WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

TWO-STAGE-PROGRAMMING UNDER RISK WITH DISCRETE OR DISCRETE APPROXIMATED CONTINUOUS DISTRIBUTION FUNCTIONS

by

Michael Werner

October 1974 WP 743-74

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139

TWO-STAGE-PROGRAMMING UNDER RISK WITH
DISCRETE OR DISCRETE APPROXIMATED
CONTINUOUS DISTRIBUTION FUNCTIONS

by

Michael Werner

October 1974 WP 743-74

ABSTRACT

To solve the complete problem of two-stage-programming under risk involving random variables with discrete distribution functions only, this paper presents an efficient algorithm by using the Dantzig/Wolfe decomposition principle. If the recourse matrix is a positive or negative identy matrix applying this algorithm an efficient approach is derived for this structure of the problem. By using this approach, under the assumptions

i) only the second stage matrix is a continuous random variable,

ii) either the expectation of the random matrix is known or there is a good estimate at least,

iii) every row of the random matrix is a transformation of only one continuous random variable with known range,

it is shown, that the solution of the two-stage-programming problem under risk with an identy as a recourse matrix can be approximated arbitrary exactly by generating a discrete random matrix with unknown distribution function.

1. Problem Formulation

The problem of two-stage programming under risk only with random variables with discrete distribution functions is

$$
\begin{aligned}
& c' v + \sum_{q=1}^{Q} (P_q \cdot \overset{o}{z}_q(v,\zeta^q,\gamma^q)) \rightarrow \min \\
& A \; v \geq b \qquad\qquad (1)\\
& \quad v \geq 0 \qquad .
\end{aligned}
$$

A is an I·H matrix $[a_{ih}]$. c, v are vectors with H and b is a vector with I components. v is a decision vector. $\overset{o}{z}_q(v,\zeta^q,\gamma^q)$ is the optimal value function of the second stage (recourse) program

$$
\begin{aligned}
\overset{o}{z}_q(v,\zeta^q,\gamma^q) := s' \, y^q & \rightarrow \min \\
M \; y^q & = \zeta^q - \gamma^q \cdot v \\
y^q & \geq 0 \qquad .
\end{aligned}
$$

M is a G·L matrix $[m_{gl}]$ and $\gamma=[\gamma_{gh}]$ is a stochastic G·H matrix. s, y^q are vectors with L and ζ^q is a vector with G components. y^q denotes a decision vector . $[\gamma|\zeta]$ is a G·(H+1) dimensional random variable with discrete distribution function, the Q realizations $[\gamma^q|\zeta^q]$ with the probabulities $P_q>0$ and $\Sigma q \; (P_q) = 1$.

For the complete problem is $M=[E|-E]$. E means a G·G identy matrix (L=2G) and the second stage matrix becomes

$$
\begin{aligned}
\overset{o}{z}_q(v,\zeta^q,\gamma^q) := s_1' \, y_1^q + s_2' \, y_2^q & \rightarrow \min \\
E \, y_1^q - E \, y_2^q & = \zeta^q - \gamma^q \cdot v \qquad\qquad (2)\\
y_1^q , \quad y_2^q & \geq 0 \qquad .
\end{aligned}
$$

If $(\gamma_{gk}',\zeta_{gk})$ denotes the different realizations of the random vector (γ_g',ζ_g) with $k=1,2,...,K_g$ (γ_g' means the row vector g of the stochastic matrix γ) and the pro-

babilities $P_{gk}>0$, $\Sigma k\ (P_{gk}) = 1$, then

$$P_{gk} = \Sigma q\ (J^q_{gk} \cdot P_q) \tag{3}$$

with

$$J^q_{gk} = \begin{cases} 1 \text{ if } (\gamma'_{gk}, \zeta_{gk}) = (\gamma^{q'}_g, \zeta^q_g) \\ 0 \text{ else} \end{cases} \tag{4}$$

By using (3)-(4) the complete problem (1)-(2) is equivalent to [El Agizy]

$$\begin{aligned} & c' \ v + \Sigma g \Sigma k \ (P_{gk} \cdot \overset{o}{z}_{gk}(v, \zeta_{gk}, \gamma_{gk})) \rightarrow \min \\ & A \ \ v \geq b \\ & \quad \ \ v \geq 0 \end{aligned} \tag{5}$$

$\overset{o}{z}_{gk}(v, \zeta_{gk}, \gamma_{gk})$ means the optimal value function ot the second stage program

$$\begin{aligned} \overset{o}{z}_{gk}(v, \zeta_{gk}, \gamma_{gk}) := s_{1g} \cdot y_{1gk} &+ s_{2g} \cdot y_{2gk} \rightarrow \min \\ y_{1gk} &- y_{2gk} = \zeta_{gk} - \gamma'_{gk} \cdot v \\ y_{1gk} &, \ y_{2gk} \geq 0 \end{aligned} \tag{6}$$

2. An approach for solving the complete two-stage-programming problem

(5)-(6) is equivalent to [Dantzig/Madansky]

$$\begin{aligned} c' \ v + \bar{s}'_1 \ y_1 + \bar{s}'_2 \ y_2 &\rightarrow \min \\ A \ \ v \qquad\qquad\qquad &\geq b \\ \bar{\gamma} \ \ v + \ E \ y_1 - \ E \ y_2 &= \bar{\zeta} \\ v \ , \quad y_1 \ , \quad y_2 &\geq 0 \end{aligned} \tag{7}$$

with

$$\bar{s}_1' = (s_{11}\cdot P_{11},\ldots,s_{11}\cdot P_{1K_1},\ldots,s_{1G}\cdot P_{G1},\ldots,s_{1G}\cdot P_{GK_G}) ,$$
$$\bar{s}_2' = (s_{21}\cdot P_{11},\ldots,s_{21}\cdot P_{1K_1},\ldots,s_{2G}\cdot P_{G1},\ldots,s_{2G}\cdot P_{GK_G}) ,$$
$$y_1' = (y_{111},\ldots,y_{11K_1},\ldots,y_{1G1},\ldots,y_{1GK_G}) ,$$
$$y_2' = (y_{211},\ldots,y_{21K_1},\ldots,y_{2G1},\ldots,y_{2GK_G}) ,$$
$$\bar{\zeta}' = (\zeta_{11},\ldots,\zeta_{1K_1},\ldots,\zeta_{G1},\ldots,\zeta_{GK_G}) ,$$

$$\bar{\gamma} = \begin{bmatrix} \gamma_{11}' \\ \gamma_{12}' \\ \vdots \\ \gamma_{1K_1}' \\ \gamma_{21}' \\ \gamma_{22}' \\ \vdots \\ \gamma_{2K_2}' \\ \vdots \\ \gamma_{G1}' \\ \gamma_{G2}' \\ \vdots \\ \gamma_{GK_G}' \end{bmatrix} .$$

The dual of (7) is

$$\begin{aligned} b'\,w + \bar{\zeta}'\,u_1 - \bar{\zeta}'\,u_2 &\rightarrow \max \\ A'\,w + \bar{\gamma}'\,u_1 - \bar{\gamma}'\,u_2 &\leq c \\ E\,u_1 - E\,u_2 &\leq \bar{s}_1 \\ E\,u_1 - E\,u_2 &\geq -\bar{s}_2 \\ w,\quad u_1,\quad u_2 &\geq 0 . \end{aligned} \tag{8}$$

w, u_1 and u_2 mean the vectors of dual variables.

By using the Dantzig/Wolfe decomposion principle (8) written as the master program is

$$\begin{aligned} b' w + (\bar\zeta', -\bar\zeta')\cdot R\cdot x + (\bar\zeta', -\bar\zeta')\cdot C\cdot t &\rightarrow \max \\ A' w + [\bar\gamma', -\bar\gamma']\cdot R\cdot x + [\bar\gamma', -\bar\gamma']\cdot C\cdot t &\leq c \\ e'x &= 1 \\ w, \quad x, \quad t &\geq 0 \end{aligned} \tag{9}$$

with the subprogram

$$\begin{aligned} (\bar\zeta' + v'\bar\gamma')\cdot u_1 - (\bar\zeta' + v'\bar\gamma')\cdot u_2 &\rightarrow \max \\ E\, u_1 - E\, u_2 &\leq \bar s_1 \\ E\, u_1 - E\, u_2 &\geq -\bar s_2 \\ u_1, \quad u_2 &\geq 0 . \end{aligned} \tag{10}$$

The columns of R (C) define optimal basic (homogenous) solutions of the subprogram (10). v means the vector of simplexmultipliers (negative dual variables) for the first H constraints of (9).

If f denotes the index of the Σg (K_g) -vectors $\bar\zeta$, $\bar s_1$, $\bar s_2$, u_1, u_2 and $\bar\gamma_f$ for the f-th columnvector of $\bar\gamma$, then (10) defines Σg (K_g) isolated problems

$$\begin{aligned} (\bar\zeta_f + v'\bar\gamma_f)\cdot u_{1f} - (\bar\zeta_f + v'\bar\gamma_f)\cdot u_{2f} &\rightarrow \max \\ u_{1f} - u_{2f} &\leq \bar s_{1f} \\ u_{1f} - u_{2f} &\geq -\bar s_{2f} \\ u_{1f} - u_{2f} &\geq 0 . \end{aligned} \tag{11}$$

(11) and then (9)-(10) have a feasible solution only if $\bar s_{1f} \geq -\bar s_{2f}$ $\forall$ f. For $\bar s_{1f} \geq -\bar s_{2f}$ there always exists an optimal basic solution for (11) and then for (10) with

$$\begin{aligned} u_{1f} &:= \max(\bar s_{1f}, 0), \quad u_{2f} := \max(-\bar s_{1f}, 0) \quad \text{if } (\bar\zeta_f + v'\bar\gamma_f) \geq 0 \\ u_{1f} &:= \max(-\bar s_{2f}, 0), \quad u_{2f} := \max(\bar s_{2f}, 0) \quad \text{if } (\bar\zeta_f + v'\bar\gamma_f) < 0 \end{aligned} \tag{12}$$

because of the identy of the coefficients in the objective function for u_{1f}, u_{2f}.

(9) can be reduced to [Werner]

$$\begin{aligned} b' w + (\bar{\zeta}', -\bar{\zeta}')\cdot R\cdot x &\rightarrow \max \\ A' w + [\bar{\gamma}', -\bar{\gamma}']\cdot R\cdot x &\leq c \\ e' x &= 1 \\ w, \quad x &\geq 0 . \end{aligned} \tag{13}$$

The optimal basic solution $(\overset{j}{u}_1, \overset{j}{u}_2)$ of (10), which defines a column of the matrix R, results immediately from (12) with the vector $\overset{j}{v}$ of simplexmultipliers in the j-th basic solution of (13).

Defining $\overset{j}{v}_{H+1}$ for simplexmultiplier of the constraint H+1 in (13) the optimal basic solution $(\overset{j}{u}_1, \overset{j}{u}_2)$ in the j-th iteration is to introduce in the basic solution j+1 of (13) if [Dantzig/Wolfe]

$$(\Sigma f[(\bar{\zeta}_f + \overset{j}{v}'\bar{\gamma}_f)\cdot\overset{j}{u}_{1f} - (\bar{\zeta}_f + \overset{j}{v}'\bar{\gamma}_f)\cdot\overset{j}{u}_{2f}] + \overset{j}{v}_{H+1}) > 0 \tag{14}$$

else the j-th basic solution of (13) is optimal. The negative vector $\overset{j}{v}$ of simplexmultipliers $(-\overset{j}{v})$ defines the solution vector of the primal problem (7).

3. An approach for the two-stage problem with M=E or M=-E

With M=E we obtain for (7)

$$\begin{aligned} c' v + \bar{s}_1' y_1 &\rightarrow \min \\ A v &\geq b \\ \bar{\gamma} v + E y_1 &= \bar{\zeta} \\ v, \quad y_1 &\geq 0 \end{aligned} \tag{15}$$

with the dual

$$\begin{aligned} b' w + \bar{\zeta}' u_1 - \bar{\zeta}' u_2 &\rightarrow \max \\ A' w + \bar{\gamma}' u_1 - \bar{\gamma}' u_2 &\leq c \\ E u_1 - E u_2 &\leq \bar{s}_1 \\ w , \quad u_1 , \quad u_2 &\geq 0 \end{aligned} \tag{16}$$

and the subprogram for (9) is

$$\begin{aligned} (\bar{\zeta}' + v'\bar{\gamma}')\cdot u_1 - (\bar{\zeta}' + v'\bar{\gamma}')\cdot u_2 &\rightarrow \max \\ E u_1 - E u_2 &\leq \bar{s}_1 \\ u_1 , \quad u_2 &\geq 0 \ . \end{aligned} \tag{17}$$

Analogous to (10) problem (17) defines $\Sigma g \ (K_g)$ isolated problems

$$\begin{aligned} (\bar{\zeta}_f + v'\bar{\gamma}_f)\cdot u_{1f} - (\bar{\zeta}_f + v'\bar{\gamma}_f)\cdot u_{2f} &\rightarrow \max \\ u_{1f} - u_{2f} &\leq \bar{s}_{1f} \\ u_{1f} , \quad u_{2f} &\geq 0 \ . \end{aligned} \tag{18}$$

According to the fundamental theorem of linear programming, (18) has an optimal basic solution only then, if the corresponding dual has a feasible solution, that is

$$(\bar{\zeta}_f + v'\bar{\gamma}_f) \geq 0 \ . \tag{19}$$

The optimal basic solution of (18) therefore is given by

$$u_{1f} := \max(\bar{s}_{1f}, 0) \quad , \quad u_{2f} := \max(-\bar{s}_{1f}, 0) \ . \tag{20}$$

If for at least one of the problems (18)

$$(\bar{\zeta}_f + v'\bar{\gamma}_f) < 0$$

is valid, then there exists no optimal basic solution for (17). The homogenous solution of (17) results from

$$\begin{aligned}(\bar{\zeta}' + v'\bar{\gamma}')\cdot u_1 - (\bar{\zeta}' + v'\bar{\gamma}')\cdot u_2 &\rightarrow \max \\ E\,u_1 - \qquad E\,u_2 &\leq 0 \\ e'u_1 + \qquad e'u_2 &= 1 \\ u_1\,, \qquad u_2 &\geq 0\end{aligned}$$

as

$$u_{1f} = 0 \quad \forall\, f\,,$$

$$u_{2f} = \begin{cases} 1 & \text{for} \quad (\bar{\zeta}_{\bar{f}} + v'\bar{\gamma}_{\bar{f}}) := \min_f (\bar{\zeta}_f + v'\bar{\gamma}_f) \\ 0 & \text{else} \end{cases} \quad .$$

The matric C in (9) is

$$C = \left[\frac{0}{E}\right]$$

with a $[\Sigma g(K_g)]\cdot[\Sigma g(K_g)]$ zero- and identymatrix.

Moreover there exist one and only one optimal basic solution for (17) if (19) is valid $\forall$ f. R is a column vector with the components defined by (20).

(9) can be reduced to

$$\begin{aligned} b'\,w - \bar{\zeta}'\,t + D1 &\rightarrow \max \\ A'\,w - \bar{\gamma}'\,t \quad &\leq c - D2 \\ w\,, \quad t \quad &\geq 0 \end{aligned} \tag{21}$$

with the constant terms

$$\begin{aligned} D1 &= (\bar{\zeta}', -\bar{\zeta}')\cdot R = \bar{\zeta}'\,\bar{s}_1\,, \\ D2 &= [\bar{\gamma}' | -\bar{\gamma}']\cdot R = \bar{\gamma}'\,\bar{s}_1\,. \end{aligned}$$

Solving (21) with the revised simplexmethod (because of the large number of columns of the matrix $\bar{\gamma}'$) the vector of simplexmultipliers defines the negative solution

vector -v of the primal problem (15).

With M=-E we obtain for (21)

$$\begin{array}{llll} b' w + \bar{\zeta}' t + D1 & \rightarrow & \max \\ A \; w + \bar{\gamma}' t & \leq & c - D2 \\ \quad w , \quad t & \geq & 0 \end{array}$$

with the components

$$u_{1f} := \max(-\bar{s}_{2f},0) \quad , \quad u_{2f} := \max(\bar{s}_{2f},0)$$

of the columnvector R and

$$C = \begin{bmatrix} E \\ \hline 0 \end{bmatrix} .$$

4. Discrete approximation of continuous random variables for the two-stage problem with M=E or M=-E

We consider the two-stage-programming problem under risk

$$\begin{array}{ll} c' v + \int \overset{o}{z}(v,\gamma) \, dF(\gamma) \rightarrow \min & \\ A \quad v \geq b & \\ \quad\; v \geq 0 & \end{array} \tag{22}$$

with

$$\begin{array}{llll} \overset{o}{z}(v,\gamma) := s' y & \rightarrow & \min \\ \qquad E \quad y & = & d - \gamma v \\ \qquad\quad\; y & \geq & 0 & . \end{array} \tag{23}$$

$\gamma = [\gamma_{gh}]$ is a (G·H)-dimensional, continuous random variable with the distribution function $F(\gamma)$.

By approximating the continuous by a discrete random variable we obtain analogous to (15)

$$
\begin{aligned}
& c' v + \sum_{g=1}^{G} s_g \cdot \Big(\sum_{k=1}^{K_g} P_{gk}\, y_{gk} \Big) \to \min \\
& A\, v \geq b \\
& \left.\begin{aligned} & \gamma_{gk}' v + y_{gk} = d_g \\ & v,\ y_{gk} \geq 0 \end{aligned}\right\} g = 1,2,\dots,G;\ k=1,2,\dots,K_g
\end{aligned}
\tag{24}
$$

with its dual

$$
\begin{aligned}
& b' w + \sum_{g=1}^{G} d_g \cdot \Big(\sum_{k=1}^{K_g} u_{gk} \Big) \to \max \\
& A' w + \sum_{g=1}^{G} \sum_{k=1}^{K_g} (\gamma_{gk}\, u_{gk}) \leq c \\
& u_{gk} \leq s_g \cdot P_{gk} \ ;\ g=1,2,\dots,G;\ k=1,2,\dots,K_g \\
& w \geq 0 \ .
\end{aligned}
\tag{25}
$$

For simplifying the transformation of (25) analogous to (15)/(21) we define

$$
t_{gk} := s_g \cdot P_{gk} - u_{gk}
$$

and (25) becomes [see (21)]

$$
\begin{aligned}
& b' w - \sum_{g=1}^{G} d_g \cdot \Big(\sum_{k=1}^{K_g} t_{gk} \Big) + D1 \to \max \\
& A' w - \sum_{g=1}^{G} \sum_{k=1}^{K_g} (\gamma_{gk}\, t_{gk}) \leq c - D2 \\
& w,\ t_{gk} \geq 0 \ ;\ g=1,2,\dots,G;\ k=1,2,\dots,K_g
\end{aligned}
\tag{26}
$$

with the constant terms

$$
\begin{aligned}
& D1 = \sum_{g=1}^{G} d_g \cdot s_g \\
& D2 = \sum_{g=1}^{G} s_g \cdot \Big(\sum_{k=1}^{K_g} \gamma_{gk}\, P_{gk} \Big) = \sum_{g=1}^{G} s_g \cdot \varepsilon(\gamma_g) \ .
\end{aligned}
$$

$\varepsilon(\gamma_g)$ means the transposed known expectation of the row vector γ_g' [see assumption ii)].

There are no random variables in the objective function of (26).Only the number of variables t_{gk} and the related vectors of outcomes γ_{gk} depends on the quality of approximation of the continuous by a discrete variable. The larger Q and then K_g $\forall$ g the better the approximation of (22) by (24) and its related dual (26).The best approximation would be $Q \to +\infty$, but the number of variables t_{gk} and related vectors γ_{gk} would become too large to handle the problem. To avoid the explicit enumeration of Q columns γ_{gk} with $Q \to +\infty$, we start with some columns γ_{gk} - for example $K_g=1$ $\forall$ g - and solve (26).The solution r=1 of (26)supplies the vector of simplexmultipliers α_r. Then we have to check up whether it is possible to generate any γ_{g,K_g+1} with

$$- d_g - \alpha_r' \gamma_{g,K_g+1} > 0 \tag{27}$$

which is not yet considered in the solution r=1 of (26):

By assumption iii)the random vector γ_g is a transformation $\gamma_g(\zeta_g)$ of only one random variable (for example ζ_g) with known range $[\zeta_{g1},\zeta_{gu}]$. So we have to solve the G problems

$$\begin{aligned} \overset{or}{c}_g := d_g + \alpha_r' \cdot \gamma_g(\zeta_g) \;\to\; \min \\ \zeta_{gl} \leq \zeta_g \leq \zeta_{gu} \end{aligned} \tag{28}$$

which means the minimizing of a linear or nonlinear function defined over a given range.

We define

$$\gamma_{g,K_g+1} := \gamma_g(\overset{o}{\zeta}_g)$$

with

$$\overset{or}{c}_g = d_g + \alpha_r' \cdot \gamma_g(\overset{o}{\zeta}_g)$$

and determine

$$\min_g \{ \overset{or}{c}_g \mid g=1,2,\ldots,G \} = \overset{or}{c}_{\bar{g}} \quad . \tag{29}$$

If $c^{or}_{\bar{g}} < 0$, then

$$K_{\bar{g}} := K_{\bar{g}} + 1$$

and the vector $\gamma_{\bar{g},K_{\bar{g}}}$ has to be considered in (26).We obtain the solution r:=r+1 with the vector of simplexmultipliers α_r and start again to solve (28) $\forall$ g.

If $c^{or}_{\bar{g}} \geq 0$ there is no vector γ_{g,K_g+1} which is able to improve the last solution of (26).The last solution of (26)is optimal and feasible for any outcome of the continuous random variable γ.

By using the dual variables of (26) we can generate the solution $\overset{o}{v}$ of (24) and (22) which is also feasible for any outcome of the continuous random variable γ.

If $\gamma_g(\zeta_g)$ means a linear function of ζ_g, then only the two points ζ_{gl}, ζ_{gu} with the related outcomes of γ_g have to be considered. However, if $\gamma_g(\zeta_g)$ means a nonlinear function the number of iterations and then the number of columns that are to be introduced in (26)may become very large. But applying the above described algorithm only those columns are generated which improve the value of the objective function. If r becomes too large, the procedure may be stopped by an approximation criterion which has to be defined. However, in this case the solution of (26) [(24)] is not optimal [feasible] for any not yet considered outcome of the random variable γ.

If there is only an estimate of the lower and upper bound of the expectation $\varepsilon(\gamma_g)$ the variation of the optimal solution of (26) and then (24) can be determined by using a sensitivity analysis or the parametric linear programming.

References:

Dantzig,G.B. and A.Madansky: On the solution of two-stage programs under uncertainty, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability 1961, pp.165-176

Dantzig,G.B. and P.Wolfe: Decomposition principle for linear programs, Operations Research, Vol.8(60), pp.101-111

El Agizy,M.: Two-stage-programming under uncertainty with discrete distribution function, Operations Research, Vol.15(67), pp.55-70

Werner,M.: Ein Loesungsansatz fuer ein spezielles zweistufiges stochastisches Optimierungsproblem, Zeitschrift fuer Operations Research, Vol.17(73), pp.119-128