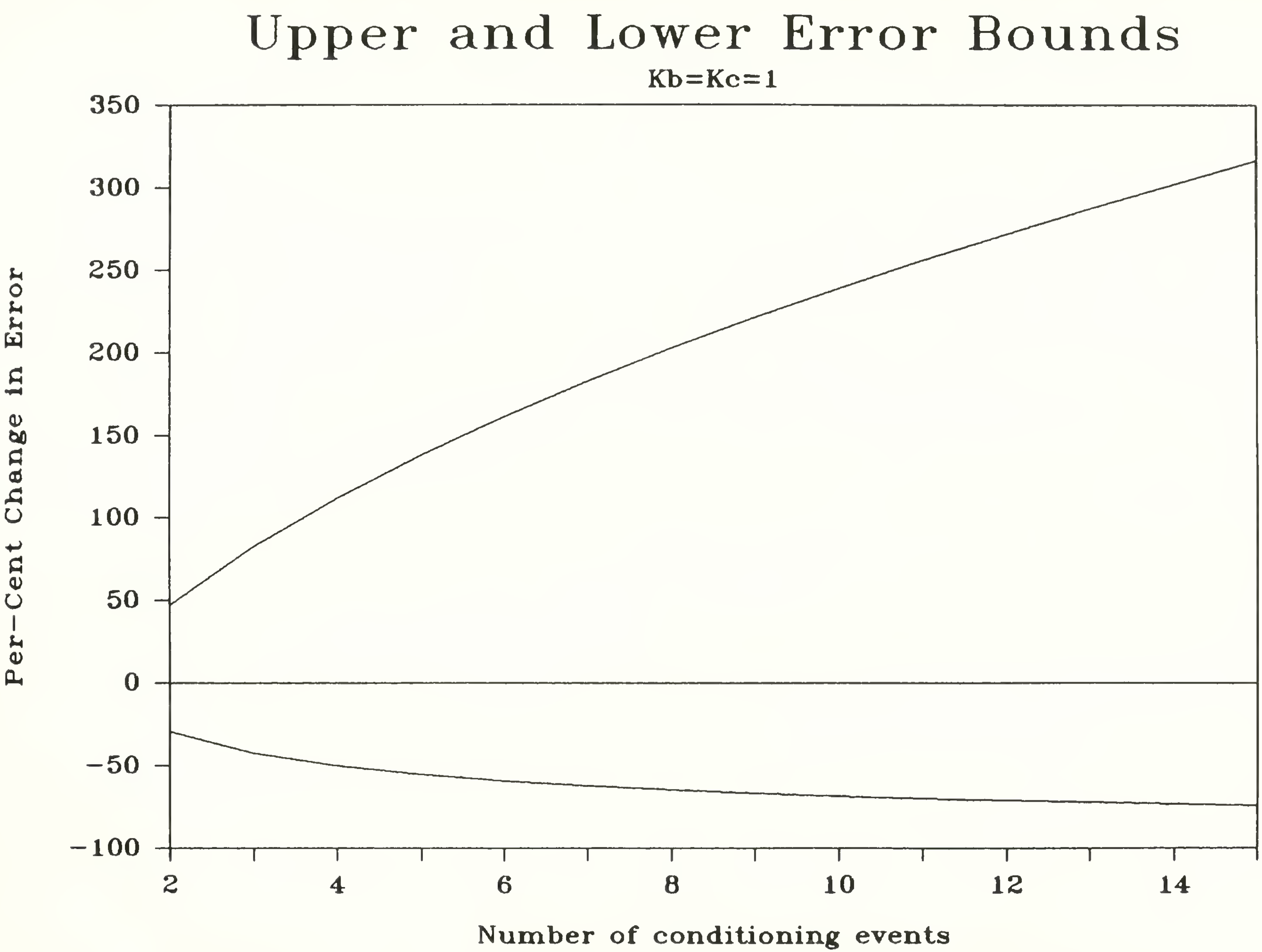
Figure 2: Upper and Lower Error Bounds—$k_b = k_c = 1.0$

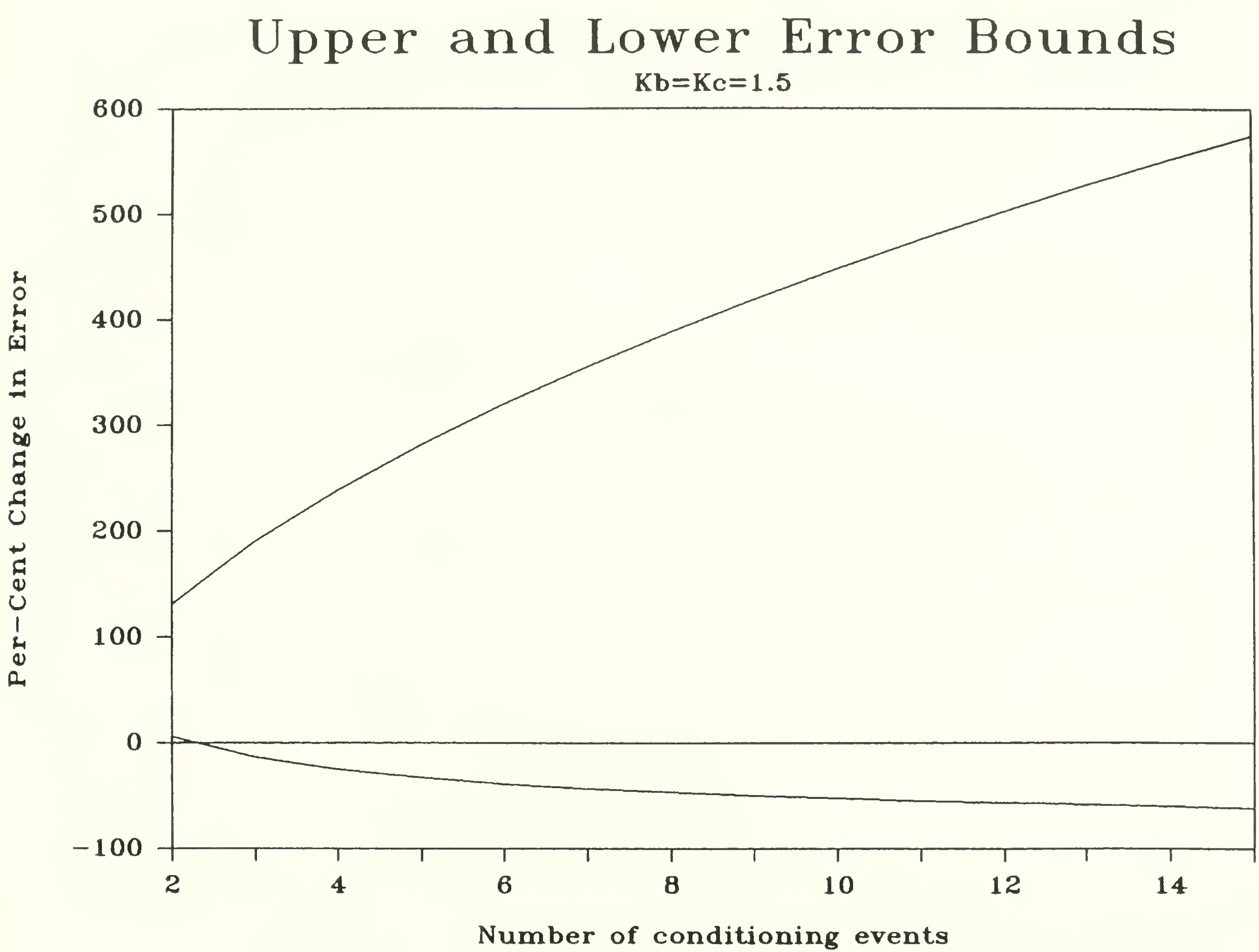
Figure 3: Upper and Lower Error Bounds—$k_b = k_c = 1.5$

Figure 4: Upper and Lower Error Bounds—$k_b = k_c = 0.5$

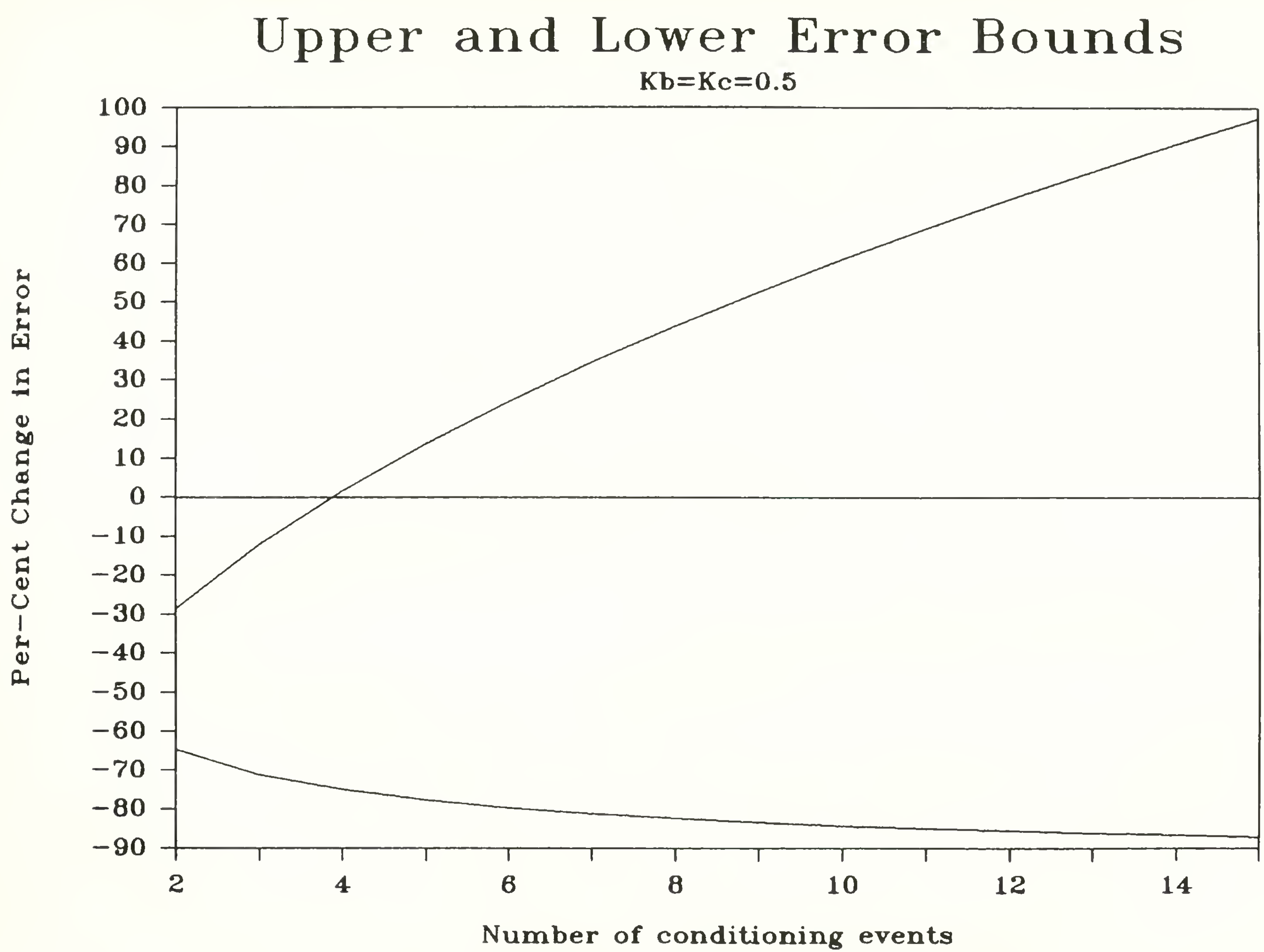
Figure 5: Revised Error Bounds—$k_b = k_c = 1.0$

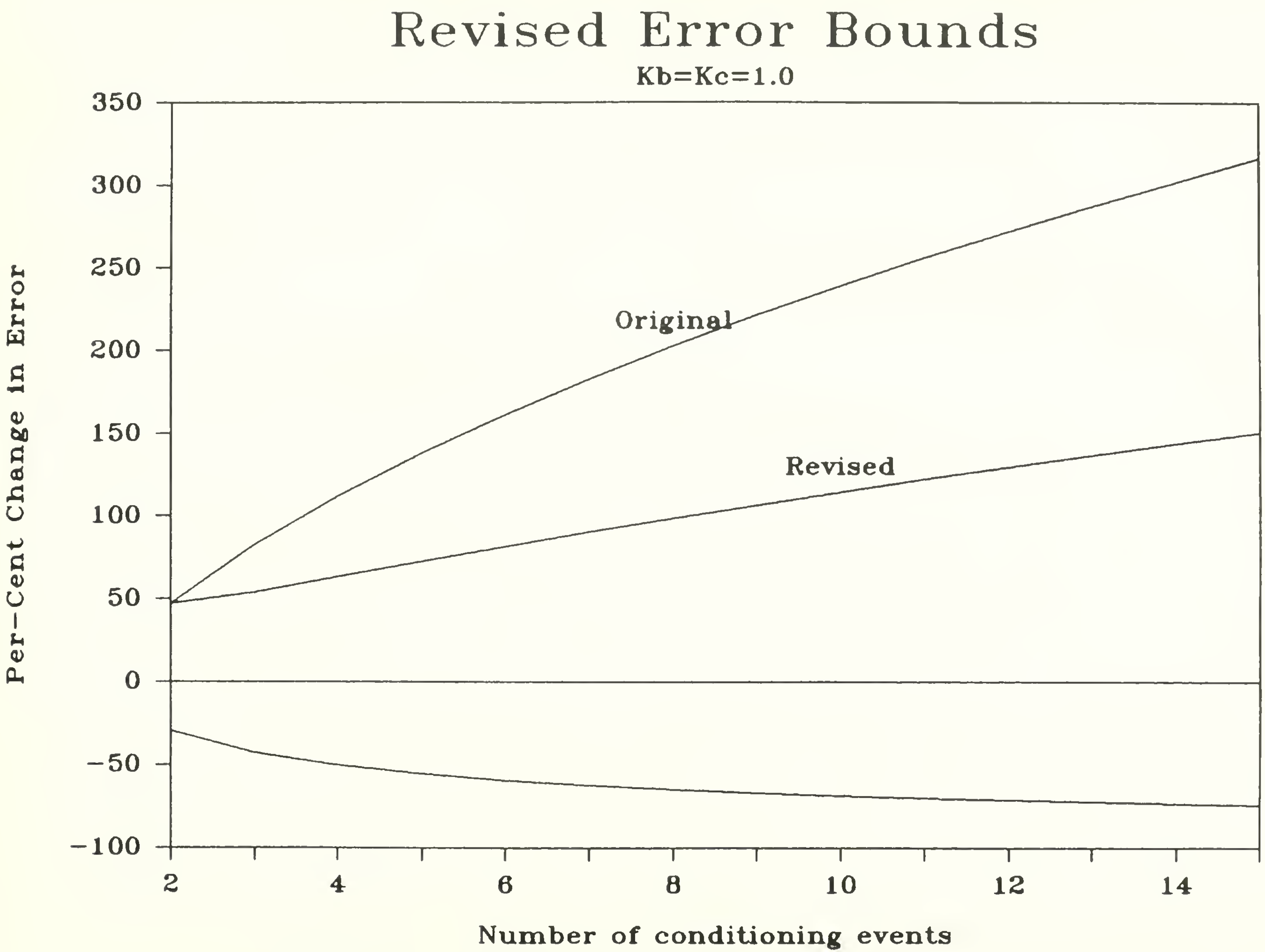

Decomposition Error as Function of n
Component error SEM=.02-.06
SEM of Decomposition
0.045
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
0.06
0.02
2
5
8
11
14
17
20
23
26
29
32
35
38
Number of Conditioning Events


Upper and Lower Error Bounds
Kb=Kc=1
Per-Cent Change in Error
350
300
250
200
150
100
50
0
-50
-100
2
4
6
8
10
12
14
Number of conditioning events



Upper and Lower Error Bounds
Kb=Kc=1.5
Per-Cent Change in Error
600
500
400
300
200
100
0
-100
2
4
6
8
10
12
14
Number of conditioning events


Upper and Lower Error Bounds
Kb=Kc=0.5
100
90
80
70
60
50
40
30
20
10
0
−10
−20
−30
−40
−50
−60
−70
−80
−90
2
4
6
8
10
12
14
Per−Cent Change in Error
Number of conditioning events


Revised Error Bounds
Kb=Kc=1.0
Original
Revised
Per-Cent Change in Error
350
300
250
200
150
100
50
0
-50
-100
2
4
6
8
10
12
14
Number of conditioning events

Date Due
