THE STOCHASTIC CASH BALANCE PROBLEM WITH AVERAGE COMPENSATING-BALANCE REQUIREMENTS

Warren H. Hausman and Antonio Sanchez-Bell

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ABSTRACT

We consider the problem of managing two assets, cash and an earning asset, when net cash flows are stochastic and when there are transfer costs for transferring assets from one form to the other. Previous work on the stochastic cash-balance problem has assumed holding costs for holding excess cash and penalty costs for holding insufficient cash, with these costs assessed per period (the same period in which there is a single decision or transfer opportunity and a single random cash flow). This formulation is appropriate when a firm faces minimum (or zero) compensating-balance requirements, but not when the compensating-balance requirement involves an average deposit balance over a number of decision periods. A dynamic programming model is presented which appropriately represents the relevant cost function for a firm facing an average compensating-balance requirement. The dynamic programming solution to a numerical example is compared to that of a static two-sided (s, S) policy; the optimal dynamic programming solution represents an 18% reduction in relevant costs in the example.
I. INTRODUCTION

The problem under consideration involves the management of cash and short-term financial assets for a firm facing a compensating-balance requirement specified as an average balance over a number of days (e.g., weekly, bi-weekly, or monthly). Daily net cash flows are partially unpredictable and are treated as stochastic (specifically, as independent random variables). We consider only two assets: cash, and some interest-bearing asset.

At the end of a period, cash holdings in excess of the compensating-balance requirement incur an opportunity cost, in that they could have been invested in the interest-bearing asset. A cash level below the requirement presumably incurs some penalty cost, which will be assumed to be proportional to the shortfall. Transactions costs of converting excess cash into the earning asset and vice-versa make it uneconomic to "even up" daily, and create the management decision problem studied (in variations) here and in the references.

Previous Work

Early work [1,26] on the cash management problem involved deterministic models closely related to inventory models. Further work on deterministic models has been performed using linear programming to model various constraints [16,17,19,25,28]. Assuming perfect forecasts of net cash flows in future

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1 The mean of the distribution of net cash flow may differ from day to day, reflecting the possibility of forecasting seasonal systematic effects. However, no forecast revision process of the type described in [10] is allowed for in this model.
time periods, Sethi [24] has derived a planning horizon theorem indicating that optimal decisions can be made using only (perfect) forecasts through some period \( N^* \); cash flows beyond period \( N^* \) do not affect the first-period decision.

White and Norman [29] consider a stochastic version of the problem, omitting any costs of transferring cash to the earning asset and vice-versa. They obtain a modified "newsboy problem" solution to their stated formulation. Miller and Orr [13, 14, 18] present an analysis of the stochastic problem when transfer costs are fixed, independent of the size of the transfer, and when opportunity costs of holding excess cash and penalty costs of holding insufficient cash are charged daily (or each decision-making period). Assuming the form of the optimal policy is of the \((0, z, z, h)\) type (i.e., return the cash balance to \( z \) if the balance falls below zero or above \( h; 0 < z < h \)), they derive optimal values for \( h \) and \( z \) as a function of the various cost coefficients; interestingly, the optimum \( z = 1/3 \) \( h \). Eppen and Fama [4] consider the same problem with transfer costs strictly proportional to the size of the transfer; they derive the form of the optimal policy as a \((T, T, U, U)\) policy (i.e., return to levels \( T \) or \( U, T < U \), if the cash balance level falls below \( T \) or above \( U \) respectively). They also [6] expand the proportional-transfer cost model to the case of three assets, and are able to characterize the form of the optimal policy for that case.

Girgis [9] considers the two-asset problem with proportional transfer costs in one direction and fixed plus proportional costs in the other (and vice-versa), proving the form of the optimal policy for these cases is of the so-called "simple" type \((t, T, U, u)\), i.e., return the cash balance to \( T \) or \( U \) if the balance falls below \( t \) or above \( u \) respectively, (see Figure 1), with \( t \leq T \leq U \leq u \). Girgis conjectures the form of the optimal policy for the case with fixed plus proportional transfer costs in both directions to be the simple form just mentioned.

Eppen and Fama [5] study the fixed-plus-proportional transfer costs
Figure 1

The Simple Static Policy \((t, T, U, u)\)
I want to make a graph...
cost case by using linear programming to find optimal numerical policies for the infinite-horizon Markovian decision problem. Although they do not prove that the optimal policy is of the simple form, in every numerical example solved by LP, the actual optimal policy is of that form, thereby providing "...a strong suggestion of what that form might be" [5, p. 95].

Neave [15] also studies the above problem, drawing on earlier results from mathematical inventory theory [23]. He shows that, with transfer costs which include both fixed and variable components, the form of the optimal policy is not generally of the simple \((t, T, U, u)\) form. There may be a region \((t, t^+)\) in which, for some values of the cash balance within that region, it is optimal to bring the cash balance to \(T\), but for other (lower or higher) values it is optimal to do nothing; a similar complication occurs on the high side, in a range \((u^-, u)\) in his notation. Neave presents conditions under which the optimal policy is of simple form; they are quite restrictive.\(^3\)

Daellenbach and Archer [2] formulate a dynamic programming version of the problem in which penalty costs for insufficient cash are replaced by a chance constraint keeping the probability of insufficient cash below some predetermined level.

All of the above studies assess holding and penalty costs per decision period; i.e., over the same period in which there is presumed only one opportunity to transfer cash to assets or vice-versa. Such a formulation is appropriate when the actual cash requirement is zero (no compensating

\(^3\)Porteus and Neave [20,21] perform a related analysis for the case in which transfer costs and penalty costs are actually assessed against the balance being controlled; all other work (and the work presented here) assumes implicitly that such costs and charges are assessed against some other account.
balances are required) or when the compensating-balance requirement is a minimum requirement each decision period. However, that type of formulation is not appropriate when the compensating-balance requirement is for an average deposit balance over some multi-decision-period interval of time (e.g., weekly, bi-weekly or monthly).

Frost [7] explores the stochastic cash balance problem when banking services are a function of both the average balance and the minimum balance. Unfortunately, Frost implicitly assumes that the Miller-Orr form of operating policy \((0, z, z, h)\) is the optimal form; he derives optimal parameter values for \(z\) and \(h\) under the conditions stated.

Homonoff and Mullins [11] also focus on the average balance but again restrict the form of operating policy to the stationary Miller-Orr type: "The decision rule ... is simply the same as the two-asset Miller-Orr model" [11, p. 87]. However, once the change is made from a minimum balance to an average balance over a number of decision periods, it will, in general, not\(^4\) be optimal to operate under a stationary policy of either the Miller-Orr form \((0, z, z, h)\) or the more general "simple" form \((t, T, U, u)\).

Intuitively, as the end of the averaging period approaches, one can compare the average balance maintained so far with the required target and plan to take action which will bring the expected average balance over the averaging period quite close to the target. Thus a type of "end game" can be played, which is not possible in the above formulations since for them, every decision period is an end game.

\(^4\)Orr [18] makes this point: "...if [average] compensating balance requirements are constraining, does some other policy form offer a superior alternative to the \((h, z)\) policy? ... If an averaging period is specified, some technique of dynamic programming must be adopted" [18, p. 100]. Orr himself views compensating balances as being non-constraining; see [18, p. 100].
This paper will present a dynamic programming formulation of the stochastic cash balance problem under an average compensating-balance requirement. The problem is relevant because, as Gibson states, "... balance requirements often apply to average, not daily, balances" [8, p. 387]. Other writers, including Frost [7] and Miller and Orr [14,18], make the same point. Section II of the paper presents the dynamic programming model. Section III explores what can be said concerning the form of the optimal policy, drawing on previous work by Neave [15]. Section IV describes a realistic numerical example which has been solved by both the dynamic programming model and by a static "simple" policy of the (t,T,U,u) form.
II. THE DYNAMIC PROGRAMMING MODEL

Time periods (days) are numbered backwards from some horizon N days away. Let $X_n$ represent the opening cash balance on day $n$, prior to any transfer decision. Let $Y_n$ represent the cash balance immediately after a transfer action (if any). We allow for both fixed and variable components of transfer costs $A(X_n, Y_n)$:

$$A(X_n, Y_n) = \begin{cases} K + k(Y_n - X_n) & \text{if } Y_n > X_n \\ 0 & \text{if } Y_n = X_n \\ Q + q(X_n - Y_n) & \text{if } Y_n < X_n \end{cases}$$

(1)

The average compensating balance requirement is given exogenously as $R$ per day over an $N$-day averaging period, or $NR$ dollar-days. The opportunity cost of holding a greater-than-$R$ average balance is $c_o$ per dollar per $N$-day period; we presume there is a per-dollar penalty cost $c_u$ per $N$-day period for holding less than the "required" average $R$ balance.

Random daily net cash inflows (+ or -) are denoted by $\xi_n$, independent random variables with known probability density function $p_n(\xi_n)$. These random flows occur after transfer action (if any) has been taken. Thus the equation relating successive daily cash balances for days $n$ and $(n-1)$ is:

$$X_{n-1} = X_n + (Y_n - X_n) + \xi_n = Y_n + \xi_n$$

(2)

Let $S_n$ represent the cumulative sum of daily closing balances from the first day ($N$) through day $n+1$ inclusive; then

$$S_n = \sum_{i=n+1}^{N} (Y_i + \xi_i).$$

(3)
The state vector will be \((S_{n'}, X_{n'})\). The usual dynamic programming return function will be defined:

\[
f_n (S_{n'}, X_{n'}) = \text{minimum expected cost from} \\
day n \text{ through day 1, given } (S_{n'}, X_{n'}) \text{ and} \\
\text{assuming optimal decisions are made from} \\
day n \text{ through day 1.}
\]

One-Period Problem

Now consider the last day (day 1). Action must be decided upon at the beginning of the day concerning a potential transfer; subsequently, random cash flow \(\xi_1\) will occur; and, since this is the last day of the \(N\)-day averaging period, overage and underage costs will be assessed as appropriate. We may write the following dynamic programming return function for period 1:
\[
f_1(S_1, X_1) = \text{Minimum} \left\{ \begin{array}{c}
A(X_1, Y_1) \\
Y_1 \geq W \\

\text{NR} - S_1 - Y_1 \\
+ c_u \int_{-\infty}^{\xi_1} [\text{NR} - S_1 - Y_1 - \xi_1] p_1(\xi_1) d\xi_1 \\
+ c_o \int_{\text{NR} - S_1 - Y_1}^{\infty} [\xi_1 + S_1 + Y_1 - \text{NR}] p_1(\xi_1) d\xi_1 \\
\end{array} \right. 
\]

where \( W \) represents a lower bound (\( W > 0 \)) on the ending balance, so as to keep the probability of a negative balance sufficiently low.

Denote the two integral terms in equation (4) by the symbol \( L_1(Y_1 | S_1) \). Then the function \( L_1(\cdot | \cdot) \) is convex in \( Y_1 \), and the required minimization\(^5\) of equation

\(^5\) Neave [15, pp. 477-479] offers a rigorous proof that for the one-period problem stated, the form of the optimal solution is "simple" \( (t_1, T_1, U_1, u_1) \) as will be shown graphically here.
may be shown graphically as in Figure 2.

Figure 2 indicates that, if transfers were free, the best place at which to aim on day 1 would be $Y_1^*$; this quantity would be the solution to the newsboy problem of equation (4) with transfer costs omitted (the $L_1(\cdot|p)$ function). However, given both fixed and variable components of transfer costs, the optimal trade-off must be found between no action and the correct action, given a transfer in the correct direction. Suppose $x_1$ were below $Y_1^*$; then, since there is a variable cost of amount $k$, the point on $L_1(\cdot|p)$ with slope of $-k$ is the farthest toward $Y_1^*$ that one would wish to move; this point is designated $T_1$ on Figure 2. Any further move toward $Y_1^*$ would cost more in additional proportional transfer costs than its expected incremental benefit. Moreover, any upward move at all will involve the fixed cost $K$, so the point $t_1$ is found by requiring $L_1(t_1|S_1)$ to be the point of indifference between action (moving to $T_1$) and no action (remaining

![Figure 2](image-url)
at $t_1$). Similar analysis in the other direction produces values for $u_1$ and $U_1$, and the optimal policy for this one-period problem is of the simple $(t_1, T_1, U_1, u_1)$ form.

**n-Period Problem**

The general recursion relation corresponding to equation (4) for period $n$ requires the expression of the state vector at period $n-1$ in terms of the state vector and decision variable at period $n$.

By definition,

$$S_{n-1} = S_n + Y_n + \xi_n$$

and

$$X_{n-1} = Y_n + \xi_n.$$

Now equation (4) may be modified in the usual manner to represent the return function for day $n$ as follows:

$$f_n(S_n, X_n) = \min_{Y_n \geq W} \left\{ A(X_n, Y_n) \right\}$$

$$+ \int_{-\infty}^{\xi_n} f_{n-1}(S_{n-1} + Y_n + \xi_n, Y_n + \xi_n) P(\xi_n) d\xi_n$$

$$\xi_n = -\infty$$
Then, apart from theoretical questions concerning the form of the optimal policy, equations (4) and (5) for $n = 1, 2, \ldots, N$ may be solved recursively to yield the optimal policy $Y_n^* (S_n, X_n)$ as a tabled function of the two-dimensional state vector at each decision stage.
III. FORM OF OPTIMAL POLICY

Neave [15, pp. 477-486] presents a careful analysis of a problem closely related to the one under consideration here. His one-period problem, with $K = Q$ and other parameters reinterpreted, can be made identical to the one-period problem presented above. The return function $f_1(S_1, X_1)$ is $C_1(x)$ in his notation ($x$ is the cash balance prior to action), and "... $C_1$ is not $K$-convex" [15, p. 481]. Neave allows for daily holding and penalty costs represented as $L_n(x)$; our case simply involves setting all these functions except $L_n(\cdot)$ equal to zero. Thus a reinterpretation of his problem allows us to use his results for our problem. He proves that in general for this problem, the form of the optimal policy is $(t_n, t_n^+, T_n, U_n, u_n^-)$; i.e., if $x < t_n$ move to $T_n$; if $x > u_n$ move to $U_n$; if $x$ in $(t_n^+, u_n^-)$ make no transfer; but if $t_n < x < t_n^+$, then a transfer to $T_n$ may be desirable, depending on the value of $x$; and similarly for $x$ in $(u_n^-, u_n)$. This is a complex policy rather than a simple policy; for $x$ falling in the latter two ranges, a separate analysis or test must be performed to ascertain whether one should move to the corresponding $T_n$ or $U_n$ or make no change.

For computational purposes the "complexity" of the form of the optimal policy is not necessarily a major hindrance, as a table of optimal actions $Y_n^*(S_n, X_n)$ will implicitly handle all values of the state vector. However, knowledge that the form of the optimal policy is complex is important in that the manner in which the minimization operation of equations (4) and (5) is performed will be affected by such knowledge. In particular, simply searching for the optimal parameters of the corresponding simple policy, $(t_n, T_n, U_n, u_n)$ is not sufficient; in the troublesome ranges cited, each potential value of the state vector (to a discrete approximation) must be evaluated separately to find the optimal action.
IV. **A NUMERICAL EXAMPLE**

In this section, a numerical example is described which has been solved both by the dynamic programming model of Section II and by a simple static \((t,T,U,u)\) policy for comparison purposes.

The numbers and parameter values selected for the example represent our estimates of a realistic problem. The distribution of daily net cash inflows was taken directly from research by Homonoff and Mullins [11] to be Normal\(^6\) with mean of $4,000 and standard deviation of $580,000; this distribution and its parameters closely approximated the daily net cash inflows of an actual U.S. corporation over an 11-month period (see [11]). That corporation was required to maintain an average compensating balance of $3 million; we selected this identical requirement for our example. Other data required for the example is described in Table 1.

**Dynamic Programming Model**

The continuous state space \((S_n, X_n)\) of the dynamic programming model of Section II was modified to a discrete two-dimensional grid for computational purposes. The accuracy of the discrete approximation to the underlying continuous process is dependent on the size of the grid and the number of points on it. A large grid with many points (i.e., small step size from one point to the next) will be more accurate than a smaller grid with fewer grid points, since interpolation between the two-dimensional grid points will increase in accuracy as the distance between the points decreases. However, the

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\(^6\)For computational purposes, the normal distribution was approximated by a discrete probability function.
## TABLE 1

DATA FOR NUMERICAL EXAMPLE

<table>
<thead>
<tr>
<th>Transfer Costs</th>
<th>Fixed (dollars)</th>
<th>Variable ($ per thousand dollars transferred)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost of transfer from interest earning asset to cash</td>
<td>$20.00</td>
<td>$0.5</td>
</tr>
<tr>
<td>Cost of transfer from cash to interest earning asset</td>
<td>$20.00</td>
<td>$0.5</td>
</tr>
</tbody>
</table>

Overage and underage costs (in $ per thousand of deviation from requirement)

Cost per day of being above balance requirement: \( \frac{c_o}{N} = 0.25 \)

Cost per day of being below balance requirement: \( \frac{c_u}{N} = 0.375 \)

Time Horizon: \( N = 20 \) days (period over which the average balance is computed).
computer program to compute equations (4) and (5) will take increasingly larger amounts of time to run as the grid is made more dense, and a tradeoff between accuracy and computational expense must be made.

Table 2 presents three alternative grid sizes and corresponding dynamic programming results for our numerical example. Figure 3 illustrates the tradeoff between increased grid size (and computation cost) and improvements in cost from increased accuracy.

For comparison purposes, a simulation was developed to ascertain near-optimal parameter values for a static policy of the simple \((t,T,U,u)\) form. Table 3 contains results from a 100-run simulation for a number of sets of \((t,T,U,u)\) parameter values. The lowest expected cost, $3089.4, was associated with a symmetric policy: \(t = 2250, T = 2500, U = 3500, u = 3750\) (see last row in Table 3). Assuming the computer run costs for running the simulations of Table 3 (run time only) were $25, the best "benchmark" static policy is also plotted in Figure 3.

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7A simulation was necessary since there was no way of solving analytically for optimum parameter values for our version of the problem. For details of the simulation, see [22].
### TABLE 2
RESULTS OF DYNAMIC PROGRAMMING MODEL

<table>
<thead>
<tr>
<th>Cash Balances $X_n$</th>
<th>Cumulative Cash Balances $S_n$</th>
<th>Expected Total Cost ($ per month)</th>
<th>Cost of Computer Run ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{min}$</td>
<td>$X_{max}$</td>
<td># of intervals</td>
<td>$S_{min}$</td>
</tr>
<tr>
<td>0</td>
<td>9000</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>9500</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>9750</td>
<td>40</td>
<td>0</td>
</tr>
</tbody>
</table>

### TABLE 3
SIMULATION OF STATIC SIMPLE (t,T,U,u) POLICIES
(100 simulations per policy tested)

<table>
<thead>
<tr>
<th>Parameters (in thousands of dollars)</th>
<th>Total Expected Cost ($ per month)</th>
<th>Standard Deviation of Total Cost (in dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>T</td>
<td>U</td>
</tr>
<tr>
<td>2500</td>
<td>3000</td>
<td>3500</td>
</tr>
<tr>
<td>2500</td>
<td>2750</td>
<td>3750</td>
</tr>
<tr>
<td>2400</td>
<td>2800</td>
<td>3625</td>
</tr>
<tr>
<td>2400</td>
<td>2800</td>
<td>2675</td>
</tr>
<tr>
<td>2345</td>
<td>2790</td>
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<td>2340</td>
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<td>3569</td>
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<tr>
<td>2300</td>
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<td>3500</td>
</tr>
<tr>
<td>2345</td>
<td>2800</td>
<td>3575</td>
</tr>
<tr>
<td>2250</td>
<td>2500</td>
<td>3500</td>
</tr>
</tbody>
</table>
FIGURE 3
RESULTS OF DYNAMIC PROGRAMMING MODEL

EXPECTED COST/MONTH ($)

3,216-

BEST SIMPLE STATIC (t,T,U,u) POLICY

3,089-

2,921-

DYNAMIC PROGRAMMING POLICY (40 x 40 GRID)

2,519-

COMPUTER RUN COST ($)

5.6 12.7 25
Conclusions

From Figure 3, the most accurate dynamic programming model produced the lowest expected costs ($2518.65); this cost is 18% less than the best static \((t,T,U,u)\) policy, which cost $3089.40. It should be emphasized that once the optimal dynamic programming solution has been obtained, it is available in tabular form and can be used without further computer cost as long as the problem description remains essentially unchanged with respect to the model. Thus, even though computer programming (as opposed to the solution run) of the dynamic programming model was certainly more involved and more costly than that of the simulation, the initial programming represents a one-shot investment, and the initial major run (costing $146.20 in Figure 2) represents an investment which need not be repeated until elements of the problem change significantly.

Further Research

We have totally ignored the question of maturity of the earning asset. Unless one is purchasing overnight Repurchase Agreements, then using our model one faces the problem of possibly selling an asset prior to its maturity, which is undesirable for a number of reasons. Direct inclusion of maturity life would quickly make the dynamic programming formulation computationally infeasible; some innovative way of incorporating maturities is needed to cope fully with this aspect of the problem.

Also, Homonoff and Mullins [11], in their study of actual daily net cash inflows, found that two distinct patterns were present in mean cash flows:
a day-of-week pattern, and a separate pattern based on dividing the month into three 10-day periods (for details see [11]). The model presented here was analyzed using a data-generating process for daily net cash inflows which did not contain these time dependencies. While equations (4) and (5) could readily incorporate different mean cash flows based on the day of the month n, the question of a benchmark alternative policy is harder to answer in this more complex world. Finally, in [11] the corporate officer responsible for cash and earning asset transfers felt that major influences on cash flows were known to him through his general business experience. The current research has not shed light on the question of whether the latter factor can in fact dominate the actual results of a cash balance problem.
REFERENCES


