METHODS OF MEASURING THE PROPERTIES OF IONIZED GASES AT MICROWAVE FREQUENCIES

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ABSTRACT

Microwave measurement techniques, devised especially for microwave gas discharge measurements, but applicable generally, are discussed. Measurement of cavity Q's from standing-wave patterns, including the effect of series loss in the input lead, is described. In addition to the method involving measurement of the resonance curve half-width, a method is given for obtaining the unloaded Q from the slope of the curve of the position of the voltage minimum vs. wavelength. Methods of determining the electric field from measurements of power and cavity parameters are discussed, and these methods are extended to low-Q cases, encountered with heavy discharge currents, by means of a four-terminal network theory. The discharge admittance from the cavity and four-terminal network points of view is discussed for low and high discharge current densities. The discharge admittance is then related to the complex conductivity of the ionized gas. Measurements of electron density, or of the decay of density from an initially ionized gas, are described in terms of the shift of resonant wavelength of the cavity enclosing the electrons. An electronic method of measurement of transient standing-wave phenomena is also presented.

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AT MICROWAVE FREQUENCIES

Introduction

This report has been written primarily to introduce to workers in the
field of gas discharges the techniques used in measuring the properties of
ionized gases at microwave frequencies. It is to replace Technical Report
No. 66; the contents of that report have been revised and corrected, and
some new material is added.

The phenomena associated with gas discharges at microwave frequencies
are in general much simpler than those corresponding to d-c discharges. On
the other hand, the microwave techniques are not so well known and may there-
fore discourage workers from entering the field. Experience has shown that
the readily available texts on ultra-high frequency measurements do not
include the particular calculations of the Q, the electric field, or the
conductivity in the most convenient or workable form for gas-discharge
measurements. To fill this need, the present report was prepared. It does
not start at an elementary level but assumes familiarity with the basic con-
cepts and techniques of the microwave field.

In the study of ionization processes, breakdown, and low current-density
discharge phenomena, high-Q devices may be used. For high current-density
discharges, low-Q devices often have to be used. Thus, methods for deter-
ming physical quantities, such as field strength and complex conductivity,
are discussed for both high- and low-Q devices.

In the problem of calculating the Q of resonant cavities used in the
gas-discharge measurements, the necessary accuracy is not achieved if one
neglects the series losses between the cavity and the reference plane to
which the measurements are referred. Also, since the total loading of the
cavity determines the width of the resonance curve, the loaded Q gives the
best coverage of the parameters. For these reasons, a method has been
worked out for determining the Q with a series loss included, in terms of
the loaded Q. This method extends the Q determination by Lawson (1) in
which the calculations neglected the series loss, and differs from the method
developed by Slater (2) which included the series loss but was calculated
in terms of the external Q. When the Q's of the cavities are very low,
accurate determinations of the parameters are best obtained by calculating
the Q from the slope of the phase curve; therefore, this method is also
described.

One of the important parameters which control the physical behavior of
the gas discharge is the value of the electric field in which the electrons
are accelerated. The ease with which the electric field configuration in a
resonant cavity can be calculated depends to a large extent on the cavity design. Since the requirements of the gas discharge are not necessarily consistent with simple field configuration requirements, three ways have been developed for determining the field in a cavity.

The simplest method for calculating the electric field applies to those cases in which the mode of oscillation is sufficiently well known to allow a quantity like the capacitance to be calculated. The ordinary capacitance can be defined as the ratio of stored energy to the square of the voltage; this capacitance is found useful in calculations provided that a modified concept of voltage is introduced. For many purposes, it is convenient to define an analogous quantity which is the ratio of stored energy to the square of the electric field.

For cavities having more complicated shapes, a method of calculating the electric field is described which involves measuring the shift in the cavity resonant wavelength by inserting a small metal plug into the high-field region. For cases in which the Q of the cavity is so low that a simple cavity theory does not apply, a network theory method of determining the electric field is also derived.

In the study of gas discharges, the admittance and associated complex conductivity of the discharge are also of importance. Methods of determination of admittance are outlined. Cavity theory is conveniently used for discharges contained in a cavity of high loaded Q. On the other hand, when the loaded Q is low, and the discharge region is coupled strongly to the transmission line, complications introduced by other modes in the cavity require the use of a four-terminal network method. The principles used in calculating the discharge admittance from the complex conductivity are outlined.

Microwave techniques offer a simple method of measuring electron densities in gases. Two methods have been developed for measuring a constant density, or alternatively, a decay of density from an initially ionized gas. These methods involve measurement of the resonant wavelength shift of a high-Q cavity: one, by a reflection method; and the other, by transmission.

In addition to the method above, a method of studying transient phenomena by measurement of transient standing-wave patterns is presented. Various details of equipment modifications and procedure are given.

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LIST OF SYMBOLS MOST COMMONLY USED

Where two symbols are given together, the lower case refers to the normalized quantity.

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<td>System bandwidth (Sec. 7)</td>
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<td>Velocity of light = ( 3 \times 10^8 ) meters/sec</td>
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<td>( d )</td>
<td>Position of the minimum of the standing-wave pattern</td>
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<td>R_vo</td>
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<td>V_o</td>
<td>rms voltage at PP'</td>
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<td>Phase angle of $\Gamma$</td>
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<td>$\omega$</td>
<td>Radian frequency of applied electric field</td>
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<td>$\omega_0$</td>
<td>The L-C cavity resonant frequency</td>
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Section 1

THE LUMPED-CIRCUIT REPRESENTATION OF A CAVITY

The problem arises of representing the characteristics of a resonant cavity coupled to the end of a line by loops, glass seals, etc. by the characteristics of a single lumped circuit. In general, these characteristics are a resonance curve of db standing-wave ratio $R_v$ vs. wavelength $\lambda$ such as that of Fig. 1, and a phase curve such as that of either Figs. 2(a) or 2(b), all measured some distance ahead of the cavity.

In order to be complete, the equivalent circuit should contain sufficient elements to account for all phenomena presented by the cavity. Practically, such a circuit is very difficult to work with in detail. The next few paragraphs are therefore concerned with the general characteristics of a fairly complete equivalent circuit; and, following certain restrictions, its simplification to usable form. One is then led to consider the circuit of Fig. 3(a); measurements are made at the points PP' on the line. The cavity, across TT', is characterized by parallel reactance $L'$ and $C'$, and conductance $G'$; it is connected to the line by a transformer of admittance ratio $A^2:1$. The reactance of the loop is represented by the lossless network $X_8$ which is often treated as a series

Fig. 1 Typical experimental resonance curve.

Fig. 2 Typical phase curves.

(a) Undercoupled $g_o = R_v > 1, R_o < 0$
(b) Overcoupled $g_o = \frac{1}{R_v} < 1, R_o > 0$
inductance; and the losses in the loop, seals and line back to PP' are represented by the conductance $G_s$. In the following analysis, it is assumed that the line of characteristic admittance $Y_c$ is matched at PP' looking toward the generator.

Following the general method given by Slater (3), one may now demonstrate the characteristics of this circuit. For the sake of discussion, the network $X_s$ is replaced by a series inductance $L_s$. If the series reactance $jL_s\omega$ is a much more slowly varying function of frequency than the cavity reactance, the impedance $Z$ at PP' is approximately a circle on the complex impedance plane, as shown in Fig. 4. Point A corresponds to the cavity resonance, at $\omega = \omega'$; point B, nominally at $\omega = 0$ and $\omega = \infty$ may be approached very closely, a few percent off resonance; its position is at $Z = 1/G_s + jL_s\omega'$, insofar as $\omega \approx \omega'$ throughout the range of interest. The reflection coefficient $\Gamma$, defined as the complex ratio of reflected voltage to incident voltage at the reference plane, is

$$\Gamma = \frac{Y_cZ - 1}{Y_cZ + 1} = \rho e^{i\phi},$$

where $\rho = |\Gamma|$, and $\phi$ is the phase angle of $\Gamma$. The reflection coefficient for this circuit is shown in the curves of Fig. 5. Two cases arise, depending on whether the locus of $\Gamma$ encircles the origin (the overcoupled case), or does not (the undercoupled case). The position $d$ of the standing-wave

\[ \text{Fig. 3 Equivalent circuits for a resonant cavity.} \]
minimum is given by the relation
\[
d = \frac{2k + 1}{4} \lambda_g + \frac{\phi^2 \lambda g}{4\pi}
\]
where \( k \) is an integer and \( \lambda_g \) is the guide wavelength. The \((2k + 1)/4\) term represents an odd number of quarter wavelengths between the cavity and the point at which the phase measurements are being made. It may be seen that the case shown in Fig. 5(a) leads to the phase curve of Fig. 2(a), and that of Fig. 5(b) to Fig. 2(b).

The voltage standing-wave ratio \( R_V \) is given by
\[
R_V = \frac{1 + \rho}{1 - \rho}
\]
further, the voltage ratio \( R_V \) and the db ratio \( R \) are related by
\[
R = 20 \log_{10} R_V
\]
Both cases shown in Fig. 5 give a resonance curve of the sort shown in Fig. 1.

A simple analysis near resonance is possible only if the energy storage in the loop circuit is negligible compared with that in the cavity at resonance. If the loop is non-resonant, an approximately equivalent criterion is that the ratio of energy stored off resonance to energy stored on resonance is negligible. Since the rate of change of susceptance with \( \omega \) is linearly related to the energy storage \((4)\), it is seen that the experimental condition to be satisfied is that the wings of the phase curves must have the asymptotic form shown in Figs. 2(a) or 2(b). Under these assumptions, the loop elements are a small perturbation, and may be taken into account.
in the following manner. The network $X_s$ is replaced by a length of line, a transformer, and a susceptance $B_s$, giving the circuit of Fig. 3(b). The elements are then transformed to those of Fig. 3(c), in which the loop reactance, which is the principal complication to further analysis, has been removed. This simplified circuit is the basis for all subsequent analysis in this report. The new cavity parameters are $L$, $G$, and $C$; and the new transformer admittance ratio, obtained from combining the two transformers, is $a^2$:1. It is seen that the principal effect of the loop is to shift the resonant frequency to $\omega_0$, where

$$\omega_0 = \frac{1}{\sqrt{LC}},$$ (5)

and to change the effective line length between the cavity and PP'.

For many purposes, this circuit may again be replaced by that of Fig. 3(d), wherein the line admittance is normalized to unity. Here,

$$g_s = \frac{G_s}{Y_c}, \quad \ell = \frac{LY_c}{a^2}, \quad g = \frac{a^2G}{Y_c}, \quad \text{and} \quad c = \frac{Ca^2}{Y_c}. \quad (6)$$

At resonance, the susceptance of the parallel $\ell$-$c$ combination is zero. It is now convenient to define two quantities, the resonant conductance across PP' as $g_0$, where

$$g = \frac{g_0}{1 - \left(\frac{g_0}{g_s}\right)},$$ (7)

and a dimensionless quantity $\beta$, akin to a susceptance, given by

$$\beta = \sqrt{\frac{c}{\ell}} = \frac{a^2}{Y_c\sqrt{L}} \cdot \quad (8)$$
Section 2

THE MEASUREMENT OF Q AND \( \beta \)

The quantity \( Q \) is defined as

\[
Q = \omega_0 \frac{U}{P}
\]

where \( U \) is the energy stored, and \( P \) the energy dissipated per second. It may be noted that the concept of a cavity \( Q \) is valid only if the stored energy takes many cycles to decay after excitation is removed; i.e., only if \( Q \gg 1 \). Various kinds of \( Q \)'s may be discussed, depending on where, in a particular problem, it is convenient to consider the energy to be dissipated. The \( Q \) of principal interest in this report is that obtained by considering the cavity represented in Fig. 3(d) to be excited externally, and calculating the ratio of energy stored in the cavity to that dissipated in \( g_s \) and \( g \). This ratio determines the unloaded \( Q, Q_U \), which is, at resonance,

\[
Q_U = \sqrt{\frac{c}{R}} \frac{1}{g(1 + \frac{g}{g_s})} = \frac{\beta(g_s - g_0)^2}{g_s g_0}.
\]

Since the quantities \( g_s \) and \( g_0 \) can be expressed directly in terms of the resonant dB standing-wave ratio \( R_0 \) and the off-resonance ratio \( R_0 \), the calculations of \( \beta \) and \( Q_U \) are closely allied. Two methods are now presented for determining \( Q_U \).

Resonance Curve Method

This method is applicable if the resonance curve is sufficiently sharp so that a certain width \( \Delta \lambda \) as shown in Fig. 1 can conveniently be measured on it. In order that this width may be measured where the curve is, in general, most accurately known, the calculation of \( Q_U \) is broken into two parts.

The cavity and line are viewed from the terminals \( tt' \) and a loaded \( Q, Q_L \), is defined in terms of the ratio of energy stored and energy dissipated through the parallel conductances \( g \) and \( g_s/(g_s + 1) \). Then in terms of the quantities \( g_s \) and \( g_0 \) given by Eq. (7)

\[
Q_L = \frac{\beta(g_s - g_0)(g_s + 1)}{g_s^2(g_0 + 1)} = \frac{\lambda_0}{\Delta \lambda}.
\]

This relation also defines the width \( \Delta \lambda \) to be measured. Now the \( \ell \)-c susceptance at \( tt' \) is

\[
b = \beta \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right).
\]
and at Δλ/2 off resonance, the quantity ω/ω₀ - ω₀/ω is equal to Δλ/λ₀; at these points, the susceptance is equal to the conductance across tt'. Under these conditions, the impedance zₜ at PP' is

$$zₜ = \frac{1}{g_s} + \frac{1}{g_s (g_o + 1)}$$

(13)

The reflection coefficient r for the circuit of Fig. 3(d) is

$$r = \frac{z - 1}{z + 1}$$

(1a)

and at the points in question, from Eqs. (13) and (1a),

$$ρₜ = \frac{\sqrt{g_s^2 + g_o^2 + g_o^2 - 4g_sg_o + 1}}{(g_s + 1)(g_o + 1)}$$

(14)

Rₜ is then found from Eqs. (3) and (4). Measurements of the db ratios R₀ and Rₜ yield values of g₀ and g_s. The value of Rₜ so determined is that at which the width Δλ is to be measured.

Fig. 6 Standing-wave ratio Rₜ db as a function of the standing-wave ratio on resonance.
Fig. 6 (cont'd.)
Fig. 6 (cont'd.)
These results may be put into the form of Fig. 6, where curves of $R_h$ vs. $R_o$ are plotted for various values of $R_m$. In order to distinguish the undercoupled and overcoupled cases, the values of $R_o$ are termed positive for $g_o < 1$, and negative for $g_o > 1$; but these curves are symmetrical about the origin and may be used for either positive or negative values of $R_o$.

The value of $Q_L$ is then determined in the following manner. The ratios $R_o$ and $R_m$, and the resonant wavelength $\lambda_o$, are found from the experimental resonance curve as in Fig. 1; then from Fig. 6, the value of $R_h$ is determined. The width of the resonance curve at $R = R_h$ is the required quantity $\Delta \lambda$, and $Q_L$ is found from Eq. (11).

From Eqs. (10) and (11), the unloaded $Q$ is seen to be

$$Q_u = \frac{(g_s - g_o)(g_o + 1)}{g_o(g_s + 1)} Q_L.$$  

---

**Fig. 7** Ratio of unloaded to loaded $Q$'s plotted as a function of the standing-wave ratio on resonance.
Fig. 8 Ratio of $\beta$ to the loaded $Q$ plotted as a function of the standing-wave ratio on resonance.
Fig. 8 (cont'd.)

\[
\frac{\beta}{Q_L}
\]

versus \( R_0 \) db.
The required ratio $Q_u/Q_L$ may thus be plotted vs. $R_o$ for various values of $R_m$. It is shown in Fig. 7. Comparison of the phase curve with Fig. 2 will determine the proper sign of $R_o$.

The quantity $\beta$ is most conveniently calculated from Eq. (10) as

$$\frac{\beta}{Q_L} = \frac{g_s^2(g_o + 1)}{(g_s - g_o)(g_s + 1)} . \tag{16}$$

The ratio $\beta/Q_L$ vs. $R_o$ is plotted for various values of $R_m$ in Fig. 8; then $\beta$ is determined from these curves and the value of $Q_L$.

If the series loss can be neglected, the calculations are greatly simplified, for the ratio $R_m$ is essentially infinite. In such a case

$$Q_L = \frac{\beta}{g_o + 1} \tag{11a}$$

and

$$Q_u = \frac{g_o + 1}{g_o} Q_L . \tag{15a}$$

Phase Curve Method

When $\beta$ is quite low, it is often difficult to perform the measurements indicated above. If, for example, $\beta$ is 5, the required wavelength separation $\Delta \lambda$ is enormous. In this case, however, it is sufficient to measure the slope of the phase curve and the standing-wave ratio on resonance. Invariably for this case, it will turn out that $g << 1$, so that it is permissible near resonance to neglect the series resistance $1/g_s$, and set $g_s = g$. If desired, the value of $g_s$ may be found by the method used for determining the reference points PP'. The circuit takes the form of Fig. 9.

In the vicinity of resonance, the susceptance $b$, Eq. (12), will be much smaller than $g$. It is then permissible to neglect all second order terms in subsequent manipulations. Near resonance, the reflection coefficient $\Gamma$ is then

$$r = \frac{1 - g - jb}{1 + g + jb} \approx \frac{(1 - g^2)^2 - j2b}{(1 + g)^2} . \tag{17}$$

The small phase angle $\phi$ is very closely

$$\phi = \frac{2b}{g^2 - 1} . \tag{18}$$

The slope of the phase curve $d(d)/d\lambda$ from Eq. (2) is

-18-
\[ \frac{d(d)}{d\lambda} = \frac{2k + 1}{4} \frac{d\lambda g}{d\lambda} + \frac{\lambda g}{\pi} \frac{d\phi}{d\lambda} + \frac{\phi}{4\pi} \frac{d\lambda g}{d\lambda}. \]  \hspace{1cm} (19)

This derivative is to be evaluated at \( \phi = 0, \lambda = \lambda_0, \lambda_g = \lambda_{go} \), using the relation

\[ \frac{d\lambda g}{d\lambda} = \left( \frac{\lambda}{\lambda_g} \right)^3. \]

There results

\[ \left. \frac{d(d)}{d\lambda} \right|_{\phi = 0} = \frac{2k + 1}{4} \left( \frac{\lambda_{go}}{\lambda_0} \right)^3 + \frac{\lambda_{go}}{\pi} \frac{d\phi}{db} \frac{db}{d\lambda} \left|_{\phi = 0} \right. \]  \hspace{1cm} (20)

But \( \frac{d\phi}{db} \left|_{\phi = 0} \right. \) is given by Eq. (18), and \( \frac{db}{d\lambda} \left|_{\phi = 0} \right. \) is simply \( \frac{2\beta}{\lambda_0} \) by definition of \( b \). With these insertions,

\[ \left. \frac{d(d)}{d\lambda} \right|_{\phi = 0} = \frac{2k + 1}{4} \left( \frac{\lambda_{go}}{\lambda_0} \right)^3 + \frac{\beta \lambda_{go}}{\pi (1 - g^2) \lambda_0}, \]  \hspace{1cm} (21)

whence

\[ \beta = \pi (1 - g^2) \left[ \frac{\lambda_0}{\lambda_{go}} \left. \frac{d(d)}{d\lambda} \right|_{\phi = 0} - \frac{2k + 1}{4} \left( \frac{\lambda_{go}}{\lambda_0} \right)^2 \right]. \]  \hspace{1cm} (22)

Here, \( g \) is the reciprocal of \( R_{vo} \), the voltage standing-wave ratio on resonance; the \( (1 - g^2) \) term is usually a minor correction when \( \beta \) is small; \( \left. \frac{d(d)}{d\lambda} \right|_{\phi = 0} \) is simply the slope of the phase curve at resonance; the factor \( \frac{2k + 1}{4} \) can usually be found by measuring the physical distance in guide wavelengths between the cavity and the point of measurement. The unloaded \( Q \) is given in this case by

\[ Q_u = \frac{\beta}{g}. \]  \hspace{1cm} (23)
Fig. 10 Ratio of absorbed to incident power as a function of db standing-wave ratio R.
Section 3

THE MEASUREMENT OF THE ELECTRIC FIELD

In this section, the electric field \( E \) is determined in terms of the incident power on the cavity and the standing-wave pattern on the line leading to it. For some of the methods of determination, it will be found convenient to introduce the concept of voltage across terminals in the cavity in order to facilitate calculation of the field.

It is generally useful to know the power \( P \) absorbed by the cavity in terms of the incident power \( P_i \). The relation is

\[
\frac{P}{P_i} = 1 - \rho^2 = \frac{4R_v}{(1 + R_v)^2}
\]

This ratio is plotted in Fig. 10 as \( P/P_i \) vs. the db standing-wave ratio \( R \).

Three general procedures of field measurement are described. The first group of methods is applicable to simple mode cavities, and presumes a knowledge of the field configuration within the cavity. The other two are applicable when the exact configuration is not known; the second method is best adapted to high-\( Q_L \) cavities, and the third, to low-\( Q_L \) cavities.

**Simple Mode Methods**

Consider the magnitude \( E \) of the rms electric field at some chosen reference point in the empty cavity. The stored energy can be expressed as

\[
U = \eta E^2
\]

where \( \eta \) is a parameter fixed for a given mode of oscillation and reference point. From the relation

\[
U = CV^2
\]

where \( C \) is the capacitance of a lumped-constant resonant circuit and \( V \) is the rms voltage across it, the parameter \( \eta \) may be seen to be a capacitance-like quantity, except that the energy is referred to the electric field instead of to the voltage.

By the definition of \( \eta \) and \( Q_U \)

\[
E = \frac{PQ_U}{\eta \omega_o}
\]

Since \( P \) and \( Q_U \) are known, it remains to compute \( \eta \) for a given cavity. This calculation is illustrated below for the cylindrical \( \text{TM}_{010} \)-mode cavity, where the reference point chosen is at the center. The results for several other cavities are given in Table I. When the cavity has a complicated
<table>
<thead>
<tr>
<th>CAVITY</th>
<th>MODE</th>
<th>CONFIGURATION OF ELECTRIC FIELD</th>
</tr>
</thead>
<tbody>
<tr>
<td>CYLINDRICAL</td>
<td>TM</td>
<td>$E_x = \frac{E_{\text{max}}}{\psi_{\text{TM}}(r_0)} \left( 2J_y(x_m r_0) \cos \theta \cos \frac{\pi}{h} z \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_y = \frac{E_{\text{max}}}{\psi_{\text{TM}}(r_0)} \left( \frac{\partial}{\partial x_m} J_y(x_m r_0) \cos \theta \sin \frac{\pi}{h} z \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_z = \frac{E_{\text{max}}}{\psi_{\text{TM}}(r_0)} \left( \frac{\partial}{\partial y_m} J_y(x_m r_0) \sin \theta \sin \frac{\pi}{h} z \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\eta = \frac{U}{E_{\text{max}}} \left[ \frac{1 + \delta(\phi, 0)}{1 + \delta(\phi, 0)} \right] e_s(\pi r_0^2 h) x$</td>
</tr>
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<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td></td>
<td>$F_{\text{max}}$ is the largest rms value of $E$ in the $z$ direction and occurs at $r_{\text{max}}$. This is not necessarily the largest field within cavity.</td>
</tr>
<tr>
<td>INDEX DIMENSION</td>
<td>$m \theta (r)$</td>
<td>$n h (z)$</td>
</tr>
<tr>
<td>RECTANGULAR</td>
<td>TM</td>
<td>$E_x = E_{\text{max}} \sin \frac{\omega}{a} x \sin \frac{\pi}{b} y \sin \frac{\pi}{h} z$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_y = E_{\text{max}} \cos \frac{\omega}{a} x \sin \frac{\pi}{b} y \sin \frac{\pi}{h} z$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_z = E_{\text{max}} \frac{\partial}{\partial y_m} \left( \frac{\partial}{\partial y_m} \cos \frac{\omega}{a} x \sin \frac{\pi}{b} y \sin \frac{\pi}{h} z \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\eta = \frac{U}{E_{\text{max}}} \left( \frac{\omega}{a} \right) e_{\text{inh}} \left( \frac{\partial}{\partial y_m} \left( \frac{\partial}{\partial y_m} \cos \frac{\omega}{a} x \sin \frac{\pi}{b} y \sin \frac{\pi}{h} z \right) \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$F_{\text{max}}$ is the largest rms value of $E$ in the $y$ direction, but is not necessarily the largest field within cavity.</td>
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<td>$n h (z)$</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>$F_{\text{max}}$ is the largest rms value of $E$ in the $x$ direction, but is not necessarily the largest field within cavity. If $a \neq 0, m = 0$, only $E_y$ exists. This case is covered by a rotation of $90^\circ$ in the $x, y$ plane.</td>
</tr>
</tbody>
</table>

$x_m$ is the root of $\psi_{\text{TM}}(r)$ |

$\delta(\phi, 0) = 1$ when $\phi > 0$ |

$\delta(\phi, 0) = 0$ when $\phi < 0$
shape, it is not feasible to compute $\eta$, but a method of measuring it is given below.

For the cylindrical $T_{M_{010}}$-mode

$$E_r = E_\phi = 0$$

$$E_z = \sqrt{2} E_0 J_0 \left(\frac{2.405r}{r_0}\right) e^{i\omega t}$$

where $r_0$ is the radius of the cavity. At the instant the electric field is a maximum, the magnetic field (not given above) is zero, and the stored energy is

$$U = \frac{1}{2} \varepsilon_0 \int_0^{r_0} \left[\sqrt{2} E_0 J_0 \left(\frac{2.405r}{r_0}\right)\right]^2 2\pi rh \, dr = 0.2695 \varepsilon_0 r_0^2 h E^2$$

where $h$ is the height of the cavity and $\varepsilon_0$ is the permittivity of free space; the volume of the cavity is $\pi r_0^2 h$. Thus

$$\eta = 0.2695 \varepsilon_0 r_0^2 h \quad .$$

A second general method can be described which applies to simple-mode cavities when measurements are not necessarily made at the resonant frequency and when there may be a discharge within the cavity. It is convenient here to perform the calculations in terms of voltages, rather than fields, using the circuit of Fig. 3(c). Consider the rms voltage $V$ across $TT'$; by judicious choice of the transformer ratio, $V$ may be defined as the integral of the field $E$ between any two points in the cavity. It is obvious that $V$ and $E$ are linearly related for any given mode and choice of reference points.

If the conductance $G_s$ is neglected, the voltage $V$ can be expressed in terms of the voltage $V_0$ across $PP'$, as

$$V = aV_0 \quad .$$

From Eqs. (5) and (8), the ratio $a$ can be expressed as

$$a = \sqrt{\frac{Y_0 \beta}{\omega_0 C}}$$

where $\beta$ and $\omega_0$ refer to the empty cavity. The capacity $C$ is that associated with the chosen voltage $V$ and the stored energy, through Eq. (26). Physically, the ratio $a$ depends on magnetic field flux linkages, especially in the vicinity of the loop. Equation (28) is then applicable to the loaded cavity insofar as the linkages are not sensibly altered by the discharge.

It is necessary to relate $V_0$ to the power flowing down the transmission line and to the standing-wave pattern on the line. By choice of reference
points PP', as detailed below,

\[ V_o = V_i(1 + \rho e^{j\phi}) \]

whence

\[ V_o^2 = V_i^2(1 + 2\rho \cos \phi + \rho^2) \]  \hspace{1cm} (31)

where \( V_i \) is the incident rms voltage. For this choice of reference planes, the angle \( \phi \) is given by

\[ \phi = \frac{4\pi d_o}{\lambda g} \]  \hspace{1cm} (32)

where \( \lambda_g/4 - d_o \) is the distance from PP' to the nearest minimum of the standing-wave pattern to the right or left.

The incident power \( P_i \) is

\[ P_i = V_i^2 \frac{V_o}{E} \]  \hspace{1cm} (33)

thus,

\[ V^2 = \frac{\beta}{\omega_0 C} P_i (1 + 2\rho \cos \phi + \rho^2). \]  \hspace{1cm} (34)

The field \( E \) is then calculated at any point desired by standard methods.

It remains to calculate \( C \). From the definition of \( C \) and \( \eta \), it is apparent that

\[ C = \eta \frac{E_n^2}{\int E \text{d}s} ^2 \]  \hspace{1cm} (35)

where \( E_n \) is the field at the reference point chosen for the calculation of \( E \) through the use of \( \eta \), and \( \int E \text{d}s \) is the voltage \( V \) across the chosen path. For example, for the TM\textit{010}-mode cavity, \( V \) is conveniently taken as the voltage across the parallel plates at the center of the cavity. Thus, \( V = Eh \), and from Eq. (28)

\[ C = 0.2695 \frac{\epsilon_0 \eta r_c^2}{h} \]  \hspace{1cm} (36)

Determination of the Reference Points PP'

For later use, it is convenient to outline here a method to determine the physical location of the reference points PP', which are an integral number of electrical half-wave lengths from the terminals of the cavity. The position is therefore a function of wavelength. If the entire phase curve of Figs. 2(a) or 2(b) can be plotted, the determination is simple; the points PP' are the asymptotes of the phase curve. If \( \beta \) is not large,
however, the method given below is applicable.

This method consists of physically shorting the cavity across the high field region with a substantial metallic conductor, taking care not to place the shorting conductor too near the input loop. Any minimum position on the standing-wave pattern is then an integral number of half wavelengths from PP'. The minimum should be determined as a function of frequency in order to yield a section of the asymptotes of Fig. 2. If this measurement is necessary, the cavity is probably overcoupled. In such a case, the angular frequency $\omega_0$ may be accurately defined as that frequency at which the reference point determined above lies exactly $\lambda_g/4$ from the minimum position recorded during measurement of the resonance curve. It may be noted that the voltage standing-wave ratio $R_V$ obtained on the shorted cavity is $g_8$.

**Tuning-Rate Method**

For a cavity having a more complicated shape, it is often difficult and sometimes impossible to compute $\eta$ or $C$ accurately. For these cases, an experimental method is available for measuring the electric field directly at any point on the cavity surface where the magnetic field is zero. At such points, the electric field is a maximum. This method is applicable only for measurements on the empty cavity at resonance, in cases when $\beta > 100$.

Suppose a small metallic body of variable volume is inserted into the cavity in a region of maximum electric field; this volume might well take the form of a plug, inserted through a hole in one of the surfaces.

An insertion into the cavity of volume $dv$ will result in a shift $d\omega$ of the resonant frequency $\omega_0$ given by (5)

$$d\omega = -\frac{\omega_0 E_1^2 dv}{2 \int_{cavity} E_0^2 dv}$$  \hspace{1cm} (37)

where $E_1$ is the peak field at the point in question, and $E_0$ is the peak field at any point in the cavity. One recognizes that

$$\int E_0^2 dv = \frac{2U}{\epsilon_0}$$

where $U$ is the total stored energy. With this substitution, and the use of Eq. (9), Eq. (37) may be put in the form

$$E_1^2 = \frac{2}{\pi} \frac{d\lambda}{dv} QU \sqrt{\frac{\mu_0}{\epsilon_0}}$$
where \( \sqrt{\frac{\mu_o}{\varepsilon_o}} = 377 \) ohms. In terms of the rms field, \( E = \frac{E_1}{\sqrt{2}} \),

\[
E^2 = \frac{d\lambda}{dv} \frac{Q_0 P}{\pi} \sqrt{\frac{\mu_o}{\varepsilon_o}} .
\]  

(38)

This relation is equivalent to defining \( \eta \), Eq. (27), as

\[
\eta = \frac{dv}{d\lambda} \cdot \frac{\pi}{\omega_0} \sqrt{\frac{\varepsilon_o}{\mu_o}} .
\]

The quantity \( d\lambda/dv \) may easily be measured; it is to be evaluated at the point where the plug is flush.

As a matter of interest, it is possible to determine the capacity \( C \) from Eq. (38), provided that the field distribution across the cavity gap (at one end of which \( E \) is measured) is known. For example, if the field is constant across the gap of height \( h \), the rms voltage is

\[
V^2 = E^2 h^2 = \frac{U}{C} .
\]

Thus, using Eq. (9) again

\[
C = \frac{dv}{d\lambda} \frac{\pi}{\omega_0 \omega_0} \sqrt{\frac{\varepsilon_o}{\mu_o}} .
\]  

(39)

Four-Terminal Network Method

When the resonant region is strongly coupled to the transmission line, the cavity concept is no longer applicable. In this case, it is convenient to use the four-terminal network method to be described in Section 4; the conditions which limit the validity of the formulae should be noted. If the series conductance \( G_s \) can be neglected, a slight extension of Eq. (51) gives the formula for the electric field as

\[
E^2 = \frac{a^2 v^2}{h^2} = \frac{\mu_o}{\varepsilon_o \lambda_g} \frac{dx}{dv} |(1 - g^2)Y_o v^2| .
\]

in the notation of that section. Substitution from Eqs. (31) and (33) yields

\[
E^2 = \sqrt{\frac{\mu_o}{\varepsilon_o}} \frac{\lambda_o}{\lambda_g} \frac{dx}{dv} |(1 - g^2)P_1(1 + 2p \cos \phi + p^2) .
\]  

(40)

If \( G_s \) is not negligible, a more exact formula can be derived from Eq. (51), and consideration of the circuit of Fig. 13.
THE MEASUREMENT OF DISCHARGE ADMITTANCE

The standing-wave ratio and phase on the input line to a cavity containing a discharge may be used to calculate the conductance $G_d$ and susceptance $B_d$ of the discharge. The discharge itself may be maintained either by the microwave field or by other means. Two methods are presented for measuring the admittance $G_d + jB_d$. The first method is applicable to the case of high-Q cavities and is essentially a cavity method. The second is a four-terminal network method and is applicable in general to any type of discharge whose physical extent is small compared to a wavelength.

Cavity Method

When the discharge is located at the center of a cavity resonator for which $\beta > 100$, it is convenient to use cavity theory to determine $G_d$ and $B_d$. The cavity is used as a load at the end of a transmission line. It is usually advantageous to overcouple the cavity so that $\beta < Q_U$. In fact, $\beta$ may well be as low as 100 for certain measurements.

The parameters of the empty cavity, represented by the circuits of Figs. 3(c) and 3(d), must first be determined. The quantities $Q_U$, $\beta$, $g$, $g_s$, and $G_s$ are determined by the methods of Section 2 and Eqs. (6). It is necessary to determine the actual value of $C$, and incidentally, of $G$. The capacity $C$ must be determined either by direct calculation or by the tuning-rate method. Both methods are outlined in Section 3. $G$ is given, from Eqs. (6) and (10) as

$$G = \frac{G_s}{2a^2} \left[ -1 + \sqrt{1 + \frac{4a^2 \omega C}{g_s g_s}} \right]. \quad (41)$$

If $G_s$ can be neglected, Eq. (41) assumes the simple form

$$G = \frac{\omega C}{Q_U}. \quad (41a)$$

The transformer ratio $a$ is given with sufficient accuracy by Eq. (30).

A discharge may now be introduced into the cavity in order to measure $G_d$ and $B_d$. Since the discharge is placed across the center of the cavity, at which position the parameters of the empty cavity were measured, the discharge may be represented as parallel elements across TT' of Fig. 3(c). The equivalent circuit is that of Fig. 11(a); it is normalized to $Y_c = 1$ in Fig. 11(b) where the cavity $\ell$-c susceptance is replaced by the single element $b$.

A measurement of standing-wave ratio and phase referred to the reference
points PP' uniquely determines $g_d$ and $b_d$ inasmuch as all other elements are known. It is possible to write formulae for $b_d$ and $g_d$ in terms of the standing-wave ratio and phase at PP' but these are so complicated that in practice it is much easier to use an impedance chart.

![Equivalent circuits of a cavity with discharge.](image)

It is evident that

$$\frac{g_d}{G_d} = \frac{b_d}{B_d} = \frac{g}{G} = \frac{c}{C}. \quad (42)$$

The quantities $G_d$ and $B_d$ may therefore be determined. If $G_s$ may be neglected, $G_d$ and $B_d$ may be expressed, through Eq. (41a), as

$$G_d = \frac{g_d \omega_0 C}{GQU} \quad (43)$$

$$B_d = \frac{b_d \omega_0 C}{GQU}. \quad (44)$$

### Four-Terminal Network Method

In cases where $\beta$ is very low, for example from 0.5 to 20, it is not accurate to apply the cavity theory because of the complications introduced by the modes of the cavity. As a matter of fact, the discharge need not be located in a cavity at all; it might, for example, be placed across terminals in a waveguide. In such cases an accurate method exists for measuring $G_d$ and $B_d$, subject to the restriction that the region of the discharge is small compared to a wavelength.

Let PP' be terminals located at some arbitrary point on the transmission line, and TT' be the terminals across which the discharge will take place. It is permissible to postulate these terminals, for the transmission line supports but one mode of propagation, and the discharge region is small compared to a wavelength. The equivalent circuit of Fig. 12 represents the physical state of affairs analogous to the circuits of Fig. 1. The series loss $G_s$ and the shunt loss $G$ include the transmission line and cavity.
losses as usual. The reactive elements are represented for convenience as a series reactance $X_s$, a shunt reactance $B$, and an ideal transformer of admittance ratio $a^2$·1. Different elements could be used but any loss-less three-element set can be represented as shown.

These circuit parameters are now determined experimentally by shorting TT' and measuring the standing-wave pattern on the line as a function of frequency, and by opening TT' and repeating the measurements. The position PP' is now chosen as the position of the standing-wave minimum with TT' shorted, at the frequency where the minimum with TT' open is exactly $\lambda g/4$ away. This procedure also defines an angular frequency $\omega_0$ characteristic of the empty cavity and the line. The fact that PP' is a minimum when TT' is shorted means that $X_s = 0$ under these conditions; the fact that PP' is $\lambda g/4$ away from a minimum when TT' is open means that $B$ is zero also. Thus at $\omega = \omega_0$ and this choice of PP', the circuit may be simplified to that of Fig. 13. The voltage standing-wave ratio $R_{Vw}$ with TT' shorted is

$$R_{Vw} = \frac{G_s}{Y_c}$$

and the ratio $R_{Vo}$ with TT' open is

$$R_{Vo} = \frac{Y_c}{G_s} + \frac{Y_c}{a^2 G}.$$  

The separate experimental determination of $a^2$ is necessary. This is carried out by altering the capacity across the terminals TT' and noting the impedance change at PP'. Now the terminals at TT' are the actual posts across which the discharge takes place; for ease in computation, they are assumed to be of parallel-plate structure as in Fig. 14. The capacity might be altered by adding dielectric or metallic disks, or better still, by means of a small movable metal plug set in one of the plates. This is similar to the measurement of electric field in the tuning-rate method of Section 3. Such a plug, whose depth must be much less than its lateral dimensions, is shown in Fig. 14.
In its normal position, the plug is flush to the surface of one of the plates. If a small volume dv of it is inserted, the additional capacity across TT' is $\varepsilon_0 dv/h^2$ where $h$ is the gap spacing. Now at $\omega = \omega_0$, all the reactive elements of the circuit cancel at PP' except for this small added capacity. The normalized circuit at PP' then takes the form of Fig. 15 where

$$g = \frac{a_0^2 G}{Y_c}, \quad b' = \frac{a_0^2 \varepsilon_0 dv}{Y_c h^2} \quad (47)$$

In most cases the series conductance $g_s$ can be neglected; it is included here for the sake of completeness. The impedance $Z_{PP'}$ is

$$Z_{PP'} = \frac{g + g_s + jb'}{g + jb'} \quad (48)$$

which, from Eq. (1a), leads to a reflection phase angle

$$\phi \approx \frac{-2b'}{\left(1 + \frac{g_s}{g_s}\right) - g^2} \quad (49)$$

This shift in $\phi$ results in a small change $dx$ in the position of the minimum at PP', and the ratio $dx/dv$ may be measured. Since

$$\frac{dx}{dv} = \frac{dx}{d\psi} \cdot \frac{d\psi}{db'} \cdot \frac{db'}{dv}$$

$$= \frac{\lambda g}{4\pi} \cdot \frac{-2}{\left(1 + \frac{g_s}{g_s}\right) - g^2} \frac{a_0^2 \varepsilon_0}{Y_c h^2}$$

then

$$a^2 = \frac{2\pi}{\lambda g} \left| \frac{(1 + \frac{g_s}{g_s})^2 - g^2}{\varepsilon_0 \omega_0} \right| h^2 Y_c$$

$$= \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\lambda g}{\lambda g} \cdot \left| \frac{dx}{dv} \left[ (1 + \frac{g_s}{g_s})^2 - g^2 \right] h^2 Y_c \right| \quad (51)$$

If the series conductance $g_s$ is neglected, Eq. (51) becomes
The quantity $\left| \frac{dx}{dv} \right|$ must be evaluated at the point where the plug is flush.

From Eqs. (45-47) and Eq. (51), all the necessary circuit parameters may be determined. Impedance measurements at PP' in Fig. 13 will then give the value of any impedance added across TT'.

Fig. 15 Normalized equivalent circuit for the addition of a small capacity at TT'.

\[
a^2 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\lambda_0}{g} \cdot \left| \frac{dx}{dv} \right| (1 - g^2)h^2Y_c
\]  

(51a)
Section 5

THE MEASUREMENT OF DISCHARGE COMPLEX CONDUCTIVITY

If a high-frequency electric field $E_0 e^{j\omega t}$ is impressed on an ionized gas, two current densities result. These are the displacement current density $D = j\omega \varepsilon_0 E_0 e^{j\omega t}$, and the electron current density $J$. The latter is composed of $J_r$, the real electron current density in phase with the field, and $J_1$, the reactive electron current density out of phase with the field, so that

$$J = J_r + jJ_1.$$ 

The complex conductivity $\sigma$ is defined as

$$\sigma = \frac{J}{E} = \frac{J_r}{E} + j \frac{J_1}{E} = \sigma_r + j\sigma_1.$$ \hspace{1cm} (53)

Thus $\sigma$, a property of the gas, and $\varepsilon_0$, a characteristic of empty space, completely define the discharge space. This is the viewpoint adopted in this section.

An alternative and equivalent viewpoint is to characterize the ionized medium by a complex permittivity $\varepsilon = \varepsilon_r - j\varepsilon_1$ in which the effect of the real conductivity is included in $\varepsilon_r$. According to this alternative viewpoint, Maxwell's equation $\nabla \times H = D + J$ becomes

$$\nabla \times H = j\omega (\varepsilon_r - j\varepsilon_1)E_0 e^{j\omega t},$$ \hspace{1cm} (54)

whereas in terms of the first viewpoint, the equation becomes

$$\nabla \times H = (j\omega \varepsilon_0 + \sigma_r + j\sigma_1)E_0 e^{j\omega t} = j\omega \left[ (\varepsilon_0 + \frac{\sigma_1}{\omega}) - j \frac{\sigma_r}{\omega} \right] E_0 e^{j\omega t}. \hspace{1cm} (55)$$

Comparison of Eqs. (54) and (55) shows that the complex dielectric constant $K = \varepsilon/\varepsilon_0$ is given by

$$K = \frac{\varepsilon_r - j\varepsilon_1}{\varepsilon_0} = \left( 1 + \frac{\sigma_1}{\omega\varepsilon_0} \right) - j \frac{\sigma_r}{\omega\varepsilon_0}. \hspace{1cm} (56)$$

In this matter, one may convert $\sigma$ into the equivalent complex dielectric constant.

As might be expected, the gas discharge conductivity $G_d$ and susceptance $B_d$ can be calculated from $\sigma_r$ and $\sigma_1$. The method of calculation is difficult and depends on the spatial distribution of the discharge; therefore, in practice, the simplest possible geometric arrangement is chosen. An example will be given.
Suppose a discharge exists between circular parallel plates of radius $r_0$ and spacing $h \ll \lambda$, as in Fig. 16. The electron concentration will in general be a function of both $x$ and $r$; but, for simplicity, assume the concentration to be independent of $r$ for $0 \leq r \leq r_0$ and to be zero for $r > r_0$. For $r < r_0$, then, the concentration is a function of $x$ alone. Since $h$ is small compared to a wavelength, the current will be continuous across the gap if both $J$ and $D$ are included. Fringing fields will be neglected, which means that the equipotentials are parallel. Under these circumstances the following computation is justified.

The admittance associated with a slab of thickness $dx$ is

$$\frac{\pi r_0^2 \left\{ \sigma_r(x) + j \left[ \sigma_i(x) + \omega \varepsilon_0 \right] \right\}}{dx}$$

where the $j \omega \varepsilon_0$ term is the admittance of the empty slab and is related to the displacement current density. The impedance of the slab is the reciprocal of this admittance, so that the impedance of the whole gap is

$$z = \int_0^h \frac{\pi r_0^2 \left\{ \sigma_r(x) + j \left[ \sigma_i(x) + \omega \varepsilon_0 \right] \right\}}{dx}$$

This impedance is not quite the reciprocal of the admittance of the discharge measured by the method of Section 3. In that measurement, a susceptibility $B_c$ arising from the capacity of the empty plates was automatically subtracted, where

$$B_c = \frac{\pi r_0^2 \varepsilon_0}{h}$$

Thus

$$\frac{1}{G_d + J(B_d + B_c)} = \int_0^h \frac{dx}{\pi r_0^2 \left\{ \sigma_r(x) + j \left[ \sigma_i(x) + \omega \varepsilon_0 \right] \right\}}$$

gives the relationship between the measured values of $G_d$, $B_d$, and $\sigma_r$, $\sigma_i$. Unfortunately, the functional form of $\sigma_r$ and $\sigma_i$ must be independently
derived from kinetic theory. One might expect $\sigma_1$ to be more or less proportional to $G_d$, and $\sigma_i$ to $B_d$, but this is seen to be generally not the case, from the form of Eq. (57).

In the limit of low densities, where $\sigma_r, \sigma_i << \omega \epsilon_0$

$$G_d, B_d << B_o$$

Eq. (57) may be split into real and imaginary parts, becoming

$$G_d \approx \frac{\pi r^2}{\hbar^2} \int_0^h \sigma_r(x)dx$$

(58)

$$B_d \approx \frac{\pi r^2}{\hbar^2} \int_0^h \sigma_i(x)dx$$

(59)

It must be borne in mind that Eqs. (57)-(59) were derived under very restrictive assumptions concerning the spatial distribution of the electrons.
Section 6

THE MEASUREMENT OF ELECTRON DENSITY

In this section, the method is outlined of measuring a certain range of electron density in a gas within a microwave cavity. The density measurement may be either of an active discharge or of a decaying plasma after ionization ceases. The microwave field used for measurement is generally a low level signal introduced specifically for the purpose; with certain precautions regarding stability of the electron density the maintaining field can be used. Two methods of measurement are presented below.

The free electrons in a gas under the influence of an electric field \( E_0 e^{i\omega t} \) give rise to the complex conductivity of the medium, and as a result produce two principal effects in the properties of the cavity. First, their oscillatory motion changes the resonant wavelength of the cavity; and second, their collisions with the gas atoms produce an energy loss which appears as a lowering of the cavity \( Q \). Either of these phenomena can be used as the basis of density measurements, but the first is easier to handle mathematically and is the one used in this section. In addition, the existing treatment (6) of the lowering of the \( Q \) involves measurement of transmission through the plasma, and does not take into account the phase retardation within it.

Provided that the imaginary part \( \sigma_i \) of the complex conductivity \( \ll \omega \varepsilon_0 \), the change in resonant wavelength of a cavity due to \( \sigma_i \), is given by (7)

\[
\Delta \lambda = \frac{\lambda - \lambda_0}{\lambda} = \frac{\lambda_0}{\lambda} \int \frac{\sigma_i E^2 dv}{\text{Vol}} \int \frac{4\pi \varepsilon_0 c E^2 dv}{\text{Vol}}.
\]  

(60)

Here, \( \lambda \) is the resonant wavelength of the cavity with electrons, \( \lambda_0 \) is the resonant wavelength in the absence of electrons, and \( c \) is the velocity of light. If \( v_c \ll \omega \), where \( v_c \) is the collision frequency of electrons with gas atoms, then \( \sigma_r \ll \sigma_i \), and \( \sigma_i \) is independent of \( v_c \). Under these conditions,

\[
\sigma_i = -\frac{n e^2 \lambda}{2\pi m c}
\]  

(61)

where \( e \) and \( m \) are the charge and mass of the electron, and \( n \) is the electron density.

The appropriate functional form for the electron density must be substituted into Eq. (61), then into Eq. (60). Division by \( \lambda \), in order to remove the dependence of the right-hand side of Eq. (60) on \( \lambda \), and
## Table II

Change of resonant wavelength as a function of the electron density in a container placed within a cylindrical TM_{010} cavity of radius \( r' \) and height \( h \)

<table>
<thead>
<tr>
<th>Container</th>
<th>Electron Distribution</th>
<th>Formula</th>
<th>Remarks</th>
</tr>
</thead>
</table>
| Cylindrical | Uniform \( n \) | \[
\Delta \lambda = 1.85 \left( \frac{ne^2 \lambda_0}{m_e (2\pi c)^2} \right) \frac{h}{\rho^2} \left[ J_0^2(2.4\rho) + J_1^2(2.4\rho) \right] \text{ cm}^{-1}
\] | \( \rho = \frac{r_0}{h} \) ELECTRON DISTRIBUTION CHARACTERISTIC OF RECOMBINATION LOSS. |
| | \( n = n_0 J_0(2.4 \frac{r_0}{h}) \cos(\frac{\pi r}{h}) \) | \[
\Delta \lambda = \left( \frac{n_0 e^2 \lambda_0}{m_e (2\pi c)^2} \right) \frac{h}{\rho^2} \left[ 1.6 \rho - 1.42 \rho^4 + 2.58 \rho^6 \right] \text{ cm}^{-1}
\] | ACCURATE FOR \( \rho \leq \frac{1}{2} \) ELECTRON DISTRIBUTION CHARACTERISTIC OF DIFFUSION LOSS. |
| Spherical | Any distribution with average electron density \( \frac{n}{\pi} \) | \[
\Delta \lambda = 1.85 \left( \frac{ne^2 \lambda_0}{m_e (2\pi c)^2} \right) \frac{\text{VOLUME OF SPHERE}}{\text{VOLUME OF CAVITY}} \text{ cm}^{-1}
\] | ACCURATE FOR \( r_0 \ll r' \) |
The positive factor $A$ is a geometrical coefficient which takes into account the distribution of electron density and electric field within the cavity, and $\bar{n}$ is the average electron density. Examples of Eq. (62) are worked out in Table II for a few electronic arrangements within a $TM_{010}$-mode cavity. Thus the change in resonant wavelength of a cavity is related to the electron density within it.

Reflection Method

In this method, the microwave cavity containing the discharge is generally overcoupled to the end of the transmission line. The asymptotes of the phase curve of Fig. 17 are then obtained either by direct measurement of the position of standing-wave minima vs. $\lambda$, or by the method outlined in Section 3. The line midway between these asymptotes, marked 0-0 on Fig. 17 is then determined. This line is the locus of the voltage minimum vs. resonant wavelength if the resonant wavelength could be perturbed.

Consider the effect of the electrons in the cavity. The resonant wavelength is shifted from $\lambda_0$ to $\lambda$, and the position of the resonant voltage minimum moves from $x_0$ to $x$. If, then, the probe position and wavelength are shifted along the locus 0-0, the position of minimum probe signal indicates the new resonant wavelength of the cavity. The shift $\Delta \lambda$ corresponding to the electron density is then determined. For measurements of electron decay (8), a modification of this procedure is very useful. Here, the probe signal is fed into a receiver and an oscilloscope whose sweep is synchronized with the start of electron decay. The probe position and wavelength corresponding to some electron density are fixed on the locus 0-0; and the appearance of the minimum of the probe signal on the oscilloscope then indicates the time at which this density is attained.
Transmission Method

In this method the cavity forms a load on the transmission line as before, but in addition, a small signal is coupled out of the cavity into a receiver, and, if required, into a synchronized oscilloscope as before. The components may be arranged so that, on resonance, transmission through the cavity is a maximum. The resonant wavelength of the cavity with electrons can then be found by noting the wavelength of maximum transmission.

This method has the advantage that one less experimental variable is required, but suffers from the possibility of error due to incorrect matching of line between the generator and cavity, and between the output coupling and detector.
Section 7

TRANSIENT MEASUREMENTS OF THE STANDING-WAVE PATTERN

Measurement of the transient behavior of a discharge after the exciting voltage is removed may be made by using circuits that measure a current or voltage during a selected short time interval. For this purpose, the standing-wave pattern in a waveguide or other transmission system is a convenient parameter, as it may be related to the desired properties of the discharge. Consequently, a method has been devised to permit measurement of the standing-wave pattern during a short time interval; the time coordinate of this interval is adjustable over a considerable range. It is assumed that the properties of the discharge do not change appreciably during the time of measurement.

The method is shown schematically in Fig. 18. A gas discharge is formed in a cavity, section of transmission line, or other container arranged to permit examination of its dielectric properties. In the method illustrated, a microwave oscillator delivers a weak signal to the discharge through a slotted section. This allows exploration of the standing-wave pattern set up by the discharge. A modified spectrum analyzer (TSS-43E) is used as a detector of this standing-wave pattern, and its video output is displayed on the screen of the A/R scope (Dumont type 256).

Because the discharge is pulsed, the pattern is a function of time, and the spectrum analyzer must now be arranged to indicate
voltage during a short time interval which can be adjusted to cover the range of time of interest. This is accomplished by use of the delay multivibrator circuits in the A/R scope, which are triggered by the same timing pulse that triggers the discharge. The length of the delay is determined by the delay control on the A/R scope. At the end of the delay period, a sawtooth sweep voltage is applied to the horizontal plates of the cathode-ray tube in the A/R scope, an intensifier pulse is applied to the grid of the tube, and a small fraction of the sweep voltage is applied to the reflector electrode of the 707-B oscillator in the spectrum analyzer. The average reflector voltage and the sweep voltage are adjusted in such a way that the local oscillator reaches the frequency necessary to pass a signal through the spectrum analyzer while the reflector electrode is being swept (and thus while the cathode-ray tube receives an intensifier pulse and horizontal sweep voltage).

Because the local oscillator frequency sweeps past the frequency necessary to send a signal through the i-f amplifier, the output of the spectrum analyzer appears as a pulse on the A/R scope screen. Sweep lengths of approximately 5, 12, and 24 µsec are available, and these can be obtained with delays up to nearly 1200 µsec. Because these sweeps are so rapid, a bandwidth problem now occurs, for the normal half-power bandwidth in the spectrum analyzer of about 75 kc/sec cannot transmit a pulse created by sweeping through this bandwidth in 24 µsec without considerable attenuation and broadening of the pulse. An i-f amplifier with a bandwidth of 184 kc/sec has been found acceptable with the 24-µsec sweep, and one of 3 Mc/sec is more than sufficient for the 5-µsec sweep.

The relations between bandwidth, sweep speed and transmitted pulse characteristics may be summarized in the following approximate relations. Let \( F \) be the total frequency range over which the local oscillator frequency is changed linearly during a time interval \( T \), and let the half-power bandwidth be \( B \). Then the smallest transmitted pulse width obtainable with a given rate of frequency sweep, \( F/T \), occurs when (9)(10)

\[
F/T = 2B^2
\]

If the ratio of the half-power width in frequency of the transmitted pulse to \( F \), the total frequency range swept, is \( W \), then

\[
FT = (k/W)^2
\]

where \( k \) is found experimentally to be approximately 1.3. From these equations, one obtains directly,

\[
T = \frac{1}{\sqrt{2 B}} \left( \frac{k}{W} \right)
\]
Thus, the frequency interval and sweep duration for minimum pulse width are
determined separately from \( W \), the fraction of the sweep length occupied by
the transmitted pulse, and from the bandwidth. If \( W \) is greater than about
1/4, the pulse occupies too large a fraction of the sweep to permit easy
measurement; whereas if it is very much smaller, the required accuracy of
measurement of time on the sweep increases, and no compensating advantages
appear. A good working range for \( k/W \) appears to be from about 6 to 10.

**Circuit Diagram**

The details of the circuit modifications are shown in Fig. 19. In
addition to the wiring changes shown it was necessary to take special pre-
cautions to eliminate very small 60-cycle modulation of the frequency of
the oscillator. Special shielding of some leads was required, and as a
final step it was found necessary to use a repetition frequency of 60 cps,
synchronized with the power line voltage. The video amplifier is a single
6AC7 with a 15K plate load resistor.

![Circuit Diagram](image)

**Fig. 19** Circuit diagram showing modification of
spectrum analyzer and A/R
scope for fast sweep.

**Method of Operation**

The time at which the standing-wave ratio is to be measured is chosen,
and the delay time \( T \) is set by the range dial on the A/R scope to slightly
less than this value (see Fig. 18). The d-c reflector voltage is then
varied until the sawtooth sweep voltage passes through the value that re-
sults in the correct intermediate frequency (this value is indicated by
the horizontal dotted line). The total delay time \( T \) is then the time \( T \) plus
the time from initiation of the sweep to the maximum of the pulse. The
standing-wave ratio or other customary transmission line measurements are

\[
F = \sqrt{2B \left( \frac{k}{W} \right)}
\]
made for this setting of time delay; then the delay may be readjusted and the measurements repeated.

The assumption underlying the sweep method is that the phenomenon to be observed does not change its characteristics appreciably during the sweep time $T$. If extremely fast discharge decay is to be observed, $T$ will be required to become so small that more elaborate circuit arrangements will be necessary. It is further assumed that the properties of the discharge repeat after each pulse.

References


(2) J. C. Slater, Rev. Mod. Phys. 18, pp. 473 et seq (1946).


(5) J. C. Slater, loc. cit., p. 482. The relation is obtained by noting that in this reference, Slater's cavity field function $E_a$ is normalized so that

$$\int_{\text{Vol}} E_a^2 dv = 1$$

Equation (37) can also be derived easily from the electromagnetic variational principle.


(7) J. C. Slater, loc. cit., p. 481.


(9) E. N. Williams, Proc. I.R.E. 34, 18P (1946). The subsequent equations are discussed further.


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