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SOME DISTRIBUTIONS INVOLVING BESSEL FUNCTIONS

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Some Distributions Involving Bessel Functions

by

G. M. Kaufman [†]

The Lognormal frequency function provides a surprisingly good fit to a wide variety of frequency histograms of observational data from widely disparate sources. As is pointed out in [1], any process that generates data according to the Law of Proportionate Effect is a strong potential candidate for characterization by a Lognormal probability law, and its ubiquitousness in nature is one reason to expect the Lognormal probability law to appear often as a stochastic model of observational data. This led us to consider in [6] economic decision problems in which the mean of a Lognormal process plays a central role, but in which the mean is not known with certainty prior to decision.

There we assume that we are sampling from an independent Lognormal process; i.e. a process generating value of mutually independent random variables $\tilde{x}_1, \ldots, x_i, \ldots$ with identical densities

$$\left(\frac{h}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}h(\log x - \mu)^2} \frac{1}{x} , \quad \begin{array}{c} h > 0, \\ -\infty < \mu < +\infty , \\ x > 0, \end{array} \tag{1}$$

and with mean $E(\tilde{x}|\mu,h) \equiv \mu_L = \exp\{\mu + \frac{1}{2h}\}$. In [6] we restrict attention to decision problems for which the expected (monetary) value criterion is appropriate and for which the expected value of each of two (terminal) acts under consideration is a linear function of μ_L. Assuming that h is known

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[†] I wish to thank John Bishop for rummaging through the attic of classical analysis with me and pointing out several interesting artifacts.

while μ is not known with certainty, the natural conjugate density of $\tilde{\mu}_L$ is then stated, and prior to posterior and preposterior analysis is done. Finally, [6] presents necessary and sufficient conditions for a sample size n to be optimal under the above assumptions about sampling, value, and acts, when in addition the cost of sampling is a linear function of n.

A natural generalization is to repeat this pattern of analysis when neither μ nor h are known with certainty.

We develop here all the distribution theory necessary to do Bayesian inference in this more general case: the prior to posterior and preposterior analysis just mentioned. In the course of the analysis we show that following the natural path of natural conjugate analysis leads us into a cul-de-sac insofar as analysis of some standard decision problems is concerned. For if one assigns a natural conjugate density to $(\tilde{\mu}, \tilde{h})$, then the derived density of $\tilde{\mu}_L$ has no finite moments of order q > 0. Thus under these distributional assumptions, a complete analysis of the optimal sample size problem mentioned above is not possible; this is also true for many of the common (terminal) infinite action problems treated in, for example [7], Chapter 6 .

The key to the distribution theory is the observation that the density of $\tilde{z} = \tilde{\mu} + \frac{1}{2\tilde{h}}$ when $(\tilde{\mu}, \tilde{h})$ have a natural conjugate density involves an integral that is an integral representation of a Bessel function $K_\nu(z)$ of purely imaginary argument. Section 2 is devoted to a cataloguing of the properties of this function that we need. In section 3 we derive the density of $\tilde{z}$ when $\tilde{\mu}$ given $\tilde{h} = h$ is Normal with mean m and variance $(hn)^{-1}$, and $\tilde{h}$ is Gamma, showing that the natural conjugate density of $\tilde{z}$ is essentially a product of an exponential term $\exp\{\tilde{z}\}$, a Student density with argument z, and a factor involving a Bessel function $K_\nu(z)$. We then state two

propositions about sums of such random variables. After computing the characteristic function of $\tilde{z}$, we find its first two moments, and by examining the characteristic function note that for some values of the parameters of the density, the density is <u>infinitely divisible</u>. Section 4 is devoted to exploring the relation of the density of $\tilde{z}$ to some well known densities. We examine relative behavior in the extreme right tail in particular, using the notion of regular variation as developed in Feller [3]. This leads to some useful statements about approximations to the right tail of a sum of mutually independent random variables $\tilde{z}_i$ with common density (3.3) below, and gives us an easy way of seeing that $\tilde{\mu}_L = \exp\{\tilde{z}\}$ has no finite moments of order q > 0. Section 5 then outlines the facts needed to do Bayesian inference about $\tilde{\mu}_L$.

Section 6 takes off in a completely different direction. By appropriate interpretation of the integrand in a particular integral representation of the Bessel function $K_\nu(z)$, we can define the probability law of a compound random process. Then putting on the probabilists' spectacles, we derive almost by inspection three identities (at least one of which is well known) involving Bessel functions of the third kind that appear involved from any other point of view.

For several additional examples of distributions involving Bessel functions the reader is referred to Feller [3], Chapter 2. He shows that the Bessel function $I_\nu(z)$ of purely imaginary argument as defined in (2.2) below appears in the analysis of randomized Gamma densities, randomized random walks, and first passage problems. Bessel functions also appear naturally in distributions of radial error; see Durand and Greenwood [2], for example.

2. **An Integral Representation of a Bessel Function of Purely Imaginary Argument †**

The integral of primary interest here is

$$\int_0^\infty e^{-(k^2 t + \frac{a^2}{4t})} t^{\nu-1} dt \quad , \tag{2.1}$$

where k and ν are real and a^2 is (possibly) complex. This integral is closely related to a Bessel function of purely imaginary argument. Such functions are real and denoted by $I_\nu(z)$, where

$$I_\nu(z) \equiv \sum_{j=0}^\infty \frac{(\frac{1}{2}z)^{\nu+2j}}{j! \; \Gamma(\nu+j+1)} \tag{2.2}$$

when ν is not a negative integer. As shown in (2.7) below, the integral (2.1) is $2(a/2k)^\nu$ times the modified Bessel function

$$K_\nu(ak) = \frac{1}{2}\pi \frac{I_{-\nu}(ak) - I_\nu(ak)}{\sin \nu\pi} \tag{2.3}$$

a function well defined for all values of ν. For any integer n,

$$K_n(ak) \equiv \frac{1}{2}\pi \lim_{\nu \to n} \frac{I_{-\nu}(ak) - I_\nu(ak)}{\sin \nu\pi} \quad . \tag{2.4}$$

In particular, †† letting $\gamma = .5772157\ldots$ denote Euler's constant and for $i=1,2,\ldots$ $\psi(i+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{i} - \gamma$,

$$K_0(z) = -\log(\frac{1}{2}z) \cdot I_0(z) + \sum_{j=0}^\infty \frac{(\frac{1}{2}z)^{2j}}{(j!)^2} \psi(j+1) \tag{2.5a}$$

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† All of the formulae in this subsection may be found in Watson [8].

†† Ibid, p. 80.

and

$$K_n(z) = \frac{1}{2} \sum_{\ell=0}^{n-1} \frac{(-1)^\ell (n-j-1)!}{j! (\frac{1}{2}z)^{n-2j}}$$

$$+ (-1)^{n+1} \sum_{j=0}^{\infty} \frac{(\frac{1}{2}z)^{n+2j}}{j! (n+j)!} \{\log(\frac{1}{2}z) - \frac{1}{2}\psi(j+1) - \frac{1}{2}\psi(n+j+1)\} . \qquad (2.5b)$$

For half-integer values of ν,

$$K_{\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \qquad (2.5c)$$

$$K_{\frac{3}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left(1 + \frac{1}{z}\right) \qquad (2.5d)$$

$$K_{\frac{5}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left(1 + \frac{3}{z} + \frac{3}{z^2}\right) \qquad (2.5e)$$

$$\vdots$$

$$K_{n+\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{j=0}^{n} \frac{(n+j)! (2z)^{-j}}{j! (n-j)!} . \qquad (2.5f)$$

A relation of primary interest here between the integral (2.1) and $K_\nu(z)$ that we need to compute the density of certain sums of random variables and the characteristic functions of these sums follows directly from the integral representation (see Watson [8] p. 183) defined for unrestricted values of ν and for $\mathrm{Re}(z^2) > 0$, [†]

$$K_\nu(z) = \frac{1}{2}(\frac{1}{2}z)^\nu \int_0^\infty e^{-(t + \frac{z^2}{4t})} t^{-\nu-1} dt. \qquad (2.6)$$

By a simple integrand transform, (2.6) leads to

[†] $\mathrm{Re}(z^2)$ denotes the real part of (possibly complex) z^2.

$$2\left(\frac{2k}{a}\right)^{\nu} K_{\nu}(ak) = \int_{0}^{\infty} e^{-(k^2 t + \frac{a^2}{4t})} t^{-\nu-1} dt \qquad (2.7)$$

defined here for unrestricted values of ν, k^2 real, and $Re(a^2) > 0$. Here we

shall be concerned only with real ν; $\,$ remark that in (2.3), $K_{-\nu}(ak) = K_{\nu}(ak)$

so that we can write $K_{\nu}(ak)$ in (2.7) as $K_{|\nu|}(ak)$, where $|\nu|$ is the absolute value

of ν. An asymptotic expansion (Watson [8] p. 202)

$$K_{\nu}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{j=0}^{p-1} \frac{(\nu, j)}{(2z)^j} + O(z^{-p})$$

$$= \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left[1 + \frac{4\nu^2 - 1}{8z} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2(8z^2)} + O(z^{-3})\right] \qquad (2.8a)$$

where $O(z^{-p})$ denotes a function of order of magnitude of z^{-p} as $z \to +\infty$, and

$$(\nu, j) \equiv \frac{\Gamma(\nu + j + \frac{1}{2})}{j! \ \Gamma(\nu - j + \frac{1}{2})} = \frac{(4\nu^2 - 1)(4\nu^2 - 3^2)\dots(4\nu^2 - (2j-1)^2)}{2^{2j} \ j!} \qquad (2.8b)$$

shows that $K_{\nu}(z)$ tends exponentially to zero as $z \to +\infty$ through positive

values. This property is of critical importance here, as we shall be

concerned with real values of $z = ak > 0$ when $K_{\nu}(z)$ forms part of a density

function--one that we wish to be "well-behaved".

An alternate expression for $K_{\nu}(z)$ is (Watson [8] p. 206)

$$K_{\nu}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \frac{e^{-z}}{\Gamma(\nu + \frac{1}{2})} \int_{0}^{\infty} e^{-u} u^{\nu - \frac{1}{2}} \left(1 + \frac{u}{2z}\right)^{\nu - \frac{1}{2}} du . \qquad (2.9)$$

By analysis of remainder terms in the expansion of $\left(1 + \frac{u}{2z}\right)^{\nu - \frac{1}{2}}$, Watson

obtains a more exact result than (2.8a): when $p \geq \nu - \frac{1}{2}$,

$$K_{\nu}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left[\sum_{j=0}^{p-1} \frac{(\nu, j)}{(2z)^j} + \theta_2 \frac{(\nu, p)}{(2z)^p}\right] \qquad (2.10)$$

where $0 \leq \theta_2 \leq 1$.

3. The Density of $\tilde{z} = \tilde{\mu} + \frac{1}{2h}$

The representation (2.7) enables us to derive the density of the sum of a Normal and the reciprocal of a Gamma random variable when their joint density is Normal-gamma.

Lemma 3.1: Let $\tilde{\mu}|h$ and $\tilde{h}$ have densities

$$\frac{\sqrt{n}}{\sqrt{2\pi}} \; e^{-\frac{1}{2}hn(\mu-m)^2} \, h^{\frac{1}{2}} \; , \qquad \begin{array}{l} n, \, h > 0 \;\; , \\ -\infty < m < +\infty, \\ -\infty < \mu < +\infty, \end{array} \tag{3.1}$$

and

$$\frac{\frac{1}{2}\varepsilon}{\Gamma(\frac{1}{2}\alpha)} \; e^{-\frac{1}{2}h\varepsilon} \, (h\varepsilon)^{\frac{1}{2}\alpha-1} \qquad \begin{array}{l} \varepsilon > 0 \;\; , \\ \alpha > 0 \;\; , \\ h > 0 \;\; , \end{array} \tag{3.2}$$

respectively. Then the density of $\tilde{z} \equiv \tilde{\mu} + \frac{1}{2h}$ is, letting $\nu = \alpha+1$, $\xi(z) \equiv n(z-m)^2 + \varepsilon$ and $c = n^{\frac{1}{4}(\nu+2)} \, \varepsilon^{\frac{1}{2}(\nu-1)} / \, 2^{\nu-1} \, \sqrt{\pi} \; \Gamma(\frac{1}{2}(\nu-1))$,

$$c \; e^{\frac{1}{2}n(z-m)} \; [\xi(z)]^{-\frac{1}{4}\nu} \, K_{\frac{1}{2}\nu}([n\xi(z)/4]^{\frac{1}{2}}), \qquad -\infty < z < +\infty. \tag{3.3}$$

Proof: Simply multiply (3.1) and (3.2), substitute $z - \frac{1}{2h}$ for μ in (3.1) and integrate over $h > 0$. The integrand is that of (2.7) with $-\frac{1}{2}\nu$ in place of ν, $k^2 = \xi(z)/2$ and $a^2 = n/2$. For notational convenience, we shall henceforth write a generic member of the family of densities characterized by (3.3) as f_z.

The density of sums of random variables $\tilde{z}_i = \tilde{\mu}_i + \frac{1}{2h}$ with common $\tilde{h}$ is easy to derive in the same fashion when the $\tilde{\mu}_i$ are mutually independent and Normal:

Corollary 3.1: Let $\tilde{z}_i \equiv \tilde{\mu}_i + \frac{1}{2h}$, $i=1,2,\ldots,p$ where the $\tilde{\mu}_i$ are mutually independent given $\tilde{h} = h$ and $\tilde{\mu}_i$ is Normal with mean m_i and variance $(hn_i)^{-1}$. Let $\tilde{h}$ have density (3.2). Then $\tilde{s} \equiv \Sigma z_i$ has density

$$c' \; e^{\frac{1}{2}Np(s-M)} \; [\xi(s)]^{-\frac{1}{4}\nu} \, K_{\frac{1}{2}\nu}(\frac{1}{2}p[N\xi(s)]^{\frac{1}{2}}) \tag{3.4}$$

where

$$M = \Sigma m_i \ , \quad N = (\Sigma \frac{1}{n_i})^{-1} \ , \quad \xi(s) = N(s-M)^2 + \epsilon \ ,$$

and

$$c' = N^{\frac{1}{4}(\nu+2)} \ \epsilon^{\frac{1}{2}(\nu-1)} \ p^{\frac{1}{2}\nu}/2^{\nu-1} \ \sqrt{\pi} \ \Gamma(\frac{1}{2}(\nu-1)) \ .$$

<u>Proof</u>: Since $\Sigma\tilde{\mu}_i$ given $\tilde{h} = h$ is Normal with mean M and variance $(hN)^{-1}$, $\tilde{s}$ given $\tilde{h} = h$ is Normal with mean $M + \frac{p}{2h}$ and variance $(hN)^{-1}$. Replacing n with N and m with M in $\xi(\cdot)$, setting $k^2 = \xi(s)/2$ and $a^2 = Np^2/2$, and proceeding as in the proof of (3.3) gives (3.4).

3.2 <u>Characteristic Function of f_z, Momenta, and Infinite Divisibility</u>

The characteristic function of a random variable $\tilde{z}$ with density f_z is

$$\frac{2^{1-\frac{1}{2}\alpha} \ e^{i\Psi m}}{\Gamma(\frac{1}{2}\alpha)} \ [\epsilon(\frac{\Psi^2}{n} - i\Psi)]^{\frac{1}{4}\alpha} \ K_{|\frac{1}{2}\alpha|}(\ [\epsilon(\frac{\Psi^2}{n} - i\Psi)]^{\frac{1}{2}}) \ . \tag{3.5}$$

By differentiating (3.5) and then letting $\Psi = 0$, we can then compute the mean $E(\tilde{z})$ and variance $V(\tilde{z})$ of a random variable $\tilde{z}$ with density f_z. Alternately, we can compute them directly from the intermediate formulae shown below:

$$E(\tilde{z}) = E(\tilde{\mu}) + \frac{1}{2}E(\tilde{h}^{-1}) = \begin{cases} m + \frac{\epsilon}{\alpha-2} & \text{if } \alpha > 2 \\ \\ +\infty & \text{otherwise} \end{cases} \tag{3.6a}$$

and

$$V(\tilde{z}) = E_h \ V(\tilde{z}|h) + V_h \ E(\tilde{z}|h) = \frac{1}{n} E(\tilde{h}^{-1}) + V(\frac{1}{2}\tilde{h}^{-1})$$

$$= \begin{cases} \frac{\epsilon}{n(\alpha-2)} + \frac{\epsilon^2}{2(\alpha-2)^2(\alpha-4)} & \text{if } \alpha > 4 \\ \\ +\infty & \text{otherwise} \ . \end{cases} \tag{3.6b}$$

<u>Proof of (3.5)</u>:   Using the fact that if the random variable $\tilde{y}$ is Normal with mean m and variance $(hn)^{-1}$, its characteristic function is

$$E(e^{i\Psi\tilde{y}}) = e^{i\Psi m - \frac{\Psi^2}{2hn}} \tag{3.7}$$

the characteristic function of $\tilde{z}$ with density f_z is

$$E_z(e^{i\Psi\tilde{z}}) = E_h \; E_{\mu|h} \; (e^{i\Psi\tilde{\mu} + \frac{i\Psi}{2\tilde{h}}}) = e^{i\Psi m} \; E(e^{-a^2/4\tilde{h}}) \tag{3.8}$$

where $a^2 = 2\left(\frac{\Psi^2}{n} - i\Psi\right)$.   As $\tilde{h}$ is Gamma as in (3.2),

$$E(e^{-\frac{a^2}{\tilde{h}}}) = \frac{(\frac{1}{2}\epsilon)^{\frac{1}{2}\alpha}}{\Gamma(\frac{1}{2}\alpha)} \int_0^\infty e^{-(\frac{1}{2}h\epsilon + \frac{a^2}{4h})} \; h^{\frac{1}{2}\alpha-1} \; dh \quad . \tag{3.8}$$

The integral in (3.8) is that of (2.7) with $k^2 = \frac{1}{2}\epsilon$ and $-\frac{1}{2}\alpha = \nu$; substituting in (3.8) according to (2.7) and multiplying by $\exp\{i\Psi m\}$ gives (3.5).

In the special case $\alpha = 1$, the characteristic function (3.5) is

$$e^{i\Psi m} \; e^{-\sqrt{\epsilon} \; (\frac{\Psi^2}{n} - i\Psi)^{\frac{1}{2}}} \tag{3.9}$$

Now a characteristic function w, say, is <u>infinitely divisible</u> if and only if for each n=1,2,... there exists a characteristic function w_n such that $w_n^n = w$.   The nth positive root of (3.9) is of precisely the same form as (3.9) with m replaced by m/n and $\sqrt{\epsilon}$ replaced by $\sqrt{\epsilon}/n$.   Thus, when $\alpha = 1$, f_z is infinitely divisible.

Notice that when $\alpha = 1$, if $(\tilde{\mu}, \tilde{h})$ have the Normal-gamma density given by (3.1) and (3.2), the marginal density of $\tilde{\mu}$ is Cauchy, centered at m.

As is well known, the Cauchy density is infinitely divisible, and this is suggestively close to what we just found above.

It is still open as to whether or not f_z is infinitely divisible with arbitrary $\alpha \neq 1$. We conjecture that f_z is infinitely divisible only for $\alpha = 1$.

We may use (3.9) to generalize Collary 3.1 to certain sums of independent $\tilde{z}_i$s.

Corollary 3.2: Let $\tilde{z}_i$, i=1,2,...,m be mutually independent, and assume $\tilde{z}_i$ has density f_z with parameters $\alpha = 1$, n, ϵ_i, m_i. Then $\tilde{W} \equiv \Sigma \tilde{z}_i$ has characteristic function

$$e^{i\Psi M} - \sqrt{E} \left(\frac{\Psi^2}{\Pi} - i\Psi\right)^{\frac{1}{2}}$$

where

$$M = \Sigma m_i \quad \text{and} \quad \sqrt{E} = \Sigma \sqrt{\epsilon_i} \ .$$

Consequently $\tilde{W}$ has density f_z with parameters $\alpha = 1$, n, E, and M. This follows from multiplying the characteristic functions of the $\tilde{z}_i$ and noting that the product is of the same functional form as (3.9) with m replaced by M and ϵ replaced by E.

3.3 Approximation of $P(\tilde{z} > z_0)$

For $z_0 > m$ probabilities $P(\tilde{z} > z_0)$ may be expressed as a weighted sum of Bessel functions ~~of the third kind~~ plus a remainder term: define $y_0 = \sqrt{n}(z_0 - m)$ and write, using (3.1) and (3.2),

$$P(\tilde{z} > z_0) = \int_0^\infty G_{N*}(h^{\frac{1}{2}}y_0) \, e^{-(\frac{1}{2}h\epsilon + \frac{n}{8h})} \, h^{\frac{1}{2}\alpha - 1} \, dh, \tag{3.10}$$

where G_{N*} is a standardized Normal right tail. For $u > 0$, $G_{N*}(u)$ has the expansion

$$\frac{1}{u} f_{N*}(u) \left[1 - \frac{1}{u^2} + \frac{1 \cdot 3}{u^4} + \ldots + \frac{(-1)^j \, 1 \cdot 3 \ldots (2j-1)}{u^{2j}} \right] + R_j(u), \tag{3.11a}$$

where

$$R_j(u) = (-1)^{j+1} \, 1 \cdot 3 \ldots (2j+1) \int_u^\infty \frac{1}{t^{2j+2}} \, f_{N*}(t) \, dt \tag{3.11b}$$

and the absolute value of $R_j(u)$ is less than that of the first neglected term. (See [4] 26.2.12). Substituting (3.11a) and (3.11b) in (3.10) with $u = h^{\frac{1}{2}}y_0$, $\omega = y_0^2 + \epsilon$, $c_0 = \frac{(\frac{1}{2}\epsilon)^{\frac{1}{2}\alpha}}{\Gamma(\frac{1}{2}\alpha)}$, and integrating,

$$\frac{1}{c_0} P(\tilde{z} > z_0) = \left(\frac{n}{2\omega}\right)^{\frac{1}{4}(\alpha-1)} K_{-\frac{1}{2}(\alpha-1)} \left(\frac{1}{2}\sqrt{n\omega}\right)$$

$$- \left(\frac{n}{2\omega}\right)^{\frac{1}{4}(\alpha-3)} K_{-\frac{1}{2}(\alpha-3)} \left(\frac{1}{2}\sqrt{n\omega}\right)$$

$$+ 1 \cdot 3 \left(\frac{n}{2\omega}\right)^{\frac{1}{4}(\alpha-5)} K_{-\frac{1}{2}(\alpha-5)} \left(\frac{1}{2}\sqrt{n\omega}\right) + \ldots \tag{3.12}$$

$$+ (-1)^j \, 1 \cdot 3 \ldots (2j-1) \left(\frac{n}{2\omega}\right)^{\frac{1}{4}(\alpha-2j-1)} K_{-\frac{1}{2}(\alpha-2j-1)} \left(\frac{1}{2}\sqrt{n\omega}\right)$$

$$+ \int_0^\infty R_j(h^{\frac{1}{2}}y_0) \, e^{-(\frac{1}{2}h\epsilon + \frac{n}{8h})} \, h^{\frac{1}{2}\alpha - 1} \, dh \ .$$

Since $R_j(u)$ for fixed u is less in absolute value than the absolute value of the first neglected term, the integral of $R_j(h^{\frac{1}{2}}y_0)$ is less than

$$\frac{1 \cdot 3 \ldots (2j+1)}{y_o^{2j+2}} \left(\frac{n}{\omega}\right)^{\frac{1}{2}(\alpha-2j-3)} K_{-\frac{1}{2}(\alpha-2j-3)} \left(\tfrac{1}{2}\sqrt{n\omega}\right) . \tag{3.13}$$

One way of approximating $P(\bar{z} > z_0)$ when $z_0 > m$ is to use (3.12) but neglect the remainder term. If j is selected less than $\frac{1}{2}(\alpha-3)$, the accuracy of this approximation increases rapidly with increasing z_0 since for large z_0 (3.13) behaves roughly like

$$[\exp\{-\tfrac{1}{2}n^{\frac{1}{2}}(y_0^2 + \epsilon)^{\frac{1}{2}}\}]/y_0^{2j+2} (y_0^2 + \epsilon)^{\frac{1}{4}(\alpha-2j-3)} .$$

4.3 Regular Variation of f_z

In order to relate the behavior of the density (3.3) in the tails to that of better known densities such as the Normal and the Student densities, we use the notion of <u>regular variation</u> as exposited in Feller [3], Vol. II. It enables us to state facts about the right tail of the density of sums of random variables $\tilde{z}$ with common density (3.3) that seem difficult to prove by other means. In the course of the discussion we recast a Lemma of Feller's to provide a useful criterion for determining whether or not the qth moment of a density defined on $(0, \infty)$ is finite. This Lemma is just a restatement of a well known result in analysis.[†]

We then go on to show that the density (3.3) is regularly varying and that it has finite moments of order $0 \leq q < \tfrac{1}{2}\alpha$. Using a new result of Feller's, we give an expression for the right tail of the density of sums of independent random variables with common density f_z.

4.3.1 Regular Variation

The idea of regular variation is contained in the following lemma and definitions from Feller [3]. [††]

<u>Lemma</u>: Let U be a positive monotone function on $(0, \infty)$ such that

$$\frac{U(tx)}{U(t)} \to \psi(x) \leq \infty \tag{4.1}$$

as $t \to \infty$ at a dense set A of points. Then

$$\psi(x) = x^\rho \quad , \quad -\infty \leq \rho \leq \infty \quad . \tag{4.2}$$

(Here x^∞ is interpreted as ∞ if $x > 1$, and as 0 if $x < 1$).

<u>Definition</u>: For a positive function U satisfying (4.3) and (4.4) with <u>finite</u> ρ, set $U(x) = x^\rho L(x)$. Then for each $x > 0$ as $t \to \infty$,

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[†] See Hobson [5] p. 505.

[††] Chapter VIII, Section 8.

$$\frac{L(tx)}{L(t)} \to 1 \qquad\qquad (4.3)$$

and so L satisfies (4.2) with $\rho = 0$.

Definition: A positive function L defined on $(0, \infty)$ <u>varies slowly</u> at infinity if and only if (4.3) is true. The function U <u>varies regularly with exponent ρ</u> if and only if $U(x) = x^{\rho} L(x)$ with $-\infty < \rho < \infty$ and L varying slowly.

Since the asymptotic behavior at infinity of unimodal densities are not affected by behavior near the mode <u>or</u> near the origin, we can use these notions to characterize the extreme right tails of such densities.

Easy calculations show these facts about some well-known densities. We assume $x \gg 1$ and h, r, λ, n, and ε greater than 0:

	Density	$\psi(x)$
Normal:	$c_1 e^{-\frac{1}{2}h(x-\mu)^2}$	$x^{-\infty}$
Lognormal:	$c_2 e^{-\frac{1}{2}h(\log x - \mu)^2}/x$	$x^{-\infty}$
Gamma:	$c_3 e^{-x} x^{r-1}$	$x^{-\infty}$
Exponential:	$c_4 e^{-\lambda x}$	$x^{-\infty}$
Student:	$c_5 (n(x-\mu)^2 + \varepsilon)^{-\frac{1}{2}(\alpha+1)}$	$x^{-(\alpha+1)}$
Strong Pareto:	$c_6 x^{-(\alpha+1)}$	$x^{-(\alpha+1)}$

While the exponent ρ in x^{ρ} serves to order partially the above densities according to their rate of approach to 0 as $x \to \infty$, this is not an essential fact here. Of more importance is the following Lemma establishing necessary and sufficient conditions in terms of ρ for the finiteness of the qth moment of a density defined on $(0, \infty)$. It can be proved by a slight modification of the proof of a Lemma on page 272 of Feller [3]:

Lemma 4.1: Let $U(x)$ be a density defined on $(0, \infty)$ that varies regularly with exponent ρ. Then $\lim_{x \to 0} x^q U(x) = 0$ and $q + \rho + 1 < 0$ are necessary and sufficient conditions for moments of U of order q to be finite. If $\psi(x) = x^{-\infty}$ for U, and $\lim_{x \to 0} x^q U(x) = 0$ for all q, then the qth moment of U is finite for all $-\infty < q < \infty$.

Applying the lemma to the densities listed above, it shows that the exponential density has finite moments of order q for $0 \leq q < \infty$, the Gamma density has finite moments of order q for $-r < q < \infty$, and the strong Pareto density has finite moments of order q for $-\infty < q < \alpha$. Remark also, that we can apply these ideas to densities defined on $(-\infty, \infty)$ by examining both left and right tails; e.g. a necessary and sufficient condition that a density defined on $(-\infty, \infty)$ have a finite moment of order $q \geq 0$ is that it varys regularly in the right tail with exponent ρ_L and in the left tail with exponent ρ_R and $q + \rho_L + 1 < 0$ and $q + \rho_R + 1 < 0$. As the Normal and Student densities shown above are symmetric about μ, we see that these conditions imply that the Normal density has finite moments of order $q \geq 0$ for all $q < \infty$, while the Student density has moments of order $q \geq 0$ only for $q < \alpha$.

The following theorem and corollary due to Feller[†] gives us a simple asymptotic expression for the extreme right (left) tail of the density of the sum of mutually independent random variables with common density that varies regularly in the right (left) tail:

Theorem: Let F_1 and F_2 be two distribution functions such that as $x \to \infty$

$$1 - F_i(x) \approx \frac{a_i}{x^\rho} L(x) \tag{4.6}$$

with L varying slowly. Then the convolution $G = F_1 * F_2$ of F_1 and F_2 has

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[†] [3], p. 271

a regularly varying tail such that

$$1 - G(x) \simeq \frac{a_1 + a_2}{x^\rho} \, L(x) \quad .$$ (4.7)

(here $\simeq$ means that as $x \to \infty$ the ratio of $1 - G(x)$ and $(a_1 + a_2) \, L(x)/x^\rho$ approaches 1).

As two examples of application, one can show the following:

(i) Let $\tilde{x}$ and $\tilde{y}$ be independent with strong Pareto densities

$$c_1 \, x^{-(\alpha+1)}, \; x > x_0 > 0, \text{ and } c_2 \, y^{-(\alpha+1)}, \; y > y_0 > 0.$$

Then letting $s = \tilde{x} + \tilde{y}$,

$$1 - G(x) \simeq \frac{c_1 + c_2}{s^\alpha} \quad .$$

(ii) Let $\tilde{u}$ and $\tilde{v}$ be independent with common Student density

$$c_5 ((x-\mu)^2 + \varepsilon)^{-\frac{1}{2}(\alpha+1)}, \; \varepsilon > 0, \; \alpha \geq 1. \text{ Then letting } \tilde{s}' = \tilde{u} + \tilde{v},$$

$$1 - G(s) \simeq \int_s^\infty \frac{2c_5}{((x-\mu)^2 + \varepsilon)^{-\frac{1}{2}(\alpha+1)}} \, dx \quad .$$

The first proposition is obvious. To show that the sum of two independent Student random variables with identical densities possess the property stated in (ii) is only slightly more involved. Examine the right tail $P(\tilde{u} > x)$ of the density. This suffices since the density is symmetric about μ. The right tail is regularly varying with exponent $\rho = -\alpha$, for by L'Hospital's rule, for fixed $x < \infty$,

$$\lim_{t \to \infty} \frac{P(\tilde{u} > tx)}{P(\tilde{u} > t)} = \lim_{t \to \infty} \frac{x((tx-\mu)^2 + \varepsilon)^{-\frac{1}{2}(\alpha+1)}}{((t-\mu)^2 + \varepsilon)^{-\frac{1}{2}(\alpha+1)}} = x^{-\alpha} \quad .$$

Applying Feller's theorem gives (ii).

With the above explanatory prelude completed, we now state three facts about f_z:

(iii) The density (3.3) is regularly varying with exponent $\rho_R = -\frac{1}{2}\alpha$
in the right tail and $\rho_L = -\infty$ in the left tail;

(iv) If $\tilde{z}$ has density f_z then $P(\tilde{z} > z_0) \equiv 1 - F_z(z_0)$ varies slowly;

(v) If $\tilde{z}_1,\ldots,\tilde{z}_\ell$ are mutually independent with common density f_z
then $s_\ell \equiv \Sigma\ \tilde{z}_i$ has a density that varies regularly in both
tails with exponent $\rho_R = -\frac{1}{2}\alpha$, and $\rho_L = -\infty$, and

$$P(\tilde{s}_\ell > s) \sim \ell c[1 - F_z(s)]$$

where

$$F_z(s) = P(\tilde{z}_1 < s) .$$

(vi) F_z and F_s vary slowly.

Notice that once we show (iii), (v) follows directly from Feller's theorem
and its corollary; (iv) and (vi) follow from consideration of the 0th moment
of $\tilde{z}$ and of $\tilde{s}$ using Lemma 4.1.

Regular variation of (3.3) with exponent $\rho = -\frac{1}{2}\alpha$ implies that densities
f_z are "Student-like" in the extreme tails in the sense that if f_z has
parameter set (m,n,ε,α), the ratio of f_z to a Student density $c_5(n(z-m)^2+\varepsilon)^{-\frac{1}{2}\alpha+\frac{1}{2}}$
with $\frac{1}{2}\alpha$ degrees of freedom approaches 1 as $z \to \pm\infty$.

<u>Proof of (iii)</u>: The ratio of (3.3) with argument tx to (3.3) with argument
t may be expressed using (2.9) as

$$e^{\frac{1}{2}n(tx-t) - \frac{1}{2}[n\xi(tx)]^{\frac{1}{2}} + \frac{1}{2}[n\xi(t)]^{\frac{1}{2}}} \left(\frac{\xi(t)}{\xi(tx)}\right)^{\frac{1}{4}(\nu+1)}$$

$$\frac{\int_0^\infty e^{-u}\ u^{\frac{1}{2}(\nu-1)}\ [1 + \dfrac{u}{\frac{1}{2}[n\xi(tx)]^{\frac{1}{2}}}]^{\frac{1}{2}(\nu-1)}\ du}{\int_0^\infty e^{-u}\ u^{\frac{1}{2}(\nu-1)}\ [1 + \dfrac{u}{\frac{1}{2}[n\xi(t)]^{\frac{1}{2}}}]^{\frac{1}{2}(\nu-1)}\ du} . \tag{4.8}$$

The limit of the logarithm of (4.8) as $t\to+\infty$ is $-\frac{1}{2}(\nu+1)\log x$, so since $\nu\equiv\alpha+1$,
$\rho_R = -\frac{1}{2}(\alpha+2)$. Repeating the argument for $t\to-\infty$ shows that $\rho_L = -\infty$.

By the same method used to establish (iii), we can show that when $\tilde{z}$ has density f_z,

(vi)   The right tail of the density of $\tilde{\mu}_L \equiv \exp\{\tilde{z}\}$ varies regularly with exponent $\rho = -1$.

The density of $\tilde{\mu}_L$ is that of a Lognormal mean $\exp\{\tilde{\mu} + \frac{1}{2\tilde{h}}\}$ when $\tilde{\mu}$ and $\tilde{h}$ are jointly distributed as in (3.1) and (3.2). Together with Lemma 4.1, (vi) implies that $\tilde{\mu}_L$ has no finite moments of order $q > 0$. We will show in detail in the next section that the density of $\tilde{\mu}_L$ has finite moments of order q only for $-n \leq q \leq 0$.

5.   An Application to Lognormal Distribution Theory

5.1   The Density of $\exp\{\tilde{z}\}$

As mentioned at the outset, if we wish to derive the density of the mean $\tilde{\mu}_L \equiv \exp(\tilde{\mu} + \frac{1}{2\tilde{h}})$ of a Lognormal process when neither μ nor h are known with certainty, but are regarded as random variables to which we have assigned a natural-conjugate (Normal-gamma) prior, then Lemma 3.1 is clearly applicable. For letting $\tilde{z} \equiv \log \tilde{\mu}_L = \tilde{\mu} + \frac{1}{2\tilde{h}}$, it gives the density of $\tilde{\mu}_L$ as

$$c \; e^{-\frac{1}{2}nm} \; [\xi(\log \mu_L)]^{-\frac{1}{4}\nu} \; K_{\frac{1}{2}\nu}([n\xi(\log \mu_L)/4]^{\frac{1}{2}}) \cdot \mu_L^{\frac{1}{2}n-1}, \tag{5.1}$$

$$\mu_L > 0 \;.$$

The density (5.1) is of course proper, but it nevertheless has a property that is distressing in the applications: moments of order $q > 0$ do not exist. This is implied by the fact that the density (5.1) is regularly varying with exponent $\rho = -1$ together with Lemma 4.1. The exact statement about moments is in

Lemma 5.1: If $\tilde{\mu}_L$ has density (5.1), then letting $\phi = \frac{1}{2}(q + \frac{q^2}{n})$,

$$E(\tilde{\mu}_L^q) = \begin{cases} \dfrac{2e^{mq}}{\Gamma(\frac{1}{2}\alpha)} \; [\frac{1}{2}\epsilon \, \phi]^{\frac{1}{4}\alpha} \; K_{\frac{1}{2}\alpha}([2\phi\epsilon]^{\frac{1}{2}}) & \\ & \text{if} \quad -n \leq q \leq 0 \\ +\infty & \text{if} \quad q > 0 \;. \end{cases} \tag{5.2}$$

Proof: We first prove (5.2) by appeal to (2.7). Starting with the densities (3.1) and (3.2) of $\tilde{\mu}|h$ and $\tilde{h}$, it is easy to show that

$$E_{\mu|h}(\tilde{\mu}_L^q) = e^{\frac{q}{2h}} \; E_{\mu|h}(e^{q\tilde{\mu}}) = e^{mq+\frac{1}{2h}(q+\frac{q^2}{n})} \;.$$

Consequently, letting $\Phi = \frac{1}{2}(q + \frac{q^2}{n})$,

$$E(\tilde{\mu}_L^q) = \frac{(\frac{1}{2}\epsilon)^{\frac{1}{2}\alpha} e^{mq}}{\Gamma(\frac{1}{2}\alpha)} \int_0^\infty e^{\frac{\Phi}{h} - \frac{1}{2}h\epsilon} h^{\frac{1}{2}\alpha-1} dh \quad . \tag{5.3}$$

By a limit test, (5.3) diverges for $\Phi > 0$. When $\Phi \leq 0$, that is $-n \leq q \leq 0$,
the integral (5.2) is by (2.7) as shown in (5.2).

Alternate Proof: A way of proving (5.2) that reveals more about the behavior
of the density (5.1) of $\tilde{\mu}_L$ is to examine the expectation of $\tilde{\mu}_L^q = e^{q\tilde{z}}$ when $\tilde{z}$
has the density (3.3).

Use the asymptotic expansion (2.10) for large values of z to write
$K_{\frac{1}{2}\nu}([n\xi(z)/4]^{\frac{1}{2}})$ as proportional to

$$[\xi(z)]^{-\frac{1}{4}} e^{-\frac{1}{2}[n\xi(z)]^{\frac{1}{2}}} \zeta([n\xi(z)/4]^{\frac{1}{2}}) \tag{5.4}$$

where $\zeta(\cdot)$ denotes the summand in $\xi(z)$ of (2.10). From (3.3) the density
of z times e^{qz} may be written using (2.10) as proportional to

$$e^{(\frac{1}{2}n+q)z} [\xi(z)]^{-\frac{1}{4}(\nu+1)} e^{-\frac{1}{2}n[(z-m)^2+\frac{\epsilon}{n}]^{\frac{1}{2}}} \zeta([n\xi(z)/4]^{\frac{1}{2}}) \quad . \tag{5.5}$$

Since $\zeta(\cdot)$ times $[\xi(z)]^{-\frac{1}{4}(\nu+1)}$ is a function of order of magnitude $[1/z^2]^{\frac{1}{4}(\nu+2)}$
as $z \to \pm\infty$, the convergence of (5.5) depends on the exponent

$$(\frac{1}{2}n + q)z - \frac{1}{2}n[(z-m)^2 + \frac{\epsilon}{n}]^{\frac{1}{2}} \quad . \tag{5.6}$$

For any $q > 0$, (5.6) approaches $+\infty$ as $z \to +\infty$. And so the function (5.2) is
divergent to $+\infty$ if $q > 0$. When $q < -n$, (5.2) diverges to $+\infty$ as $z \to -\infty$.

The proof just given yields useful information about partial moments of
$\tilde{\mu}_L$: if $q \geq -n$, (5.6) is bounded for all $z < z_o < +\infty$ and so the partial moment
of $\exp\{q\tilde{z}\}$ from $-\infty$ to z_o exists if $q \geq -n$. Consequently, in two action decision

problems with acts whose expected values are linear in $\mu_L = \exp\{z\}$, if $0 \leq b < +\infty$ then although $E(\tilde{\mu}_L) = +\infty$ and $E \max\{0, \tilde{\mu}_L - b\} = +\infty$, the expectation of the terminal loss function $\max\{0, b - \tilde{\mu}_L\}$ is bounded.

5.2 Prior to Posterior Analysis of the Lognormal Process

Suppose we observe a process generating mutually independent random variables $\tilde{x}_1,\ldots,\tilde{x}_i,\ldots$ identically distributed according to

$$\frac{1}{x_i} \, e^{-\frac{1}{2}hn(\log x_i - \mu)^2} \, \frac{h^{\frac{1}{2}}}{\sqrt{2\pi}} \, , \qquad \begin{array}{l} h,\ n > 0 \, , \\ -\infty < \mu < +\infty \, , \\ x_i > 0, \text{ all } i. \end{array} \tag{5.7}$$

If we observe a sample $x_1,\ldots,x_n$ generated according to (5.2) and neither μ nor h are known with certainty but are regarded as random variables and are assigned a Normal-gamma prior consisting of the product of (3.1) and (3.2) with parameter set $(m',\ n',\ \epsilon',\ \nu')$, then we may carry out prior to posterior analysis of $\tilde{\mu}$ and $\tilde{h}$ exactly as in Raiffa and Schlaifer [7] once we have defined the sufficient statistics n, $\nu = n-1$,

$$\bar{g} = \frac{1}{n} \, \Sigma \log x_i \quad \text{and} \quad \epsilon = \Sigma(\log x_i)^2 - n\bar{g}^2 \, . \tag{5.8}$$

Since the Normal-gamma prior (3.1) x (3.2) is a natural conjugate density, the posterior is of the same form with m', n', ϵ', and ν' replaced by

$$n'' = n' + n \, , \quad m'' = \frac{1}{n''} \, (n'm' + nm) \, , \quad \epsilon'' = \epsilon' + \epsilon + \frac{n'n}{n''} \, (\bar{g}-m'')^2 \tag{5.9}$$

and

$$\nu'' = \nu' + \nu + 1 = \nu' + n \, .$$

As stated earlier, however, in most economic decision problems, not $\tilde{\mu}$ and $\tilde{h}$ but the mean $\tilde{\mu}_L \equiv \exp(\tilde{\mu} + \frac{1}{2\tilde{h}})$ of the density of the $\tilde{x}_i$s is of central concern.

Posterior to a sample yielding the set of sufficient statistics $(n,\ \bar{g},\ \epsilon)$, the density of $\tilde{\mu}_L$ is just (5.1) with the set $(m',\ n',\ \epsilon',\ \nu')$ replaced by $(m'',\ n'',\ \epsilon'',\ \nu'')$, consequently the posterior density of $\tilde{\mu}_L$ is of the form (5.1) with this new parameter set.

5.3 Limiting Behavior of $\tilde{z}$ and $\tilde{\mu}_L$

Suppose the values of the parameters of the Lognormal sampling density (5.7) are μ and h. Since $\tilde{\mu}_L = \exp\{\tilde{z}\}$ has no finite positive moments when $\tilde{z}$ is assigned a prior density of the functional form of f_z it is natural to inquire whether or not the density of $\tilde{\mu}_L$ posterior to a sample of size n "squeezes down" about $\exp\{\mu + \frac{1}{2h}\}$.

To this end suppose a prior f_z' of form (3.3) is assigned to $\tilde{z}$. Consider the sequence $\{\tilde{z}_n\}$ of random variables, where $\tilde{z}_n$ denotes the random variable $\tilde{z}$ posterior to a sample of size n. Let $\bar{z}_n''$ and z_n'' denote the mean and variance of $\tilde{z}_n$ respectively. We show below that prior to observing a sample of size n,

I   The sequence $\{\tilde{z}_n\}$ converges in mean square to $\mu + \frac{1}{2h}$ in the sense that

$$\lim_{n \to +\infty} E(\tilde{z}_n | \mu, h) = \mu + \frac{1}{2h}$$

and

$$\lim_{n \to +\infty} \mathrm{Var}(\tilde{z}_n | \mu, h) = 0.$$

II   It follows from I that the sequence $\{\exp\{\tilde{z}_n\}\}$ converges in probability to $\exp\{\mu + \frac{1}{2h}\}$.

To show I remark that if we define $\upsilon = \varepsilon/n-1$, then prior to observing the outcome of a sample of size n,

$$E(\tilde{m}'' | \mu, h) = \frac{n'm'}{n''} + \frac{n}{n''}, \quad E(\tilde{\bar{g}} | \mu, h) = \frac{n'm'}{n''} + \frac{n\mu}{n''}$$

and

$$E(\tilde{\varepsilon}'' | \mu, h) = \varepsilon' + (n-1) E(\tilde{\upsilon} | \mu, h) + \frac{n'n}{n''} E((\tilde{\bar{g}} - m')^2 | \mu, h)$$

or since $E((\tilde{\bar{g}} - m')^2 | \mu, h) = E((\tilde{\bar{g}} - \mu)^2 | \mu, h) + (\mu - m')^2 = \frac{1}{hn} + (\mu - m')^2$,

$$E(\tilde{\varepsilon}'' | \mu, h) = \varepsilon' + \frac{n-1}{h} + \frac{n'}{hn''} + (\mu - m')^2.$$

Thus

$$E(\tilde{z}_n | \mu, h) = \frac{n'm'}{n''} + \frac{n\mu}{n''} + \frac{\varepsilon'}{2(\nu''-2)} + \frac{1}{2h}\left(\frac{n-1}{\nu''-2}\right) + \frac{1}{2h}\left(\frac{n'}{n''(\nu''-2)}\right)$$

$$+ \frac{(\mu-m')^2}{2(\nu''-2)}$$

and so $\lim_{n \to +\infty} E(\tilde{z}_n | \mu, h) = \mu + \frac{1}{2h}$. It can be shown in a similar fashion that $\lim_{n \to +\infty} V(\tilde{z}_n | \mu, h) = 0$.

To see that I also implies that the sequence $\{\exp\{\tilde{z}_n\}\}$ converges in probability to $\exp\{\mu + \frac{1}{2h}\}$, first apply the Chebyshev inequality to the sequence $\{\tilde{z}_n\}$. This shows that $\{\tilde{z}_n\}$ converges in probability to $z_0 \equiv \mu + \frac{1}{2h}$; i.e. for every $\delta > 0$, $\lim_{n \to +\infty} P(|\tilde{z}_n - z_0| < \delta) = 1$. Now consider for small $\phi > 0$ and fixed z_0

$$P(|e^{\tilde{z}_n} - e^{z_0}| < \phi) = P(|e^{\tilde{z}_n - z_0} - 1| < \phi e^{-z_0}) = P(1-\phi e^{-z_0} < e^{\tilde{z}_n - z_0} < 1+\phi e^{-z_0}).$$

For very small $\phi > 0$, $\log(1 \pm \phi e^{-z_0}) = \pm \phi e^{-z_0} \mp o(\phi e^{-z_0})$ so that if we define $\delta = \phi e^{-z_0} > 0$,

$$P(\log(1 - \phi e^{-z_0}) < \tilde{z} - z_0 < \log(1 + \phi e^{-z_0})) < P(-\delta < \tilde{z}-z_0 < +\delta) ,$$

whereupon the convergence in probability of $\{\tilde{z}_n\}$ to z_0 implies that of $\{\exp\{\tilde{z}_n\}\}$ to $\exp\{z_0\}$.

6. A Compound Process

Let $\tilde{r}$ be a Poisson random variable with probability mass function

$$\frac{1}{\Gamma(r+1)} \; e^{-(t/h)} \; (t/h)^r \quad , \quad r = 0,1,2,\ldots, \tag{6.1}$$

$$t, \, h > 0 \quad .$$

Then we may informally interpret the exponential density

$$\left(\frac{1}{h}\right) \; e^{-(t/h)} \qquad , \quad \begin{array}{l} t > 0 \\ h > 0 \end{array} \qquad , \tag{6.2}$$

as the probability density of waiting time t between "events"; i.e. we let $\tilde{t}_1,\ldots,\tilde{t}_1,\ldots$ be a set of mutually independent random variables with common density (6.2), and define $P(\tilde{r} = r | t, h)$ as the probability of the event: the number $\tilde{r}$ such that $\tilde{t}_1 + \tilde{t}_2 + \ldots + \tilde{t}_r \le t$ and $\tilde{t}_1 + \tilde{t}_2 + \ldots + \tilde{t}_{\tilde{r}+1} > t$ equals r. Then $P(\tilde{r} = r | t, h)$ is (6.1). Call the process generating $\tilde{r}$ "Process I", and remark that given $\tilde{r} = r$, the density of $\tilde{t}$, the waiting time to the rth event is

$$\tilde{t} \; : \; \frac{(1/h)^r}{\Gamma(r)} \; e^{-t/h} \; (t/h)^{r-1}$$

In a similar fashion, consider another Poisson process in which the waiting time to the sth event is $\tilde{h}$, and $\tilde{h}$ has density

$$\frac{\varepsilon}{\Gamma(s)} \; e^{-h\varepsilon} \; (\varepsilon h)^{s-1} \qquad \begin{array}{l} s > 0 \; , \\ h > 0 \; , \end{array} \tag{6.3}$$

call this "Process II".

Now suppose the mechanism generating values of $\tilde{t}$ and $\tilde{r}$ acts in two steps: first, a value of the mean waiting time $1/\tilde{h}$ between events for process I is generated according to (6.3); then a value of $\tilde{r} | h$ and a value of $\tilde{t} | h$ are generated according to (6.1) and (6.2) respectively.

Formula (2.7) immediately yields the probability that $\tilde{r} = r$ uncondi-tional as regards $\tilde{h}$ and the probability density of $\tilde{t}_r$ unconditional as regards h:

$$\tilde{r} \sim \mathbf{1} \frac{2(\epsilon t)^{\frac{1}{2}(s+r)}}{\Gamma(r+1)\Gamma(s)} \; K_{|s-r|} \; (2\sqrt{t\epsilon}) \quad , \quad r = 0,1,2,\ldots \tag{6.4}$$

and

$$\tilde{t} \sim [\frac{2\epsilon^{\frac{1}{2}(s+r)}}{\Gamma(r)\,\Gamma(s)}] \; t^{\frac{1}{2}(r+s)-1} \; K_{|s-r|} \; (2\sqrt{t\epsilon}) \quad , \quad t > 0 \; . \tag{6.5}$$

(Notice that in (6.4) t, ϵ, and s are fixed and r is the argument of the function, while in (6.5) ϵ, s, and r are fixed and t is the argument.)

Switching our point of view, we have as immediate consequences of the fact that (6.4) and (6.5) are probability mass and density functions respectively, the interesting identities:

$$\sum_{r=0}^{\infty} \frac{t^{\frac{1}{2}r}}{\Gamma(r+1)} \; K_{|s-r|} \; (2\sqrt{t\epsilon}) = \frac{\Gamma(s)}{2(t\epsilon)^{\frac{1}{2}s}} \tag{6.6}$$

and

$$\int_0^{\infty} x^{\frac{1}{2}(r+s)-1} \; K_{|s-r|} \; (2\sqrt{x\epsilon}) \; dx = \frac{\Gamma(r)\Gamma(s)}{2\epsilon^{\frac{1}{2}(s+r)}} \quad . \tag{6.7}$$

The latter identity is essentially formula 11.4.22 of the <u>Handbook of Mathematical Functions</u> [4] (p. 486). By comparing right hand sides we obtain a further identity between the integral (6.7) and the infinite sum (6.6). Setting t = 1 in (6.6),

$$\int_0^{\infty} x^{\frac{1}{2}(r+a)-1} \; K_{|s-r|} \; (2\sqrt{x\epsilon}) \; dx = \frac{2\Gamma(r)}{\epsilon^{\frac{1}{2}r}} \sum_{j=0}^{\infty} \frac{1}{\Gamma(j+1)} K_{|s-j|} \; (2\sqrt{\epsilon}) \; . \tag{6.8}$$

While (6.6) and (6.8) do not appear in the explicit form shown here in Watson [ ] or in [ ], they are undoubtedly classical relations (?) The ease of their proof via probabilistic interpretation is worth explicit statement.

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