PROPERTIES OF GUIDED WAVES ON INHOMOGENEOUS CYLINDRICAL STRUCTURES

R. B. ADLER

TECHNICAL REPORT NO. 102

MAY 27, 1949

RESEARCH LABORATORY OF ELECTRONICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

The research reported in this document was made possible through support extended the Massachusetts Institute of Technology, Research Laboratory of Electronics, jointly by the Army Signal Corps, the Navy Department (Office of Naval Research) and the Air Force (Air Materiel Command), under Signal Corps Contract No. W36-039-sc-32037, Project No. 102B; Department of the Army Project No. 3-99-10-022.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Research Laboratory of Electronics

Technical Report No. 102 May 27, 1949

PROPERTIES OF GUIDED WAVES ON INHOMOGENEOUS CYLINDRICAL STRUCTURES*

R. B. Adler

ABSTRACT

An analysis is given of some basic properties of exponential modes on passive cylindrical structures, in which ϵ, μ and σ vary over the cross section and the bounding surface is not completely opaque. Major, but not exclusive, consideration is directed to lossless structures. Each mode is generally a TE-TM mixture. Conventional orthogonality conditions do not all remain valid, but some are retained. Conditions are discussed under which the instantaneous-, vector-, or double-frequency power flows along the structure are additive among the modes. Stored and dissipated energies generally are not additive. It is shown that the propagation constant for modes on a lossless structure cannot be complex; when the lossless structure has no confining boundary (like a dielectric rod), the modes cannot even possess a true cutoff. Consideration is given to the relation between the direction of real power flow and that of the phase and group velocities. The frequency dependence of the field distribution is also interpreted. Examples are included in the Appendices.

---

* This report is identical with a thesis of the same title submitted by the author in partial fulfillment of the requirements for the degree of Doctor of Science in Electrical Engineering at the Massachusetts Institute of Technology.

TABLE OF CONTENTS

I.	Introduction	1
II.	Formulation and Preliminary Analysis of Problem	6
	2.1 Coordinates and Notation	6
	2.2 Reduction of Maxwell Equations to Cylindrical Form	8
	2.3 Detailed Formulation of the Problem	13
	2.4 Boundary Conditions	15
III.	Basic Properties of the Modes	21
	3.1 TE-TM Properties of the Modes	22
	3.2 Incident and Reflected Waves	25
	3.3 Orthogonality Conditions	28
	3.4 Power and Energy Consequences of the Orthogonality Conditions	33
	3.5 Characteristics of the Propagation Constant γ	41
IV.	Physical Characteristics of the Modes	47
	4.1 Mode Properties at a Single Frequency	48
	4.2 Frequency Behavior of the Propagation Constant	55
	4.3 Frequency Behavior of the Transverse Field Distribution	64
	4.4 Polarization of the Fields	70
V.	"Open-Boundary" Structures	77
	5.1 Coordinates and Boundary Conditions	78
	5.2 Characteristics of the Propagation Constant	78
	5.3 Physical Interpretation of the Free Modes	85
VI.	Conclusion	90
Appendices		91
	A. Circular Guide with Reactive Wall	91
	B. Rectangular Guide Partially Filled with Dielectric	104
	C. "TE-TM" Waves on a Rectangular Waveguide	114
	D. Dielectric Rod in Free Space, Driven by Axial Dipole	117
Acknowledgment		136
References		137

PROPERTIES OF GUIDED WAVES ON INHOMOGENEOUS CYLINDRICAL STRUCTURES

I. INTRODUCTION

Among the simplest of common waveguide structures are those which consist of an electromagnetically opaque tube, filled uniformly with a substantially dissipationless dielectric material. The wall, or tube, is usually fashioned from a metal of virtually infinite conductivity, and the cross-sectional shape of the enclosed space may take many forms, of which only a few are both structurally practical and analytically simple.

In any case, however, the first step often taken, in order to develop an understanding of the behavior of the electromagnetic fields which may be propagated along the tube, is to consider those waves which can exist in the absence of sources within the guide. More precisely, attention is directed toward the case of an infinite tube whose longitudinal axis is designated as the z-axis; a solution is then found for fields having harmonic time dependence $e^{j\omega t}$, and exponential behavior in the z-direction ($e^{-\gamma z}$). Although there are no sources within any finite length of the structure, these solutions or "free modes" may sometimes most conveniently be thought of in the steady state as being produced by sources located at $z = \pm\infty$.

Under the physical conditions described above, the modes in question have interesting general properties, with which the reader is assumed to be familiar (1); such properties become useful not only from the point of view of understanding the basic phenomenae involved, but for the calculation of more complicated problems involving the junction of dissimilar guides, or the effects of transverse discontinuities in a given structure.

Within the past few years a considerable number of practical problems have arisen which require an understanding of the propagation of electromagnetic waves along cylindrical structures of more complicated varieties than those previously mentioned. The extent to which these structures differ from ordinary waveguides has not been entirely clear. Examples of the problems in question are: the waveguide phase shifter, comprising a section of ordinary guide partially filled with lossless dielectric; various delay lines employed for the purpose of obtaining slow velocities of wave propagation, and comprising a loaded guide or helical wire; the polyrod antenna, involving a dielectric rod as waveguide and antenna; velocity-modulated tubes, which make use of a drift space either partially or completely filled with an electron beam; and the traveling-wave tube, employing a loaded-guide or helical-wire delay line, surrounding an electron beam.

It is common among these physical situations that the "modes" encountered therein no longer possess some of the usual waveguide mode properties. In particular, the modes found by Hansen (2) for the delay line with a "reactive wall" are not orthogonal in the manner characteristic of standard waveguide modes. A similar comment applies to the modes obtained by Pincherle (3) in the waveguide partially filled with dielectric. Hahn (4,5) has employed a set of modes, applicable to the normal waveguide containing an electron beam, which again fail to be orthogonal in the conventional sense; he has suggested (5), however, that the conservation of longitudinal time-average power flow along a lossless guide may be used to furnish an orthogonality condition in dissipationless structures. Pierce (6) and Chu (7) have encountered modes for the helix type of traveling-wave tube; these modes also lack the conventional orthogonality property.

In another direction, so-called "open-boundary" problems have been attacked on the mode basis. Examples of these are the treatments of the dielectric rod given by many authors (8,9,10,11,12), as well as various approximate studies of the helical-wire guide(13). The difficulty in such open-boundary problems is that at any radian frequency $\omega < \infty$ there may be only a finite number of the discrete free modes which can exist on a given structure. Whether or not an orthogonality condition of some sort exists between these discrete free modes, the fact that they are finite in number at any particular frequency evidently means that they are not a complete set. It is therefore indicated that a consideration of only such free modes leaves much to be desired from the point of view of acquiring an understanding about the general electromagnetic properties of open-boundary structures.

In most of the earlier engineering investigations of the more complicated problems outlined above, it has nevertheless been implied that the modes encountered therein have essentially the same important properties as those uncovered in the solution to the simpler waveguide problems mentioned at the outset of this discussion. Yet it has already been made clear that there are some significant differences between the mode properties in the two categories; for simplicity, the conventional waveguide problems with an opaque wall may be classed as "homogeneous problems", and all the rest (broadly) as "inhomogeneous problems".

It is not practical in this work to cover quite as wide a range of inhomogeneous problems as has thus far been suggested. A convenient division can be made, however, into active systems (with an electron beam present) and passive systems (with a dielectric medium present, which is, at most, dissipative). The discussion will henceforth be limited to passive cylindrical structures. They will be termed "inhomogeneous" if either the bounding wall is not perfectly opaque, or the dielectric medium is not distributed uniformly in the cross section, or both.

Even with this further subdivision of the inhomogeneous problems, it will be found, upon the more detailed examination in the sequel, that if any of the mode properties are substantially the same as those for the homogeneous problems, the reasons therefor are likely to be misunderstood at first glance. Moreover, there are also some significant differences; consequently it is deserving of further consideration to discover the sources of these similarities and differences, in order to enhance and extend the engineer's understanding of these more complicated problems.

In the ensuing work, therefore, an attempt will be made to point out and analyze the most important physical properties of exponential modes on inhomogeneous cylindrical structures, in which the material constants of the enclosed (passive) medium may vary in the transverse plane, and in which the bounding surface is not absolutely opaque. The general direction of the investigation will be to determine which of the most significant properties of the familiar modes for homogeneous structures can be carried over into these passive inhomogeneous problems. The analysis in the main body of the work can be broadly divided into two major headings, the first of which deals primarily with "closed-boundary" structures (Parts II-IV inclusive), while the second (Part IV) considers "open-boundary" structures. Although the admittance boundary conditions (Section 2.4) are intermediate between opaque boundaries and open boundaries (Section 5.1), it seemed advisable to include problems involving an admittance wall under the "closed" heading. The Appendices are illustrative problems, of which the first three (Appendices A,B and C) amplify and verify matters discussed under the closed-boundary heading, while Appendix D treats a typical open-boundary problem.

After a preliminary reduction of the Maxwell equations to cylindrical form, and a discussion of the dyadic-admittance boundary conditions, (Part II), the mode properties on closed structures are discussed in Parts III and IV. While Part III is called "Basic Properties of the Modes" and Part IV "Physical Characteristics" thereof, the dividing line between them is not sharp. It was desirable to make the separation primarily for purposes of logical order.

Part III takes up the need for combined "TE-TM" modes in the general inhomogeneous structure (Section 3.1), followed by an indication of the $\pm z$-symmetry in the entire problem, which leads to the presence of "incident" and "reflected" waves for each mode (Section 3.2). These considerations lead to the main development of the orthogonality conditions (Section 3.3), which is then followed by a discussion of the various consequences thereof in terms of power flow and stored energy when two modes are present on the structure simultaneously (Section 3.4). The final section of Part III reviews briefly some pertinent properties of the propagation constant γ on homogeneous structures, pointing out the fact that γ^2 is entirely real when the structure is lossless, and giving correlations between the algebraic sign of α and β ($\gamma=\alpha+j\beta$) and the direction of power flow along the guide. The main object of this section (Section 3.5) is, however, the ensuing proof that γ^2 must also be pure real on a lossless inhomogeneous structure.

Part IV then proceeds with a study of vector-power flow in a single mode, emphasizing the point that the correlation between the algebraic sign of β and the direction of power flow down the guide is no longer so simple for inhomogeneous problems as for homogeneous ones (Section 4.1). Sections 4.2 and 4.3 deal primarily with the physical interpretation of the fact that the field distribution in a single TE-TM mode generally changes with frequency; and Part IV concludes with Section 4.4 on the polarization of the fields in these mixed TE-TM modes. It has been advisable to restrict most of the discussion in Part IV to lossless cases.

Part V on "Open-Boundary" problems draws upon the material in the preceding work, but develops the additional conclusions that an open structure cannot support either a free exponential mode below cutoff, or one which has a phase velocity greater than that of plane waves in the externally surrounding space. A brief discussion is then given of the consequent fact that these free modes may be finite in number at any given frequency, and therefore cannot be a complete set. In particular, they cannot account for radiation from a dielectric-rod antenna, and the actual mechanism of such radiation is touched upon.

Following a short conclusion, and some suggestions for further work, the four Appendices are attached. Sufficient idea of their content can be gained from their titles in the Table of Contents; they supply a small background of experience to substantiate the general discussions outlined above.

It requires emphasis at the outset that mathematical rigor in the derivations is far less important to a discussion of this nature than are the fundamental physical ideas behind the analysis. The purpose of the present work is to improve the engineer's intuition, rather than his technique.

II. FORMULATION AND PRELIMINARY ANALYSIS OF THE PROBLEM

Preliminary to the main topics under consideration is the introduction of the coordinate system and notation. A brief analysis will then be required to convert the basic complex Maxwell equations into a form particularly suitable to cylindrical coordinates. The problems to be considered can then be stated in more precise form. In particular, it will be desirable to make a few remarks about the form of the boundary conditions which will be included in the term "closed-boundary structure".

2.1 Coordinates and Notation

With reference to Figure 2.1 (page 7), the following notation will be clear:

$E(t), H(t), B(t), D(t)$ - Real field vectors; functions of (x,y,z,t).

$\hat{E}, \hat{H}, \hat{B}, \hat{D}$ - Complex field vectors; functions of (x,y,z,ω).

E, H, B, D - Complex field vectors; functions of (x,y,ω) only.

$E_T(t), \hat{E}_T, E_T$, etc. - Vector functions as above, but having space components only in the transverse (T) plane (x,y).

$\hat{E}_n, \hat{E}_\tau, \hat{E}_z$, etc. - Complex scalar components; functions of (x,y,z,ω).

E_n, E_τ, E_z, etc. - Complex scalar components; functions of (x,y,ω) only.

$E_s(t), \hat{E}_s, E_s$, etc. - Vector functions as above, but having space components tangential to some particularly designated surface (s).

For example, in the particular case of an electric-field vector which is harmonic in time and exponential in z, the following relations will hold:

$$E(t) = \mathrm{Re}(\hat{E}e^{j\omega t}) = \mathrm{Re}[(Ee^{-\gamma z})e^{j\omega t}]$$
$$= \mathrm{Re}(Ee^{j\omega t-\gamma z}) = E_T(t) + i_z E_z(t), \tag{2.1}$$

where

$$\hat{E} = \hat{E}_T + i_z\hat{E}_z = Ee^{-\gamma z} = (E_T + i_z E_z)e^{-\gamma z}, \tag{2.2}$$

and therefore

$$\text{(a)} \quad E_T(t) = \mathrm{Re}(\hat{E}_T e^{j\omega t}) = \mathrm{Re}(E_T e^{j\omega t-\gamma z}),$$
$$\text{(b)} \quad E_z(t) = \mathrm{Re}(\hat{E}_z e^{j\omega t}) = \mathrm{Re}(E_z e^{j\omega t-\gamma z}). \tag{2.3}$$

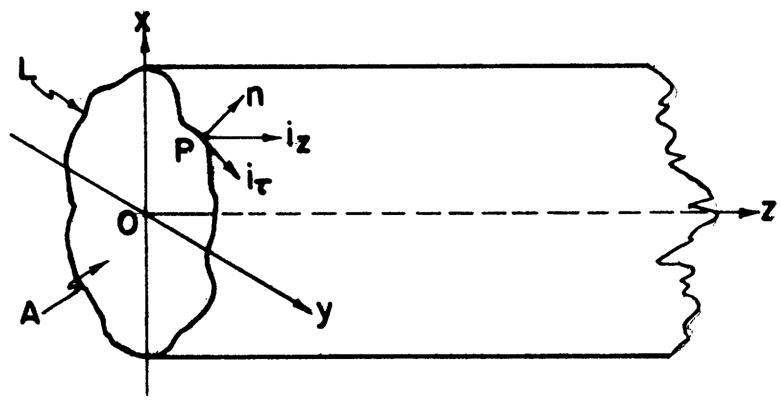

Fig. 2.1. Coordinate system for cylindrical structure.

P - Any point on the bounding wall.

A - Any cross sectional area of guide.

L - Any bounding contour line of the guide wall.

n - A real unit vector normal to the wall and directed outward; independent of z.

i_τ - A real unit vector in the transverse (x,y) plane, tangent to the wall and independent of z.

i_z - A real unit vector along the +z direction, independent of x,y and z.

The positive direction of i_τ is such that at any point P on the wall n, i_τ and i_z form a right-handed system of orthogonal unit base vectors, in that order.

The area A and contour L may lie in any plane normal to the z-axis.

If the component of the vector tangent to the bounding wall in Figure 2.1 (page 7) is desired, the required component would be a vector given by

$$E_s(t) = Re(\hat{E}_s e^{j\omega t}) = Re(E_s e^{j\omega t-\gamma z}), \tag{2.4}$$

with

$$\begin{aligned}\hat{E}_s &= i_r \hat{E}_r + i_z \hat{E}_z = E_s e^{-\gamma z} \\ &= (i_r E_r + i_z E_z) e^{-\gamma z}.\end{aligned} \tag{2.5}$$

Additional detailed notation will be introduced as required, with MKS Rationalized Units employed throughout.

2.2 Reduction of Maxwell Equations to Cylindrical Form

When the time variation of the fields is taken to be harmonic ($e^{j\omega t}$), the appropriate form of the Maxwell equations applicable to the cylindrical system of Figure 2.1 (page 7), in the absence of sources, is

$$\begin{aligned}&(a) \quad \nabla \times \hat{E} = -j\omega\mu\hat{H}, \\ &(b) \quad \nabla \times \hat{H} = j\omega\epsilon'\hat{E},\end{aligned} \tag{2.6}$$

with $j\omega\epsilon' = \sigma + j\omega\epsilon$. It is to be recalled that ϵ, μ, and σ, the (real) dielectric, permeability and conductivity constants of the medium within the guide, may be functions of the transverse coordinates (x,y), but not functions of z. For the sake of simplicity, these parameters have also been taken independent of frequency ω, although in the majority of the results which follow an extension can easily be made to include such frequency dependence.

Since the problem is cylindrically symmetric, it is natural to search for solutions which have the cylindrical behavior

$$\hat{E} = E\varphi(z,\omega) \quad ; \quad \hat{H} = H\varphi(z,\omega) \; ; \tag{2.7}$$

in which $\varphi(z,\omega)$ is a complex scalar function of (z,ω). The introduction of Eq. (2.7) into Eq. (2.6) results in the relations

$$\begin{aligned}&(a) \quad \nabla\varphi\times E + \varphi\nabla\times E = -j\omega\mu\varphi H, \\ &(b) \quad \nabla\varphi\times H + \varphi\nabla\times H = j\omega\epsilon'\varphi E.\end{aligned} \tag{2.8}$$

To select only the transverse part of Eq. (2.8), take the cross product of both sides with i_z as follows:

$$\text{(a)} \quad [i_z\times(i_z\times E_T)] \frac{d\varphi}{dz} + \varphi \nabla_T E_z = -j\omega\mu\varphi(i_z\times H_T), \qquad (2.9)$$

$$\text{(b)} \quad [i_z\times(i_z\times H_T)] \frac{d\varphi}{dz} + \varphi \nabla_T H_z = j\omega\epsilon'\varphi(i_z\times E_T),$$

in which ∇_T is the gradient operator confined to the transverse plane. A dot multiplication of Eq. (2.9a) by E_T^*, and of Eq. (2.9b) by H_T^*, brings forth the new forms

$$\text{(a)} \quad -(E_T\cdot E_T^*) \frac{d\varphi}{dz} + \varphi E_T^*\cdot \nabla_T E_z = -j\omega\mu\varphi i_z\cdot(H_T \times E_T^*), \qquad (2.10)$$

$$\text{(b)} \quad -(H_T\cdot H_T^*) \frac{d\varphi}{dz} + \varphi H_T^*\cdot \nabla_T H_z = j\omega\epsilon'\varphi i_z\cdot(E_T \times H_T^*).$$

The star (*) represents the complex conjugate of the function to which it is applied. Division of Eq. (2.10a) by the scalar function $\varphi(E_T\cdot E_T^*)$, and similar division of Eq. (2.10b) by $\varphi(H_T\cdot H_T^*)$, accomplishes a separation of both equations, as indicated by the results

$$\text{(a)} \quad -\frac{1}{\varphi}\frac{d\varphi}{dz} = -j\omega\mu \frac{i_z\cdot(H_T \times E_T^*)}{(E_T\cdot E_T^*)} - \frac{E_T^*\cdot\nabla_T E_z}{(E_T\cdot E_T^*)}, \qquad (2.11)$$

$$\text{(b)} \quad -\frac{1}{\varphi}\frac{d\varphi}{dz} = j\omega\epsilon' \frac{i_z\cdot(E_T \times H_T^*)}{(H_T\cdot H_T^*)} - \frac{H_T^*\cdot\nabla_T H_z}{(H_T\cdot H_T^*)}.$$

Since the left sides of both equations in Eq. (2.11) above are functions only of z, while their right sides are functions only of (x,y), the conclusion must be that

$$\frac{1}{\varphi}\frac{d\varphi}{dz} = -\gamma(\omega), \qquad (2.12)$$

in which $\gamma(\omega)$ is a complex constant, independent of x,y and z, but generally a function of ω.

Before drawing final conclusions about this separation property of the Maxwell equations, it is necessary to be certain that Eq. (2.12) is consistent with the longitudinal parts of Eqs. (2.8a) and (2.8b), namely the dot product of Eq. (2.8) with the unit vector i_z:

$$\text{(a)} \quad i_z\cdot(\nabla \times E_T) = -j\omega\mu H_z, \qquad (2.13)$$

$$\text{(b)} \quad i_z\cdot(\nabla \times H_T) = j\omega\epsilon' E_z.$$

The resultant cancellation of the function $\varphi(z,\omega)$ means that Eq. (2.13) allows the separation of the fields in the form selected, without imposing further restrictions on φ.

It is now possible to conclude from Eq. (2.12) that if a solution of the form chosen in Eq. (2.7) is at all possible, then

$$\varphi(z,\omega) = e^{-\gamma(\omega)z}.$$

The complex "propagation constant" γ will presumably be determined at any frequency ω from the boundary conditions. In fact, it is of primary importance to recognize that γ is a function of frequency, and further consideration will be directed subsequently toward this frequency dependence.

Equation (2.9) may be rewritten in a new form, appropriate to the exponential solution found above for φ:

$$\begin{aligned} &\text{(a)} \quad \gamma E_T + \nabla_T E_z = -j\omega\mu\,(i_z \times H_T), \\ &\text{(b)} \quad \gamma H_T + \nabla_T H_z = j\omega\epsilon'\,(i_z \times E_T). \end{aligned} \qquad (2.14)$$

Solution for H_T in terms of $\nabla_T E_z$ and $\nabla_T H_z$ may be made from Eq.(2.14) with a cross multiplication of Eq.(2.14a) by $(j\omega\epsilon' i_z/\gamma)$, and a subsequent addition of Eqs.(2.14a) and (2.14b). Similar steps yield a solution for E_T, and the results will be

$$\begin{aligned} &\text{(a)} \quad H_T = \frac{\gamma}{p^2}\nabla_T H_z + \frac{j\omega\epsilon'}{p^2}\, i_z \times \nabla_T E_z\,, \\ &\text{(b)} \quad E_T = \frac{\gamma}{p^2}\nabla_T E_z - \frac{j\omega\mu}{p^2}\, i_z \times \nabla_T H_z\,. \end{aligned} \qquad (2.15)$$

The function p^2 introduced in Eq.(2.15) is defined by the relations

$$\text{(a)} \quad p^2 = -(\gamma^2 + k^2)$$

and (2.16)

$$\begin{aligned} \text{(b)} \quad k &= \omega\sqrt{\epsilon'\mu}\,, \text{ or} \\ k^2 &= \omega^2\epsilon'\mu = -j\omega\mu\,(\sigma + j\omega\epsilon). \end{aligned}$$

By reason of the dependence of ϵ' and μ upon the transverse coordinates, k^2 (or k) is also a function of position in the guide cross section. Then p^2 also becomes a function of position in the transverse plane, as well as a function of frequency.

Equation (2.15) should be looked upon merely as a restatement of the transverse parts of the two Maxwell equations

[Eq.(2.6)]; a restatement, however, which makes them specially applicable to cylindrical systems, and places in evidence the fact that the longitudinal field components E_z and H_z are in the nature of a pair of scalar potentials from which the transverse fields may be derived.

It is natural to ask next for the equations governing the behavior of E_z and H_z. Such equations can most expeditiously be found by returning to Eq.(2.14) and taking the divergence of both sides:

$$\begin{aligned} &(a) \quad \gamma \nabla_T \cdot E_T + \nabla_T^2 E_z = j\omega\mu i_z \cdot (\nabla \times H_T) \\ &\qquad\qquad - j\omega \nabla_T \mu \cdot (i_z \times H_T), \\ &(b) \quad \gamma \nabla_T \cdot H_T + \nabla_T^2 H_z = -j\omega\epsilon' i_z \cdot (\nabla \times E_T) \\ &\qquad\qquad + j\omega \nabla_T \epsilon' \cdot (i_z \times E_T). \end{aligned} \tag{2.17}$$

With reference to Eqs.(2.13) and (2.16b), this result may be rewritten in a simpler form, namely

$$\begin{aligned} &(a) \quad \gamma \nabla_T \cdot E_T + \nabla_T^2 E_z = -k^2 E_z - j\omega \nabla_T \mu \cdot (i_z \times H_T) , \\ &(b) \quad \gamma \nabla_T \cdot H_T + \nabla_T^2 H_z = -k^2 H_z + j\omega \nabla_T \epsilon' \cdot (i_z \times E_T) . \end{aligned} \tag{2.18}$$

The divergence terms in Eq.(2.18) can be removed most easily by returning to the Maxwell equations (2.6), and taking the divergence of both sides:

$$\begin{aligned} &(a) \quad \nabla \cdot (\mu \hat{H}) = \mu \nabla \cdot \hat{H} + \hat{H} \cdot \nabla_T \mu = 0 , \\ &(b) \quad \nabla \cdot (\epsilon' \hat{E}) = \epsilon' \nabla \cdot \hat{E} + \hat{E} \cdot \nabla_T \epsilon' = 0 . \end{aligned} \tag{2.19}$$

Now $\nabla_T \mu$ and $\nabla_T \epsilon'$ are vectors in the transverse plane, while according to Eq.(2.2)

$$\begin{aligned} \hat{H} &= He^{-\gamma z} = (H_T + i_z H_z)e^{-\gamma z} , \\ \hat{E} &= Ee^{-\gamma z} = (E_T + i_z E_z)e^{-\gamma z} . \end{aligned} \tag{2.20}$$

Therefore Eq.(2.19) leads to the conclusion that

$$\begin{aligned} &(a) \quad \nabla_T \cdot H_T = \gamma H_z - \frac{\nabla_T \mu}{\mu} \cdot H_T , \\ &(b) \quad \nabla_T \cdot E_T = \gamma E_z - \frac{\nabla_T \epsilon'}{\epsilon'} \cdot E_T . \end{aligned} \tag{2.21}$$

As a result of Eqs.(2.21) and (2.16a), Eq.(2.18) becomes

$$\begin{aligned}
&(a) \quad \nabla_T^2 E_z - p^2 E_z = \gamma\left(\frac{\nabla_T \epsilon'}{\epsilon'}\right)\cdot E_T - j\omega(\nabla_T \mu)\cdot(i_z \times H_T) \ , \\
&(b) \quad \nabla_T^2 H_z - p^2 H_z = \gamma\left(\frac{\nabla_T \mu}{\epsilon}\right)\cdot H_T + j\omega(\nabla_T \epsilon')\cdot(i_z \times E_T) \ .
\end{aligned} \qquad (2.22)$$

Substitution of $(i_z \times H_T)$ from Eq.(2.14a) into Eq. (2.22a), and of $(i_z \times E_T)$ from Eq.(2.14b) into Eq.(2.22b) yields

$$\begin{aligned}
&(a) \quad \nabla_T^2 E_z - p^2 E_z = \gamma E_T \cdot \frac{\nabla_T k^2}{k^2} + \frac{\nabla_T \mu}{\mu}\cdot \nabla_T E_z \ , \\
&(b) \quad \nabla_T^2 H_z - p^2 H_z = \gamma H_T \cdot \frac{\nabla_T k^2}{k^2} + \frac{\nabla_T \epsilon'}{\epsilon'}\cdot \nabla_T H_z \ ,
\end{aligned} \qquad (2.23)$$

where it should be noticed that

$$\frac{\nabla_T k^2}{k^2} = \frac{\nabla_T \mu}{\mu} + \frac{\nabla_T \epsilon'}{\epsilon'} \ . \qquad (2.24)$$

The transverse fields are given in terms of $\nabla_T E_z$ and $\nabla_T H_z$ by Eq.(2.15). Use of the latter equation in Eq.(2.23) results in the final relations:

$$\begin{aligned}
&(a) \quad \nabla_T^2 E_z - p^2 E_z = \frac{1}{p^2}\left\{\left[\gamma^2 \frac{\nabla_T \epsilon'}{\epsilon'} - k^2 \frac{\nabla_T \mu}{\mu}\right]\cdot \nabla_T E_z \right. \\
&\qquad\qquad \left. + j\omega\mu\gamma i_z \cdot \left[\frac{\nabla_T k^2}{k^2} \times \nabla_T H_z\right]\right\} , \\
&(b) \quad \nabla_T^2 H_z - p^2 H_z = \frac{1}{p^2}\left\{\left[\gamma^2 \frac{\nabla_T \mu}{\mu} - k^2 \frac{\nabla_T \epsilon'}{\epsilon'}\right]\cdot \nabla_T H_z \right. \\
&\qquad\qquad \left. - j\omega\epsilon'\gamma i_z \cdot \left[\frac{\nabla_T k^2}{k^2} \times \nabla_T E_z\right]\right\} .
\end{aligned} \qquad (2.25)$$

These last equations between E_z and H_z can be considered as replacing the longitudinal parts of the Maxwell equations, just as Eq.(2.14) (or 2.15) replaces the transverse parts therof.

"Equations (2.14) or (2.15) along with Eq.(2.25) or Eq.(2.23) are a complete restatement of the complex Maxwell equations for a source-free cylindrical system in which all field components have harmonic time dependence

$$e^{j\omega t},$$

and a separated z-dependence

$$e^{-\gamma(\omega)z}."$$

2.3 Detailed Formulation of the Problem

In order to solve any particular problem, the solutions of Eq.(2.25) must be expressed in terms of the transverse coordinates (x,y) and the unknown value of γ. Equation (2.15) determines the transverse fields, and application of the boundary conditions leads to a functional equation which will select the appropriate values of γ at each frequency. It is to be expected that in some cases the relative amplitudes of E_z and H_z on the boundary will also be fixed by these same boundary conditions.

It should be emphasized again, however, that according to Eq.(2.16a) p^2 is a function of the transverse coordinates. As a result, it does not have the significance of an eigenvalue in these inhomogeneous problems. For any particular frequency, the set of allowed values of γ form the eigenvalues. In general, the functional equations determining γ will be transcendental, and the various branches of the functions will designate the "modes". Since p^2 is a function of both the frequency ω and the coordinates (x,y), it is to be anticipated that the field distribution in the transverse plane, governed by Eq.(2.25), will in general change with frequency. This fact is in marked contrast with the situation in homogeneous guides, where p^2 is a constant for each mode, and Eqs.(2.25) do not contain any coefficients dependent upon ω. In homogeneous cases, the field distribution for any particular mode remains the same over the entire frequency range $0<\omega<\infty$, and the modes themselves may in fact be designated by the various allowed values of p^2.

When the problem is not homogeneous, the variation of the field distribution with frequency makes it much harder to identify the different modes.

It is not the function of the following portions of this paper either to solve Eqs.(2.25), or to prove that allowed values of γ must exist under the particular boundary conditions to be prescribed later in Section 2.4. Rather, an investigation will be conducted to determine some of the general properties which are to be expected of those modes which do exist, in order that some insight may be gained to guide the engineer in his search for solutions to any particular problem. The importance of such aids can be appreciated only when the mathematical complications of even the simplest inhomogeneous problems have been examined through various specific examples. It is particularly important to know some of the very elementary properties of those eigenvalues $\gamma(\omega)$ which do exist, because otherwise much effort can be expended uselessly in looking for solutions to any specific multi-valued eigenvalue equation on a branch thereof where, on more general grounds, such solutions could a priori be ruled out.

Perhaps it is pertinent to point out, however, that it would be strange, indeed, if in some inhomogeneous cylindrical problem there were no allowed values of $\gamma(\omega)$; for it has been shown already that if there is any cylindrical solution at all, it must have exponential z-dependence. If no values of γ were permissible, it would follow that some problem with cylindrical symmetry would have no solutions with cylindrical symmetry.

But even granting the existence of some propagation constants and associated modes, there is still a severe question about the completeness of the entire set of modes (for the purpose of representing any given transverse field distribution, for example). This question of completeness is a difficult one, and the discussion contained in the present work will not touch upon it significantly. Yet the results of this analysis of mode properties, along with the examples in the Appendices, do indicate one interesting point connected therewith; the open boundary structure has modes which never even reach cutoff (γ=0). Each mode simply ceases to exist below a certain frequency. As a result, at any given frequency, and for any particular circular variation, only a finite number of modes are available. It is clear that such a limited set cannot be complete, and this fact is illustrated in Appendix D. The physical reason for this mode behavior is quite understandable in such problems, as outlined in Part V.

It is in fact hard to avoid the belief that when any modes among a given set individually cannot exist over the entire frequency range $0<\omega<\infty$, then the set of modes at a particular frequency cannot be complete; but this matter is still in the realm of conjecture.

In this connection, however, some remarks should be made about the circular guide with a reactive wall, treated in Appendix A. A detailed study of the eigenvalue equation in that problem has been made, but is not fully presented in Appendix A. It was assumed, when that study was undertaken, that the wall admittances were independent of frequency. Such an assumption is not in accordance with the restrictions for physical realizability given in Eq.(2.40), Section 2.4; and the curious results to which it leads suggest that a less idealized example ought to be treated. The peculiarities encountered consisted chiefly in the fact that, for certain choices of the wall parameters, modes which were not axially symmetric suddenly "broke off" discontinuously. The break did not occur in the understandable way characteristic of open-boundary structures, but took place either at or below cutoff. For any particular $n>0$ (circular-variation index), a finite number of modes possessed this "break off" property, while the (infinite) remaining set did not.

Without a further study of the problem, making more appropriate choices of the boundary admittances, it would be unwise to draw conclusions from such an anomalous result. A little more discussion on the subject is included in Appendix A, but the major treatment will be postponed pending further work on the problem.

2.4 Boundary Conditions

In order to deal with a bounding surface which shall not be entirely opaque, but which shall at the same time eliminate the need for any detailed consideration of the fields outside the structure, the boundary conditions at each point on the wall of the guide will be taken in the form of a dyadic admittance (1b,14)

$$\text{(a)} \quad n \times \hat{H}_s = \overline{Y}\cdot\hat{E}_s \;,$$

or (2.26)

$$\text{(b)} \quad n \times H_s = \overline{Y}\cdot E_s \;.$$

The dyadic $\overline{Y}$ is independent of z and, in fact, is taken for simplicity to be entirely independent of position on the wall. It is therefore not a function of (x,y,z). When written out, the dyadic $\overline{Y}$ has the general representation

$$\overline{Y} = \left\{ \begin{array}{l} y_{\tau\tau} i_\tau i_\tau + y_{\tau z} i_\tau i_z \\ + y_{z\tau} i_z i_\tau + y_{zz} i_z i_z \end{array} \right\} , \tag{2.27}$$

in which the various elements $y_{\mu\nu}$ of the dyadic are, in general, complex scalars, having the physical dimensions of admittance. For the purposes of this paper, a somewhat more specialized form of the dyadic $\overline{Y}$ will be assumed:

$$\overline{Y} = \left\{ \begin{array}{ll} y_{\tau\tau} i_\tau i_\tau & + 0 \\ + 0 & + y_{zz} i_z i_z \end{array} \right\} . \tag{2.28}$$

While the restriction of $\overline{Y}$ to this "Normal" form will shortly be shown to entail no real loss of generality insofar as the desired physical properties of the wall are concerned, it is not premature to mention that a symmetry property to be discussed later (Section 3.2) would be considerably modified if the dyadic $\overline{Y}$ were left in the more general form (2.27). Besides, the desirability of obtaining a symmetric dyadic boundary condition ($y_{z\tau} = y_{\tau z}$) will also become apparent in the ensuing pages.

An expansion of the dot product in Eq.(2.26b) can now be made in the light of Eq.(2.28),

$$n \times (i_\tau H_\tau + i_z H_z) = i_\tau y_{\tau\tau} E_\tau + i_z y_{zz} E_z \quad . \tag{2.29}$$

A further expansion of the cross product on the left yields the two scalar relations

$$\text{(a)} \quad H_\tau = y_{zz}E_z \quad ,$$
$$\text{(b)} \quad H_z = -y_{\tau\tau}E \quad , \tag{2.30}$$

after similar vector components have been equated on each side of Eq.(2.29). The resulting boundary condition Eq.(2.30) places in evidence the admittance character of $y_{\tau\tau}$ and y_{zz}. It also shows that the admittances which describe the wall properties can be chosen in such a way that H_s may have any desired magnitude, space angle, or time phase with respect to E_s. These admittances could even be chosen to make H_s represent an elliptically polarized vector $H_s(t)$ when E_s represents a linearly polarized vector $E_s(t)$, or vice versa. There is actually more freedom allowed by even the normal form of $\overline{Y}$ than will be used in the sequel of this discussion.

It will be assumed here that while y_{zz} and $y_{\gamma\gamma}$ are functions of the frequency ω, they are definitely not functions of γ (or the guide wavelengths) for the various modes which may exist at any particular frequency. The fact that the admittances are assumed to be independent of the modes (or γ's) which may exist at a given frequency is roughly tantamount to the assumption that the admittance of the wall material to plane waves is independent of the angle of incidence. Such would be the case, for example, if the wall were constructed of metal with a large, but finite, conductivity. Examples of lossless walls with these same admittance properties are not easy to visualize generally, although Hansen (2) has approximated an iris-loaded circular waveguide operating in the axially symmetric modes by using such a susceptance concept. The approximation is based upon the assumption that the spacing between successive irises is very small compared to the guide wavelength of the lowest propagating mode at the frequency involved. In the limit of differentially small iris spacing the approximation becomes better, but further question may be raised about its validity for those higher modes in which the fields no longer have axial symmetry. More recently, attention has been given to the electromagnetic behavior of metals at extremely low temperatures. Since the phenomenon of superconductivity takes place at such temperatures, it has been convenient to consider a metal wall as a reactance when resonant cavaties are constructed therefrom. But even if only for purposes of generality, it is both easy and desirable to include boundary condition (2.30) in these general discussions.

The special cases in which the bounding wall has been referred to as "opaque" are included in Eq.(2.30) when

$$\left.\begin{array}{l} y_{zz} = y_{\tau\tau} = 0 \\ \text{or} \quad n \times H_s = 0 \end{array}\right\} , \tag{2.31}$$

and when

$$\left.\begin{array}{l} y_{zz} = y_{\tau\tau} \rightarrow \infty \\ \text{or} \quad n \times E_s = 0 \end{array}\right\} . \tag{2.32}$$

Condition (2.31) refers to a "magnetic wall", while condition (2.32) refers to the more common "electric wall", or perfect conductor.

Equations (2.15) and (2.25) inside the guide, along with Eq.(2.30) on the wall, completely characterize the boundary-value problem presented by the structure. Of course, it must be hastily added that the solutions for E_z and H_z from Eqs.(2.25) must first be chosen to make physical sense; which requires that certain finiteness, single-valuedness, and continuity conditions be imposed upon the functions and their space derivatives (of first and second orders) at each point within the guide. Moreover, for the present purposes, it will be well to consider that the functions $\epsilon'(x,y)$ and $\mu(x,y)$ are continuous, with continuous first derivatives. Any discontinuities actually present in these functions can be replaced by regions of rapid but continuous variation. This assumption will be made throughout, unless otherwise specifically stated. In the examples (included in the Appendices), discontinuous distributions have been considered for reasons of simplicity. It is important to observe, however, that since a limiting form of the Maxwell equations is applied at each such discontinuity, these situations are simply limiting cases for more idealized functions ϵ' and μ.

Further interpretation of the boundary condition Eq.(2.26b) requires a consideration of that component of the complex Poynting vector

$$S = \tfrac{1}{2}(E \times H^*)$$

which is directed <u>into</u> the wall, to wit:

$$2n\cdot S = n\cdot(E_s \times H_s^*) = -E_s\cdot(n \times H_s^*)$$
$$= -E_s\cdot\overline{Y}^*\cdot E_s^* \quad . \tag{2.33}$$

With the stipulation that

$$\overline{Y} = \overline{G} + j\overline{B} \quad ,$$

Eq.(2.33) becomes

$$2n\cdot S = -E_s\cdot\overline{G}\cdot E_s^* + jE_s\cdot\overline{B}\cdot E_s^* \tag{2.34}$$

Now in view of the symmetric form of $\overline{Y}$ in Eq.(2.28), and the consequent symmetry of the two real dyadics $\overline{G}$ and $\overline{B}$ in Eq.(2.34), it follows that the first term on the right of Eq.(2.34) is purely real, while the second term is purely imaginary. In fact, if

$$y_{\mu\nu} = g_{\mu\nu} + jb_{\mu\nu} \quad ,$$

then

$$E_s\cdot\overline{G}\cdot E_s^* = g_{\tau\tau}E_\tau E_\tau^* + g_{zz}E_zE_z^* = -2\mathrm{Re}(n\cdot S) \quad , \tag{2.35}$$

from which the expression $E_s\cdot\overline{G}\cdot E_s^*$ is seen to be a real quadratic form with coefficients $g_{\tau\tau}$ and g_{zz}. If, then, the wall is to be truly passive, it must not cause real power to flow into the guide, regardless of the orientation of E_s. In order that this be true generally, the quadratic form in Eq.(2.35) must remain negative for all orientations of E_s; which in turn requires that the elements $g_{\mu\nu}$ of $\overline{G}$ shall be the coefficients of a negative definite quadratic form. In the special case at hand, where $\overline{G}$ is in Normal form, the requirement for a passive wall may be stated in the relations

$$g_{zz} \leq 0$$

and

$$g_{\tau\tau} \leq 0 \tag{2.36}$$

It may seem curious that the $y_{\mu\nu}$ have <u>negative</u> real parts when they represent the admittance of a passive wall. Equation (2.34) also yields the additional disconcerting result that when b_{zz} and $b_{\tau\tau}$ are both > 0, the wall abstracts primarily <u>magnetic</u>

energy from the region which it surrounds. That is, an inductive wall has an admittance with a positive imaginary part. But the two peculiarities together mean simply that the admittances $y_{\mu\nu}$ are defined with a sign opposite to that normally associated with ordinary circuit admittance. The root of the difficulty lies in using $(n \times H_s)$ instead of $(H_s \times n)$ in the defining relation (2.26) for the boundary conditions. It is consequently necessary to consider the $y_{\mu\nu}$ as the negatives of ordinary circuit admittances.

It will be required, in the course of this text, to consider the properties of the modes as functions of the frequency. Some statement about the properties of the boundary conditions, qua functions of ω, must therefore be included here. Since the major part of the development in this connection will concern itself with lossless systems, the boundary conditions will become

$$n \times H_s = j\overline{B} \cdot E_s \quad , \tag{2.37}$$

with

$$\overline{B} = \begin{Bmatrix} b_{\tau\tau} i_\tau i_\tau + 0 & \\ + 0 & + b_{zz} i_z i_z \end{Bmatrix}, \tag{2.38}$$

and $y_{\mu\nu} = jb_{\mu\nu}$. If the analogy to circuit susceptances is to be preserved (with the previously mentioned change in sign) it will be necessary to specify that

$$E_s \cdot \frac{\partial \overline{B}}{\partial \omega} \cdot E_s^*$$

is a negative definite quadratic form. In terms of the Normal form of $\overline{B}$, this stipulation becomes simply

$$\left. \begin{aligned} \frac{\partial b_{\tau\tau}}{\partial \omega} &\leq 0 \\ \frac{\partial b_{zz}}{\partial \omega} &\leq 0 \end{aligned} \right\} , \tag{2.39}$$

or, the slope of the susceptances versus ω is always negative. This restriction is not, however, made solely by analogy with the familiar circuit properties of susceptance. For Schwinger (1b) has shown that in an entirely closed lossless system, the only admittance boundary conditions under which a desirable uniqueness theorem may be deduced for the fields inside, are those for which the considerations leading to Eq.(2.39) apply. To be sure, this uniqueness theorem for closed systems precludes the existence of two solutions to a given lossless boundary-value problem if the difference between the solutions is required to be a continuous function of frequency. That the same theorem cannot be true in cylindrical structures follows from the fact that in ordinary waveguides, for example, each mode is itself a continuous function of ω; whence the difference between any two of them is also continuous in ω. Nevertheless, it still seems advisable to consider the dyadic suspectance as a property characteristic of the wall material itself, and to retain for that material in a cylindrical structure those same properties which would be required of it in an entirely closed system.

In addition to Eq.(2.39), another restriction should be mentioned which also comes from the network analogy, as well as from considerations underlying the uniqueness proof mentioned above. It may be most easily stated for present purposes in the form

$$\text{(a)} \quad \left|\frac{\partial b_{\tau\tau}}{\partial \omega}\right| \geq \left|\frac{b_{\tau\tau}}{\omega}\right| ,$$

$$\text{(b)} \quad \left|\frac{\partial b_{zz}}{\partial \omega}\right| \geq \left|\frac{b_{zz}}{\omega}\right| . \qquad (2.40)$$

Because of Eq.(2.40), it would appear that problems involving a reactive wall cannot be expected to make sense, over a wide range of frequencies, if the admittances b_{zz} and $b_{\tau\tau}$ are assumed independent of ω.

III. BASIC PROPERTIES OF THE MODES

One of the most outstanding differences between modes in homogeneous problems and those connected with inhomogeneous problems lies in the fact that TE and TM modes are independent in the former, and dependent in the latter. Therefore some discussion is necessary with regard to mixture of TE and TM modes in the cases where the boundary is not opaque, or the internal medium is not uniform. Moreover, the consequences of this mixture make it necessary to re-examine the orthogonality conditions between modes, as well as the proof that γ^2 must be real in a lossless system. Such examination will be the primary concern of Part III.

3.1 TE-TM Properties of the Modes

The discussion of TE-TM mixture may most conveniently be pursued by considering the effects of the boundary and the internal medium separately. When the guide is uniformly filled with material, $\nabla_T \epsilon' = \nabla_T \mu \equiv 0$. Then Eq.(2.25) reduces to

$$\begin{aligned} &(a) \quad \nabla_T^2 E_z - p^2 E_z = 0 \quad , \\ &(b) \quad \nabla_T^2 H_z - p^2 H_z = 0 \quad . \end{aligned} \qquad (3.1)$$

As far as the medium inside is concerned, therefore, one solution with $H_z = 0$ (TM) and one with $E_z = 0$ (TE) are independently possible. The transverse fields given by Eqs.(2.15) can similarly be split into two groups, in which a superscript 1 denotes the TM fields, and 2 the TE fields:

(a) TM ($H_z \equiv 0$)

$$E_T^{(1)} = \frac{\gamma}{p^2} \nabla_T E_z \quad ,$$

$$Z_{TM} H_T^{(1)} = i_z \times E_T^{(1)} \quad ,$$

$$Z_{TM} = \frac{Z_o \gamma}{jk} \quad ;$$

(b) TE ($E_z \equiv 0$) (3.2)

$$H_T^{(2)} = \frac{\gamma}{p^2} \nabla_T H_z \quad ,$$

$$E_T^{(2)} = -Z_{TE} \, i_z \times H_T^{(2)} \quad ,$$

$$Z_{TE} = \frac{jkZ_o}{\gamma} \quad ;$$

where

$$Z_o = \sqrt{\frac{\mu}{\epsilon'}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \sqrt{\frac{\mu}{\epsilon - j(\sigma/\omega)}} \quad . \tag{3.3}$$

Equations (3.1) and (3.2) are the conventional set, as applied to ordinary waveguides, and very complete discussions of the solutions under the conditions of an opaque boundary surface have been given in many places (1,15). But even in the simplest cases, where the solutions for E_z and H_z are separable functions of the transverse coordinates, the boundary conditions (2.30) do not usually allow separation of TE and TM modes. For, suppose the bounding contour L (Figure 2.1, page 7) is one of a family of orthogonal curves (η, τ) in the transverse plane; in particular, the one at $\eta = \eta_o$. Let it be supposed that a TE solution is required ($E_z \equiv 0$), and that $H_z = N(\eta)T(\tau)$ is a separable solution to Eq.(3.16), where η represents a "radial" coordinate and τ an "angular" coordinate. Then the boundary conditions (2.30) become

$$\text{(a)} \quad \frac{1}{h_\tau}\left(\frac{\partial H_z}{\partial \tau}\right)_{\eta_o} = \frac{N(\eta_o)}{h_\tau}\left(\frac{dT}{d\tau}\right) = 0 \quad ,$$

$$\text{(b)} \quad N(\eta_o) = y_{\tau\tau}\frac{j\omega\mu}{p^2 h_\eta}\left(\frac{dN}{d\eta}\right)_{\eta_o} \quad , \tag{3.4}$$

in which h_τ and h_η are the metric coefficients appropriate to the coordinates τ and η respectively.

The only non-trivial solution to Eq.(3.4) occurs when $(dT/d\tau) = 0$, in which case H_z and all the fields derived from it would be everywhere independent of τ. That is, the supposition that a TE solution is possible in a separable problem, with the boundary conditions (2.30), is equivalent to the requirement that the solution be axially symmetric. However, in order that a solution independent of τ exist, the geometric and electric properties of the cross section (including the wall) must be independent of τ. Even then, <u>all</u> the solutions to the problem will not necessarily have to be independent of τ, and any others will involve TE-TM mixtures. In any event, even if such axially symmetric solutions do exist in any particular case, they cannot form a complete set ; for it is perfectly possible to specify, by appropriate location of sources, that the transverse field in an axially symmetric structure shall not itself possess axial symmetry.

Since the TM modes can be treated in manner similar to the preceding, it is to be concluded that, even for the separable cases, the boundary conditions (2.30) do not admit a complete set of modes which are either TE or TM.

If there is a complete set at all, TE-TM mixtures must be considered, and these will be made up of combinations of the solutions to Eqs.(3.1a) and (3.1b). The boundary conditions (2.30) will then fix not only γ for the combined TE-TM mode, but also the relative amplitudes of E_z and H_z at any point on the boundary wall.

In Appendix A will be found the example of a circular waveguide with admittance wall. Because of the geometric and electric symmetry of the boundary with respect to the polar-coordinate angle φ, there are some solutions which break down into TE and TM waves. These occur only when axial symmetry of the fields is specified by taking $\frac{\partial}{\partial\varphi} \equiv 0$. As soon as the fields are allowed to vary in the angular direction, the modes become TE-TM combinations.

Incidentally, if the guide were elliptic in cross section there would be no solutions which were independent of the "angular" coordinate, because the geometry of the cross section would no longer be axially symmetric.

Not only the boundary conditions, but also the inhomogeneities in the internal medium will produce a TE-TM mixture. It is apparent from Eq.(2.25) that E_z and H_z are dependent in the general case, and it is only under very special circumstances of symmetry that a TE or TM solution is possible alone. For example, if a TE solution is required ($E_z \doteq 0$), then Eq.(2.23a) demands that

$$\nabla k^2 \cdot E_T^{(2)} = 0 \quad , \tag{3.5}$$

in which the superscript 2 refers to the TE wave of Eq.(3.2b).

H_z, however, is determined from Eq.(2.25b), with $E_z = 0$:

$$\nabla_T^2 H_z - p^2 H_z = \frac{1}{p^2}\left[\gamma^2 \frac{\nabla_T \mu}{\mu} - k^2 \frac{\nabla_T \epsilon'}{\epsilon'}\right] \cdot \nabla_T H_z \quad . \qquad (3.6)$$

Now if the structure is lossless, ∇k^2 is a purely real vector. Then condition (3.5) states that the polarization of the transverse electric field must be linear, and in a direction perpendicular to ∇k^2. Once H_z is determined from Eq.(3.6), on the other hand, the transverse electric field is specified by Eq.(3.2b), and there is no guarantee tnat the two conditions will be compatible. Even if they should be, however, it is clear that the polarization of the transverse electric field is entirely fixed by the internal medium, in virtue of Eq.(3.5); and there is no assurance that the boundary condition (even if it is homogeneous) will also be compatible with that restriction. Similar comments apply to a TM wave.

In Appendix B is included an example in which the polarization requirements of Eq.(3.5) can be met, along with the other requirements mentioned above. But only the lowest modes of the structure can satisfy all the conditions for TE and TM separation; higher modes being necessarily TE-TM combinations.

Once again it should be clear that a complete set, if it exists at all, cannot be made up of only those modes which possess TE and TM character alone, because a transverse field can easily be given, the polarization of which simply does not agree with the demands of Eq.(3.5).

> "It is to be concluded from the foregoing that any complete set of modes for an inhomogeneous problem must include those of mixed TE-TM character. If there are any which possess either TE or TM properties alone, they are the result of fortuitous symmetries and will not in general constitute the complete set by themselves."

3.2 Incident and Reflected Waves

Preparatory to the main derivations of the orthogonality conditions and the properties of γ on a lossless structure, it is necessary to exhibit a useful symmetry property of the boundary value problem posed by the guide structure. This symmetry amounts merely to the fact that for every mode which can exist on the structure there is always a second one which travels in the opposite direction.

The proof can start most conveniently from a slightly altered form of Eqs.(2.14), (2.13) and (2.30), which together characterize the guide problem:

$$\left.\begin{aligned} &(a) \quad \nabla_T E_z + \gamma E_T = -j\omega\mu(i_z \times H_T) \\ &(b) \quad \nabla_T H_z + \gamma H_T = j\omega\epsilon'(i_z \times E_T) \end{aligned}\right\} \text{Transverse Parts of Maxwell Equations},$$

$$\left.\begin{aligned} &(c) \quad \nabla\cdot(i_z \times E_T) = j\omega\mu H_z \\ &(d) \quad \nabla\cdot(i_z \times H_T) = -j\omega\epsilon' E_z \end{aligned}\right\} \text{Longitudinal Parts of Maxwell Equations}, \tag{3.7}$$

$$\left.\begin{aligned} &(e) \quad H = y_{zz} E_z \\ &(f) \quad H_z = -y_{\gamma\gamma} E \end{aligned}\right\} \text{Boundary Conditions on the wall}.$$

Suppose an appropriate solution to the first four equations has been found at a particular frequency ω_o. Suppose also that the application of the boundary conditions (3.7e,f) yields at least one value of γ at the specified frequency ω_o. In other words, the field $(E_{zo}, E_{To}, H_{zo}, H_{To}, \gamma_o)$ is a solution to the boundary value problem as a whole.

Next, consider a new field denoted by $(E_z', E_T', H_z', H_T', \gamma')$, in which the following relations hold:

$$\begin{aligned} &(a) \quad E_z' = -E_{zo}, \\ &(b) \quad E_T' = E_{To}, \\ &(c) \quad H_z' = H_{zo}, \\ &(d) \quad H_T' = -H_{To}, \\ &(e) \quad \gamma' = -\gamma_o. \end{aligned} \tag{3.8}$$

A substitution of Eqs.(3.8) into Eqs.(3.7a,b,c,d) shows that the latter remain unchanged, except for the addition of primes on all the appropriate variables. Hence the solutions for the new fields may be taken to be exactly the same functions of

(x,y,γ') as the old fields were of (x,y,γ_o). But it is also true that the boundary conditions (3.7e,f) remain unchanged when Eqs.(3.8a,b,c,d) are substituted therein, so that the functional equations which determine γ' are exactly the same as those which determined γ_o before. It follows that γ' and γ_o are solutions to the <u>same</u> set of equations, or that the boundary conditions give solutions for <u>both</u> γ_o and $-\gamma_o$. The wall conditions, therefore, cannot distinguish γ_o from $-\gamma_o$, and may consequently be said to determine only γ_o^2.

Observe that the field in Eq.(3.8) could have been defined in a second way, which differs but slightly from the actual definitions employed there:

$$\begin{aligned}
&(a) \quad E_z'' = E_{zo} \;, \\
&(b) \quad E_T'' = -E_{To} \;, \\
&(c) \quad H_z'' = -H_{zo} \;, \\
&(d) \quad H_T'' = H_{To} \;, \\
&(e) \quad \gamma'' = -\gamma_o \;.
\end{aligned} \tag{3.9}$$

The discussion showing that γ'' is determined from the same functional equation as γ_o goes through as before, and no essentially new information is obtained.

The alternate wave (3.8) or (3.9) may be referred to as the "reflected" wave corresponding to the "incident" wave given originally. The reflected field $(E_z', E_T', H_z', H_T', \gamma')$ moves along the z-axis in a direction opposite to that of the incident field $(E_{zo}, E_{To}, H_{zo}, H_{To}, \gamma_o)$, in view of Eq.(3.8e). Moreover, the complex Poynting vectors for the two fields are related as follows:

$$\begin{aligned}
S_z' &= S_z'' = -S_{oz} \;, \\
S_T' &= S_T'' = S_{oT} \;,
\end{aligned} \tag{3.10}$$

so that only the longitudinal components of S reverses upon "reflection".

The physical significance of the fact that the boundary conditions can determine only γ^2 is now made clear, because, as indicated earlier, it is merely another way of stating that:

"For every wave which can propagate down the structure, there is always another similar wave moving in the opposite direction."

Such a result is by no means surprising upon consideration of the fact that the system has cylindrical symmetry. Nevertheless, this symmetry property is quite important, and will be used a number of times in the rest of the work.

3.3 Orthogonality Conditions

Enough preliminary work has now been completed to allow the development of the orthogonality conditions which remain valid for inhomogeneous structures. It is helpful to review this matter rapidly in terms of homogeneous problems first, and then proceed to the more general case.

In the usual homogeneous cylindrical problems, a number of orthogonality relations are known to hold. If the subscripts 1 and 2 refer to any two exponential modes, for which $\gamma_1 \pm \gamma_2 \neq 0$, then it is true that (15) at any particular frequency ω:

$$\int_A \epsilon' E_{z1} E_{z2} d\sigma = \int_A \epsilon' E_{T1} \cdot E_{T2} d\sigma = \int_A \epsilon' E_1 \cdot E_2 d\sigma$$
$$= \int_A \mu H_{z1} H_{z2} d\sigma = \int_A \mu H_{T1} \cdot H_{T2} d\sigma \qquad (3.11)$$
$$= \int_A \mu H_1 \cdot H_2 d\sigma = 0 \quad .$$

Also

$$\int_A i_z \cdot E_{T1} \times H_{T2}) \, d\sigma = 0 \quad . \qquad (3.12)$$

In Eqs.(3.11) and (3.12) the integral is taken over the cross-sectional area A of the guide, with the recollection that all the quantities concerned are functions of only the transverse coordinates.

As long as the wall remains opaque, and therefore lossless, the validity of Eqs.(3.11) and (3.12) is not impaired by the presence of losses in the internal medium, provided that such losses are also uniformly distributed in the cross section.

It is interesting that under the same conditions (including possible loss in the medium), the fields in a homogeneous problem also have the properties:

$$\int_A \epsilon'^* E_{z1} E_{z2}^* d\sigma = \int_A \epsilon'^* E_{T1} \cdot E_{T2}^* d\sigma = \int_A \epsilon'^* E_1 \cdot E_2^* d\sigma$$
$$= \int_A \mu H_{z1} H_{z2}^* d\sigma = \int_A \mu H_{T1} \cdot H_{T2}^* d\sigma \quad (3.13)$$
$$= \int_A \mu H_1 \cdot H_2^* d\sigma = 0 \quad ,$$

as well as

$$\int_A i_z \cdot (E_{T1} \times H_{T2}^*) \, d\sigma = 0 \quad , \qquad (3.14)$$

where, however, $\gamma_1 \pm \gamma_2^* \neq 0$ in addition to $\gamma_1 \pm \gamma_2 \neq 0$. The second restriction on γ is not really physically significant because: for lossless homogeneous problems γ_1 and γ_2 are each either pure real or pure imaginary (1c); while for dissipative problems either γ or γ^* represents a wave which becomes infinite in the direction of propagation, and would have been rejected as a solution at the outset. More will be said about matters pertaining to the nature of γ in Section 3.5.

With reference to Eqs.(3.11) and (3.13), it is convenient to refer to the properties described by them as "energy orthogonality" conditions, while the properties expressed in Eqs.(3.12) and (3.14) may be referred to simply as "power orthogonality" conditions. The proofs of these various orthogonality properties are usually given from the nature of the differential equations (3.1) under the homogeneous boundary conditions (2.31) or (2.32).

It is a matter of experience that most of these orthogonality conditions do not hold when the problem is inhomogeneous. The standard procedures for proving them apparently break down when applied to Eqs.(2.25) and (2.15) under the boundary conditions (2.30). Nevertheless, it is possible to show that Eq.(3.12) remains true for inhomogeneous problems of the type being considered here, even if loss is present in both the dielectric material and the wall. Equation (3.14) is applicable along with Eq. (3.12), however, only when the entire system is dissipationless.

The reciprocity theorem forms the basis of the required proof, and may be written in two convenient ways for any region in which there are no sources. It is supposed that (ϵ, μ, σ) are reasonable functions of the coordinates, and that two linearly independent fields $(\hat{E}_1, \hat{H}_1)$ and $(\hat{E}_2, \hat{H}_2)$ are solutions to the Maxwell equations at the same frequency ω. Then

$$\begin{aligned} &(a) \quad \nabla \cdot (\hat{E}_1 \times \hat{H}_2 - \hat{E}_2 \times \hat{H}_1) = 0 \quad , \\ &(b) \quad \nabla \cdot (\hat{E}_1 \times \hat{H}_2^* + \hat{E}_2^* \times \hat{H}_1) = -2\sigma \hat{E}_1 \cdot \hat{E}_2^* \quad . \end{aligned} \tag{3.15}$$

Application of Eq.(3.15a) is now made to a pair of exponential modes on a cylindrical structure of the type in Figure 2.1 (page 7) where

$$\begin{aligned} \hat{E}_1 &= E_1 \, e^{-\gamma_1 z} \quad ; \qquad \hat{H}_1 = H_1 \, e^{-\gamma_1 z} \quad , \\ \hat{E}_2 &= E_2 \, e^{-\gamma_2 z} \quad ; \qquad \hat{H}_2 = H_2 \, e^{-\gamma_2 z} \quad . \end{aligned} \tag{3.16}$$

The result is that

$$\nabla \cdot \left[(E_1 \times H_2 - E_2 \times H_1) \, e^{-(\gamma_1 + \gamma_2) z} \right] = 0 \quad , \tag{3.17}$$

or

$$\begin{aligned} &\nabla \cdot (E_1 \times H_2 - E_2 \times H_1) \\ &= (\gamma_1 + \gamma_2) \, i_z \cdot (E_{T1} \times H_{T2} - E_{T2} \times H_{T1}) \quad . \end{aligned} \tag{3.18}$$

This last expression is next integrated over the cross section A of the guide, and the two-dimensional form of Gauss' theorem is applied on the left side of the equation;

$$\begin{aligned} &\int_L n \cdot (E_1 \times H_2 - E_2 \times H_1) \, d\ell \\ &= (\gamma_1 + \gamma_2) \int_A i_z \cdot (E_{T1} \times H_{T2} - E_{T2} \times H_{T1}) \, d\sigma \quad . \end{aligned} \tag{3.19}$$

But since each of the fields satisfies the boundary conditions (2.26), with the dyadic $\overline{Y}$ in the symmetric form (2.28), it follows that on the contour L

$$\begin{aligned} n \cdot (E_1 \times H_2 - E_2 \times H_1) &= E_2 \cdot \overline{Y} \cdot E_1 - E_1 \cdot \overline{Y} \cdot E_2 \\ &= 0 \quad . \end{aligned} \tag{3.20}$$

As a result, Eq.(3.19) states that

$$\gamma_1 + \gamma_2 \neq 0 \longrightarrow$$

$$\int_A \mathbf{i}_z \cdot (E_{T1} \times H_{T2} - E_{T2} \times H_{T1})\, d\sigma = 0 \quad . \tag{3.21}$$

Now it has been shown in Section 3.2, Eq.(3.9), that corresponding to any given solution, such as field 2 above, there is always another solution $(-E_{T2}, H_{T2}, -\gamma_2)$ which satisfies all the conditions of the problem. For the latter, Eq.(3.21) reads

$$\gamma_1 - \gamma_2 \neq 0 \longrightarrow$$

$$\int_A \mathbf{i}_z \cdot (E_{T1} \times H_{T2} + E_{T2} \times H_{T1})\, d\sigma = 0 \quad . \tag{3.22}$$

Addition of Eqs.(3.21) and (3.22) completes the analysis, with the conclusion

$$\gamma_1 \pm \gamma_2 \neq 0 \longrightarrow$$

$$\int_A \mathbf{i}_z \cdot (E_{T1} \times H_{T2})\, d\sigma = 0 \quad . \tag{3.23}$$

> "Equation (3.23) is the formal statement of an orthogonality condition between any two different exponential modes on an inhomogeneous cylindrical structure of the 'closed' variety. The only exclusions occur when both waves have the same γ (and hence are essentially the same in the transverse plane), or if either is the 'reflected' counterpart of the other."

When the entire system is lossless ($\sigma \equiv 0$), Eq.(3.15b) becomes

$$\nabla \cdot (\hat{E}_1 \times \hat{H}_2^* + \hat{E}_2^* \times \hat{H}_1) = 0 \quad , \tag{3.24}$$

and the boundary conditions are

$$\begin{aligned} &(a) \quad n \times H_1 = j\overline{B} \cdot E_1 \quad , \\ &(b) \quad n \times H_2 = j\overline{B} \cdot E_2 \quad , \end{aligned} \tag{3.25}$$

with $\overline{B}$ entirely real. By steps similar to those in Eqs.(3.16) through (3.23), the resulting new orthogonality condition

$$\left.\begin{aligned} &\overline{Y} = j\overline{B} \\ &\sigma \equiv 0 \\ &\gamma_1 \pm \gamma_2 \neq 0 \end{aligned}\right\} \longrightarrow \int_A \mathbf{i}_z \cdot (E_{T1} \times H_{T2}^*)\, d\sigma = 0 \tag{3.26}$$

follows readily.

"Emphasis must be placed upon the fact that Eq.(3.23) holds for both dissipative and non-dissipative structures. When the structure is non-dissipative, however, Eqs.(3.26) and (3.23) become valid together."

Since condition (3.23) holds more generally than Eq.(3.26), it is the one which acts most effectively as an orthogonality condition. Equation (3.26) is useful primarily for the purpose of understanding energy relations in a dissipationless cylindrical guide on which several modes are present together.

It is interesting to mention that the present search for orthogonality properties was originally instituted with the thought that they might be of the form (3.26), and would be valid only for lossless structures. The reasoning was based upon the fact that in a lossless structure the time-average power flowing across every section of the guide must be the same, i.e., independent of z (5). Since, in a rough way, the cross terms between two different modes propagating simultaneously along the guide would involve exponentials of $(\gamma_1-\gamma_2^*)z$, with coefficients similar to the expression in Eq.(3.26), it was felt that these coefficients would have to vanish. Actually, it is possible to derive Eq.(3.26), as it stands, from a consideration of the Poynting theorem applied to a lossless structure with two modes on it; but the derivation misses condition (3.23) completely. Apparently these power-orthogonality conditions should be looked upon as restatements of the reciprocity theorem, rather than consequences of Poynting's theorem.

The usefulness of Eq.(3.23) as an orthogonality condition arises in the problem of finding the coefficients in a transverse-field expansion. If it is assumed that the set of exponentials modes is complete, then the expression for any possible transverse field in the guide may be written in the form

$$\text{(a)} \quad \hat{E}_T = \sum_n A_n E_{Tn} e^{-\gamma_n z} + \sum_n B_n E_{Tn} e^{\gamma_n z} ,$$

$$\text{(b)} \quad \hat{H}_T = \sum_n A_n H_{Tn} e^{-\gamma_n z} - \sum_n B_n H_{Tn} e^{\gamma_n z} , \qquad (3.27)$$

in which (E_{Tn}, H_{Tn}) are the transverse fields appropriate to the propagation constant γ_n. If the fields $\hat{E}_T$ and $\hat{H}_T$ are given over a particular cross section z = 0, then A_n and B_n must be found from the equations

$$\begin{aligned} &\text{(a)} \quad E_T = \sum_n (A_n + B_n) E_{Tn} , \\ &\text{(b)} \quad H_T = \sum_n (A_n - B_n) H_{Tn} . \end{aligned} \tag{3.28}$$

Equation (3.28a) may be cross-multiplied by H_{Tn}, and then dot-multiplied by the unit vector i_z. From Eq.(3.23), a cross-sectional integration of the resulting equation yields

$$A_n + B_n = \frac{\int_A i_z \cdot (E_T \times H_{Tn})\, d\sigma}{\int_A i_z \cdot (E_{Tn} \times H_{Tn})\, d\sigma} . \tag{3.29}$$

By similar steps, Eq.(3.28b) furnishes the expression

$$A_n - B_n = \frac{\int_A i_z \cdot (E_{Tn} \times H_T)\, d\sigma}{\int_A i_z \cdot (E_{Tn} \times H_{Tn})\, d\sigma} . \tag{3.30}$$

It is a simple matter to solve Eqs.(3.29) and (3.30) for the coefficients A_n and B_n.

While a determination of these coefficients by no means proves the completeness of the set of free modes for the expansion of given transverse fields, it is an aid to such expansions once the completeness of the set is known.

3.4 Power and Energy Consequences of the Orthogonality Conditions

In spite of the fact that Eqs.(3.23) and (3.26) spring from the reciprocity theorem, it is profitable to examine the consequences of these equations in terms of energy propagation when two modes exist simultaneously on the given structure.

Let the two modes have transverse fields whose instantaneous values are given by:

$$\left.\begin{aligned} E_{T1}(t) &= \tfrac{1}{2}\left[E_{T1}e^{j\omega t-\gamma_1 z} + E_{T1}^{*}e^{-j\omega t-\gamma_1^{*}z}\right] \\ H_{T1}(t) &= \tfrac{1}{2}\left[H_{T1}e^{j\omega t-\gamma_1 z} + H_{T1}^{*}e^{-j\omega t-\gamma_1^{*}z}\right] \end{aligned}\right\},$$

$$\left.\begin{aligned} E_{T2}(t) &= \tfrac{1}{2}\left[E_{T2}e^{j\omega t-\gamma_2 z} + E_{T2}^{*}e^{-j\omega t-\gamma_2^{*}z}\right] \\ H_{T2}(t) &= \tfrac{1}{2}\left[H_{T2}e^{j\omega t-\gamma_2 z} + H_{T2}^{*}e^{-j\omega t-\gamma_2^{*}z}\right] \end{aligned}\right\}, \qquad (3.31)$$

where the first group represents mode 1, and the second mode 2. It is assumed that $\gamma_1 \pm \gamma_2 \neq 0$ and $\gamma_1 \pm \gamma_2^{*} \neq 0$.

The total instantaneous Poynting vector has a longitudinal component $S_z(t)$ given by

$$S_z(t) = S_{z11}(t) + S_{z22}(t) + S_{zc}(t) \quad . \qquad (3.32)$$

The terms $S_{z11}(t)$ and $S_{z22}(t)$ are instantaneous longitudinal power flows for modes 1 and 2, respectively, as though each were propagating alone. The general form for such "self power", $S_{z\nu\nu}(t)$, in terms of the complex fields, is obtained from Eq. (3.31):

$$\begin{aligned} 2e^{2\alpha_\nu z}\, S_{z\nu\nu}(t) &= \mathrm{Re}\Big[\, i_z\cdot(E_{T\nu}\times H_{T\nu}^{*}) \\ &\qquad + i_z\cdot(E_{T\nu}\times H_{T\nu})\, e^{j2(\omega t-\beta_\nu z)}\Big] , \end{aligned} \qquad (3.33)$$

where the notation $\gamma_\nu = \alpha_\nu + j\beta_\nu$ has been employed. $S_{z\nu\nu}(t)$ therefore contains the familiar time-average part and the usual double-frequency, or time-dependent part.

The remaining term in Eq.(3.32) represents a "cross term", and actually comprises two factors, condensed into the combined form $S_{zc}(t)$. It is in fact the presence of two cross terms in the total cross power which makes the derivation of the orthogonality condition (3.26) from Poynting's theorem somewhat more difficult than might first be anticipated. The combination of these terms, represented by $S_{zc}(t)$, is written:

$$2e^{(\alpha_1+\alpha_2)z} S_{zc}(t) =$$

$$\mathrm{Re}\left\{ i_z\cdot\left(E_{T1}\times H^*_{T2} + E^*_{T2}\times H_{T1}\right) e^{-j(\beta_1-\beta_2)z} + i_z\cdot\left(E_{T1}\times H_{T2} + E_{T2}\times H_{T1}\right) e^{j[2\omega t-(\beta_1+\beta_2)z]}\right\}. \quad (3.34)$$

$S_{zc}(t)$ also contains a part which is independent of time, and a double-frequency part.

The essence of Eq.(3.23), therefore, is that the time-dependent part of $S_{zc}(t)$ integrates to zero over the cross section. This orthogonality condition therefore can be interpreted to state that:

> "When two modes are present together, the time-varying part of the integrated longitudinal power flow along even a dissipative inhomogeneous guide can be computed as though each mode were propagating by itself."

On the other hand, Eq.(3.26) does not hold generally in an inhomogeneous system with loss, so that in such cases the time-average power can be expected to contain additional terms due to mutual interaction between the modes.

When the system is lossless, both Eq.(3.23) and Eq.(3.26) are valid together. As a result, the entire instantaneous cross power $S_{zc}(t)$ integrates to zero over any cross section:

> "The total instantaneous longitudinal power flow down the guide is the simple sum of the corresponding flows for each mode alone, provided that the structure is without loss."

Insofar as the vector power is concerned, the longitudinal component of the complex Poynting vector must be examined. When two modes are present simultaneously, the form thereof will be

$$2S_z = i_z\cdot(E_{T1}\times H^*_{T1})e^{-2\alpha_1 z} + i_z\cdot(E_{T2}\times H^*_{T2})e^{-2\alpha_2 z} + i_z\cdot(E_{T1}\times H^*_{T2} + E^*_{T2}\times H_{T1})e^{-[(\alpha_1+\alpha_2)+j(\beta_1-\beta_2)]z} \quad (3.35)$$

When the system contains loss, the orthogonality condition (3.23) gives no information about the vector power. It is to be expected, therefore, that cross terms will appear in both the average (or active) power flow and the reactive power flow. But if the structure is lossless, the validity of Eq.(3.26) under these special circumstances means that the third term of Eq.(3.35) integrates to zero over the guide cross section. Then the conclusion must be:

> "The total vector power flowing down a lossless inhomogeneous guide can also be calculated as a simple sum of the corresponding flows for each mode separately."

As regards the energy orthogonalities in Eqs.(3.11) and (3.13), it is possible to obtain relations somewhat similar to these for the inhomogeneous structure. It will be seen, however, that in general the integrals do not vanish correspondingly.

In order to develop the desired analogy of Eq.(3.11), it is convenient to consider first a modified form of Poynting's theorem. For lack of a common name, it may be called the "double-frequency" Poynting theorem. The derivation of this theorem follows closely the method pursued in developing the usual complex Poynting theorem, and the result becomes

$$\nabla\cdot(\hat{E}\times\hat{H}) = -j\omega(\epsilon'\hat{E}\cdot\hat{E} + \mu\hat{H}\cdot\hat{H}) \quad . \qquad (3.36)$$

When two modes are simultaneously present, $\hat{E} = \hat{E}_1 + \hat{E}_2$ and $\hat{H} = \hat{H}_1 + \hat{H}_2$. It is assumed that each of the fields 1 and 2 is itself a solution to the Maxwell equations, and hence each satisfies Eq.(3.36) when the other is absent. Therefore Eq.(3.36) becomes

$$\nabla\cdot(\hat{E}_1\times\hat{H}_2 + \hat{E}_2\times\hat{H}_1) = -2j\omega(\epsilon'\hat{E}_1\cdot\hat{E}_2 + \mu\hat{H}_1\cdot\hat{H}_2) \quad . \qquad (3.37)$$

But the reciprocity theorem (3.15a) may be used to reduce the left side of Eq.(3.37) to a single term, so that

$$\nabla\cdot(\hat{E}_1\times\hat{H}_2) = -j\omega(\epsilon'\hat{E}_1\cdot\hat{E}_2 + \mu\hat{H}_1\cdot\hat{H}_2) \quad . \qquad (3.38)$$

When both modes are exponential, an expansion of the divergence term yields the result

$$\nabla\cdot(E_1\times H_2) - (\gamma_1+\gamma_2)\, i_z\cdot(E_{T1}\times H_{T2})$$
$$= -j\omega(\epsilon'E_1\cdot E_2 + \mu H_1\cdot H_2) \quad . \qquad (3.39)$$

which, integrated over the cross section A, becomes

$$\int_L n\cdot(E_1\times H_2)d\ell - (\gamma_1+\gamma_2)\int_A i_z\cdot(E_{T1}\times H_{T2})d\sigma$$
$$= -j\omega\int_A(\epsilon' E_1\cdot E_2 + \mu H_1\cdot H_2)d\sigma \quad . \qquad (3.40)$$

In view of the boundary conditions (2.26) and (2.28), as well as the restriction (3.23) when $\gamma_1\pm\gamma_2 \neq 0$, the above equation becomes

$$\gamma_1\pm\gamma_2 \neq 0 \longrightarrow$$
$$\int_L E_1\cdot\overline{Y}\cdot E_2 d\ell = j\omega\int_A(\epsilon' E_1\cdot E_2 + \mu H_1\cdot H_2)d\sigma \quad , \qquad (3.41)$$

or, in the more expanded form,

$$\gamma_1\pm\gamma_2 \neq 0 \longrightarrow$$
$$\int_L y_{\tau\tau}E_{\tau1}E_{\tau2}d\ell + \int_L y_{zz}E_{z1}E_{z2}d\ell$$
$$= j\omega\int_A(\epsilon' E_{T1}\cdot E_{T2}+\epsilon' E_{z1}E_{z2}+\mu H_{T1}\cdot H_{T2}+\mu H_{z1}H_{z2})d\sigma \quad . \qquad (3.42)$$

Since, however, Eq.(3.42) is valid for any two fields under the indicated restrictions on γ, the alternate field of Eq.(3.9) can be substituted for field 2 in the former, with the result that

$$-\int_L(y_{\tau\tau}E_{\tau1}E_{\tau2} + y_{zz}E_{z1}E_{z2})d\ell$$
$$= j\omega\int_A(-\epsilon' E_{T1}\cdot E_{T2}+\epsilon' E_{z1}E_{z2}+\mu H_{T1}\cdot H_{T2}-\mu H_{z1}H_{z2})d\sigma \quad ; \qquad (3.43)$$

whence addition and subtraction of Eqs.(3.42) and (3.43) yield respectively:

$$\text{(a)} \quad \int_A\mu H_{T1}\cdot H_{T2}d\sigma = -\int_A\epsilon' E_{z1}E_{z2}d\sigma + \frac{1}{j\omega}\int_L y_{zz}E_{z1}E_{z2}d\ell \quad ,$$
$$(3.44)$$
$$\text{(b)} \quad \int_A\epsilon' E_{T1}\cdot E_{T2}d\sigma = -\int_A\mu H_{z1}H_{z2}d\sigma + \frac{1}{j\omega}\int_L y_{\tau\tau}E_{\tau1}E_{\tau2}d\ell \quad .$$

Equation (3.44) is the more general analogy of Eq.(3.11), which was valid only for homogeneous problems. Unfortunately, there is no guarantee that any of the terms are zero in the more general case. It is also unfortunate that Eq.(3.41) requires cross terms, even in the time-dependent or double-frequency part of the total stored plus dissipated energy per unit length, when two modes are present together. This remains true even when the structure is lossless.

When, however, the wall is <u>opaque</u>, but the internal medium not necessarily homogeneous, Eq(3.41) becomes

$$\left.\begin{matrix}\overline{Y} = 0 \\ \text{or } \overline{Y} = \infty\end{matrix}\right\} \rightarrow \int_A (j\omega\epsilon' E_1 \cdot E_2 + j\omega\mu H_1 \cdot H_2) d\sigma = 0 \quad ; \qquad (3.45)$$

while Eq.(3.44) yields

$$\begin{aligned} &(a) \quad \int_A \mu H_{T1} \cdot H_{T2} d\sigma = -\int_A \epsilon' E_{z1} E_{z2} d\sigma \quad , \\ &(b) \quad \int_A \epsilon' E_{T1} \cdot E_{T2} d\sigma = -\int_A \mu H_{z1} H_{z2} d\sigma \quad . \end{aligned} \qquad (3.46)$$

In this case, then, Eq.(3.45) shows that the time-dependent part of the total stored plus dissipated energy per unit length can be computed as the sum of those contributions provided by the individual modes. Note that the time-dependent part of the stored electric, magnetic, or dissipated energies cannot <u>individually</u> be so computed because Eq.(3.46) does not guarantee the vanishing of the individual cross terms.

It might be assumed, from experience with membrane problems in accoustics, that corresponding to the orthogonality conditions

$$\begin{aligned} &(a) \quad \int_A E_{z1} E_{z2} d\sigma = 0 \quad , \\ &(b) \quad \int_A H_{z1} H_{z2} d\sigma = 0 \quad , \end{aligned} \qquad (3.47)$$

which are known to be valid in homogeneous problems, there ought to follow some analogous pair of "weighted" orthogonality conditions like

$$\left.\begin{aligned} &(a) \quad \int_A \epsilon E_{z1} E_{z2} d\sigma = 0 \quad , \\ &(b) \quad \int_A \mu H_{z1} H_{z2} d\sigma = 0 \quad , \end{aligned}\right\} \quad \text{not generally true} \quad .$$

which would be valid at least in lossless structures with opaque walls. But it is not generally possible to obtain such a result from Eqs.(2.25). The reason apparently lies in the fact that the

TE-TM mixture takes the problem out of the purely scalar class, and there is no a priori reason to suppose, therefore, that such analogies with membrane problems in accoustics can be pushed so far.

The previous considerations have been directed toward the time-dependent, or double-frequency, parts of the various energies, in order to obtain results which would be valid for both systems with and without loss. The analogies of Eq.(3.11) were found, insofar as it was possible. There remains the problem of time-average energies, or the analogies of Eq.(3.13). Since there are no such analogies for an inhomogeneous problem with losses, discussion will be limited here to cases without loss.

From the conventional form of the complex Poynting theorem for a lossless system

$$\nabla\cdot(\hat{E}\times\hat{H}^*) = j\omega(\epsilon\hat{E}\cdot\hat{E}^* - \mu\hat{H}\cdot\hat{H}^*) \quad , \tag{3.48}$$

reasoning similar to that preceding Eq.(3.37) will lead to

$$\nabla\cdot(\hat{E}_1\times\hat{H}_2^* + \hat{E}_2\times\hat{H}_1^*) = j\omega\epsilon(\hat{E}_1\cdot\hat{E}_2^* + \hat{E}_1^*\cdot\hat{E}_2) - j\omega\mu(\hat{H}_1\cdot\hat{H}_2^* + \hat{H}_1^*\cdot\hat{H}_2) \quad , \tag{3.49}$$

when two modes are present in the guide at the same time. But Eq.(3.24) allows the following alteration of Eq.(3.49):

$$\nabla\cdot(\hat{E}_1\times\hat{H}_2^* - \hat{E}_1^*\times\hat{H}_2) = j\omega\epsilon(\hat{E}_1\cdot\hat{E}_2^* + \hat{E}_1^*\cdot\hat{E}_2) - j\omega\mu(\hat{H}_1\cdot\hat{H}_2^* + \hat{H}_1^*\cdot\hat{H}_2) \quad , \tag{3.50}$$

which is equivalent to

$$\mathrm{Im}\left[\nabla\cdot(\hat{E}_1\times\hat{H}_2^*)\right] = \mathrm{Im}\left[j\omega(\epsilon\hat{E}_1\cdot\hat{E}_2^* - \mu\hat{H}_1\cdot\hat{H}_2^*)\right] \quad . \tag{3.51}$$

Now the field $(\hat{E}_1,\hat{H}_1)$ is linearly independent of $(\hat{E}_2,\hat{H}_2)$, and may therefore be taken with any complex amplitude desired. In particular, Eq.(3.51) must remain true when $(\hat{E}_2,\hat{H}_2)$ is present with a new field $(j\hat{E}_1, j\hat{H}_1)$, just 90° (time phase) in advance of $(\hat{E}_1,\hat{H}_1)$. But then Eq.(3.51) would read

$$\mathrm{Im}\left[j\nabla\cdot(\hat{E}_1\times\hat{H}_2^*)\right] = \mathrm{Im}\left[j\omega(\epsilon j\hat{E}_1\cdot\hat{E}_2^* - \mu j\hat{H}_1\cdot\hat{H}_2^*)\right] \quad , \tag{3.52}$$

or

$$\mathrm{Re}\left[\nabla\cdot(\hat{E}_1\times\hat{H}_2^*)\right] = \mathrm{Re}\left[j\omega(\epsilon\hat{E}_1\cdot\hat{E}_2^* - \mu\hat{H}_1\cdot\hat{H}_2^*)\right] \quad . \tag{3.53}$$

It is concluded, therefore, that actually

$$\nabla\cdot(\hat{E}_1 \times \hat{H}_2^*) = j\omega(\epsilon\hat{E}_1\cdot\hat{E}_2^* - \mu\hat{H}_1\cdot\hat{H}_2^*) \quad . \tag{3.54}$$

When the modes are exponential, and $\gamma_1 \pm \gamma_2^* \neq 0$, it is possible to use the boundary conditions (2.37) and Eq.(3.26) in order to proceed from Eq.(3.54) to the result:

$$\int_L E_1\cdot\overline{B}\cdot E_2^*\, d\ell = \omega\int_A(\epsilon E_1\cdot E_2^* - \mu H_1\cdot H_2^*)d\sigma \quad . \tag{3.55}$$

Steps similar to Eq.(3.42) and (3.43) then establish the formulas

$$\begin{aligned}
&\text{(a)} \quad \int_A \mu H_{T1}\cdot H_{T2}^* d\sigma = \int_A \epsilon E_{z1}E_{z2}^* d\sigma - \frac{1}{\omega}\int_L b_{zz}E_{z1}E_{z2}^* d\ell \quad , \\
&\text{(b)} \quad \int_A \epsilon E_{T1}\cdot E_{T2}^* d\sigma = \int_A \mu H_{z1}H_{z2}^* d\sigma + \frac{1}{\omega}\int_L b_{\tau\tau}E_{\tau1}E_{\tau2}^* d\ell \quad ,
\end{aligned} \tag{3.56}$$

which are the desired analogies of Eq.(3.13), but are now restricted to lossless problems only.

Once again, the nature of the inhomogeneous problem prevents the possibility of finding any of the time-average individual stored energies by simply summing over those for each mode; for the cross terms do not vanish in general.

Even when the wall is opaque, and all the terms involving $\overline{B}$ go to zero, the best to be said, according to Eq.(3.55), is that the time-average difference between electric and magnetic stored energies (per unit length) is summable over the individual modes. If only the weighted orthogonality properties suggested on page 38 were actually true, then at least Eq.(3.56) would lead to "average-energy" orthogonality when the guide is bounded by opaque walls. But the examples in Appendices A and B will show that the hoped-for weighted orthogonalities are not true in general, and the matter must be left as it stands.

In summary then, an extension of the power-orthogonality conditions, found in Section (3.3), to the various energy orthogonalities mentioned here cannot generally be accomplished. It appears that the power orthogonalities are properties of the Maxwell equations and symmetries of the structure; in particular, they are consequences of the reciprocity theorem. They are therefore common to both homogeneous and inhomogeneous problems. The energy orthogonalities, however, depend essentially upon the scalar functions (E_z, H_z), and the particular differential equations and boundary conditions to which they are solutions. These equations lead to orthogonal scalar functions (E_z, H_z) for homogeneous problems, but counter examples show that they do not always behave similarly in the inhomogeneous cases. In other words;

> "the change from an homogeneous to an inhomogeneous structure must generally be paid for by giving up the 'energy summation' properties of the modes, although the 'power-summation' properties are at least partially retained."

3.5 Characteristics of the Propagation Constant γ

It has been observed in the previous section that some of the familiar orthogonality properties of modes on homogeneous structures are connected very directly with the Maxwell equations and the symmetries of the system. Other such properties depended upon the more special nature of the equations for E_z and H_z. The former properties were carried over to inhomogeneous structures, while the latter could not be so extended.

The purpose of the present section is to carry on a similar analysis with respect to additional mode properties, namely, some of the properties of γ. The point of departure is once again a brief statement about these matters with reference to homogeneous problems.

One of the most important facts about the modes in homogeneous problems is that the propagation constant γ must be either pure real or pure imaginary when the structure is nondissipative. Normally the proof(1c) depends upon an application of Green's theorem to the wave equations (3.1), with consequent demonstration that p^2 must be real. Actually, the proof is also valid for a homogeneous system with loss, so that the reality of p^2 is a consequence of only the opaque boundary conditions and the fact that the internal medium is uniformly distributed over the cross section. In the homogeneous problems, then, $\gamma^2 + k^2 = -p^2$ is always real, so that when k^2 is real, γ^2

is also real. If, however, k^2 is complex, then, since $\gamma = \alpha + j\beta$,

$$(\alpha^2-\beta^2) + 2j\alpha\beta + \omega^2\epsilon\mu - j\omega\mu\sigma = -p^2 \quad , \qquad (3.57)$$

which is entirely real. The imaginary part of Eq.(3.57) must be zero, therefore, and

$$\alpha\beta = \tfrac{1}{2}\omega\mu\sigma \qquad (3.58)$$

Hence α and β have the same sign, as long as $\sigma > 0$. It is interesting to observe here that the exclusion of waves which grow in the direction of propagation (when the system is passive, or $\sigma \geqq 0$) is not a separately imposed boundary condition for $z \longrightarrow \pm\infty$, but follows from the wave equations (3.1) for homogeneous problems.

Furthermore, when TE and TM waves are considered separately, the longitudinal component of the complex Poynting vector is given by

(a) TM ($H_z \equiv 0$)

$$2S_z^{(1)} = i_z \cdot (E_T^{(1)} \times H_T^{(1)*}) = \frac{-j\omega\epsilon'^*\gamma}{|p|^4}(\nabla_T E_z \cdot \nabla_T E_z^*)$$

$$= \frac{(\sigma - j\omega\epsilon)\gamma}{|p|^4} \left\|\nabla_T E_z\right\|^2 \quad ,$$

(3.59)

(b) TE ($E_z \equiv 0$)

$$2S_z^{(2)} = i_z \cdot (E_T^{(2)} \times H_T^{(2)*}) = \frac{j\omega\mu\gamma^*}{|p|^4} \left\|\nabla_T H_z\right\|^2 \quad ,$$

in which $|p|$ is the time magnitude of the complex scalar p, and $\left\|\nabla_T E_z\right\|$ is the space and time magnitude of the complex vector $\nabla_T E_z$, etc. It follows from Eqs.(3.58) and (3.59) that the algebraic sign of Re $S_z^{(1)}$ [and Re $S_z^{(2)}$] is always the same as that of β, whether or not the structure contains internal losses. The direction in which longitudinal time-average power flows at each point of the cross section is the same, and corresponds to the direction of the phase velocity. Of course, the integrated value of the time-average longitudinal power flow over the entire cross section then has the same property. In this connection it should be pointed out that when the structure is lossless, and the mode is "below cutoff" ($\gamma = \alpha$), neither $S_z^{(1)}$ nor $S_z^{(2)}$ in Eq.(3.59) has any real part. There can be no time-average longitudinal power flow at any point of the guide cross section when a single TE or TM mode is below cutoff. As a result, there is certainly no integrated value thereof over any cross section.

In considering inhomogeneous problems, the elementary facts presented above can no longer be obtained so easily from the nature of p^2, since it is a function of the coordinates in the transverse plane. An approach to them through Eqs.(2.25), (2.15), and (2.26) cannot easily be made in the same manner as is done for homogeneous problems; yet it must be felt intuitively that some of these facts are still true, and that more fundamental reasons than the particular form of the E_z - H_z equations should exist to prove them.

The primary concern of this section will be to prove that γ is either pure real or pure imaginary when the inhomogeneous structure is lossless; discussion of correlations similar to Eqs.(3.58) and (3.59)(between power flow and γ) will be considered in Part IV.

First of all, a general property of the complex fields $\hat{E}(\omega)$ and $\hat{H}(\omega)$ must be emphasized (1b). It is, in fact, independent of whether or not the structure has loss. In Section 2.1, page 6, the time-dependent fields E(t) and H(t) were required to be <u>real</u> vectors in space. Therefore the complex Maxwell equations in $\hat{E}(\omega)$ and $\hat{H}(\omega)$ are nothing but the Fourier transforms of the time-dependent Maxwell equations in E(t) and H(t), which means that $\hat{E}(\omega)$, for example, must, as a function of ω, be the Fourier transform of a <u>real</u> time function. Therefore

$$E(t) = \int_{-\infty}^{\infty} \hat{E}(\omega) e^{j\omega t}\, d\omega \quad . \qquad (3.60)$$

The substitution of $-\omega$ for ω in Eq.(3.60) results in the relation

$$E(t) = \int_{-\infty}^{\infty} \hat{E}(-\omega) e^{-j\omega t}\, d\omega \quad . \qquad (3.61)$$

But, from Eq.(3.60),

$$E^*(t) = \int_{-\infty}^{\infty} \hat{E}^*(\omega) e^{-j\omega t} \quad , \tag{3.62}$$

and, since E(t) is entirely real,

$$E^*(t) \equiv E(t) \tag{3.63}$$

for all values of t. Thus the integrands (or transforms) in Eqs.(3.61) and (3.62) must be equal, or

$$\hat{E}(-\omega) = \hat{E}^*(\omega) \quad . \tag{3.64}$$

With $\hat{E}(\omega) = \hat{E}_R(\omega) + j\hat{E}_i(\omega)$, the result (3.64) means that

$$\begin{aligned} &(a) \quad \hat{E}_R(-\omega) = \hat{E}_R(\omega) \quad , \\ &(b) \quad \hat{E}_i(-\omega) = -\hat{E}_i(\omega) \quad , \end{aligned} \tag{3.65}$$

or the real part $\hat{E}_R(\omega)$ of $\hat{E}(\omega)$ is an even function of ω, while the imaginary part $\hat{E}_i(\omega)$ is odd in ω. Similar conclusions can be drawn about $\hat{H}(\omega)$ and the other complex field vectors; these conclusions are true for all values of (x,y,z) in the system.

If, then, a cylindrical system is under consideration, so that, for example,

$$\hat{E}(x,y,z,\omega) = E(x,y,\omega) e^{-\gamma(\omega) z} \quad , \tag{3.66}$$

then

$$\begin{aligned} &(a) \quad \hat{E}(-\omega) = E(x,y,-\omega) e^{-\gamma(-\omega) z} \quad , \\ &(b) \quad \hat{E}^*(\omega) = E^*(x,y,\omega) e^{-\gamma^*(\omega) z} \quad . \end{aligned} \tag{3.67}$$

But at z = 0, Eqs.(3.64) and (3.67) require that

$$E(-\omega) = E^*(\omega) \quad , \tag{3.68}$$

and since Eq.(3.64) must be true for all values of z, it follows [using Eqs.(3.67) and (3.68)] that

$$\gamma(-\omega) = \gamma^*(\omega) \quad . \tag{3.69}$$

In the notation $\gamma(\omega) = \alpha(\omega) + j\beta(\omega)$, Eq.(3.69) shows that α is an even function of frequency, while β is an odd function of frequency. Similar conclusions, of course, follow from Eq.(3.68) with respect to the fields E,H,etc., all of which have the property that their real parts are even functions of ω, while their imaginary parts are odd functions thereof.

That $E(\omega)$ and $H(\omega)$ are Fourier transforms of real time functions must also be true on the wall of the cylindrical structure. But the boundary conditions are

$$n \times H(\omega) = \overline{Y}(\omega) \cdot E(\omega) \quad , \tag{3.70}$$

whence the conjugate of Eq.(3.70) states that

$$n \times H^*(\omega) = \overline{Y}^*(\omega) \cdot E^*(\omega) \quad , \tag{3.71}$$

and the substitution of $-\omega$ for ω in Eq.(3.70) makes

$$n \times H(-\omega) = \overline{Y}(-\omega) \cdot E(-\omega) \quad . \tag{3.72}$$

If the boundary condition is to hold at all frequencies, and for all orientations of E, use of Eq.(3.68) for both E and H, along with Eqs.(3.71) and (3.72), shows that

$$\overline{Y}(-\omega) = \overline{Y}^*(\omega) \quad . \tag{3.73}$$

Then

$$\begin{aligned} &(a) \quad \overline{G}(-\omega) = \overline{G}(\omega) \quad , \\ &(b) \quad \overline{B}(-\omega) = -\overline{B}(\omega) \quad . \end{aligned} \tag{3.74}$$

Under the assumption that the structure is lossless, the complex Maxwell equations and the boundary conditions may be written

$$\begin{aligned} &(a) \quad \nabla \times E(\omega) - \gamma(\omega)\, i_z \times E(\omega) = -j\omega\mu H(\omega) \quad , \\ &(b) \quad \nabla \times H(\omega) - \gamma(\omega)\, i_z \times H(\omega) = j\omega\epsilon E(\omega) \quad , \\ &(c) \quad n \times H(\omega) = j\overline{B}(\omega) \cdot E(\omega) \quad , \end{aligned} \tag{3.75}$$

in which exponential z-dependence has been assumed. A solution for E and H from Eqs.(3.75a) and (3.75b) is inserted into Eq.(3.75c) to determine $\gamma(\omega)$. Consider that a solution for (E,H,γ) has been found at a frequency ω. These quantities obey Eq.(3.75), along with finiteness, single-valuedness, and continuity conditions mentioned previously. Recall also that the boundary condition (3.75c) determines only $\pm\gamma(\omega)$, or $\gamma^2(\omega)$, in accordance with the discussion of Section 3.2. The change of variable $\omega \rightarrow -\omega$ is now made in Eq.(3.75). Use of Eq.(3.74b) in Eq.(3.75c) will then lead to the result:

$$\text{(a)}\quad \nabla \times E(-\omega) - \gamma(-\omega)\, i_z \times E(-\omega) = j\omega\mu H(-\omega) \ ,$$

$$\text{(b)}\quad \nabla \times H(-\omega) - \gamma(-\omega)\, i_z \times H(-\omega) = -j\omega\epsilon E(-\omega) \ , \qquad (3.76)$$

$$\text{(c)}\quad n \times H(-\omega) = j\overline{B}(-\omega)\cdot E(-\omega) = -j\overline{B}(\omega)\cdot E(-\omega) \ .$$

Let a new field be defined as follows:

$$\text{(a)}\quad E' = E(-\omega) \ ,$$
$$\text{(b)}\quad H' = -H(-\omega) \ , \qquad (3.77)$$
$$\text{(c)}\quad \gamma' = \gamma(-\omega) \ .$$

Then the equations and boundary conditions satisfied by this new field can be found from Eq.(3.76), and are given by:

$$\text{(a)}\quad \nabla \times E' - \gamma' i_z \times E' = -j\omega\mu H' \ ,$$

$$\text{(b)}\quad \nabla \times H' - \gamma' i_z \times H' = j\omega\epsilon E' \ , \qquad (3.78)$$

$$\text{(c)}\quad n \times H' = +j\overline{B}\cdot E' \ .$$

In other words, the new primed field satisfies the same continuity conditions, and the same equations (3.78a) and (3.78b) as did the original unprimed field. Moreover, the boundary condition (3.78c) is exactly the same, too. Therefore, the functional equation which determines γ'^2 is exactly the same as that which determined γ^2 originally, and it follows that

$$\gamma'^2 = \gamma^2 \ ; \qquad (3.79)$$

or, from Eqs.(3.77) and (3.79),

$$\gamma(-\omega) = \pm\gamma(\omega) \ . \qquad (3.80)$$

But the result (3.80) can be taken with Eq.(3.69) to prove the desired theorem, because in combination they state that

$$\gamma(\omega) = \pm\gamma^*(\omega) \ . \qquad (3.81)$$

> "The propagation constant for a lossless cylindrical structure of the type considered in this paper must therefore be either pure real, or pure imaginary. It cannot be complex."

In any particular case, a study of the eigenvalue equation would normally be required to establish that there were no complex γ-roots thereof. Such a study is often tedious and difficult because of the transcendental functions involved. Once the theorem of Eq.(3.81) is established, however, studies of the above variety are not necessary.

There will be found in Appendices A and B two examples in which the conclusion of Eq.(3.81) is verified. They will suggest a method by means of which this corroboration can be made in particular problems without making a detailed study of the eigenvalue equation.

Perhaps it seems curious at first that the proof presented above depends in no way upon the law of conservation of energy (Poynting's theorem). It is probably natural to believe at first that energy conservation ought somehow to lie at the base of a theorem on the character of attenuation and phase shift. Yet further examination shows that conservation of energy is not a distinguishing factor between dissipative and non-dissipative systems; it is a common factor. More to the point, then, is the distinction that electric and magnetic energies are irreversibly transformed into heat when loss is present, and not so transformed when loss is absent. When heat is generated in the process, the "orderliness" of the system decreases with time. The state of affairs in the system "now" is no longer sufficient to determine what happened previously, although its future degenerations can be predicted therefrom. When, however, the structure is dissipationless, it must be possible to extrapolate from the present to both the past and the future, the criterion for these extrapolations being a reversal of the time coordinate. In the derivation of Eqs.(3.81) it was, in fact, necessary to consider the transformation $\omega \rightarrow -\omega$, which is the same as a time reversal when the time dependence is harmonic. The invariance of the lossless Maxwell equations and boundary conditions under a reversal of the time or frequency coordinates then forms the real basis for the distinction between structures with and without loss.

IV. PHYSICAL CHARACTERISTICS OF THE MODES

Further important properties of the individual exponential modes can now be investigated by means of the Poynting and energy theorems (1b).

Some correlations between γ and the longitudinal power flow will first be examined (Section 4.1) by applying Poynting's theorem to the given structure at a single frequency. These correlations differ somewhat from those discussed in Section 3.5 [Eqs.(3.58) and (3.59)], which were pertinent only to homogeneous problems.

A brief study of the frequency behavior of $\gamma(\omega)$ will next be undertaken (Section 4.2), mainly as a preliminary to the ensuing discussion of the behavior of the parameter p^2, and the physical significance of the space and frequency dependence thereof (Section 4.3).

Finally, some remarks will be made relative to the polarization of the transverse fields, leading to a short statement of the resultant difficulties encountered in trying to extend circuit concepts such as voltage, current, or impedance into the inhomogeneous waveguide problems (Section 4.47).

The major portion of these four sections will, however, be limited to consideration of lossless systems.

4.1 Mode Properties at a Single Frequency

The Poynting theorem, including loss, may be expressed in the form

$$\nabla\cdot(\hat{E}\times\hat{H}^*) = j\omega(\epsilon'^*\hat{E}\cdot\hat{E}^* - \mu\hat{H}\cdot\hat{H}^*) \quad . \tag{4.1}$$

For a single exponential mode, with $\gamma = \alpha + j\beta$, the above equation becomes

$$\nabla\cdot(E\times H^*) - 2\alpha i_z\cdot(E_T\times H_T^*) = j\omega(\epsilon'^*E\cdot E^* - \mu H\cdot H^*) \quad . \tag{4.2}$$

An integration over the cross section A, and an application of the boundary conditions, yields

$$-\int_L E\cdot\overline{Y}^*\cdot E^* d\ell \; - 2\alpha\int_A i_z\cdot(E_T\times H_T^*)d\sigma$$

$$= j\omega\int_A(\epsilon'^*E\cdot E^* - \mu H\cdot H^*)d\sigma \quad , \tag{4.3}$$

which can be split into real and imaginary parts, in accordance with the definitions $\overline{Y} = \overline{G} + j\overline{B}$ and $j\omega\epsilon' = \sigma + j\omega\epsilon$, as follows:

$$\begin{aligned}
&(a) \quad 2\alpha\int_A \text{Re } i_z\cdot(E_T \times H_T^*)da \\
&\qquad\qquad = \int_A \sigma E\cdot E^* da - \int_L E\cdot\overline{G}\cdot E^* d\ell = 0 \quad , \\
&(b) \quad 2\alpha\int_A \text{Im } i_z\cdot(E_T \times H_T^*)d\sigma \\
&\qquad\qquad = \omega\int_A(\mu H\cdot H^* - \epsilon E\cdot E^*)d\sigma + \int_L E\cdot\overline{B}\cdot E^* d\ell \quad .
\end{aligned} \tag{4.4}$$

The element of area in Eq.(4.4a) has been written as da instead of dσ to avoid confusion with the conductivity σ appearing explicity therein. It was previously stated in connection with Eqs.(2.35) and (2.36) that $E\cdot\overline{G}\cdot E^*$ is always negative, whence Eq.(4.4a) gives the assurance that, when any loss is present, α and the <u>total</u> time-average longitudinal power flow

$$\frac{1}{2}\int_A \text{Re } i_z\cdot(E_T \times H_T^*)d\sigma$$

have the same algebraic sign. In fact, 2α is merely the total real power loss per meter (in both the volume and the wall of the guide) per unit of total longitudinal real power flow through the cross section.

Equation (4.4b) shows explicity that a positive value of $E\cdot\overline{B}\cdot E^*$ on the wall is equivalent to additional <u>magnetic</u> stored energy within the volume, and further substantiates the statement to that effect made earlier (page 19).

Attention will now be directed to the case of a dissipationless structure, for which $\sigma \equiv 0$ and $\overline{G} \equiv 0$. Equation (4.4a) therefore requires that

$$2\alpha\int_A \text{Re } i_z\cdot(E_T \times H_T^*)d\sigma = 0 \quad . \tag{4.5}$$

This leaves two possibilities: either

$$\begin{aligned}
&(a) \quad \alpha = 0 \quad , \\
&\text{or} \\
&(b) \quad \alpha \neq 0 \text{ , but } \int_A \text{Re } i_z\cdot(E_T \times H_T^*)d\sigma = 0 \quad .
\end{aligned} \tag{4.6}$$

The first choice (4.6a) should correspond to a propagating wave, with $\gamma = j\beta$. Under such conditions (4.4b) requires that

$$\alpha = 0 \longrightarrow$$

$$\omega \int_A (\mu H \cdot H^* - \epsilon E \cdot E^*) d\sigma + \int_L E \cdot \overline{B} \cdot E^* d\ell = 0 \quad . \qquad (4.7)$$

The total time-average magnetic and electric stored energies per unit length (including that stored in the wall) must be equal for a purely propagating wave on a lossless structure. Reference to Eq.(3.55) will show that if two purely propagating modes ($\alpha_1 = \alpha_2 = 0$) are present simultaneously, the same remains true. It is possible to say, therefore, that:

> "No matter how many <u>purely propagating</u> modes are present at once, the total time-average electric and magnetic stored energies per unit length of lossless guide must be equal, provided that the wall is included in the calculation."

The second choice in Eq.(4.6) corresponds to $\alpha \neq 0$, with a corresponding damping of the mode. Observe, however, that Poynting's theorem supplies <u>no information at all about β</u>, although it does show that there is no <u>total</u> time-average power flow down the guide.

Under these conditions, the wave is below cutoff, for it has been shown already in Section 3.5 that $\beta = 0$ when $\alpha \neq 0$. Equation (4.4b) now gives a convenient interpretation to α, viz: α is just the time-average difference between magnetic and electric stored energies per unit length of guide and wall, per unit total reactive power flow along the guide.

A word of caution is in order here, lest it be assumed that

$$\mathrm{Re}\ \left[1_z \cdot (E_T \times H_T^*) \right]$$

<u>must</u> be zero at each point of the cross section, merely because the wave is below cutoff. It is true that the <u>integrated value</u> must vanish, according to Eq.(4.6b). It is also true that when either TE or TM modes exist alone, the longitudinal component of S_z also becomes imaginary at every point of the cross section for a wave below cutoff (Eq.(3.59) ff.). But when the problem is inhomogeneous, TE and TM modes are generally mixed, and it

will be shown below that the extrapolation from the properties of the total power flow to the power flow at a point may no longer be possible.

In these more general circumstances, the longitudinal component of the Poynting vector may be calculated from Eq.(2.15), to yield

$$2S_z = i_z \cdot (E_T \times H_T^*)$$

$$= \frac{1}{|p|^4}\Big[|\gamma|^2 i_z \cdot (\nabla_T E_z \times \nabla_T H_z^*) + k^2 i_z \cdot (\nabla_T E_z^* \times \nabla_T H_z)$$

$$+ j\omega\mu\gamma^* \|\nabla_T H_z\|^2 - j\omega\epsilon\gamma \|\nabla_T E_z\|^2\Big] \quad . \qquad (4.8)$$

The real part becomes

$$2\mathrm{Re}\ S_z = \frac{1}{|p|^4}\Big[\left(|\gamma|^2 + k^2\right) \mathrm{Re}\ i_z \cdot (\nabla_T E_z \times \nabla_T H_z^*)$$

$$+ \omega\beta\left(\mu\|\nabla_T H_z\|^2 + \epsilon\|\nabla_T E_z\|^2\right)\Big] \quad , \qquad (4.9)$$

which for a wave below cutoff reduces to

$$\left.\begin{matrix}\alpha \neq 0 \\ \beta = 0\end{matrix}\right\} \longrightarrow \mathrm{Re}\ S_z = \frac{1}{2}\,\frac{\mathrm{Re}\left[i_z \cdot (\nabla_T E_z \times \nabla_T H_z^*)\right]}{(\alpha^2 + k^2)} \quad . \qquad (4.10)$$

It is shown later (Section 4.4) that it is always possible to choose E_z and H_z 90° out of phase below cutoff; if such a choice is elected, then the Re S_z will vanish everywhere, along with its integrated value. But it is also shown that in many symmetrical problems such choice is not necessary. Therefore Eqs.(4.6) and (4.10) show that unless Re S_z is identically zero for frequencies below cutoff, it must necessarily be positive over some portions of the cross section and negative over others; otherwise the integrated power could not vanish. In Appendix C appears a very simple example of a mixed TE-TM mode illustrating this behavior below cutoff. A somewhat more satisfactory example is furnished also by Appendix A. A more thorough understanding of these matters will be gained only after the completion of Section 4.4.

With reference to Appendix C again, a second peculiarity becomes evident. It will be observed in the example that when the TE - TM mode is above cutoff, the Re S_z may still be negative over some portions of the cross section and positive over others. There is no general restriction on the integrated real power flow above cutoff, however, since it is expected that then there will be a total power flow in one direction or the other along the guide.

A return to Eq.(4.9) will show that the phenomenon in question is not too surprising. For a wave above cutoff, the latter equation becomes

$$\left.\begin{array}{l}\alpha=0\\ \beta\neq 0\end{array}\right\}\longrightarrow$$

$$2\text{Re } S_z = \frac{1}{|p|^4}\Big[(\beta^2+k^2) \text{ Re } i_z\cdot(\nabla_T E_z \times \nabla_T H_z^*) + \omega\beta\left(\mu\left\|\nabla_T H_z\right\|^2 + \epsilon\left\|\nabla_T E_z\right\|^2\right)\Big] \quad . \qquad (4.11)$$

Equation (4.11) shows that when $\beta>0$, for example, Re S_z will become negative at any point where

$$\text{Re } i_z\cdot(\nabla_T E_z \times \nabla_T H_z^*)$$

becomes negative, and where the first term exceeds the second term in magnitude. The example in Appendix C shows that this situation may indeed occur, in spite of the fact that Eq.(4.11) might appear at first glance to be restricted in sign by a special form of the Schwartz inequality. Equation (4.11), however, is not quite in the form of the inequality in question, because the latter springs from the fact that

$$(\sqrt{\omega\mu}\,\nabla_T H_z + \sqrt{\omega\epsilon}\; i_z\times\nabla_T E_z)\cdot(\sqrt{\omega\mu}\,\nabla_T H_z^* + \sqrt{\omega\epsilon}\; i_z\times\nabla_T E_z^*) \geq 0 \;. (4.12)$$

When expanded, and then multiplied through by $\beta>0$, Eq.(4.12) becomes

$$2\beta k\text{Re } i_z\cdot(\nabla_T E_z\times\nabla_T H_z^*) + \omega\beta(\mu\left\|\nabla_T H_z\right\|^2 + \epsilon\left\|\nabla_T E_z\right\|^2) \geq 0 \;. (4.13)$$

The realization that

$$(\beta-k)^2 = (\beta^2 + k^2) - 2\beta k \geq 0 \tag{4.14}$$

will show that Re $S_z < 0$ in Eq.(4.11) does not in any way contradict the general inequality (4.13).

> "It is a consequence of the essentially vector character of the TE-TM modes on inhomogeneous lossless structures that the correlation between the direction of active power flow at a point and the algebraic sign of β is no longer necessarily unique. Moreover, there may be active power flow in both directions at various points in the cross section, even when a mode is below cutoff."

Other connections between the character of γ and the flow of vector power down the guide may be obtained from the relation between any given wave and its corresponding "reflected" wave, defined in Eq.(3.9). Let the given wave be described by $E_T^{(+)}$, $H_T^{(+)}$, $\gamma^{(+)} = \gamma$, with $\beta > 0$ so that it travels to the right (+z). For the moment, assume γ is complex, even though the structure is lossless. Under the boundary conditions (2.37), the field (3.9) with

$$E_T^{(-)} = -E_T^{(+)} \quad ,$$
$$H_T^{(-)} = H_T^{(+)} \quad ,$$
$$\gamma^{(-)} = -\gamma^{(+)} = -\gamma \; ,$$

is also a solution to the guide problem. Hence the sum

$$\hat{E}_T = E_T^{(+)} \, (e^{-\gamma z} - e^{\gamma z}) \quad ,$$

$$\hat{H}_T = H_T^{(+)} \, (e^{-\gamma z} + e^{\gamma z}) \quad , \tag{4.15}$$

is a solution, too. In fact, this particular combination is appropriate to the solution of a problem involving the cylindrical structure with a perfectly conducting metal wall across the guide at z = 0. The field may be considered as existing only for $z < 0$, so that the wave traveling to the right becomes the "incident" wave, while that traveling to the left becomes the "reflected" wave.

An application of the general Poynting theorem to the lossless, source-free, volume enclosed by the shorting plane at $z = 0$, the guide wall at $z \leq 0$, and <u>any</u> cross section A for $z < 0$, will show that

$$\int_A \mathrm{Re}\ i_z \cdot (\hat{E}_T \times \hat{H}_T^*) d\sigma = 0 \quad , \tag{4.16}$$

provided the wall is lossless ($\overline{Y} = j\overline{B}$). Now from Eq.(4.15)

$$\begin{aligned} & i_z \cdot (\hat{E}_T \times \hat{H}_T^*) \\ &= i_z \cdot (E_T^{(+)} \times H_T^{(+)*}) \ (e^{-\gamma z} - e^{\gamma z})(e^{-\gamma^* z} + e^{\gamma^* z}) \\ &= -2\ i_z \cdot (E_T^{(+)} \times H_T^{(+)*}) \ (\sinh 2\alpha z + j \sin 2\beta z) \quad . \end{aligned} \tag{4.17}$$

Therefore

$$\begin{aligned} & \tfrac{1}{2} \int_A \mathrm{Re}\ i_z \cdot (\hat{E}_T \times \hat{H}_T^*) d\sigma \\ &= \sin 2\beta z \int_A \mathrm{Im}\ i_z \cdot (E_T^{(+)} \times H_T^{(+)*}) d\sigma \\ &\qquad - \sinh 2\alpha z \int_A \mathrm{Re}\ i_z \cdot (E_T^{(+)} \times H_T^{(+)*}) d\sigma = 0 \quad , \end{aligned} \tag{4.18}$$

and Eq.(4.18) must hold for all values of $z \leq 0$.

The only new conclusion resulting from Eq.(4.18) is that in a purely propagating wave ($\alpha = 0$, $\beta \neq 0$) there is no integrated reactive power flow in the longitudinal direction, or

$$\left.\begin{matrix} \alpha = 0 \\ \beta \neq 0 \end{matrix}\right\} \longrightarrow \int_A \mathrm{Im}\ i_z \cdot (E_T \times H_T^*) d\sigma = 0 \quad . \tag{4.19}$$

The superscript (+) has been dropped in Eq.(4.19) because the expression now refers to only a single wave.

Thus Eq.(4.19) essentially completes the information given previously in Eq.(4.6). Taken together, they show that:

> "On a lossless inhomogeneous guide, a mode below cutoff ($\alpha \neq 0, \beta = 0$) carries no total active power, while a mode above cutoff ($\alpha = 0, \beta \neq 0$) carries no total reactive power."

At least in this respect, the lossless homogeneous and inhomogeneous structure have similar behavior.

4.2 Frequency Behavior of the Propagation Constant

A study of mode behavior as a function of frequency will lead to further understanding of their properties. The energy theorem (1b) forms a convenient basis for such a study, and when applied to lossless structures, without sources, it may be written

$$\nabla\cdot\left(\hat{E}^*\times\frac{\partial\hat{H}}{\partial\omega}+\frac{\partial\hat{E}}{\partial\omega}\times\hat{H}^*\right)=-j\left(\epsilon\hat{E}\cdot\hat{E}^*+\mu\hat{H}\cdot\hat{H}^*\right)\quad, \tag{4.20}$$

and if $\gamma=\alpha+j\beta$, then

$$\hat{E}^*\times\frac{\partial\hat{H}}{\partial\omega}+\frac{\partial\hat{E}}{\partial\omega}\times\hat{H}^*=\left(E^*\times\frac{\partial H}{\partial\omega}+\frac{\partial E}{\partial\omega}\times H^*\right)e^{-2\alpha z}$$
$$-\frac{\partial\gamma}{\partial\omega}\left(E^*\times H+E\times H^*\right)ze^{-2\alpha z}\quad. \tag{4.21}$$

Now

$$\nabla\cdot\left[\left(E^*\times\frac{\partial H}{\partial\omega}+\frac{\partial E}{\partial\omega}\times H^*\right)e^{-2\alpha z}\right]$$
$$=\left[\nabla\cdot\left(E^*\times\frac{\partial H}{\partial\omega}+\frac{\partial E}{\partial\omega}\times H^*\right)\right.$$
$$\left.-2\alpha i_z\cdot\left(E_T^*\times\frac{\partial H_T}{\partial\omega}+\frac{\partial E_T}{\partial\omega}\times H_T^*\right)\right]e^{-2\alpha z}\quad, \tag{4.22}$$

and

$$\nabla\cdot\left[\left(E^*\times H+E\times H^*\right)ze^{-2\alpha z}\right]$$
$$=2\nabla\cdot\left\{\mathrm{Re}\left[\left(E\times H^*\right)\right]ze^{-2\alpha z}\right\}$$
$$=2\,\mathrm{Re}\left\{ze^{-2\alpha z}\left[\nabla\cdot\left(E\times H^*\right)-2\alpha i_z\cdot\left(E_T\times H_T^*\right)\right]\right.$$
$$\left.+e^{-2\alpha z}i_z\cdot\left(E_T\times H_T^*\right)\right\}\quad. \tag{4.23}$$

But from the real part of the complex Poynting theorem, Eq.(4.2), the first term on the right side of Eq.(4.23) reduces to zero.

Therefore a substitution of Eq.(4.22) and the modified Eq.(4.23) into Eq.(4.21), makes the energy theorem assume the new form

$$\nabla\cdot\left(E^*\times\frac{\partial H}{\partial\omega}+\frac{\partial E}{\partial\omega}\times H^*\right)-2\alpha i_z\cdot\left(E_T^*\times\frac{\partial H_T}{\partial\omega}+\frac{\partial E_T}{\partial\omega}\times H_T^*\right)$$

$$-2\frac{\partial\gamma}{\partial\omega}\,\mathrm{Re}\left[i_z\cdot(E_T\times H_T^*)\right]=-j(\epsilon E\cdot E^*+\mu H\cdot H^*)\quad. \tag{4.24}$$

In anticipation of an integration of Eq.(4.24) over the cross section of the guide, some additional relations should be derived from the boundary conditions (2.37). Since

$$\frac{\partial H}{\partial\omega}\times n=-j\left(\overline{B}\cdot\frac{\partial E}{\partial\omega}+\frac{\partial\overline{B}}{\partial\omega}\cdot E\right)\quad, \tag{4.25}$$

and since $\overline{B}$ is a symmetric dyadic, therefore

(a) $$n\cdot\left(E^*\times\frac{\partial H}{\partial\omega}\right)=E^*\cdot\left(\frac{\partial H}{\partial\omega}\times n\right)$$

$$=-j\left(E^*\cdot\overline{B}\cdot\frac{\partial E}{\partial\omega}+E^*\cdot\frac{\partial\overline{B}}{\partial\omega}\cdot E\right)\quad, \tag{4.26}$$

(b) $$n\cdot\left(\frac{\partial E}{\partial\omega}\times H^*\right)=\frac{\partial E}{\partial\omega}\cdot(H^*\times n)=jE^*\cdot\overline{B}\cdot\frac{\partial E}{\partial\omega}\quad.$$

On the boundary wall, then, addition of Eqs.(4.26a) and (4.26b) shows that the relation

$$n\cdot\left(E^*\times\frac{\partial H}{\partial\omega}+\frac{\partial E}{\partial\omega}\times H^*\right)=-jE\cdot\frac{\partial\overline{B}}{\partial\omega}\cdot E^* \tag{4.27}$$

is valid, with the consequence that the contemplated cross-sectional integration of Eq.(4.24) leads to the formula

$$j\int_L E\cdot\frac{\partial\overline{B}}{\partial\omega}\cdot E^*d\ell+2\alpha\int_A i_z\cdot\left(E_T^*\times\frac{\partial H_T}{\partial\omega}+\frac{\partial E_T}{\partial\omega}\times H_T^*\right)d\sigma$$

$$+2\frac{\partial\gamma}{\partial\omega}\int_A \mathrm{Re}\ i_z\cdot(E_T\times H_T^*)d\sigma=j\int_A(\epsilon E\cdot E^*+\mu H\cdot H^*)d\sigma\quad. \tag{4.28}$$

At any frequency for which the mode propagates, $\alpha = 0$ and $\gamma = j\beta$. In such event, the imaginary part of Eq.(4.28) states that

$$2\frac{\partial\beta}{\partial\omega}\int_A \mathrm{Re}\ i_z\cdot(E_T\times H_T^*)d\sigma$$

$$=\int_A(\epsilon E\cdot E^*+\mu H\cdot H^*)d\sigma-\int_L E\cdot\frac{\partial\overline{B}}{\partial\omega}\cdot E^*d\ell\quad. \tag{4.29}$$

With the stipulation from Eq.(2.39) that

$$E\cdot \frac{\partial \overline{B}}{\partial \omega}\cdot E^* \leq 0 \quad , \tag{4.30}$$

it may be concluded from Eq.(4.29) that:

"$(\partial\beta/\partial\omega)$ and the integrated time-average longitudinal power flow have the same algebraic sign".

In fact, the equation shows that the group velocity $(\partial\beta/\partial\omega)^{-1}$ is also the velocity of energy propagation, since it is merely the real power flow divided by the time-average total energy stored per unit length of guide.

While this correlation between the sign of $(\partial\beta/\partial\omega)$ and that of the integrated power flow holds equally well for both homogeneous and inhomogeneous lossless structures, it is to be observed that in the latter there has not been given any unique connection between the sign of β and that of the integrated power flow. It is entirely possible for the group velocity and the integrated power flow to be negative when β is positive. The investigation of the eigenvalue equation in Appendix A, under the condition that the wall admittances were independent of frequency, led , in fact, to some slow modes for which β and $(\partial\beta/\partial\omega)$ had opposite signs. While this eigenvalue study has not been included in Appendix A for reasons mentioned previously (page 15), it might eventually turn out that a proper choice of reactive wall [according to Eq.(2.40)] would nevertheless lead to this same integrated-power reversal. Since the problem in question concerns a guide with a reactive wall, it is possible that this power-reversal phenomenon is really only a special case of the previously considered correlation difficulties between β and the power flow at a point (or small region); in a sense, the guide cross section inside a reactive wall is only a part of the entire "system cross section". The wall has, in other words, merely replaced and obscured the details of what happens "outside", and may very well be imagined to conceal an "external" region in which the total power flow is oppositely directed.

It will be appreciated later in this section, on the other hand, that β and $(\partial\beta/\partial\omega)$ will always have the same sign if the bounding wall is <u>opaque</u>, even though the internal medium may not be uniform in the cross section. Then the <u>integrated</u> power flow and the algebraic sign of β will be correlated in the conventional way.

It is profitable to continue the investigation of mode properties by examining them

a) at cutoff,

b) at high frequencies.

First let it be supposed that a cutoff exists, where $\gamma = 0$ and $\omega = \omega_c > 0$. The interpretation of the resulting picture will then suggest conditions under which no true cutoff should be expected.

At such a cutoff, therefore, equations (2.25) become

$$\begin{aligned} &(a) \quad \nabla_T^2 E_z + k^2 E_z = \frac{\nabla_T \mu}{\mu} \cdot \nabla_T E_z \ , \\ &(b) \quad \nabla_T^2 H_z + k^2 H_z = \frac{\nabla_T \epsilon}{\epsilon} \cdot \nabla_T H_z \ . \end{aligned} \qquad (4.31)$$

The significant fact about Eqs.(4.31) is the absence of $E_z - H_z$ cross terms. So far as the internal medium is concerned, the TE and TM waves which normally form a single mode are now completely independent.

According to Eqs.(3.2), with $\gamma = 0$, the transverse fields are given by

$$\underset{(E_z \equiv 0)}{\text{TE}} \qquad \begin{aligned} &H_T^{(2)} \equiv 0 \\ &E_T^{(2)} = -\frac{1}{j\omega\epsilon} \, i_z \times \nabla_T H_z \ , \\ &H_z \neq 0 \end{aligned} \qquad (4.32)$$

$$\underset{(H_z \equiv 0)}{\text{TM}} \qquad \begin{aligned} &E_T^{(1)} \equiv 0 \\ &H_T^{(1)} = \frac{1}{j\omega\mu} \, i_z \times \nabla_T E_z \ . \\ &E_z \neq 0 \end{aligned} \qquad (4.33)$$

The boundary conditions (2.37) may therefore also be satisfied now by TE or TM waves alone. The entire problem of the guide reduces to one in only two dimensions. There is no z-dependence for any field component, and no total vector power flowing along the guide. The TE and TM modes in Eqs.(4.32) and (4.33) [now completely independent solutions to the problem] are really "TEM" waves with respect to some axis in the (x,y) plane, the direction of this axis depending upon the particular point in question. The fact is that the TE wave has only transverse E and longitudinal H, while the TM wave has only transverse H and longitudinal E. The mechanism of cutoff is seen to be somewhat similar to the familiar picture in simpler cases: namely, that "TEM" waves, or "fans" of plane waves, are spreading out in the transverse plane, but now are refracted by the variations in ϵ and μ as well as being reflected from the enclosing wall. Both polarizations of the "plane" waves are available, but which one is actually present at cutoff will depend upon the particular mode in question. It should be mentioned that any mode which is mixed TE-TM at other frequencies degenerates to either the form (4.32) or (4.33) at cutoff. It is commonly found, in fact, that the TE-TM modes can be split into two groups, which might be called "primary TE" and "primary TM". The former assume the character of Eq.(4.32) at cutoff, while the latter degenerate into form (4.33). Appendices A and C will illustrate these matters, and Section 4.4 contains further discussion on the subject.

It is important to note that this cutoff ($\gamma = 0$) concept of "cylindrical standing waves" in the (x,y) plane is reasonable only if the boundary is lossless, for, otherwise, power would leave the bounding surface, and a source in the transverse plane would be required by Poynting's theorem to supply this two-dimensional outward power flow.

If either the wall admittance or the internal medium is dissipative, it is to be expected that γ will remain complex over the whole range of frequencies. It will not become zero (except possibly at $\omega = 0$), since the source-free problem evidently cannot become two-dimensional ($\gamma = 0$) when any losses are present.

The open boundary structure, (Part V), even though dissipationless, will be found to suffer from a similar difficulty, because power can leave the guide system through the walls. It will not be surprising then to find that the concept of cutoff, as outlined above, simply breaks down for free modes on even a lossless open-boundary structure.

So far as the reactive-wall case is concerned, the phenomenon of cutoff is certainly understandable. Therefore, as mentioned in Section 2.3, the apparent dissappearance of some of the modes on a reactive-wall structure at, or below, cutoff would present an unusual situation. It is definitely necessary to determine whether such a phenomenon will take place when the wall admittances satisfy Eq.(2.40), and it is hoped that the results can be presented elsewhere shortly.

In any case, whenever a propagating mode approaches cutoff, there will still be fields in the guide [solutions to Eqs.(4.32), (4.33) and (4.31)]. The right side of Eq.(4.29) therefore remains finite, while the longitudinal power flow becomes zero. Hence $(\partial\beta/\partial\omega)$ must increase without limit. At cutoff, the phase velocity becomes ∞ $(\beta \rightarrow 0)$, while the group velocity $(\partial\omega/\partial\beta)$ becomes zero. The cutoff frequency is therefore a branch point of $\gamma(\omega)$, and incidently of the fields (E,H). Further discussion of the behavior of E and H in the neighborhood of cutoff is contained in Section 4.4.

At higher frequencies, above cutoff, the picture of mode behavior becomes quite different. It is to be kept clearly in mind now that at any frequency ω, $k = \omega\sqrt{\epsilon\mu}$ is a function of position in the guide cross section. The values of $(\epsilon\mu)$ range from a minimum $(\epsilon\mu)_{min}$ to a maximum $(\epsilon\mu)_{max}$. In general, there will be certain areas of the cross section in the vicinity of which $k \approx k_{max}$, and others where $k \approx k_{min}$. Remaining portions of the cross section can be considered as transition regions. This concept becomes most striking when either k_{max} or k_{min}, or both, occur within the boundary; because if k is any reasonable function of the transverse coordinates, $\nabla_T k = 0$ at the maxima and minima thereof.

It will be useful now to consider more precisely the effects of these inhomogeneities in the dielectric constant and permeability. To this end, a theorem analogous to the energy theorem (4.20) can be derived (1b), which pertains to the effect of changing ϵ or μ in a small neighborhood (A_o) of a point (x_o, y_o) in the cross section. Of course the fields on the structure satisfy the Maxwell equations:

$$\begin{aligned} &(a) \quad \nabla \times \hat{E} = -j\omega\mu\hat{H} \quad , \\ &(b) \quad \nabla \times \hat{H} = j\omega\epsilon\hat{E} \quad . \end{aligned} \tag{4.34}$$

The form of the boundary conditions and the functions $[\epsilon(x,y), \mu(x,y)]$ are regarded as given. Consider that a small change ($\delta\epsilon_o$) is made in the dielectric constant ϵ_o in the elementary neighborhood A_o. The fields E and H will change somewhat at <u>all</u> points of the cross section, but it is advisable to treat the region A_o separately. In the neighborhood A_o, Eq.(4.34) can be differentiated with respect to ϵ_o as follows:

$$\begin{aligned} &(a) \quad \nabla \times \frac{\delta\hat{E}_o}{\delta\epsilon_o} = -j\omega\mu_o \frac{\delta\hat{H}_o}{\delta\epsilon_o} \quad , \\ &(b) \quad \nabla \times \frac{\delta\hat{H}_o}{\delta\epsilon_o} = j\omega\epsilon_o \frac{\delta\hat{E}_o}{\delta\epsilon_o} + j\omega\hat{E}_o \quad . \end{aligned} \tag{4.35}$$

By appropriate dot multiplications between Eqs.(4.34) and (4.35) it is not hard to prove the relation

$$\nabla \cdot \left(\hat{E}_o^* \times \frac{\delta\hat{H}_o}{\delta\epsilon_o} + \frac{\delta\hat{E}_o}{\delta\epsilon_o} \times \hat{H}_o^*\right) = -j\omega\hat{E}_o \cdot \hat{E}_o^* \quad . \tag{4.36}$$

A similar expression results for points <u>outside</u> the region A_o, except that the second term of the right-hand side of Eq.(4.35b) is absent (since ϵ at such points is <u>not</u> a function of ϵ_o). The result, in place of Eq.(4.36), will therefore be

$$\nabla \cdot \left(\hat{E}^* \times \frac{\delta\hat{H}}{\delta\epsilon_o} + \frac{\delta\hat{E}}{\delta\epsilon_o} \times \hat{H}^*\right) = 0. \tag{4.37}$$

By steps similar to those in Eqs.(4.20) through (4.28), but employing differentiations with respect to ϵ_o rather than ω, Eqs.(4.36) and (4.37) can be put in a form analogous to Eq.(4.28), namely

$$2\alpha\int_A i_z\cdot\left(E_T^*\times\frac{\delta H_T}{\delta\epsilon_o}+\frac{\delta E_T}{\delta\epsilon_o}\times H_T^*\right)d\sigma$$

$$+\ 2\,\frac{\delta\gamma}{\delta\epsilon_o}\int_A \mathrm{Re}\ i_z\cdot(E_T\times H_T^*)d\sigma = j\omega\int_{A_o} E_o\cdot E_o^*\ d\sigma\ . \qquad (4.38)$$

The absence of the boundary term in Eq.(4.38) is explained by the fact that $\overline{B}$ is <u>not</u> a function of ϵ_o, whereas in Eq.(4.28) it is a function of ω.

When the mode is above cutoff and $\beta>0$, the imaginary part of Eq.(4.38) becomes

$$\frac{\delta\beta}{\delta\epsilon_o} = \frac{1}{2}\left\{\frac{\omega\int_{A_o} E_o\cdot E_o^*\ d\sigma}{\int_A \mathrm{Re}\ i_z\cdot(E_T\times H_T^*)\ d\sigma}\right\}\ . \qquad (4.39)$$

An entirely similar result follows for changes in μ, except that H_o replaces E_o in the numerator of Eq.(4.39).

In order to separate the effects of the boundary from those of the internal medium, assume in connection with Eq.(4.39) that the initial structure was completely homogeneous (including the requirement that the wall be opaque). Let $k = \omega\sqrt{\epsilon\mu}$ be called k_{max}, for reasons which will appear shortly. Then the general behavior of $\beta(\omega)$ is familiar, and is shown in Figure 4.1 below.

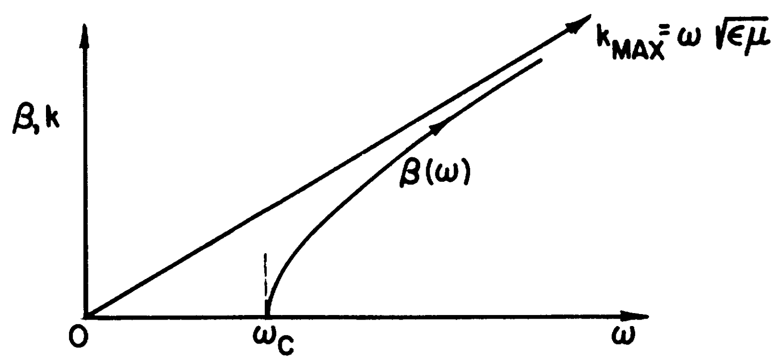

Fig. 4.1. β <u>vs.</u> ω for a lossless homogeneous structure.

At frequencies $\omega > \omega_c$, $\beta < k_{max}$; but $\beta(\omega)$ becomes asymptotic to the line $k_{max} = \omega\sqrt{(\epsilon\mu)}_{max}$ at very high frequencies. The total real power flow has the same algebraic sign as β, because Figure 4.1 shows that β and $(\partial\beta/\partial\omega)$ are both positive.

Now let a small decrease in either ϵ or μ (or both) take place in accordance with the assumptions used in deriving Eq. (4.39). The resulting plot of $\beta(\omega)$ will look very much like that in Figure 4.1, except that β will be decreased everywhere by an amount which depends upon frequency. Under these circumstances, k_{max} in the figure becomes simply the largest value of k in the cross section.

Similarly, suppose that the original guide was filled uniformly with a medium for which ϵ and μ were constants, but such that $\omega\sqrt{\epsilon\mu} = k_{min} < k_{max}$. Then the behavior of $\beta(\omega)$ would again be similar to that shown in Figure 4.1, except that $k_{min} < k_{max}$ would be the asymptote thereof.

If now a slight increase is made in ϵ, μ, or both, β will everywhere be increased. The new plot of β vs. ω will be as shown in Figure 4.2 below, where k_{min} now refers to the minimum value of k in the cross section.

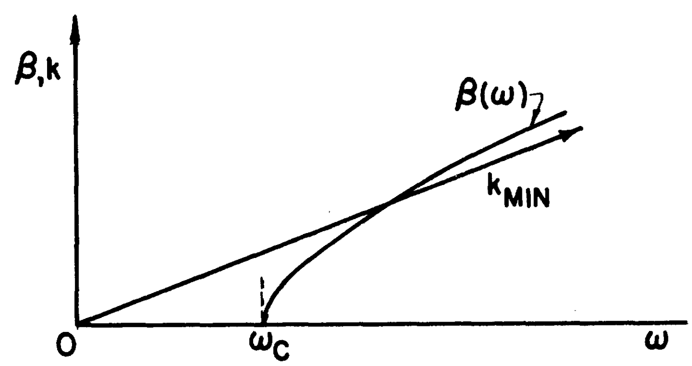

Fig. 4.2. β vs. ω for a perturbed lossless homogeneous system.

Finally, in the general case, with k_{min} and k_{max} respectively the minimum and maximum values of k in the cross section, it is to be expected that $\beta(\omega)$ will take a form similar to the illustration in Figure 4.3:

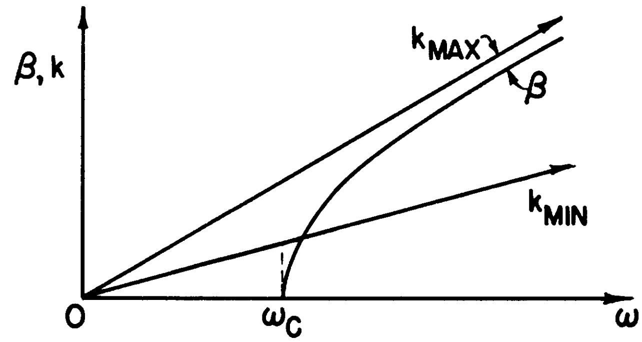

Fig. 4.3 $\beta(\omega)$ for a lossless inhomogeneous structure with opaque walls.

Justification for this figure stems from the fact that the actual distribution of k in the cross section may be considered as obtained by either starting from a uniform medium with $k = k_{max}$, and successively decreasing k by small amounts where necessary, or, alternately, by starting with $k = k_{min}$, and successively increasing k where necessary. Moreover, while this continuous "warping" process goes on, Eq.(4.39) shows that the sign relationship between β and $(\partial\beta/\partial\omega)$ cannot change. Thus the statement made previously, relative to the sign relations between β and the integrated power flow, is definitely true, viz.:

> "The integrated real power flow down the guide has the same algebraic sign as β, provided the lossless inhomogeneous structure is specialized by the requirement that it must have opaque walls."

4.3 Frequency Behavior of the Transverse Field Distribution

The preceding section sets the stage for a more detailed study of the field distribution. The factor which is primarily responsible for the frequency dependence of this distribution is the parameter p. Now $p^2 = -\gamma^2 - k^2$ is surely negative when $\omega = \omega_c$; and it is negative at all points of the cross section because γ is real (or zero). The significant fact illustrated by Figure 4.3 is that when the frequency is sufficiently far above cutoff, p^2 eventually becomes positive in at least some regions of the cross section. At any such high frequency, Figure 4.3 shows that p^2 will, loosely, be positive where k is "small" and negative where k is "large".

A somewhat more satisfactory understanding of the meaning of these sign changes in p^2 will follow from a review of the Maxwell equations in the limiting instance $\omega \rightarrow \infty$. It is not difficult to eliminate $\hat{H}$ from the Maxwell equations (4.34), in order to obtain the equation

$$-\nabla^2\hat{E} = k^2\hat{E} + \frac{\nabla\mu}{\mu} \times (\nabla \times \hat{E}) + \nabla(\frac{\nabla\epsilon}{\epsilon} \cdot \hat{E}) \quad , \qquad (4.40)$$

valid for the rectangular components of $\hat{E}$. At very high frequencies, the sensitive term $\nabla^2\hat{E}$ is most strongly affected by $k^2\hat{E} = \omega^2\epsilon\mu\hat{E}$, since all the other terms on the right have coefficients which are independent of frequency. If $\lambda = (2\pi/k)$, the above reasoning may be restated to point out that when $\lambda\,|\nabla\mu/\mu|$ and $\lambda\,|\nabla\epsilon/\epsilon|$ everywhere become $\ll 1$, the percentage changes in dielectric properties (per wavelength) are small enough that the governing equations differ only slightly from those in a homogeneous medium; except that the average value of k^2 must still be considered to change from region to region of the cross section. Therefore, as $\omega \rightarrow \infty$, Eq.(4.40) becomes

$$\nabla^2\hat{E} + k^2\hat{E} = 0 \quad , \qquad (4.41)$$

in which k^2 is still a function of the transverse coordinates.

As applied to the z-component of an exponential wave, Eq. (4.41) may be written

$$\nabla^2 E_z - p^2 E_z = 0 \qquad (\omega \rightarrow \infty) \quad . \qquad (4.42)$$

By similar reasoning

$$\nabla^2 H_z - p^2 H_z = 0 \qquad (\omega \rightarrow \infty) \quad . \qquad (4.43)$$

At very high frequencies, therefore, the TE-TM coupling due to a smooth distribution of ϵ and μ becomes negligible. It should be recalled that, since the present considerations are still limited to a guide with opaque walls, the problem has actually split into a TE and a TM problem. The effect of non-opaque walls will be briefly treated at the end of this section.

Now Eqs.(4.42) and (4.43) are in the form

$$\nabla^2\varphi = p^2\varphi \quad , \tag{4.44}$$

in which p^2 is a real function of position (x,y). Therefore $\varphi(x,y)$ may be taken as real also. Interpretation of the meaning of the algebraic sign of p^2 will be clearer when Eq.(4.44) is altered somewhat:

$$\nabla\cdot(\nabla\varphi) = p^2\varphi \quad . \tag{4.45}$$

Let Eq.(4.45) be integrated over a very small circular area "A" centered about a given point (x_o,y_o), at which point φ has the value φ_o. Reference to Figure 4.4 will explain the notation in greater detail.

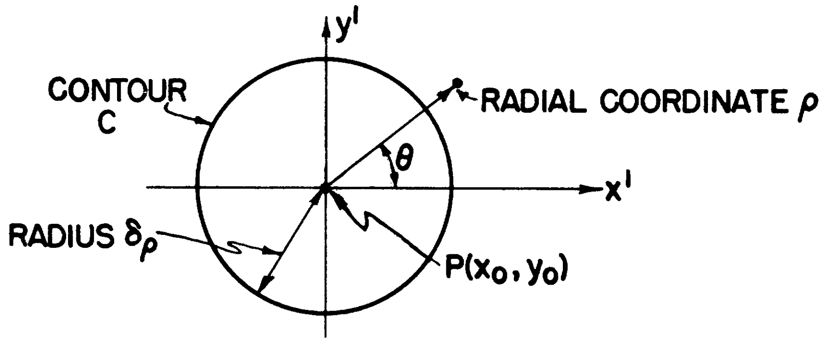

Fig. 4.4. Definition of the area "A" for interpretation of Eq.(4.45).

From Eq.(4.45)

$$\int_A \nabla\cdot(\nabla\varphi)d\sigma = \int_A p^2\varphi \; d\sigma \quad . \tag{4.46}$$

By Gauss' theorem then,

$$\int_0^{2\pi}\left[\left(\frac{\partial\varphi}{\partial\rho}\right)_\theta \delta\rho\right]d\theta \cong \pi(\delta\rho)^2 p_o^2\varphi_o \quad , \tag{4.47}$$

in which the subscript θ on $(\partial\varphi/\partial\rho)$ indicates that this is a directional derivative, and therefore varies with θ.

Let $\varphi_c(\theta)$ denote the values of φ on the contour C. Since $\delta\rho$ is infinitesimal, a Taylor expansion of φ gives for the integrand of Eq.(4.47)

$$\left(\frac{\partial\varphi}{\partial\rho}\right)_\theta \delta\rho = \varphi_c(\theta) - \varphi_o \quad . \tag{4.48}$$

A substitution of Eq.(4.48) into Eq.(4.47) accomplishes the required transformation:

$$\left[\frac{1}{2\pi}\int_0^{2\pi} \varphi_c(\theta)\, d\theta\right] - \varphi_o = \tfrac{1}{2}\, \varphi_o p_o^2 (\delta\rho)^2 \quad . \tag{4.49}$$

Eq.(4.49) makes it clear that the Laplacian of φ at a certain point represents the difference between the average values of φ in the neighborhood of the point in question, and the value of φ at that point. As a result, in regions of space where p^2 is negative, or $\nabla^2\varphi$ and φ have opposite signs, the general trend is to make $|\varphi|$ at neighboring points <u>less</u> than $|\varphi|$ at a given point. In other words, a negative value of p^2 in a region of the cross section causes φ to oscillate up and down, assuming alternately positive and negative values. It has been observed (Figure 4.3) that p^2 is negative in regions where k is "large" (near k_{max}). Hence the function φ (= E_z or H_z) has oscillatory behavior in these regions of the cross section at high frequencies.

In other regions of the cross section where k is "small" (near k_{min}), Figure 4.3 shows that p^2 becomes large and positive at sufficiently high frequencies. Then in accordance with Eq. (4.49) the average of the neighboring values of $|\varphi|$ tends to be $>|\varphi|$ at a given point, and φ has monotonic behavior over such regions of the cross section. Finally, where $p^2 \approx 0$ (somewhere in the "transition" regions of the cross section) it is necessary that $\nabla\varphi$ (= ∇E_z or ∇H_z) ≈ 0, as well as $\nabla^2\varphi \approx 0$. This restriction on $\nabla\varphi$ comes from Eqs.(2.15), with the stipulation that the transverse fields E_T and H_T remain <u>finite</u> at <u>all</u> points of the cross section (and for all finite frequencies). These transition regions, then, must form the parts of the cross section where φ has essentially "flat" behavior, connecting those regions where it is monotonic with those in which it becomes oscillatory.

It is a familiar fact that a plane wave which attempts to pass through a discontinuity, from a lossless medium of uniformly high k into one of uniformly lower k, will suffer total reflection when the angle of incidence is sufficiently far from the normal. On the high-k side of the discontinuity, the reflected and incident waves will set up oscillatory standing waves in planes normal to the boundary, while on the low-k side there will be a monotonic decrease of all field components in similar planes.

The behavior of the waveguide with opaque walls at high frequencies can now be seen to present a very similar picture. In the transverse plane, the waves become "trapped" in regions of high k, and fall off monotonically in other parts of the cross section. To be sure, the trapping is not quite a result of critical reflection, but rather of an excessive <u>refraction</u> where k varies rapidly with position. If the transition regions between those of highest and lowest k are squeezed down to almost lines of discontinuity, the trapping phenomenae become more pronounced; but in any case, the fields are always crowded into the regions of highest k when the frequency becomes sufficiently high. As a consequence, the curve of $\beta(\omega)$ in Figure 4.3 actually still becomes asymptotic to k_{max} as $\omega \rightarrow \infty$, for more and more of the field becomes crowded into corresponding regions of the guide, and the propagation constant k_{max} eventually controls virtually all of the field.

The foregoing reasoning leads to the surprising result that, at sufficiently high frequencies, a very small rod of high dielectric constant inserted into an otherwise homogeneous air-filled guide structure will eventually "suck in" most of the fields in any particular mode; the phase and group velocities for the whole structure will first approach those for the rod itself, acting in a corresponding mode in free space, and eventually will approach the values for plane waves in the rod medium. Further discussion of rod behavior will follow in Section V, and is illustrated in Appendix D.

The overall change in p^2 as a function of frequency can now be summarized in terms of the standing-wave pattern in the transverse plane. When the frequency is at or below cutoff,

p^2 is negative everywhere in the cross section; the oscillatory standing waves extend from boundary point to boundary point. As the frequency is raised above cutoff, the standing waves become more and more crowded into limited regions of the cross section where k is largest and where p^2 remains negative. The other portions of the cross section, in which p^2 becomes positive, are filled with monotonic fields connecting the high-k regions with the boundary. The regions of negative p^2 gradually contract, from the whole cross section at low frequencies, to only the high-k regions at high frequencies.

The effect of a reactive wall can now be made a little clearer. For simplicity, suppose the internal medium is homogeneous. Then Eqs.(4.42) and (4.43) are precisely applicable inside the guide at all frequencies. The influence of the wall is most strongly apparent for the slow modes, which occur when

$$p^2 = \beta^2 - k^2 > 0.$$

Since such modes do not exist at all on homogeneous structures, it is most reasonable to look for special effects of the wall in these modes. Since p^2 is, under these circumstances, positive everywhere inside the guide, the considerations following Eq. (4.49) show that the fields are concentrated near the boundary of the guide rather than inside it. Such a conclusion implies that the fields are trapped "outside" the boundary, in a loose manner of speaking, and further substantiates the suggestion made on page 57 to explain why a negative group velocity might conceivably occur in some reactive-wall structures, particularly for the slow modes.

When the mode propagates more rapidly ($\beta < k$), p^2 is negative inside the structure, and it is again reasonable to suppose that most of the field is now concentrated there rather than "in" the wall. It would not be anticipated, therefore, that the wall admittance should exercise a major influence on the propagation constant when p^2 is negative.

Since p^2 is also negative below cutoff, it is hard to understand why the reactive wall should exercise so profound an effect on the fields as, for example, to cause the modes to "break off". This question must, unfortunately, be left unanswered for the present.

4.4. Polarization of the Fields

It must be pointed out immediately that the questions of polarization to be discussed in this section are reasonably clear-cut only in the lossless problems. Attention should therefore be focused on Eq.(2.25),(2.15) and (2.37), with the recognition that $\epsilon' = \epsilon$ is pure real.

If the mode under consideration is propagating ($\gamma = j\beta$), then all the coefficients in Eqs.(2.25) are entirely real. It will therefore <u>always</u> be possible to choose solutions for E_z and H_z which are entirely real functions. Then, if desired, both E_z and H_z can be multiplied by the same complex constant $K = a + jb$; for, the real and imaginary parts of $K\ E_z$ and $K\ H_z$ will each still satisfy Eqs.(2.25). For the moment, consider the entirely real solutions (E_z, H_z), in connection with Eqs.(2.15). It appears from the latter equations that E_T and H_T are pure imaginary, since they each become just j times a real vector function. As such, the proper phase relations will exist between (H_γ, E_z) and (H_z, E_γ) for satisfying the boundary conditions (2.30) in the form (2.37). These boundary conditions then become merely <u>magnitude</u> restrictions on the field components, the phases of which are already properly fixed by Eqs.(2.25) and (2.15).

If the complex solutions ($K\ E_z, K\ H_z$) are chosen instead, then E_T and H_T become multiplied by the same complex constant K. Each becomes a real function multiplied by a complex constant jK, and as such is still capable of satisfying the boundary conditions, with $K\ E_z$ and $K\ H_z$ for the z-components.

Now it must be recalled that a complex vector whose real and imaginary parts differ only by a multiplicative real constant, represents a linearly polarized vector in the time domain. Since E_T and H_T, under the two conditions outlined above, are just two real vectors multiplied by the same complex constants (j or jK), they satisfy the required conditions for the representation of linearly polarized vectors in the time domain.

When the mode is below cutoff ($\gamma = \alpha$), the situation is slightly different. Reference to Eqs.(2.25) shows that if a new function $H_z' = -jH_z$ is substituted therein, all the coefficients again become real. In other words, below cutoff, a possible solution is that E_z shall be real, while H_z is pure imaginary. But it is not to be concluded that this is the only possibility, because the substitution $E_z' = -jE_z$ also accomplishes the reduction of the coefficients in Eqs.(2.25) to pure real functions.

Therefore, below cutoff, it may occur that either E_z is real and H_z imaginary, or E_z is imaginary and H_z is real. In either case, it is not hard to show, by reasoning similar to the above, that E_T and H_T are linearly polarized, although now they are out of time phase by 90°. Similarly also, the boundary conditions can be met as before, and the multiplication of E_z and H_z by a complex constant K does not alter the picture materially.

It follows that:

"In a lossless problem, it is always possible to choose modes in such a way that the transverse fields will be linearly polarized over the entire frequency range."

The significance of the two possible choices for the fields (E_z, H_z) below cutoff can be further elucidated; reference to Eqs.(4.31) through (4.33), and the discussion included therewith, will aid materially in the following presentation.

The reasoning upon which the real and/or imaginary character of (E_z, H_z) was based hinged upon the nature of the cross terms in Eqs.(2.25). Above cutoff, E_z and H_z could always be chosen as pure real, regardless of the mode i.e., regardless of $\beta(\omega)$. Suppose such a choice has been made for a particular $\beta_\nu(\omega)$, defining a particular mode ν. As the frequency is decreased through cutoff, $\beta_\nu(\omega)$ passes into $\alpha_\nu(\omega)$; but exactly at cutoff, Eqs. (4.31) through (4.33) show that TE or TM character alone is sufficient to describe the fields, and there is apparently no way to decide, from those equations, which would result. Now, in addition, it has been shown that below cutoff there are two possibilities (if any exponential field exists at all):

a) E_z remains real, H_z becomes imaginary;

or

b) E_z becomes imaginary, H_z remains real.

Surely only one transition is possible for a single (continuous) mode with a specified $\gamma_\nu(\omega)$. Since apparently both situations a) and b) are compatible with all the conditions of the problem, it follows that there must be two groups of modes, one corresponding to the transition a), and the other to transition b). Moreover, if transition a) takes place continuously, then H_z must pass continuously from pure real to pure imaginary, whence $H_z = 0$ at cutoff. The a)-modes might then be called "primarily TM" modes, even though they become TM only at cutoff. Similarly, the b)-modes will have $E_z = 0$ at cutoff, and may be called "primarily TE" modes.

It is perhaps necessary to emphasize somewhat more the definition of a "mode" as employed in this work. In general, the solution of the eigenvalue equation will lead to a set of γ's, each of which is a different function of frequency. The specification of a mode "ν" picks a particular $\gamma_\nu(\omega)$, and the associated fields which go with the corresponding $p_\nu(\omega)$.

The point of view taken above with regard to the definition of a mode easily leads to the curious circumstance illustrated in Appendix C. The $TE_{m,n}$ and $TM_{m,n}$ modes in an ordinary rectangular waveguide (1b) have the same $\gamma(\omega)$, so long as m and n are > 0. Not only are the propagation constants identical functions of the frequency, for a TE and a TM mode with the same indices $m,n(\neq 0)$, but the transverse-field components are also (respectively) identical functions of (x,y), even though the vectors $E_T^{(1)}$ and $E_T^{(2)}$ are neither parallel nor perpendicular. From the difinition of a mode adopted here, these $TE_{m,n}$ and $TM_{m,n}$ $(m,n \neq 0)$ must be considered as defining a single TE-TM mode, with the outstanding property that the relative amplitudes of E_z and H_z may be chosen at will.

Such a "degeneracy" is not found in those inhomogeneous problems which demand TE-TM mixture. For a given $\gamma_\nu(\omega)$, the relative amplitudes of $E_{z\nu}$ and $H_{z\nu}$ are fixed by the boundary conditions and/or the differential equations. Under normal circumstances, then, the discussion of polarization given heretofore indicates that below cutoff E_T and H_T may always be made linearly polarized, and will then be 90° out of time phase.

> "If linearly polarized solutions are chosen to define a mode, then Re $S_z \equiv 0$ at frequencies below cutoff."

It is clear from Eq.(2.15) that if E_z is real and H_z imaginary (with $\gamma = \alpha$) then H_T is imaginary and E_T is real. Correspondingly, imaginary E_z and real H_z lead to imaginary E_T and real H_T. Below cutoff, in either case, the transverse and longitudinal components of E are in phase, which guarantees that E represents a linearly polarized vector in space. The same is true of H.

> "If the transverse fields are chosen to be linearly polarized in the (x,y) plane, then at frequencies below cutoff, the entire E and H fields represent linearly polarized E(t) and H(t) vectors in space."

Also, it should not be overlooked that under these circumstances the transverse component of the complex Poynting vector is pure imaginary, since E_T and H_z (as well as H_T and E_z) are 90° out of time phase. Therefore

$$\left.\begin{matrix} \omega \leq \omega_c \\ \gamma = \alpha \end{matrix}\right\} \longrightarrow \text{Re } S_z = \text{Re } S_T = 0 \quad , \tag{4.50}$$

when the fields are linearly polarized, and the wave is below cutoff.

At frequencies above cutoff, however, the choice of linearly polarized transverse fields $E_T(t)$, $H_T(t)$ leads to total fields E(t) and H(t) which are elliptically polarized. The plane of rotation of E(t) is longitudinal, and is defined by the linearly polarized vector $E_T(t)$ along with the unit longitudinal vector i_z. Similarly, H(t) rotates in a longitudinal plane containing $H_T(t)$ and i_z. Even though now Re $S_z \neq 0$, the relative phase relations between E_T and H_z or H_T and E_z are such that

$$\left.\begin{array}{l}\omega > \omega_c \\ \gamma = j\beta\end{array}\right\} \longrightarrow \text{Re } S_T = 0 \quad . \tag{4.51}$$

"According to Eqs.(4.50) and (4.51), the choice of linearly polarized transverse fields leads to modes for which there is no time-average power flow in any direction in the transverse plane, at any frequency."

Incidentally, the conclusion in Eq.(4.51) is just one step more specialized than the following result, obtained by integrating Poynting's theorem [Eq.(4.2)] over any internal region of the guide cross section:

$$\left.\begin{array}{l}\omega \geq \omega_c \\ \gamma = j\beta\end{array}\right\} \longrightarrow \text{Re}\int_{L'} n'\cdot S_T \, d\ell' = 0 \quad , \tag{4.52}$$

in which L' and n' refer, respectively, to the contour enclosing the region and its outward normal. Equation (4.52) is valid even if the transverse fields are not linearly polarized, whereas the validity of Eq.(4.51) is limited to cases in which they are linearly polarized.

It was stated on page 73 that the choice of linearly polarized transverse fields avoids the appearance of positive and negative power flows at various points of the cross section, when a mode is below cutoff. The example in Appendix C shows that this same choice of linear polarization does not avoid power reversal when the mode is above cutoff. Therefore:

"Even when a lossless structure has opaque walls, it is possible for a TE-TM mode with linearly polarized transverse fields to exist above cutoff in such a way that Re S_z changes sign at certain points of the cross section."

The preceding discussion has emphasized the fact that linearly polarized transverse fields may always be chosen for any mode which is characterized by a particular $\gamma_\nu(\omega)$. It is therefore pertinent to indicate where any other choice is even possible. The key to the matter arises from the possibility that there may be two linearly independent solutions for the pair of real functions (E_z, H_z), with the same $\gamma_\nu(\omega)$. That is, (E_{z1}, H_{z1})

and (E_{z2},H_{z2}) may both represent possible (real) fields with identical propagation constants at all frequencies. For example, if the guide cross section has a rotational physical and electrical symmetry, as shown in Figure 4.5 below, it is to be

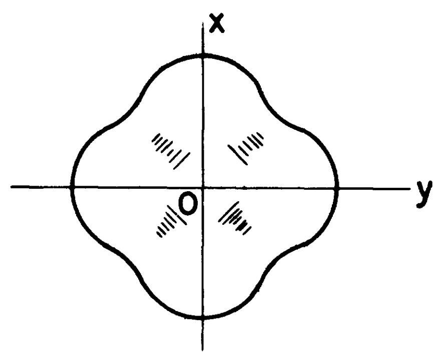

Fig. 4.5. Guide cross section with a rotational symmetry.

expected that, corresponding to any particular field solution, there will always be another one having a transverse field rotated by 90° with respect to that of the original. The first solution will correspond to the pair of functions (E_{z1},H_{z1}), and the second to the pair (E_{z2},H_{z2}). In such cases, either a real or a complex linear combination of both may arbitrarily be taken as defining the "mode" corresponding to $\gamma_{\nu}(\omega)$. According to the previous discussion, linearly polarized transverse fields will result only if some real linear combination of the two pairs of real functions is chosen, but it is by no means essential to do so. It is not hard to see that a complex linear combination $(E_{z1} + K\ E_{z2})$; $(H_{z1} + K\ H_{z2})$ leads to transverse fields which are no longer linearly polarized. One common method (1a) of handling such situations is to consider each linearly polarized solution as a subdivision of the mode $\gamma_{\nu}(\omega)$, designating one "e" for even, and the other "o" for odd. This form of degeneracy occurs, for instance, in a circular structure, where sin $n\varphi$ and cos $n\varphi$ may be used either alternatively or in two combinations. In Appendix A of this paper it has proved convenient to take the angular dependence in the particular combinations

$$e^{\pm jn\varphi} = \cos n\varphi \pm j \sin n\varphi$$

for the representation of a single mode, merely for analytical simplicity. The result is, accordingly, that the corresponding transverse fields are not linearly polarized.

Granting that an appropriate definition of the modes may always be taken to guarantee linear polarization of $E_T(t)$ and $H_T(t)$, it is still not possible to assume that the space angle between them is the same at each point of the cross section. A glance at Eq.(2.15) will show that, even with linear polarization, $(\|E_T\|)(\|H_T\|)$ is not in general the same function of (x,y) as $(E_T \cdot H_T)$; hence the space angle between $E_T(t)$ and $H_T(t)$ is generally a function of position in the transverse plane. It is this variation of angle which makes a convenient definition of impedance virtually impossible, even on a lossless structure. While a dyadic impedance might be defined, it would be a function of the transverse coordinate, and would therefore lack the simplicity usually obtained from the reduction of electromagnetic-field problems to circuit analogies. The failure of the usual impedance definition carries with it a similar failure of the familiar definitions of voltage and current, in terms of the transverse fields. The process of visualizing and utilizing mode behavior therefore becomes much more difficult for the electrical engineer than is the case with homogeneous systems.

The failure of circuit concepts in lossless inhomogeneous problems is unfortunate. It is more unfortunate that whatever simplicity does remain in the properties of modes on such structures is generally lost when dissipation is present, either in the bounding wall or the internal medium. Equations (2.25) usually develop complex coefficients as a result of the fact that ϵ' becomes complex. Then p^2 is also a complex function of (x,y) in the general case. The real and imaginary parts of E_z (and/or H_z) can no longer be taken as constant multiples of each other, since the real and imaginary parts thereof no longer satisfy the same differential equations. There is no guarantee that the polarization of the transverse fields can always be made linear, and it may necessarily be forced to vary from point to point of the cross section. Either $E_T(t)$ or $H_T(t)$ may be linearly polarized at one point, and elliptically polarized at another. Any consideration of phase angles or space angles between $E_T(t)$ and $H_T(t)$ is made difficult by this jumble of heterogeneous states of polarization; and the mode structure (if it still exists) becomes very difficult to visualize, despite the fact that the orthogonality condition (3.23) still remains to help separate one mode from another. Even if there were no other difficulties with dissipative systems, the foregoing complications would be reason enough to consider their mode properties beyond the scope of this paper.

V. "OPEN-BOUNDARY" STRUCTURES

In marked contrast to the properties of lossless "closed-boundary" cylindrical structures, it will be shown that the "open-boundary" lossless systems do not yield free exponential modes possessing true cutoff properties. In general, there is no critical frequency $\omega_c (\neq 0)$ at which $\gamma = 0$. It must be observed carefully that these comments apply only to "free" exponential modes, which exist when no sources are present within any finite region of space. But the question of source location "at ∞" is a little more difficult in these open-boundary structures than was the case in the closed-boundary problems previously discussed. For example, if a plane wave is obliquely incident upon a thin dielectric slab which lies all along the (y,z) plane, it would hardly be proper to consider the total resulting field structure as that of one or more "free" exponential modes, in spite of the fact that there are no sources within any finite region of space. The only question arising here, then, is the inadequacy of this definition of a "free" mode on the slab. A little consideration will show that the important point to be added to the earlier definition must be the stipulation that the propagation constants γ_ν are to be determined solely by the geometrical and electrical constants of the structure. Only under this added condition will the modes be "free" modes, and it is only to such modes that the following demonstrations will apply.

5.1 Coordinates and Boundary Conditions

The structure shown in Figure 5.1 represents the form of the system to which the analysis will be applied. It comprises

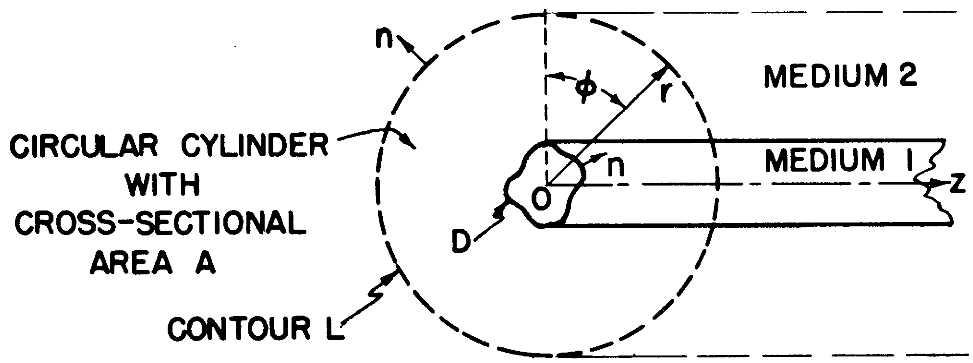

Fig. 5.1. "Open-Boundary" structure.

a rod or cylinder of dielectric material, which material need not be homogeneously distributed in the cross section; this cylinder is Medium 1. Surrounding cylinder 1 is a Medium 2 which is homogeneous, and which may be taken for convenience as free space. The boundary line between the two media is a discontinuity D, across which the continuity conditions

$$\begin{aligned} &(a) \quad n \times H_1 = n \times H_2 \ , \\ &(b) \quad n \times E_1 = n \times E_2 \end{aligned} \tag{5.1}$$

must be applied. It is through these boundary conditions of continuity that the values of γ are determined for the "free" modes, as defined earlier.

5.2 Characteristics of the Propagation Constant

When the entire system is lossless, the proof that γ^2 must be real can be carried out just as before (Section 3.2 and 3.5). The boundary conditions (5.1) are employed instead of Eq.(3.75c), and Eqs.(3.1) and (3.2) are applied in the homogeneous Medium 2.

"The discussion of this Section can therefore proceed at once on the basis that either $\gamma = \alpha$ or $\gamma = j\beta$."

It will now be demonstrated that $\gamma = \alpha$ is not possible in a source free lossless structure of the open-boundary type. Generally, a mixture of TE and TM waves will be required to solve the problem. The radial (i_r) component of the complex Poynting vector in Medium 2 is

$$2S_r = E_\varphi H_z^* - E_z H_\varphi^*$$

$$= \frac{\gamma}{p^2 r}\left(\frac{\partial E_z}{\partial \varphi}\right) H_z^* - \frac{j\omega\mu}{p^2}\left(\frac{\partial H_z}{\partial r}\right) H_z^*$$

$$- \frac{\gamma^*}{p^{*2} r}\left(\frac{\partial H_z}{\partial \varphi}\right)^* E_z + \frac{j\omega\epsilon}{p^{*2}}\left(\frac{\partial E_z}{\partial r}\right)^* E_z \quad . \qquad (5.2)$$

The subscript 2 will be omitted whenever it is clear that quantities refer to their values in Medium 2. In Medium 2, $p^2 = -(\gamma^2 + k^2)$ is not a function of the transverse coordinates, because the medium is homogeneous. Of course, γ is the same in both media.

Now the solutions for E_z and H_z in the outside medium must be expressible in terms of the solutions to Eq.(3.1). In particular, the general linear combinations of those solutions which remain finite as $r \to \infty$ may be used to express E_z and H_z for the present problem. In cylindrical coordinates, then,

$$\text{(a)} \quad E_z = \sum_n A_n K_n(pr)\, e^{jn\varphi} \quad ,$$

$$\text{(b)} \quad H_z = \sum_m B_m K_m(pr)\, e^{jm\varphi} \quad , \qquad (5.4)$$

where K_n is the integral-order modified Bessel function of the second kind. It is understood that m and n are integers, and that all summations extend from $-\infty$ to $+\infty$, unless otherwise stated. Therefore

$$\text{(a)} \quad H_z^* \frac{\partial E_z}{\partial \varphi} = j \sum_{m,n} n \, A_n B_m^* \, K_n K_m^* \, e^{j(n-m)\varphi} \quad ,$$

$$\text{(b)} \quad E_z \frac{\partial H_z^*}{\partial \varphi} = -j \sum_{m,n} m \, A_n B_m^* \, K_n K_m^* \, e^{j(n-m)\varphi} \quad ,$$

$$\text{(c)} \quad H_z^* \frac{\partial H_z}{\partial r} = p \sum_{m,n} B_n B_m^* \, K_n' \, K_m^* \, e^{j(n-m)\varphi} \quad , \qquad (5.4)$$

$$\text{(d)} \quad E_z \frac{\partial E_z^*}{\partial r} = p^* \sum_{m,n} B_n B_m^* \, K_n' \, K_m^* \, e^{j(n-m)\varphi} \quad ,$$

in which the argument of the Bessel functions is understood to be (pr), and the prime$(')$ indicates differentiation with respect to that entire argument.

Of most importance for the ensuing analysis will be the total outward flow of vector power per meter length in the z-direction, denoted by P_o and given by

$$P_o = \int_o^{2\pi} S_r r_o \, d\varphi \quad . \qquad (5.5)$$

The quantity r_o is the radius of the circular cylinder shown in Figure 5.1, page 78. To facilitate the calculation of this power, the expressions in Eq.(5.4) are integrated in accordance with Eq.(5.5). Observe in this connection that all terms in the double sums, except those for which $m = n = q$, will vanish upon such an integration, with the result that the expressions are simplified to the form

$$\int_o^{2\pi} H_z^* \frac{\partial E_z}{\partial \varphi} r_o \, d\varphi = 2\pi r_o j \sum_q q \, A_q B_q^* \left| K_q \right|^2 \quad , \qquad (5.6a)$$

$$\int_o^{2\pi} E_z \frac{\partial H_z^*}{\partial \varphi} r_o \, d\varphi = -2\pi r_o j \sum_q q \, A_q B_q^* \left| K_q \right|^2 \quad , \qquad (5.6b)$$

$$\int_0^{2\pi} H_z^* \frac{\partial H_z}{\partial r} r_o \, d\varphi = 2\pi p r_o \sum_q \left|B_q\right|^2 K_q' \, K_q^* \quad , \tag{5.6c}$$

$$\int_0^{2\pi} E_z \frac{\partial E_z^*}{\partial r} r_o \, d\varphi = 2\pi p^* r_o \sum_q \left|A_q\right|^2 K_q (K_q')^* \quad . \tag{5.6d}$$

The assumption that $\gamma = \alpha$ will now be made, and P_o computed from Eqs.(5.2), (5.5) and (5.6). Thus, since p^2 is surely negative-real,

$$\frac{P_o}{j\pi} = \frac{2\alpha}{p^2} \sum_q q \, A_q B_q^* \left|K_q\right|^2$$

$$+ \omega r_o \sum_q \left(\frac{\epsilon}{p^*} \left|A_q\right|^2 K_q K_q'^* - \frac{\mu}{p} \left|B_q\right|^2 K_q' \, K_q^* \right) . \tag{5.7}$$

But since p is pure imaginary, it is possible to set $p = j\rho$. Therefore, in accordance with well-known identities among the Bessel functions, and with the definition $Z = \rho r_o$,

$$K_q(j\rho r_o) = K_q(Ze^{j\pi/2}) = \frac{\pi}{2} \, (j)^{q+1} \, H_q^{(1)}(Ze^{j\pi}) \quad . \tag{5.8}$$

However, if ρ is taken > 0, then Z is real and > 0. Therefore

$$H_q^{(1)} \, (Ze^{j\pi}) = (-1)^{q+1} \, H_q^{*(1)}(Z) \quad . \tag{5.9}$$

Hence

$$K_q(Ze^{j\pi/2}) = \frac{\pi}{2}(-j)^{q+1} \, H_q^{*(1)}(Z) \quad , \tag{5.10}$$

also

$$K_q'(pr_o) = -j \, \frac{d}{dZ}\left[K_q(Ze^{j\pi/2})\right]$$

$$= -\frac{\pi}{2}(-j)^q \, \frac{d}{dZ}\left[H_q^{*(1)}(Z)\right] . \tag{5.11}$$

As a result

$$K_q'(pr_o) K_q^*(pr_o) = -j\frac{\pi^2}{4} \, H_q^{(1)}(Z) \, \frac{d}{dZ}\left[H_q^{*(1)}(Z)\right] , \tag{5.12a}$$

and therefore

$$K_q'(pr_o)K_q^*(pr_o)$$

$$= -j\frac{\pi^2}{4}\left[J_q(Z) + jN_q(Z)\right]\left[J_q'(Z) - jN_q'(Z)\right] \quad , \qquad (5.12b)$$

where J_q and N_q are Bessel functions of the first and second kinds, respectively.

The main point of interest about Eq.(5.7) will be its real part. Since $p = j\rho$, the real part will come from the real parts of the last two sums and the imaginary part of the first sum. But

$$\text{Re}\left[K_q'(pr_o)K_q^*(pr_o)\right] = \text{Re}\left[K_q(pr_o)\, K_q'^*(pr_o)\right]$$

$$= -\frac{\pi^2}{4}\left[J_q(Z)N_q'(Z) - J_q'(Z)N_q(Z)\right] . \qquad (5.13)$$

The bracketed expression in Eq.(5.13) is the Wronskian W of the two solutions J_q and N_q. Since

$$W(J_q;N_q) = \frac{2}{\pi Z} \quad , \qquad (5.14)$$

Eq.(5.13) becomes

$$\text{Re}\left[K_q'(pr_o)K_q^*(pr_o)\right] = -\frac{\pi}{2\rho r_o} \quad , \qquad (5.15)$$

and, therefore, from Eq.(5.7)

$$\frac{2\ \text{Re}\ P_o}{\pi^2} = \frac{4\alpha}{\pi\rho^2}\ \text{Im}\sum_q q\ A_q B_q^*\ \left|K_q\right|^2$$

$$+ \frac{\omega}{\rho^2}\sum_q \left(\epsilon\left|A_q\right|^2 + \mu\left|B_q\right|^2\right) \quad . \qquad (5.16)$$

Observe that only the first term of Eq.(5.16) is a function of r_o, because of the presence of $K_q(pr_o)$ therein. That is, Eq.(5.16) indicates a variation (with radius) of the total outward flow of real power through any circular cylinder which lies entirely in Medium 2. This variation is caused by the presence of the first term cited above. But a consideration of Poynting's theorem shows that the only possible causes of such a variation would be either sources (or sinks) located in Medium 2, or a "bending" of part of the longitudinal power into the radial direction in that medium.

The first cause is ruled out at once, because there are no sources in Medium 2, and it is lossless. The second possibility might offhand seem reasonable, because there would be attenuation in the z-direction when $\gamma = \alpha$, and it might conceivably be due to outward radiation.

For this lossless structure, however, the general discussion in Section 4.4 relative to polarization and phasing still holds: namely, it must always be possible to choose the modes in such a way that, for any allowed value of α, E_T and H_T are linearly polarized and 90° out of time phase. The present boundary conditions (5.1) in no way upset this conclusion. Hence, if $\gamma = \alpha$ is possible at all, it must be possible when Re $S_z = 0$ at each point in the transverse plane. Under such conditions, Poynting's theorem shows again that the radial power (if any) cannot vary with r_o in Medium 2. Therefore the first term in Eq.(5.16) must vanish.

If there is any radial power at all, it must have the form of the last two terms in Eq.(5.16),

$$\text{Re } P_o = \frac{\pi^2 \omega}{2\rho^2} \sum_q \left(\epsilon |A_q|^2 + \mu |B_q|^2\right) \neq 0 \quad . \qquad (5.17)$$

The real power given by Eq.(5.17) is now independent of r in Medium 2. It is not zero unless all the fields are zero. This radial power flow must terminate somewhere inside Medium 1, since it has been shown that it cannot arise from Medium 2. Once again, though, matters can always be arranged so that when $\gamma = \alpha$ there will be no Re S_z in Medium 1, at any point of its cross section. Moreover, there are no sources in that medium either, and it, too, is lossless.

There is no way of avoiding the conclusion that the assumption $\gamma = \alpha$ leads to a radial flow of power which contradicts the Poynting theorem for a source-free, lossless structure.

The existence of a wave below cutoff is in direct conflict with the concept of a free exponential mode. If such modes exist, they can do so only at frequencies where $\gamma = j\beta$. As a result, on open-boundary lossless structures:

> "There are no free exponential modes below cutoff; and the radiation properties of a lossless rod cannot be accounted for by any free exponential modes which supposedly attenuate in the z-direction by virtue of power lost through radiation in the r-direction."

The "leaky water pipe" concept of free-mode radiation is not valid, because no such mode can attenuate at all in the z-direction (when the system is lossless).

It is instructive to return to Eqs.(5.2), (5.5) and (5.6) when $\gamma = j\beta$. Then $p^2 = \beta^2 - k^2$, and two cases arise. First, suppose $\beta < k$, in which event $p = j\rho$ again and Eq.(5.7) is replaced by

$$\frac{P_o}{j\omega} = \pi r_o \sum_q \left(\frac{\epsilon}{p^*}|A_q|^2 K_q K_q'^* - \frac{\mu}{p}|B_q|^2 K_q' K_q^*\right) \quad . \tag{5.18}$$

By steps similar to Eqs.(5.8) through (5.15) it is found that

$$\mathrm{Re}\ P_o = \frac{\pi^2\omega}{2\rho^2}\sum_q \left(\epsilon|A_q|^2 + \mu|B_q|^2\right) \neq 0 \quad . \tag{5.19}$$

This result is clearly at variance with the Poynting theorem. When $\gamma = j\beta$, the <u>longitudinal</u> power flow is <u>independent</u> of z entirely. The radial power flow now exhibited by Eq.(5.19) obviously cannot be explained by bending of the longitudinal power. There are still no sources (or sinks) in any finite region of the cross section to account for this power, and the contradiction is again thrown back upon the assumption that $\beta < k$.

Fortunately, the second possibility, $\gamma = j\beta$ and $\beta > k$, does not lead to any contradiction. Under these conditions, p^2 is positive-real, and p is also real. The modified Bessel functions $K_q(pr)$ are entirely real for real values of argument, and their derivatives with respect to the argument are also real. The expression for P_o under these circumstances remains in the form of Eq.(5.18), and now becomes a purely imaginary quantity. The previous difficulties are removed. Consequently,

> "Free exponential modes on an open-boundary lossless structure must have phase velocities which are less than that of plane waves in the external medium."

5.3 Physical Interpretation of the Free Modes

The results thus far raise two questions: a) What becomes of a mode which is propagating at a high frequency ω when the frequency is lowered toward the point where a "closed" guide would normally cut off? b) If "free" modes alone cannot explain radiation from a lossless rod antenna, what part do these free modes play in the behavior of such a structure?

The answers to these questions will be appreciated most readily after some experience with a few examples has been gained. To this end, Appendix D is useful, and has been included primarily to illustrate and clarify the remarks of the present general section.

First of all, when $\gamma = j\beta$ and $\beta > k$, the form of the radial power Eq.(5.18) is reduced to

$$P_o = \frac{j\omega\pi r_o}{p} \sum_q (\epsilon|A_q|^2 - \mu|B_q|^2) K_q K_q' \quad . \tag{5.20}$$

Now

$$K_q'(pr_o) = -\frac{q}{pr_o} K_q(pr_o) - K_{q-1}(pr_o) \quad , \tag{5.21}$$

and Eq.(5.20) becomes

$$\frac{P_o}{j\omega\pi} = \sum_q \left[(|B_q|^2 - \epsilon|A_q|^2)(\frac{q}{p^2} K_q^2 + \frac{r_o}{p} K_q K_{q-1}) \right] \quad . \tag{5.22}$$

For values of $pr_o \gg 1$ and $\gg q$, the asymptotic form of $K_q(pr_o)$ is known to be

$$K_q(pr_o) \longrightarrow \sqrt{\frac{\pi}{2pr_o}} \; e^{-pr_o} \quad . \tag{5.23}$$

If the convergence of the series in Eq.(5.22) is rapid enough (and the assumption will be made here), the substitution of Eq. (5.23) can be made therein, in spite of the apparent difficulty that q becomes very large. Hence, for large values of pr_o,

$$P_o \rightarrow \frac{j\omega\pi^2}{2p^2} e^{-2pr_o} \sum_q \left[(\mu |B_q|^2 - \epsilon |A_q|^2)(\frac{q}{pr_o} + 1) \right] \; ; \quad (5.24)$$

or, since r_o is arbitrarily large,

$$P_o \rightarrow \frac{j\omega\pi^2}{2p^2} e^{-2pr_o} \sum_q (\mu |B_q|^2 - \epsilon |A_q|^2) \quad . \quad (5.25)$$

The important point to notice here is that because p is real (and <u>positive</u> if the solutions are to remain finite at large r), P_o dies out exponentially with radius. In other words, for sufficiently large r_o even the <u>reactive</u> power flowing through a unit length of large cylinder (Figure 5.1, page 78) approaches zero exponentially.

Because of this fact, much of the earlier discussion pertinent to closed-boundary structures above cutoff can be applied to the present structure, so long as the modes exist ($\gamma = j\beta$; $\beta > k_2$). In making these applications, the area A should be taken as the infinite cross section, and the exponential dependence given in Eqs.(5.23) and (5.25) effectively reduces the boundary conditions to homogeneous form at large radial distances from Medium 1. In particular, if A is considered to be the infinite cross section, Eq.(4.29) becomes modified only to the extent that the term involving $\overline{B}$ vanishes. The conclusion on page 64 can therefore be shown to remain valid ($\partial\beta/\partial\omega > 0$ when $\beta > 0$) so long as $\beta > k_2$. Similarly, the interpretation of the fact that p_2^2 is positive and real (in Medium 2) can be taken from the discussion of Eq.(4.49), and the exponential dependence of E_z and H_z found in Eq.(5.23) bears out this interpretation. Lest this monotonic behavior of the fields be thought to set in only at large distances from Medium 1, it should be added here that the functions $K_q(pr)$ are monotonically decreasing functions for all real non-zero values of the argument (18).

Since the modes new under consideration exist only when $\beta > k_2$, and since $(\partial\beta/\partial\omega) > 0$ when they do exist, it follows that the modes can always be studied at arbitrarily high frequencies. This being the case, Medium 1 can also be treated according to Eqs.(4.44) through (4.49), and if it contains inhomogeneities, p_1^2 will be a function of (x,y) in that medium. Now in order to obtain a "guided" mode, there must be some standing wave in the cross section of Medium 1. That is, p_1^2 must remain <u>negative</u> <u>real</u> at least in some regions of that cross section. Hence there must exist protions of Medium 1 for which $\beta < k_1$ at high frequencies.

It will be simplest now to restrict the discussion to cases in which Medium 1 is <u>homogeneous</u>. Let it be characterized by the intrinsic propagation constant k_1. Then if the mode is to be guided by this medium, $k_2 < \beta < k_1$. But this implies $k_1 > k_2$, or $(\epsilon\mu)_1 > (\epsilon\mu)_2$. Unless the latter condition is satisfied, <u>no</u> waves can be guided along Medium 1.

It appears that free exponential modes can be guided only by a material rod having a higher intrinsic propagation constant than the surrounding medium. When the wave is guided, the fields fall off monotonically outside the rod and must possess standing-wave character inside it. The explanation of the free-mode phenomenon becomes clearer. These waves must travel unattenuated down the rod by means of successive internal critical reflections from the bounding discontinuity. Of course, such critical reflections can take place only when the medium in which they are "trapped" has a higher $\epsilon\mu$ product than does the surrounding space.

Moreover, critical reflection can take place with plane waves only when the angle of incidence is sufficiently far from the normal. There should be a corresponding criterion for the propagation along the rod. The key to the correspondence lies in the discussion of waves <u>at</u> cutoff in closed-boundary structures (pages 58-60). It has been shown that at cutoff cylindrical waves spread out in the transverse plane and strike the bounding wall normally. Evidently this condition cannot exist on the open-boundary rod, because the discontinuity will no longer totally reflect such waves. As a matter of fact, the discontinuity generally fails to confine the waves by critical reflection long before the frequency reaches a low enough value to make $\beta = 0$. In particular, since $(\partial\beta/\partial\omega) > 0$, and β cannot be less than k_2, the mode ceases to exist at a frequency defined by the relations

$$\text{or} \qquad \left.\begin{array}{l} \beta = k_2 \\ p_2 = 0 \end{array}\right\} \quad . \qquad (5.26)$$

Occasionally this frequency has been referred to (10,11) as a "cutoff", but since the free exponential modes cannot exist at all below this "cutoff", the name is misleading. Besides, the mechanism is sufficiently different to deserve another name, and "divergence frequency" will be used in the remainder of this text.

The consequences of this lack of true cutoff among the free modes are interesting. At any particular frequency ω_o, the only free modes which can exist at all are those whose divergence frequencies are less than ω_o. It is illustrated in Appendix D that, for any given "angular" variation, there are generally only a finite number of these; perhaps none at all. It is certainly not possible to construct an arbitrary transverse field from such a finite set of modes. In other words, the fact that the individual free modes cannot exist in this problem over the entire frequency range $0 < \omega < \infty$ means that the set of these modes cannot be complete for transverse-field expansion at a single frequency.

Notwithstanding the lack of completeness for the set of free modes discussed above, it is noteworthy that the orthogonality conditions (3.23) and (3.26) are still valid when taken over the infinite cross section. This illustrates the danger of trying to infer completeness from the existence of an orthogonality condition, as was pointed out on page 33.

While the comments relative to the interpretation of free modes have been restricted to the case where Medium 1 is homogeneous, a combination of the material in Section 4.2 and 4.3 with these remarks will make the effect of inhomogeneities sufficiently clear for the purposes of this work.

The answer to question a) (page 85) has been indicated. The nature of that answer makes question b) much easier. It is now clear that the free modes play only a small part in the solution of rod problems, because these modes do not form a complete set. The resultant behavior of such a rod, when sources are specified, depends much more upon the nature of the sources than upon the free modes appropriate to the structure itself. A rather simple problem of this type is briefly considered in Appendix D, where it appears that the free modes represent natural modes for the lossless rod. Under any given source excitation, a few of these natural modes may be excited--the number thereof depending upon the source distribution, frequency, constants of the rod, and constants of the surrounding medium. There may not be any at all, in appropriate circumstances. In a sense, the field structures corresponding to these modes can be considered as "space resonances", because they are the only fields which persist for infinitely large values of z, when the source is at a finite point in space. The remainder of the field structure is related to that of the source alone, acting in free space, except that it is "bent" or diffracted by the rod. The important point to be observed is that the individual free modes do not contribute to the radiation from the rod as they travel along it, since the fields due to such modes die out rapidly in the surrounding space.

With reference again to the example in Appendix D, it can also be said that the free modes represent source power which becomes channeled into the z-direction instead of going radially outward from the dipole. If the rod were chopped off at some large distances from the dipole source, the free modes would be reflected at the far end. There would result a standing wave on the finite section of the rod, and some additional radiation from the chopped-off end; but no contribution to the radiation would come from the free-mode standing wave on the bulk of the structure. The function of these free modes is, therefore, to trap a fraction of the source power, and project it from the far end.

VI. CONCLUSION

A comparison of the mode concept in homogeneous and inhomogeneous cylindrical structures has been given in the preceding sections of this paper. Primary consideration has been directed to those mode properties which are substantially independent of the detailed configuration of the system. There are very few such properties, but they are fundamental. It is common engineering practice, in fact, to take most of them for granted. This procedure has occasionally led, either directly or indirectly, to unjustified analogies between homogeneous and inhomogeneous waveguides. It is to be hoped that the analysis presented here has clarified the conditions under which such analogies can be made, and that, thereby, a deeper understanding of the basic properties of modes on ordinary waveguides has also resulted.

Nevertheless, it is clear that considerable work remains to be done. Among passive systems, the particular lossless variety which are bounded by a reactive wall still presents some unexplained difficulties (Section 2.3 and Appendix A); those with loss have hardly been considered at all. It does appear, however, that if the walls of the latter are opaque, it may be possible to find some additional general properties of the modes which would greatly clarify the precise effect of dissipation on waveguide behavior.

A practical problem of much greater importance than those suggested above concerns active structures. The role of exponential modes in the solution to field problems arising from such structures is by no means clearly defined. In particular, the method of separating one mode from another has apparently not been satisfactorily analyzed. As an example, the properties of waveguides containing one or more electron beams deserve more careful examination from the mode point of view. These waveguide problems are becoming so important, in fact, that the work contained in the present paper should be regarded as an introduction to them, rather than as an end in itself.

APPENDIX A

Circular Guide with Reactive Wall

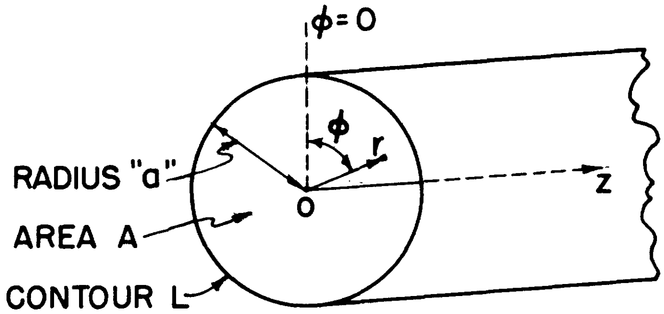

Fig. A-1. Coordinates for circular guide.

The boundary conditions at r = a are

$$\begin{aligned} &(a) \quad H_z = -j\, B_2\, E_\varphi \ , \\ &(b) \quad H_\varphi = j\, B_1\, E_z \ . \end{aligned} \qquad (1)$$

The internal medium is homogeneous, with

$$\begin{aligned} &(a) \quad Z_o = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \ , \\ &(b) \quad k = -j\sqrt{j\omega\mu\,(\sigma + j\omega\epsilon)} \ , \\ &(c) \quad p^2 = -(\gamma^2 + k^2) \ . \end{aligned} \qquad (2)$$

For the time being, the walls and internal medium will be considered as dissipative. Therefore $(B_1, B_2, Z_o, k, \gamma, p)$ will be complex.

The modes will be designated by a double subscript "ns", n being the circular-variation index, and s the radial-variation index. Although a TE-TM mixture is required, it is helpful at first to keep the TE and TM parts separated. This is accomplished by using superscripts 1 (for TM) and 2 (for TE); but each "mode" contains both, therefore the subscripts ns are the same for the two types.

The basic equations are:

TM:

$$\text{(a)}\quad \nabla_T^2 E_{zns} - p_s^2 E_{zns} = 0 \quad ;$$

$$\text{(b)}\quad E_{Tns}^{(1)} = \frac{\gamma_s}{p_s^2} \nabla_T E_{zns} \quad ; \qquad (3)$$

$$\text{(c)}\quad H_{Tns}^{(1)} = \frac{jk}{Z_o p_s^2} i_z \times \nabla_T E_{zns} \quad ;$$

and

TE:

$$\text{(a)}\quad \nabla_T^2 H_{zns} - p_s^2 H_{zns} = 0 \quad ;$$

$$\text{(b)}\quad H_{Tns}^{(2)} = \frac{\gamma_s}{p_s^2} \nabla_T H_{zns} \quad ; \qquad (4)$$

$$\text{(c)}\quad E_{Tns}^{(2)} = \frac{-jkZ_o}{p_s^2} i_z \times \nabla_T H_{zns} \quad .$$

Solutions to Eqs.(4a) and 3(a), respectively, are:

$$\text{(a)}\quad E_{zns} = a_{ns}\Psi_{ns} = a_{ns}\, I_n(p_s r)\, e^{jn\varphi} \quad ,$$

$$\qquad (5)$$

$$\text{(b)}\quad H_{zns} = b_{ns}\Psi_{ns} = b_{ns}\, I_n(p_s r)\, e^{jn\varphi} \quad ,$$

where a_{ns}, b_{ns} are complex constants independent of r and φ. $I_n(p_s r)$ is the modified Bessel function of the first kind.

Now

$$E_{\varphi ns} = E_{\varphi ns}^{(1)} + E_{\varphi ns}^{(2)} \text{ and } H_{\varphi ns} = H_{\varphi ns}^{(1)} + H_{\varphi ns}^{(2)} \quad \text{, with}$$

$$E_{\varphi ns}^{(1)} = \frac{jn\gamma_s}{p_s^2 r} a_{ns}\Psi_{ns}(\rho) \quad , \qquad (6a)$$

$$E_{\varphi ns}^{(2)} = \frac{-jkZ_o}{p_s} b_{ns}\Psi_{ns}'(\rho) \quad , \qquad (6b)$$

$$H_{\varphi ns}^{(1)} = \frac{jk}{Z_o p_s} a_{ns}\Psi_{ns}'(\rho) \quad , \qquad (6c)$$

$$H^{(2)}_{\varphi ns} = \frac{jn\gamma_s}{p_s^2 r} b_{ns}\psi_{ns}(\rho) \quad ; \tag{6d}$$

where $\rho = p_s r$, and ($'$) denotes $(d/d\rho)$.

Application of boundary conditions (1) at $r = a$, with $Z = p_s a$, yields

$$\text{(a)} \quad b_{ns}\left[p_s^2 a\psi_{ns}(Z)+kB_2 Z_o p_s a\psi'_{ns}(Z)\right] = B_2 n\gamma_s a_{ns}\psi_{ns}(Z) \quad ,$$

$$\text{(b)} \quad nb_{ns}\gamma_s\psi_{ns}(Z) = a_{ns}\left[B_1 p_s^2 a\psi_{ns}(Z)- \frac{k}{Z_o}p_s a\psi'_{ns}(Z)\right] \quad . \tag{7}$$

Thus, solving Eqs.(7a) and (7b) for $(a_{ns}/b_{ns}) = K_{ns}$ yields

$$\frac{K_{ns}}{Z_o} = \frac{1}{Z_o}\left(\frac{a_{ns}}{b_{ns}}\right) = \frac{p_s^2 a\psi_{ns}(Z) + kb_2 p_s a\psi'_{ns}(Z)}{nb_2\gamma_s\psi_{ns}(Z)}$$

$$= \frac{n\gamma_s\psi_{ns}(Z)}{b_1 p_s^2 a\psi_{ns}(Z) - kp_s a\psi'_{ns}(Z)} \quad , \tag{8}$$

where $b_1 = (B_1)(Z_o) = (B_1/Y_o)$ [normalized admittance], and similarly $b_2 = (B_2/Y_o)$.

For convenience, momentarily set $\psi = \psi_{ns}(Z)$ and $\psi' = p_s\psi'_{ns}(Z)$; so from (8)

$$\frac{K_{ns}}{Z_o} = k_{ns} = \frac{p_s^2 a}{b_2 n\gamma_s} + \frac{ka}{n\gamma_s}\left(\frac{\psi'}{\psi}\right) \quad . \tag{9}$$

From (8), also, the eigenvalue equation is

$$(ap_s^2\psi+kab_2\psi')(ab_1 p_s^2\psi-ka\psi') - n^2 b_2\gamma_s^2\psi^2 = 0 \quad , \tag{10}$$

which should be rearranged to solve for ψ' in the form

$$\psi'^2 + \psi'\psi\left[\frac{p_s^2}{k}\sqrt{\frac{b_1}{b_2}}\right]\left[\frac{1}{\sqrt{b_1 b_2}} - \sqrt{b_1 b_2}\right]$$

$$+\psi^2\left[\left(\frac{n\gamma_s}{ka}\right)^2 - \left(\frac{p_s^2}{k}\sqrt{\frac{b_1}{b_2}}\right)^2\right] = 0. \tag{11}$$

With the new symbols

$$\xi = \frac{n\gamma_s}{ka} \quad , \qquad \delta = \frac{p_s^2}{k}\sqrt{\frac{b_1}{b_2}} \quad , \qquad \zeta = \left(\sqrt{b_1 b_2} - \frac{1}{\sqrt{b_1 b_2}}\right) , \tag{12}$$

Eq.(11) becomes simply

$$\psi'^2 - \psi'\psi\delta\zeta + (\xi^2 - \delta^2)\psi^2 = 0 \quad . \tag{13}$$

The solution is

$$\frac{\psi'}{\psi} = \frac{\delta}{2}\left[\zeta \pm \sqrt{\zeta^2+4} \quad \sqrt{1 - \frac{4\xi^2}{(\zeta^2+4)\delta^2}}\right] . \tag{14}$$

But defining t' by the relation

$$\sqrt{\zeta^2+4} = \left(\sqrt{b_1 b_2} + \frac{1}{\sqrt{b_1 b_2}}\right) = t' \quad , \tag{15}$$

gives for (14)

$$\frac{\psi'}{\psi} = \frac{\delta}{2}\left[\zeta \pm t' \sqrt{1 - \left(\frac{2\xi}{\delta t'}\right)^2}\right] , \tag{16}$$

which shows that there are <u>two sets</u> of modes (for each n) corresponding, respectively, to the + and - signs in Eq.(16).

It is convenient to write two conditions in the original notation of Eq.(8), but with added <u>subscripts</u> 1 and 2, as follows:

$$\text{(a)} \qquad \frac{\psi'_{1ns}(Z_1)}{Z_1\psi_{1ns}(Z_1)} = \frac{1}{2ka}\left[(b_1 - \frac{1}{b_2}) + (b_1 + \frac{1}{b_2})\sqrt{1+\Omega^2_{1ns}}\right],$$

(17)

$$\text{(b)} \qquad \frac{\psi'_{2ns}(Z_2)}{Z_2\psi_{2ns}(Z_2)} = \frac{1}{2ka}\left[(b_1 - \frac{1}{b_2}) - (b_1 + \frac{1}{b_2})\sqrt{1+\Omega^2_{2ns}}\right],$$

where

$$\left(b_1 + \frac{1}{b_2}\right)^2 \Omega^2_{ns} = \left(\frac{2n}{Z}\right)^2 \left[1 + \left(\frac{ka}{Z}\right)^2\right] . \tag{18}$$

From Eqs.(9), (12), (15), and (16)

$$k_{1ns} = \frac{\delta_1 t'}{2\xi_1}\left[1 + \sqrt{1 - \left(\frac{2\xi_1}{\delta_1 t'}\right)^2}\right] , \tag{19}$$

and

$$k_{2ns} = \frac{\delta_2 t'}{2\xi_2}\left[1 - \sqrt{1-\left(\frac{2\xi_2}{\delta_2 t'}\right)^2}\right] . \qquad (20)$$

In the special case where n = 0, the problem becomes axially symmetric. Then $\xi = 0$, and the modes 2 and 1 become TE and TM, respectively. From Eqs.(18), (19) and (20):

(a) $n = 0 \longrightarrow k_{1os} = \infty \; ; \quad H_{z1os} \equiv 0 \; ;$

TM

$$\therefore \; \frac{\Psi'_{1os}(Z_1)}{Z_1\Psi_{1os}(Z_1)} = \frac{b_1}{ka} .$$

(21)

(b) $n = 0 \longrightarrow k_{2os} = 0 \; ; \quad E_{z2os} \equiv 0 \; ;$

TE

$$\therefore \; \frac{\Psi'_{2os}(Z_2)}{Z_2\Psi_{2os}(Z_2)} = \frac{-1}{b_2 ka} .$$

Even when $n \neq 0$, the cutoff frequency (defined by $\gamma = 0$) is characterized by $\xi = 0$. Hence modes with subscript 1 fall into the "primarily TM" category. Similarly, modes with subscript 2 may be called "primarily TE".

When n = 0, $(\partial/\partial\varphi) \equiv 0$; whence TE_{os} modes have only three field components: H_z, H_r, E_φ. TM_{os} modes have only E_z, E_r, H_φ. Hence TE_{os} modes are obviously orthogonal to TM_{os} modes in the longitudinal-power sense. There remains the question of orthogonality among TE_{os} modes, TM_{os} modes, and mixed modes $(n \neq 0)$ with different values of γ. Because of the circular variation $e^{jn\varphi}$, modes with different values of "n" are clearly orthogonal, both in the energy and power senses. Consideration need be given, therefore, only to modes with the same "n", but different γ's. Since the n becomes a common index, it will henceforth be omitted.

Let two modes be described as follows:

Mode 1 — Mode 2

$$
\begin{aligned}
&H_{z1} = \Psi_1\,, && H_{z2} = \Psi_2\,, \\
&E_{z1} = K_1\Psi_1\,, && E_{z2} = K_2\Psi_2\,, \\
&E_{T1}^{(1)} = \frac{K_1\gamma_1}{p_1^2}\nabla_T\Psi_1\,, && E_{T2}^{(1)} = \frac{K_2\gamma_2}{p_2^2}\nabla_T\Psi_2\,, \\
&E_{T1}^{(2)} = \frac{-jkZ_o}{p_1^2}1_z\times\nabla_T\Psi_1\,, && E_{T2}^{(2)} = \frac{-jkZ_o}{p_2^2}1_z\times\nabla_T\Psi_2\,, \\
&H_{T1}^{(1)} = \frac{jkK_1}{p_1^2Z_o}1_z\times\nabla_T\Psi_1\,, && H_{T2}^{(1)} = \frac{jkK_2}{p_2^2Z_o}1_z\times\nabla_T\Psi_2\,, \\
&H_{T1}^{(2)} = \frac{\gamma_1}{p_1^2}\nabla_T\Psi_1\,, && H_{T2}^{(2)} = \frac{\gamma_2}{p_2^2}\nabla_T\Psi_2\,,
\end{aligned}
\tag{22}
$$

with boundary conditions expressed from Eq.(9) in the form

$$
\begin{aligned}
&\text{(a)} \quad (n\cdot\nabla_T\Psi_1)_a = \left(\frac{\partial\Psi_1}{\partial r}\right)_a = p_1\Psi_1'(Z) = \frac{p_1^2}{k}\left[\frac{nK_1}{Z_o a p_1^2} - \frac{1}{b_2}\right]\Psi_1(Z)\,, \\
&\text{(b)} \quad (n\cdot\nabla_T\Psi_2)_a = \left(\frac{\partial\Psi_2}{\partial r}\right)_a = p_2\Psi_2'(Z) = \frac{p_2^2}{k}\left[\frac{nK_2}{Z_o a p_2^2} - \frac{1}{b_2}\right]\Psi_2(Z)\,.
\end{aligned}
\tag{23}
$$

Although the subscripts 1 and 2 have heretofore referred to primarily TM and primarily TE modes, respectively, Eqs.(22) and (23) may now refer to any two different modes, because their form is such that modes 1 and 2 merely differ by an interchange of these subscripts. The question of whether either (or both) has primarily TE or primarily TM character will be decided only when a choice of signs is made in relating K_{ns} to γ_s [Eqs.(19),(20)]. It is advisable to postpone this step until later.

In order to compute the cross terms (c) in the longitudinal power flow, the functions

$$P_c = \int_A i_z \cdot (E_{T1} \times H^*_{T2})\, d\sigma$$

$$= \int_A i_z \cdot \Big[E^{(1)}_{T1} \times H^{*(1)}_{T2} + E^{(1)}_{T1} \times H^{*(2)}_{T2}$$

$$+ E^{(2)}_{T1} \times H^{*(2)}_{T2} + E^{(2)}_{T1} \times H^{*(1)}_{T2}\Big]\, d\sigma$$

$$= P_c^{(11)} + P_c^{(12)} + P_c^{(22)} + P_c^{(21)} \qquad (24)$$

will be required. It is necessary to express these $P_c^{(\mu\nu)}$ in terms of common variables on the boundary, for which certain integral transformations will be needed. The required transformations can be obtained from Green's theorems in the following manner.

Let φ and ψ represent any two scalar functions of (r,φ) which obey appropriate continuity conditions in the (r,φ) plane. Green's second theorem requires that

$$\int_A (\varphi \nabla_T^2 \psi - \psi\, \nabla_T^2 \varphi)\, d\sigma = \int_L \left(\varphi \frac{\partial}{\partial r} - \psi \frac{\partial \varphi}{\partial r}\right) d\ell \quad . \qquad (25)$$

Equation (25), applied with the identifications

$$\text{(a)} \quad \begin{aligned} \varphi &= \psi_1 \ , \\ \psi &= \psi_2^* \ , \end{aligned} \qquad\qquad \text{(b)} \quad \begin{aligned} \nabla^2 \varphi &= p_1^2 \psi_1 \ , \\ \nabla^2 \psi &= p_2^{*2} \psi_2^* \ , \end{aligned} \qquad (26)$$

and the boundary conditions (23), yields

$$(p_2^{*2} - p_1^2) \int_A \psi_1 \psi_2^*\, d\sigma = \left[\frac{p_1^2 k^* b_2^* - p_2^{*2} k b_2}{k b_2 k^* b_2^*} + \frac{n K_2^* \gamma_2^*}{Z_o^* k^* a} - \frac{n K_1 \gamma_1}{Z_o k a}\right] \int_L \psi_1(Z) \psi_2^*(Z)\, d\ell . \qquad (27)$$

Green's first theorem states that

$$\int_A (\varphi \nabla_T^2 \psi + \nabla_T \varphi \cdot \nabla_T \psi)\, d\sigma = \int_L \varphi \frac{\partial \psi}{\partial r}\, d\ell \quad , \qquad (28)$$

or, in terms of the previous identifications (26),

$$\int_A \nabla_T\Psi_1 \cdot \nabla_T\Psi_2^* \, d\sigma$$

$$= \int_L \frac{p_2^{*2}}{k^*} \left[\frac{nK_2^*\gamma_2^*}{Z_o^* a p_2^{*2}} - \frac{1}{b_2^*}\right] \Psi_1(Z)\Psi_2^*(Z) \, d\ell$$

$$- p_2^{*2} \int_A \Psi_1\Psi_2^* \, d\sigma \quad . \tag{29}$$

Employing Eq.(27) to reduce the last integral in Eq.(29) gives,

$$(p_2^{*2}-p_1^2) \int_A \nabla_T\Psi_1 \cdot \nabla_T\Psi_2^* \, d\sigma$$

$$= \left\{\frac{n}{a}\left[\frac{p_2^{*2}K_1\gamma_1}{Z_o k} - \frac{p_1^2 K_2^*\gamma_2^*}{Z_o^* k^*}\right] + 2p_2^{*2}p_1^2 \,\text{Im}\left(\frac{1}{k^* b_2^*}\right)\right\} \int_L \Psi_1(Z)\Psi_2^*(Z) \, d\ell \quad . \tag{30}$$

The various terms in Eq.(24) can now be computed.

$$P_c^{(12)} = \int_A \left[i_z \times E_{T1}^{(1)}\right] \cdot H_{T2}^{*(2)} \, d\sigma$$

$$= \frac{\gamma_1\gamma_2^*}{p_1^2 p_2^{*2}} \int_A (i_z \times \nabla_T E_{z1}) \cdot \nabla_T H_{z2}^* \, d\sigma$$

$$= - \frac{\gamma_1\gamma_2^*}{p_1^2 p_2^{*2}} \int_A \nabla_T \cdot (E_{z1} i_z \times \nabla_T H_{z2}^*) \, d\sigma$$

$$= - \frac{\gamma_1\gamma_2^*}{p_1^2 p_2^{*2}} \int_L i_r \cdot \left[(i_z E_{z1}) \times \nabla_T H_{z2}^*\right] \, d\sigma$$

$$= \frac{\gamma_1\gamma_2^*}{p_1^2 p_2^{*2}} \int_L \frac{E_{z1}}{r} \frac{\partial H_{z2}^*}{\partial \varphi} \, d\ell \quad . \tag{31}$$

But the integral on L implies that r = a, and is therefore only an integration on φ. The integrand does not depend upon φ, however, because E_{z1} varies as $e^{jn\varphi}$, while H^*_{z2} behaves like $e^{-jn\varphi}$. Therefore

$$P_c^{(12)} = \frac{-j2\pi n\gamma_1\gamma_2^*K_1}{p_1^2 p_2^{*2}} \Psi_1(z)\Psi_2^*(z) \quad . \tag{32}$$

Similarly

$$P_c^{(21)} = \int_A i_z \cdot [E_{T1}^{(2)} \times H_{T2}^{*(1)}]\, d\sigma$$

$$= \frac{j2\pi nkk^*Z_oK_2^*}{Z_o^*p_1^2p_2^{*2}} \Psi_1(z)\Psi_2^*(z) \quad . \tag{33}$$

Also

$$P_c^{(11)} = \int_A i_z \cdot [E_{T1}^{(1)} \times H_{T2}^{*(1)}]\, d\sigma$$

$$= \frac{-jk^*K_1K_2^*\gamma_1}{Z_o^*p_1^2p_2^{*2}} \int_A \nabla_T\Psi_1 \cdot \nabla_T\Psi_2^*\, d\sigma \quad . \tag{34}$$

From Eq.(30), with the line integral evaluated as in Eq.(32), the final form of Eq.(34) becomes

$$P_c^{(11)} = \frac{-jk^*K_1K_2^*\gamma_1 2\pi a}{Z_o^*(p_2^{*2}-p_1^2)} \left\{ \frac{n}{a}\left[\frac{K_1\gamma_1}{p_1^2Z_ok} - \frac{K_2^*\gamma_2^*}{p_2^{*2}Z_o^*k^*}\right] + 2\,\mathrm{Im}\left(\frac{1}{k^*b_2^*}\right)\right\} \Psi_1(z)\Psi_2^*(z) \quad . \tag{35}$$

By similar steps

$$P_c^{(22)} = \int_A i_z \cdot [E_{T1}^{(2)} \times H_{T2}^{*(2)}]\, d\sigma$$

$$= \frac{jkZ_o\gamma_2^* 2\pi a}{(p_2^{*2} - p_1^2)} \left\{ \frac{n}{a}\left[\frac{K_1\gamma_1}{p_1^2Z_ok} - \frac{K_2^*\gamma_2^*}{p_2^{*2}Z_o^*k^*}\right] + 2\,\mathrm{Im}\left(\frac{1}{k^*b_2^*}\right)\right\} \Psi_1(z)\Psi_2^*(z) \quad . \tag{36}$$

It is next necessary to add Eqs.(32),(33),(35) and (36) in accordance with Eq.(24). A considerable amount of algebraic manipulation will be required to carry out this operation, during the course of which the substitutions $\lambda_s = (\gamma_s/k)$ and $k_{ns} = (K_{ns}/Z_o)$ are helpful. It is essential to make repeated use of the relation $\lambda_s^2 = -\left[(p_s^2/k^2) + 1\right]$, in order to obtain the following expression for P_c:

$$\frac{(p_2^{*2}-p_1^2)\ P_c}{2\pi j n Z_o |k|^2\ \psi_1(Z)\psi_2^*(Z)}$$

$$= 2(\lambda_2^*-k_1k_2^*\lambda_1)\ \mathrm{Im}\left(\frac{a}{k^*b_2^*n}\right)+ 2k_2^*\ \mathrm{Im}\left(\frac{1}{k^{*2}}\right)$$

$$+\frac{k_1\lambda_1\lambda_2^*}{p_2^{*2}}\left(1+k_2^{*2}\right) - \frac{k_2^*\lambda_1^2}{p_1^2}(1+ k_1^2) \quad . \tag{37}$$

It now becomes appropriate to complete the application of the boundary conditions by using Eqs.(19) and (20), in the form

$$k_{ns} = \chi \pm\sqrt{\chi^2-1} \quad , \tag{38}$$

where

$$\text{(a)} \quad \chi = \frac{\delta t'}{2\xi} = \frac{p_s^2 a t}{2kn\lambda_s} \quad ,$$

$$\text{(b)} \quad t = \sqrt{\frac{b_1}{b_2}}\ t' = \left(b_1 + \frac{1}{b_2}\right) \quad . \tag{39}$$

But from Eq.(38), regardless of the sign chosen there,

$$1 + k_{ns}^2 = 2\chi(1 \pm \sqrt{\chi^2-1}) = 2\chi k_{ns} \quad . \tag{40}$$

A substitution of Eq.(40) into the last two terms of Eq.(37) produces the final relation

$$\frac{(p_2^{*2}-p_1^2)\ P_c}{4\pi j n Z_o |k|^2\psi_1(Z)\psi_2^*(Z)}$$

$$= (\lambda_2^*-k_1k_2^*\lambda_1)\mathrm{Im}\left(\frac{a}{k^*b_2^*n}\right)+k_2^*\mathrm{Im}\left(\frac{1}{k^{*2}}\right)+ \frac{a}{n}\ k_1k_2^*\mathrm{Im}\left(\frac{t^*}{k^*}\right) \quad . \tag{41}$$

The only circumstance in which P_c goes to zero (if $p_2^{*2} \neq p_1^2$) is when

$$k = k^* \quad , \qquad b_2 = b_2^* \quad , \qquad t^* = t \quad .$$

The structure is lossless under these conditions.

An interesting feature of Eq.(41) is the fact that it remains valid if mode 2 and mode 1 become identical. It then gives an expression for the total vector power flowing along the guide (due to a single mode). Since the right side of that equation still vanishes identically, it must be concluded that no vector power will exist for any of the modes unless $p^{*2}=p^2$. Hence the fact that γ^2 must be real on this lossless structure is observed to be a necessary condition for the existence of modes. The fact that p must be either real or imaginary requires that (ψ'/ψ) in Eq.(9) remain real at all frequencies. Hence k_{ns} is real below cutoff ($\gamma=\alpha$), while it is imaginary above cutoff ($\gamma=j\beta$). This reversal of the usual phase relation between E_z and H_z (discussed in Section 4.4) arises from the choice $e^{jn\varphi}$ for the circular variation, instead of sin $n\varphi$ and cos $n\varphi$.

Since E_z and H_z in a single mode are in phase below cutoff, Re S_z is not identically zero everywhere in the cross section. It must therefore be alternately positive and negative, in order to assure the vanishing of the integrated real power.

Attention is directed once more to the relationships between two different modes by observing that Eq.(27) becomes

$$\int_A \psi_1 \psi_2^* d\sigma$$

$$= -2\pi a \left[\frac{1}{kb_2} + \frac{n}{ka} \frac{k_1\gamma_1 - k_2^*\gamma_2^*}{\gamma_1^2 - \gamma_2^{*2}} \right] \psi_1(Z)\psi_2^*(Z) \quad , \qquad (42)$$

which is generally not zero. Hence H_{z1} and H_{z2}^* are not orthogonal [Eq.(22)]. This remains true even if n=0, Similarly for E_{z1} and E_{z2}^*.

When the structure is lossless, a derivation similar to Eqs.(24) through (41) shows that

$$
\begin{aligned}
&\text{(a)} \quad \omega(p_2^{*2}-p_1^2)\int_A \epsilon E_{T1}\cdot E_{T2}^{*}d\sigma = 2\pi n(K_2^{*}\gamma_2^{*}-K_1\gamma_1)\psi_1(Z)\psi_2^{*}(Z) \ , \\
&\text{(b)} \quad \omega(p_2^{*2}-p_1^2)\int_A \mu H_{T1}\cdot H_{T2}^{*}d\sigma = 2\pi n(K_2^{*}\gamma_1-K_1\gamma_2^{*})\psi_1(Z)\psi_2^{*}(Z) \ .
\end{aligned}
\qquad (43)
$$

The integrals in Eq.(43) are therefore not generally zero, except in the special case n=0.

Normally, then, the average-energy orthogonality conditions present in homogeneous structures cannot be expected to hold when the boundary is not opaque (even when the system is lossless).

The present system does have the special property, however, that both energy and power are orthogonal in the double-frequency sense, even if the system contains loss. This unexpected inclusion of the energy orthogonality property again arises from the circular variation $e^{jn\varphi}$, which fails to drop out in any product where neither factor is a conjugate.

Some curious matters come to light from an examination of the eigenvalue equations (17) and (18). In order to make the examination, however, a frequency dependence must be assigned to the wall admittances b_1 and b_2, whereas in the work just completed it was not necessary to do so. Unfortunately, the only cases which have been treated in detail involve the assumption that b_1 and b_2 are independent of frequency. Various choices of the relative signs and magnitudes of these admittances have been considered;but, of course, the assumption above is contrary to the normal properties of susceptance given by Eq.(2.40) of the text. Furthermore, the results of the calculations which have been made are sufficiently bizarre so that it would be unwise to present them at length until the effect of violating Eq.(2.40) is determined first. The curious "breaking off" of fast modes at frequencies below (or at) cutoff has been mentioned in the main body of the paper. There are also some slow modes which suddenly break off too. Both varieties of modes may simply cease to exist (discontinuously) at all frequencies below certain critical values, which depend upon the choices of b_1 and b_2. Many of the modes behave quite normally at all frequencies, but when the break-off phenomenon occurs at all, it affects a finite number of the lowest modes for <u>each</u> value of $n > 0$. Moreover, the number of modes which break off discontinuously below some critical frequency in the range $0 < \omega < \infty$ is roughly proportional to n, so that the completeness of the <u>set</u> at any <u>one</u> frequency is highly questionable.

One thing is definite, however: Regardless of the assumed frequency variation, none of the fast modes break off when $b_1(\omega) \equiv b_2(\omega)$.

Further work is required to establish the effect of the frequency variation of the admittances upon the slow modes, and to investigate its effects upon the fast modes in greater detail. This work has not been completed.

APPENDIX B

Rectangular Waveguide Partially Filled with Dielectric

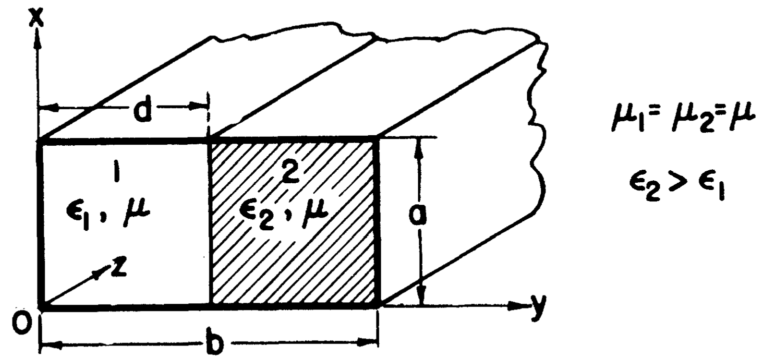

Fig. B-1. Coordinates for rectangular guide.

Figure B-1 above shows a conventional metal rectangular waveguide partially filled with a lossless dielectric (ϵ_2, μ). The remainder is filled with a second lossless material (ϵ_1, μ), which may be considered as air.

The modes on the structure will have z-dependence $e^{-\gamma z}$ in both medium 1 and medium 2. Because of the rectangular symmetry, however, the problem can be considered as cylindrical about the y-axis. All the fields can therefore be derived from E_y and H_y instead of from E_z and H_z. In each of the two media, field components of the form

$$\text{(a)} \quad E_y = F(x,z)\, e^{\pm \varkappa y} \,, \qquad \text{(b)} \quad H_y = G(x,z)\, e^{\pm \varkappa y} \,, \tag{1}$$

may be assumed, with the functions F and G further specialized as follows:

$$\text{(a)} \quad F(x,z) = f(x)\, e^{-\gamma z} \,, \qquad \text{(b)} \quad G(x,z) = g(x)\, e^{-\gamma z} \,. \tag{2}$$

In a manner analogous to that usually employed for the determination of the fields from E_z and H_z [Eqs.(3.1) and (3.2) of the text], the following equations can be derived for each of the two modes:

$$\begin{aligned} &(a) \quad \nabla_L^2 \hat{H}_y + (\varkappa^2 + k^2)\hat{H}_y = 0 \quad , \\ &(b) \quad \nabla_L^2 \hat{E}_y + (\varkappa^2 + k^2)\hat{E}_y = 0 \quad , \end{aligned} \tag{3}$$

in which ∇_L is an operator restricted to the longitudinal (L) plane (x,z).

Equations (3), as well as all boundary conditions, can be satisfied for two kinds of waves; those for which $E_y=0$ (called "LE" waves), and those for which $H_y=0$ (called "LM" waves). Field components in the longitudinal plane can be found by equations analogous to Eq.(3.2) of the text; but because there will be a standing wave in the y-direction, both $\pm\varkappa$ will be required in the y-variation. A slight alteration of Eq.(3.2) will therefore be helpful:

$$\begin{aligned} &(a) \quad \underline{\text{LE}} \; (\hat{E}_y \equiv 0) \\ &\qquad \hat{E}_L = \frac{j\omega\mu}{\varkappa^2+k^2} (i_y \times \nabla_L \hat{H}_y) \quad , \\ &\qquad \hat{H}_L = \frac{1}{\varkappa^2+k^2} \nabla_L\left(\frac{\partial \hat{H}_y}{\partial y}\right) \; ; \\ &(b) \quad \underline{\text{LM}} \; (\hat{H}_y \equiv 0) \\ &\qquad \hat{H}_L = \frac{-j\omega\epsilon}{\varkappa^2+k^2} (i_y \times \nabla_L \hat{E}_y) \quad , \\ &\qquad \hat{E}_L = \frac{1}{\varkappa^2+k^2} \nabla_L\left(\frac{\partial \hat{E}_y}{\partial y}\right) \; . \end{aligned} \tag{4}$$

Consider the LE waves first ($\hat{E}_y \equiv 0$). Equations (3b) and (2b) yield for $g_1(x)$ (in medium 1)

$$\begin{aligned} &(a) \quad \frac{d^2 g_1}{dx^2} + (\varkappa_1^2 - p_1^2) g_1 = 0 \quad , \\ &(b) \quad p_1^2 = -\gamma^2 - k_1^2 \quad . \end{aligned} \tag{5}$$

In view of the boundary condition that $E_z=0$ when $x=0$, Eq.(4a) demands that $(\partial H_y/\partial x) = 0 = (dg_1/dx)$ at $x=0$. Therefore

$$g_1(x) = A \cos x\sqrt{\varkappa_1^2-p_1^2} \quad . \tag{6}$$

At $x=a$, $E_z = 0 = (dg_1/dx)$ also. Hence

$$\sin a\sqrt{\varkappa_1^2-p_1^2} = 0 \quad ,$$

or

$$a^2(\varkappa_1^2 - p_1^2) = (n\pi)^2 \quad , \qquad n = 0,1,2,\cdots. \tag{7}$$

Because of the boundary condition that $\hat{E}_L=0$ when $y=0$, the appropriate choice of exponentials for the y-variation must be such that $H_y=0$ when $y=0$ [Eq.(4a)]. Thus

$$H_{y1} = A \cos(\frac{n\pi x}{a}) \sinh \varkappa_1 y \quad . \tag{8}$$

In medium 2, Eq.(5) is valid, except for a change in subscript, from 1 to 2. Since the boundary conditions at $x=0$ and $x=a$ also remain the same,

$$\text{(a)} \quad g_2(x) = E \cos x\sqrt{\varkappa_2^2-p_2^2} \quad ,$$

$$\text{(b)} \quad a^2(\varkappa_2^2 - p_2^2) = (n'\pi)^2 \quad , \qquad n' = 0,1,2,\cdots. \tag{9}$$

On the otherhand, $\hat{E}_L$ (or H_y) = 0 at $y=b$, so the proper combination of exponentials for the y-variation in this medium is contained in the following final expression for H_{y2}:

$$H_{y2} = E \cos(\frac{n'\pi x}{a}) \sinh \varkappa_2(b-y) \tag{10}$$

The final boundary conditions at $y=d$ are

$$\text{(a)} \quad \hat{E}_{L1}(d) = \hat{E}_{L2}(d) \quad ,$$

$$\text{(b)} \quad \hat{H}_{L1}(d) = \hat{H}_{L2}(d) \quad , \tag{11}$$

which must hold for all values of x and z. From Eq.(4a), the above conditions may be written alternatively in the form

$$(\varkappa_2^2 + k_2^2)\, H_{y1}(d) = (\varkappa_1^2 + k_1^2)\, H_{y2}(d) \quad , \tag{12a}$$

$$(\mathcal{H}_2^2 + k_2^2)\left(\frac{\partial H_{y1}}{\partial y}\right)_d = (\mathcal{H}_1^2 + k_1^2)\left(\frac{\partial H_{y2}}{\partial y}\right)_d \quad . \tag{12b}$$

Substitution of Eqs.(8) and (10) into Eq.(12a) yields

$$A(\mathcal{H}_2^2+k_2^2)\cos(\frac{n\pi x}{a})\sinh\mathcal{H}_1 d$$

$$= E(\mathcal{H}_1^2+k_1^2)\cos(\frac{n'\pi x}{a})\sinh\mathcal{H}_2(b-d) \quad , \tag{13}$$

which, according to the statement following Eq.(11), must hold for all values of x. Therefore n'=n, and Eqs.(7) and (9b) consequently demand that

$$\mathcal{H}_1^2 - p_1^2 = \mathcal{H}_2^2 - p_2^2 \quad , \tag{14}$$

or, from Eq.(5b),

$$\mathcal{H}_1^2+k_1^2 = \mathcal{H}_2^2+k_2^2 = (\frac{n\pi}{a})^2 - \gamma^2 \quad , \qquad n = 0,1,2,\cdots. \tag{15}$$

In the light of the above facts, Eq.(13) reduces to

$$A\sinh(\mathcal{H}_1 d) = E\sinh[\mathcal{H}_2(b-d)] \quad . \tag{16}$$

The application of Eq.(12b) is now simple, and leads to the result

$$\mathcal{H}_1 A\cosh(\mathcal{H}_1 d) = -\mathcal{H}_2 E\cosh[\mathcal{H}_2(b-d)] \quad . \tag{17}$$

A division of Eq.(16) by Eq.(17) results in the eigenvalue equation

$$\frac{\tanh(\mathcal{H}_1 d)}{(\mathcal{H}_1 d)} = -(\frac{b}{d} - 1)\frac{\tanh[\mathcal{H}_2(b-d)]}{[\mathcal{H}_2(b-d)]} \quad , \tag{18}$$

where $(b/d) \geq 1$. Equations (15) and (18) serve to determine γ at any frequency, while Eq.(16) or (17) determines the relative amplitudes of H_y in the two media.

The rest of the field components can be found from Eq.(4a), and are summarized below. For convenience, all the fields have been multiplied through by

$$\left[\gamma^2 - (\frac{n\pi}{a})^2\right] \quad .$$

$$
\left.\begin{aligned}
&n = 0,1,2,\cdots, \\
&E_{x1} = j\omega\mu A\gamma \cos\left(\frac{n\pi x}{a}\right) \sinh \mathcal{H}_1 y \ , \\
&E_{y1} \equiv 0 \ , \\
&E_{z1} = -j\omega\mu A\left(\frac{n\pi}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \sinh \mathcal{H}_1 y \ , \\
&H_{x1} = A\mathcal{H}_1\left(\frac{n\pi}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \cosh \mathcal{H}_1 y \ , \\
&H_{y1} = A\left[\gamma^2 - \left(\frac{n\pi}{a}\right)^2\right] \cos\left(\frac{n\pi x}{a}\right) \sinh \mathcal{H}_1 y \ , \\
&H_{z1} = A\mathcal{H}_1\gamma \cos\left(\frac{n\pi x}{a}\right) \cosh \mathcal{H}_1 y \ .
\end{aligned}\right\} \text{Medium 1 } \underline{\text{LE}} \qquad (19)
$$

For medium 2, fields can be found from Eq.(19) with the substitutions

$$
\begin{aligned}
&y \rightarrow y-b \ ; \quad \mathcal{H}_1 \rightarrow \mathcal{H}_2 \ ; \quad A \rightarrow E \ ; \\
&E_1 \rightarrow E_2 \ ; \quad H_{L1} \rightarrow -H_{L2} \ ; \quad H_{y1} \rightarrow H_{y2} \ .
\end{aligned} \qquad \text{Medium 2 } \underline{\text{LE}} \qquad (20)
$$

It should be noticed that the LE wave for n=0 is actually a TE wave. This value of n also makes $(\partial/\partial x) \equiv 0$. Since $H_{x1}=H_{x2}=0$, the vector H_T is parallel to the discontinuity between the media. In a limiting sense, therefore, it may be said that

$$H_T \cdot \text{"}\nabla k^2\text{"} = 0 \ ,$$

which is to be interpreted in the light of the statements made in Section 3.1 of the text.

The derivation of the LM waves is so similar to the previous work that only the results will be given below.

$$
\left.\begin{aligned}
&n = 1,2,\cdots, \\
&E_{x1} = B\mathcal{H}_1\left(\frac{n\pi}{a}\right) \cos\left(\frac{n\pi x}{a}\right) \sinh \mathcal{H}_1 y \ , \\
&E_{y1} = B\left[\left(\frac{n\pi}{a}\right)^2 - \gamma^2\right] \sin\left(\frac{n\pi x}{a}\right) \cosh \mathcal{H}_1 y \ , \\
&E_{z1} = -B\mathcal{H}_1\gamma \sin\left(\frac{n\pi x}{a}\right) \sinh \mathcal{H}_1 y \ ,
\end{aligned}\right\} \text{Medium 1 } \underline{\text{LM}} \qquad (21a)
$$

$$\left.\begin{aligned} H_{x1} &= j\omega\epsilon_1 B\gamma \sin\left(\frac{n\pi x}{a}\right) \cosh \mathcal{H}_1 y \quad , \\ H_{y1} &\equiv 0, \\ H_{z1} &= j\omega\epsilon_1 B\left(\frac{n\pi}{a}\right) \cos\left(\frac{n\pi x}{a}\right) \cosh \mathcal{H}_1 y \quad . \end{aligned}\right\} \begin{matrix}\text{Medium 1}\\ \underline{\text{LM}}\end{matrix} \qquad \text{(21b)}$$

The fields in medium 2 can be found from Eq.(21) with the substitutions

$$\left.\begin{aligned} & y \rightarrow y-b \ ; \quad \mathcal{H}_1 \rightarrow \mathcal{H}_2 \ ; \quad B \rightarrow F \ ; \\ & H_1 \rightarrow H_2 \ ; \quad E_{L1} \rightarrow -E_{L2} \ ; \quad E_{y1} \rightarrow E_{y2} \ ; \\ & \qquad \epsilon_1 \rightarrow \epsilon_2 \quad . \end{aligned}\right. \quad \begin{matrix}\text{Medium 2}\\ \underline{\text{LM}}\end{matrix} \qquad \text{(22)}$$

The boundary conditions at y=d yield

$$\begin{aligned} &\text{(a)} \quad \epsilon_1 B \cosh(\mathcal{H}_1 d) = \epsilon_2 F \cosh\left[\mathcal{H}_2(b-d)\right] \quad , \\ &\text{(b)} \quad \mathcal{H}_1 B \sinh(\mathcal{H}_1 d) = -\mathcal{H}_2 F \sinh\left[\mathcal{H}_2(b-d)\right] \quad , \end{aligned} \qquad \text{(23)}$$

and the eigenvalue equations are

$$\begin{aligned} &\text{(a)} \quad \mathcal{H}_1 d \tanh \mathcal{H}_1 d \\ &\qquad = -\frac{\epsilon_1}{\epsilon_2}\left(\frac{d}{b-d}\right)\left[\mathcal{H}_2(b-d)\right] \tanh\left[\mathcal{H}_2(b-d)\right] \quad , \end{aligned} \qquad \text{(24)}$$

$$\text{(b)} \quad \mathcal{H}_1^2 + k_1^2 = \mathcal{H}_2^2 + k_2^2 = \left(\frac{n\pi}{a}\right)^2 - \gamma^2 \quad ; \qquad (n = 1,2,3,\cdots.)$$

There is no LM mode when n=0.

From Eqs.(19) and (21), it appears that at cutoff (γ=0) the LE wave becomes TM ($H_z \rightarrow 0$), while the LM wave becomes TE ($E_z \rightarrow 0$). Therefore the LE wave may be called "primarily TM" and the LM wave "primarily TE".

In this particular example, it is again easy to prove, without investigating the eigenvalue equations, that γ^2 must be real. To illustrate, the integrated vector power flow in an LE wave can be computed directly. In medium 1

$$i_z \cdot (E_{T1} \times H^*_{T1}) = E_{x1} H^*_{y1}$$

$$= \left(\frac{D}{a}\right) |A|^2 \cos^2\left(\frac{n\pi x}{a}\right) \sinh \mathcal{H}_1 y \sinh \mathcal{H}^*_1 y \, , \tag{25}$$

where

$$D = j\omega\mu a\gamma \left[\gamma^{*2} - \left(\frac{n\pi}{a}\right)^2\right] \, .$$

Integrated over the range (x=0, x=a) and (y=0, y=d), Eq.(25) becomes

$$4\int_{A_1} i_z \cdot (E_{T1} \times H^*_{T1}) \, d\sigma$$

$$= D|A|^2 \left[\frac{\sinh(\mathcal{H}_1 + \mathcal{H}^*_1)d}{\mathcal{H}_1 + \mathcal{H}^*_1} - \frac{\sinh(\mathcal{H}_1 - \mathcal{H}^*_1)d}{\mathcal{H}_1 - \mathcal{H}^*_1}\right] \, , \tag{26}$$

When reduced to a common denominator and transformed by means of trigonometric identities, Eq.(26) may be written

$$\int_{A_1} S_z \, d\sigma = \frac{D|A|^2}{(\gamma^{*2} - \gamma^2)} \operatorname{Im}\left[\mathcal{H}_1 \cosh(\mathcal{H}_1 d) \sinh(\mathcal{H}^*_1 d)\right] \, . \tag{27}$$

The substitutions (20) show that the corresponding power in medium 2 is given by:

$$\int_{A_2} S_z \, d\sigma$$

$$= \frac{D|E|^2}{(\gamma^{*2} - \gamma^2)} \operatorname{Im}\left\{\mathcal{H}_2 \cosh\left[\mathcal{H}_2(b-d)\right] \sinh\left[\mathcal{H}^*_2(b-d)\right]\right\} \, . \tag{28}$$

The total power flowing along the guide is the sum of Eqs.(27)and (28). But if boundary condition (17) is multiplied by the conjugate of Eq.(16), it will become clear that the numerators in Eqs.(27) and (28) cancel upon addition. Since the total power cannot be zero for all the modes, it must be true that

$$\gamma^2 = \gamma^{*2} \, . \tag{29}$$

The same conclusion applies to LM modes, by a similar proof.

Equations (15) and (24b), along with Eq.(29), provide assurance that $\mathcal{H}_1^2$ and $\mathcal{H}_2^2$ are always real. Moreover, the former equations also guarantee that if $\epsilon_2 > \epsilon_1$,

$$\mathcal{H}_1^2 - \mathcal{H}_2^2 = k_2^2 - k_1^2 \geq 0 \quad . \tag{30}$$

In particular, $\mathcal{H}_1$, cannot be imaginary when (and if) $\mathcal{H}_2$ is real; if both are imaginary, then $|\mathcal{H}_2| \geq |\mathcal{H}_1|$. Similarly, from Eqs.(14), (15) and (24b)

$$\text{(a)} \quad p_1^2 \geq p_2^2 \quad ,$$

$$\text{(b)} \quad \mathcal{H}_1^2 - p_1^2 = \mathcal{H}_2^2 - p_2^2 = (\frac{n\pi}{a})^2 \geq 0 \quad . \tag{31}$$

Further simple restrictions on $\mathcal{H}_1$ and $\mathcal{H}_2$ can be obtained from a very brief examination of Eqs.(18) and (24a). The functions (x tanh x) and(tanh x/x) are positive, real, and even functions of (real) x. Therefore the two sides of Eq.(18) or Eq.(24a) will always be of opposite algebraic sign if $\mathcal{H}_1$ and $\mathcal{H}_2$ are assumed to be real at the same time. Consequently, such a solution is not possible.

It follows from Eqs.(29), (30), (15) and (24b) that $\mathcal{H}_2$ must be imaginary under all circumstances. Therefore Eq.(31b) shows that p_2 must also remain imaginary, which is in accordance with general matters discussed in Section 4.3 of the text. An additional conclusion from Eqs.(15) and (24b) is that no mode can be above cutoff ($\gamma = j\beta$) unless

$$k_2^2 = (\frac{n\pi}{a})^2 + K_2^2 + \beta^2 \geq (\frac{n\pi}{a})^2 \quad , \tag{32}$$

where $\mathcal{H}_2 = jK_2$. Thus the cutoff frequencies for all modes must lie above $\omega_0 = (n\pi/a\sqrt{\epsilon_2\mu})$.

The behavior of $\mathcal{H}_1$ is more complicated, and a detailed study of the eigenvalue equations is required to understand it precisely. Such studies have been made elsewhere (3). It will suffice to state here that above cutoff there must be some frequency at which $\gamma = jk_1$ or $p_1 = 0$ (Figure 4.3 of the text). Hence, according to Eq.(24b), $\mathcal{H}_1 = (\pi n/a)$ at that frequency, and $\mathcal{H}_1$ is therefore real. It remains real at all higher frequencies, too, because β remains $> k_1$. There exist lower frequencies at which $\mathcal{H}_1$ becomes imaginary, and in particular some such frequency where $\mathcal{H}_1 = 0$. The relation between the cut-off frequency ($\gamma = 0$) and the frequency at which $\mathcal{H}_1$ becomes imaginary is, however, a detailed function of the specific problem being considered.

From Eqs.(19) and (21) it can now be observed that, for either LE or LM modes, E_z and H_z are in phase when the mode is above cutoff, and 90^o out of phase when the mode is below cutoff. This checks with the fact that the transverse fields are linearly polarized.

The power orthogonality conditions are easily demonstrated directly. It is not necessary to consider waves with different values of n, since such waves are obviously orthogonal as a result of the x-integration. Moreover, all LE modes are orthogonal to all LM modes in the power sense, because of the missing field-components in each group. Therefore the only problems concern the power orthogonality among LE modes with a common value of n, and similarly among LM modes with a common value of n. The proof for the former will be considered as exemplary.

When no superscript is present, let subscripts 1 and 2 refer to the two LE modes. Otherwise, the subscript refers to the two media in the guide, while the superscript distinguishes the modes. Then, the required integrals are

$$\int_o^a \int_o^d E_{x1}^{(1)} H_{y1}^{*(2)}\, dxdy + \int_o^a \int_o^d E_{x2}^{(1)} H_{y2}^{*(2)}\, dxdy$$

$$= D\left\{\int_o^d A_1 A_2^* \sinh(\mathcal{H}_1^{(1)} y)\, \sinh(\mathcal{H}_1^{*(2)} y)\, dy \right.$$

$$\left. + \int_d^b E_1 E_2^* \sinh\left[\mathcal{H}_2^{(1)}(b-y)\right] \sinh\left[\mathcal{H}_2^{*(2)}(b-y)\right] dy\right\} . \tag{33}$$

where

$$D = \frac{j\omega\mu\gamma_1 a}{2}\left[\gamma_2^{*2} - (\frac{n\pi}{a})^2\right] .$$

The first integral, denoted by I_a, becomes,

$$(\gamma_2^{*2} - \gamma_1^2)\, I_a$$

$$= A_1 A_2^* \Big\{ \mathcal{H}_1^{(1)} \sinh(\mathcal{H}_1^{*(2)} d)\ \cosh(\mathcal{H}_1^{(1)} d) - \mathcal{H}_1^{*(2)} \sinh(\mathcal{H}_1^{(1)} d)\ \cosh(\mathcal{H}_1^{*(2)} d) \Big\}, \tag{34}$$

while the second (I_b) becomes

$$(\gamma_2^{*2} - \gamma_1^2)\, I_b$$

$$= E_1 E_2^* \Big\{ \mathcal{H}_2^{(1)} \sinh\left[\mathcal{H}_2^{*(2)} (b-d)\right]\ \cosh\left[\mathcal{H}_2^{(1)} (b-d)\right] - \mathcal{H}_2^{*(2)} \sinh\left[\mathcal{H}_2^{(1)} (b-d)\right]\ \cosh\left[\mathcal{H}_2^{*(2)} (b-d)\right] \Big\}. \tag{35}$$

If boundary conditions (16) and (17) and their conjugates are applied to each mode, it will be observed that the sum of Eqs. (34) and (35) vanishes. Therefore, if $\gamma_2^{*2} \neq \gamma_1^2$, $I_a + I_b = 0$. The LE modes are orthogonal in a power sense. A similar result follows for the LM modes.

By steps almost identical to those in Eqs.(33) through (35), it can be shown that for LE waves with the same index n,

(a) $\int_A E_{z1} E_{z2}^*\, d\sigma = 0$

(b) $\int_A \epsilon E_{z1} E_{z2}^*\, d\sigma \neq 0$ (36)

(c) $\int_A H_{z1} H_{z2}^*\, d\sigma \neq 0$

Equation (36) becomes only slightly modified for LM waves, and the conclusion is essentially unchanged: in general, neither the set of E_z's nor the set of H_z's is orthogonal; nor can a "weighting" factor ϵ or μ be relied upon to render them orthogonal.

APPENDIX C

"TE-TM" Waves on a Rectangular Waveguide

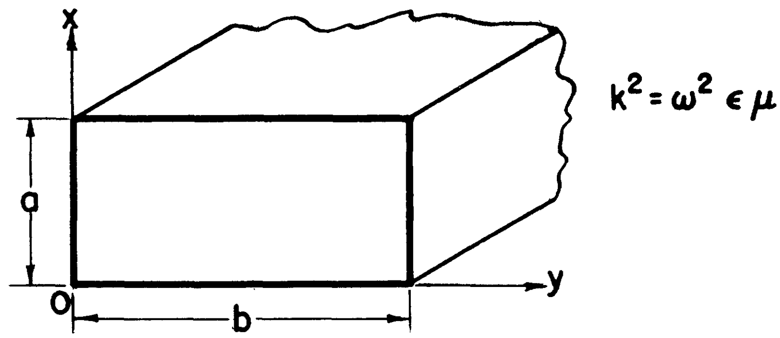

Fig. C-1. Coordinates for rectangular guide.

The above figure represents a conventional rectangular waveguide with perfectly conducting metal walls. It will be assumed that the derivation of the fields for the TE and TM modes is familiar enough to be omitted.

For TE_{mn} waves,

(a) $H_{zmn} = A_{mn} \cos(\frac{m\pi x}{a}) \cos(\frac{n\pi y}{b})$,

(b) $p_{mn} = j \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$, (1)

(c) $m,n = 0,1,2,\cdots$. (but $m^2+n^2 \neq 0$) .

For TM_{mn} waves,

(a) $E_{zmn} = B_{mn} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b})$,

(b) $p_{mn} = j \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$, (2)

(c) $m,n = 1,2,3,\cdots$. .

Since $\gamma_{mn}^2 = -(p_{mn}^2 + k^2)$, the TE_{mn} and TM_{mn} modes are degenerate when $m,n \neq 0$.

Consider a mixture of the two waves. If the frequency is below cutoff, then

$$2 \text{ Re } S_z = \frac{1}{|p|^2} \text{ Re}\left[(i_z \times \nabla_T E_z) \cdot \nabla_T H_z^*\right] \quad , \tag{3}$$

which, in the present instance, becomes

$$2 \text{ Re } S_z$$

$$= \frac{\text{Re}(A^*_{mn} B_{mn})}{|p_{mn}|^2} \left(\frac{mn\pi^2}{ab}\right) \left[\sin \pi(\frac{mx}{a} - \frac{ny}{b}) \sin \pi(\frac{mx}{a} + \frac{ny}{b})\right]. \tag{4}$$

Equation (4) shows that Re S_z reverses sign in certain parts of the cross section. This can be appreciated more easily by considering the special case m=n=1, and Re $(A^*_{11}B_{11}) > 0$. Then Re S_z = 0 along the lines

(a) $x = (\frac{a}{b})\, y$

and (5)

(b) $x = a - (\frac{a}{b})\, y$.

It is =0 along the lines y=0 and y=b. Thus the sign distribution of Re S_z will be as shown in Figure C-2 below.

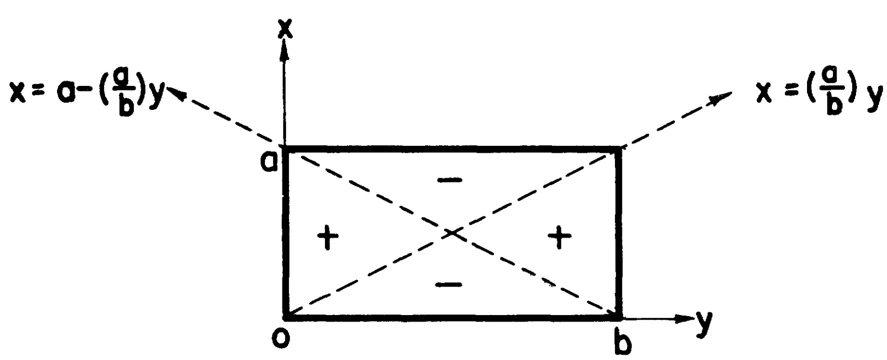

Fig. C-2. Sign distribution of
Re S_z for a $(\text{TE-TM})_{1,1}$ wave below cutoff.

By expanding Eq.(4) trigonometrically, it can be verified that

$$\int_0^a \int_0^b \text{Re } S_{zmn} dx\, dy = 0 \quad . \tag{6}$$

If the mode is above cutoff,

$2 \operatorname{Re} S_z$

$$= \frac{(\beta^2+k^2)\operatorname{Re}\left[(i_z \times \nabla_T E_z)\cdot\nabla_T H_z^*\right] + \omega\beta\left(\mu\|\nabla_T H_z\|^2 + \epsilon\|\nabla_T E_z\|^2\right)}{|p|^4}. \quad (7)$$

Equations (1) and (2) show that E_{zmn} and H_{zmn} are independent of frequency, provided that A_{mn} and B_{mn} are not functions of frequency. Therefore, if $\beta > 0$, it must be possible to find a frequency sufficiently near cutoff ($\beta \approx 0$) such that the second term in the numerator of Eq.(7) becomes arbitrarily small. If $\operatorname{Re}(A_{mn}^* B_{mn}) > 0$ however, Figure C-2 shows that the first term of the numerator in Eq.(7) is always negative over certain portions of the cross section. In view of the fact that $k^2 \neq 0$, even when β becomes zero, it follows that in these regions of the cross section, the Re S_z will necessarily become negative at frequencies sufficiently near (but, nevertheless, above) cutoff.

It should be recognized that the Poynting vector reversals indicated in the foregoing can take place above cutoff when A_{mn} and B_{mn} are entirely real, under which conditions the transverse fields would be linearly polarized.

In any event, Eq.(6) shows that the first term on the right side of Eq.(7) integrates to zero over the cross section. Hence the integrated real power flow along the guide has the same algebraic sign as β, in accordance with the general discussions given in Sections 4.2 and 4.3.

APPENDIX D

Dielectric Rod in Free Space, Driven by Axial Dipole

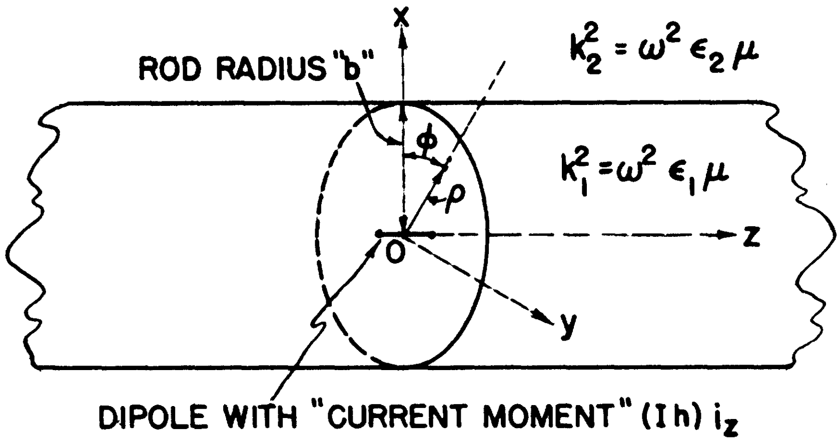

Fig. D-1. Coordinates and notation for dielectric rod.

The above figure represents a solid dielectric rod having a circular cross section and constants (ϵ_1, μ). It is surrounded by an infinite region with constants (ϵ_2, μ). An infinitesimal dipole, polarized along the z-axis, is located at the origin of the circular-cylindrical coordinates ρ, φ, z. The general method of solution will be to expand this dipole source into an infinite set of line sources, by direct Fourier transformation. The boundary-value problem can then be solved for each line source, and the results finally combined by the inverse Fourier transformation.

From the principles of Fourier analysis, any current-density distribution I(z) (amperes per meter) may be represented by the integral

$$I(z) = \int_{-\infty}^{\infty} g(\beta)\, e^{-j\beta z}\, d\beta \quad , \tag{1}$$

where

$$g(\beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I(z)\, e^{j\beta z}\, dz \quad . \tag{2}$$

For the particular problem at hand,

$$I(z) = (Ih)\, u_0(z) \quad , \tag{3}$$

in which $u_0(z)$ is a "unit impulse" at $z = 0$, while I and h are, respectively, the "effective" current density and "effective" length of the dipole. Equation (2) therefore yields

$$g(\beta) = \frac{Ih}{2\pi} \quad . \tag{4}$$

If the boundary-value problem corresponding to a <u>line</u> source can be solved, the solution to the problem with a <u>dipole</u> source can be obtained by integration as follows. Suppose that $G(\rho,\varphi;\beta)\, e^{-j\beta z}$ is any field component arising from the solution to the boundary-value problem when a line source having unit current density $e^{-j\beta z}$ is impressed upon the axis of the rod. Then the linearity of the system requires that the field component in question be multiplied by $g(\beta)$ if the current density is multiplied by the same factor. The superposition theorem demands, in turn, that the total field component $R(\rho,\varphi,z)$, due to the <u>actual</u> current-density distribution $I(z)$, be given by the integral

$$R(\rho,\varphi,z) = \int_{-\infty}^{\infty} g(\beta)\, G(\rho,\varphi;\beta)\, e^{-j\beta z}\, d\beta \quad . \tag{5}$$

Therefore it will be necessary to obtain a solution to the rod problem in the presence of a line source having unit current density $i_z e^{-j\beta z}$.

The required solution can be broken down into several parts. In medium 1, a source is present. Thus the Maxwell equations have a corresponding source term in them. The total solution in that medium can consequently be taken as the sum of a "particular solution" to the differential equations with the source term present, plus a collection of appropriate source-free solutions. Furthermore, the particular solution in medium 1 need only be a solution to the differential equations; in finding it, the boundary can therefore be entirely disregarded. Hence the desired result may be obtained by solving for the field due to the given line source acting in medium 1, as if that medium were infinite in extent. The free solutions must then be chosen to combine with this particular solution in such a way that the total field will satisfy the boundary conditions.

In medium 2, however, there are no sources. Therefore the required solutions to the Maxwell equations in that medium will be an appropriate collection of source-free modes, which are specifically required to remain finite at large distances from the rod.

The boundary conditions are the continuity requirements on the tangential E and H fields at the surface of the rod. These conditions will fix the amplitude of each free solution in relation to the strength of the source.

The aforementioned particular solution in medium 1 can now be determined. First of all, it will certainly be axially symmetric. Furthermore, the line source produces no z-component of magnetic field. The z-component of the electric field at all points in space except $\rho = 0$ must be a solution to Eq.(3.1a) of the text. At $\rho = 0$, it must have a singularity to account for the presence of the line source. The linearity of the system requires that the z-dependence of E_z correspond with that of the line source. Therefore in Eq.(3.1a)

$$p_1^2 = \beta^2 - k_1^2 \quad . \tag{6}$$

Consequently, the solution for the longitudinal component of the electric field caused by the line source acting in an infinite region with constants (ϵ_1,μ), may be written

$$E_z = C\, K_o(p_1\rho)\, e^{-j\beta z} \quad . \tag{7}$$

The constant C will be determined by the source from conditions on the magnetic field H_φ. The latter, according to Eq. (3.2a) of the text and Eq.(7) above, is given by the expression

$$H_\varphi = \frac{j\omega\epsilon_1}{p_1}\, CK_o'(p_1\rho)\, e^{-j\beta z} = \frac{-j\omega\epsilon_1}{p_1}\, CK_1(p_1\rho)\, e^{-j\beta z} \quad . \tag{8}$$

Now for any value of z (z = 0 for example), the line integral of H_φ around an infinitesimal circle of radius ρ_o must equal the enclosed current. Since the latter is equal to 1 in the present problem, this condition, along with Eq.(8) and the series expansion for the modified Bessel function K_1, requires that

$$C = \frac{jp_1^2}{2\pi\omega\epsilon_1} \quad . \tag{9}$$

The rest of the field components may be evaluated from Eq. (3.2a) of the text. With a subscript "ℓ" to denote the fact that the solutions refer to the field of a line source, they may be written

$$\begin{aligned}
&(a) \quad E_{z\ell} = \frac{j\omega\mu p_1^2}{2\pi k_1^2} K_o(p_1\rho)\ e^{-j\beta z} \quad , \\
&(b) \quad E_{\rho\ell} = \frac{\omega\mu\beta p_1}{2\pi k_1^2} K_1(p_1\rho)\ e^{-j\beta z} \quad , \\
&(c) \quad H_{\varphi\ell} = \frac{p_1}{2\pi} K_1(p_1\rho)\ e^{-j\beta z} \quad .
\end{aligned} \tag{10}$$

The remainder of the solution is made up of appropriate modes both inside and outside the rod. These must have the propagation constant β, and be axially symmetric TM modes. Therefore

$$\begin{aligned}
&(a) \quad E_{zi} = A' I_o(p_1\rho)\ e^{-j\beta z} \quad , \\
&(b) \quad E_{\rho i} = \frac{j\beta A'}{p_1} I_1(p_1\rho)\ e^{-j\beta z} \quad , \\
&(c) \quad H_{\varphi i} = \frac{j\omega\epsilon_1 A'}{p_1} I_1(p_1\rho)\ e^{-j\beta z} \quad ,
\end{aligned} \tag{11}$$

represents the internal (i) free-mode field, while

$$E_{zo} = B' K_o(p_2\rho) e^{-j\beta z} ,$$

$$E_{\rho o} = \frac{-j\beta B'}{p_2} K_1(p_2\rho) e^{-j\beta z} ,$$

$$H_{\varphi o} = \frac{-j\omega\epsilon_2 B'}{p_2} K_1(p_2\rho) e^{-j\beta z} , \qquad (12)$$

$$p_2^2 = \beta^2 - k_2^2 ,$$

represents the outside (o) free-mode field. It will be shown immediately that no other modes will be required, because the boundary conditions suffice to determine the unknown constants A' and B' in terms of the strength of the source.

The boundary conditions at $\rho = b$ are

$$\text{(a)} \quad E_{z\ell} = E_{zo} - E_{zi} ,$$

$$\text{(b)} \quad H_{\varphi\ell} = H_{\varphi o} - H_{\varphi i} , \qquad (13A)$$

which become

$$\text{(a)} \quad B' K_o(Z_2) - A' I_o(Z_1) = \alpha' Z_1^2 K_o(Z_1) ,$$

$$\text{(b)} \quad B' \lambda \frac{K_1(Z_2)}{Z_2} + A' \frac{I_1(Z_1)}{Z_1} = \alpha' Z_1 K_1(Z_1) , \qquad (13B)$$

where

$$Z_1 = p_1 b , \quad Z_2 = p_2 b ,$$

$$\lambda = \frac{\epsilon_2}{\epsilon_1} , \quad \alpha' = \frac{j}{2\pi\omega\epsilon_1 b^2} . \qquad (14)$$

Solution of the pair of equations (13B) for A' and B' can be carried out by determinants. The system determinant, Δ, is given by

$$\Delta = I_o(Z_1) K_o(Z_1) \left[\frac{I_1(Z_1)}{Z_1 I_o(Z_1)} + \lambda \frac{K_1(Z_2)}{Z_2 K_o(Z_2)} \right] , \qquad (15)$$

while the constants A' and B' are determined by

$$\text{(a)} \quad A' = \frac{\alpha'\left[Z_1 K_o(Z_2) K_1(Z_1) - \lambda Z_2 K_o(Z_1) K_1(Z_2)\right]}{\Delta} \; ,$$

$$\text{(b)} \quad B' = \frac{\alpha' Z_1 \left[I_1(Z_1) K_o(Z_1) + I_o(Z_1) K_1(Z_1)\right]}{\Delta} = \frac{\alpha'}{\Delta} \; . \qquad (16)$$

The Wronskian relation has been used to reduce the numerator of Eq.(16b).

The solutions to the unit line-source problem, which correspond to $G(\rho,\varphi,\beta)\, e^{-j\beta z}$ (page 118), may now be written:

Medium 1	Medium 2	
$E'_{z1} = E_{z\ell} + E_{zi}$,	$E'_{z2} = E_{zo}$,	
$E'_{\rho 1} = E_{\rho\ell} + E_{\rho i}$,	$E'_{\rho 2} = E_{\rho o}$,	(17)
$H'_{\varphi 1} = H_{\varphi\ell} + H_{\varphi i}$;	$H'_{\varphi 2} = H_{\varphi o}$.	

In order to form the integrand corresponding to that in Eq.(5), the field components in Eq.(17) need only be multiplied by the value of $g(\beta)$ given in Eq.(4). They must then be integrated with respect to β from $-\infty$ to $+\infty$. The resulting contribution of the integrals containing $E_{z\ell}$, $E_{\rho\ell}$ and $H_{\varphi\ell}$ to the field inside the rod can be evaluated at once. <u>It must be precisely the free-space field of a dipole in a medium with constants (μ, ϵ_1)</u>, because it represents the superposition of the free-space fields caused by <u>line-sources</u> whose integral (or superposition) is the <u>dipole source</u>. The integral involving $H_{\varphi\ell}$, for example, can indeed be shown to be the magnetic field ($H_{\varphi d}$) of a dipole. Thus, from Eqs.(10c), (6), (5) and (4)

$$H_{\varphi d} = \frac{Ih}{(2\pi)^2} \int_{-\infty}^{\infty} \sqrt{\beta^2-k_1^2}\; K_1\left(\sqrt{\beta^2-k_1^2}\;\rho\right) e^{-j\beta z}\, d\beta \; . \qquad (18)$$

The integral in Eq.(18) is evaluated in the Campbell and Foster tables (16), pair 867.5. Use of the result contained there gives for Eq.(18)

$$H_{\varphi d} = \frac{Ih\rho}{4\pi} \left[\frac{1+jk_1\sqrt{\rho^2+z^2}}{\left(\sqrt{\rho^2+z^2}\right)^3}\right] e^{-jk_1\sqrt{\rho^2+z^2}} \; , \qquad (19)$$

which in standard spherical coordinates $R = \sqrt{\rho^2+z^2}$, $\rho = R \sin \theta$, $z = z$ becomes

$$H_{\varphi d} = \frac{Ih \sin \theta}{4\pi} \left[\frac{1}{R^2} + j \frac{k_1}{R} \right] e^{-jk_1 R} \quad . \tag{20}$$

Equation (20) is the well-known form of the dipole field, and confirms the fact that none of the integrals involving the "particular" solution need actually be evaluated. They will be written simply as the dipole field E_{zd}, $E_{\rho d}$ and $H_{\varphi d}$.

The rest of the integrals involve the "i" and "o" fields. The multiplication by $g(\beta)$ merely multiplies A' and B' in Eq.(16) by $(Ih/2\pi)$, or effectively changes α' to α, where

$$\alpha = \frac{Ih\alpha'}{2\pi} = \frac{jIh}{4\pi^2 \omega \epsilon_1 b^2} \quad . \tag{21a}$$

With the notation

$$\Delta_A = Z_1 K_o(Z_2) K_1(Z_1) - \lambda Z_2 K_o(Z_1) K_1(Z_2) \quad , \tag{21b}$$

the total field solution to the problem therefore becomes:

<u>Medium 1</u>

$$E_{z1} = E_{zd} + \alpha \int_{-\infty}^{\infty} \left(\frac{\Delta_A}{\Delta} \right) I_o(p_1 \rho) \, e^{-j\beta z} \, d\beta \quad ,$$

$$E_{\rho 1} = E_{\rho d} + j\alpha \int_{-\infty}^{\infty} \left(\frac{\beta \Delta_A}{p_1 \Delta} \right) I_1(p_1 \rho) \, e^{-j\beta z} \, d\beta \quad , \tag{22a}$$

$$H_{\varphi 1} = H_{\varphi d} + j\omega \epsilon_1 \alpha \int_{-\infty}^{\infty} \left(\frac{\Delta_A}{p_1 \Delta} \right) I_1(p_1 \rho) \, e^{-j\beta z} \, d\beta \quad ;$$

and

Medium 2

$$E_{z2} = \alpha \int_{-\infty}^{\infty} \frac{K_o(p_2\rho)}{\Delta} e^{-j\beta z} d\beta \quad ,$$

$$E_{\rho 2} = -j\alpha \int_{-\infty}^{\infty} \left(\frac{\beta}{p_2\Delta}\right) K_1(p_2\rho) e^{-j\beta z} d\beta \quad , \tag{22b}$$

$$H_{\varphi 2} = -j\omega\epsilon_2\alpha \int_{-\infty}^{\infty} \frac{K_1(p_2\rho)}{p_2\Delta} e^{-j\beta z} d\beta \quad .$$

The additional relations required to connect p_1, p_2 and β are rewritten from Eqs.(6) and (12):

$$p_1^2 = \beta^2 - k_1^2 \quad ,$$
$$p_2^2 = \beta^2 - k_2^2 \quad , \tag{23}$$

and therefore

$$p_2^2 - p_1^2 = k_1^2 - k_2^2 \quad . \tag{24}$$

The integrals in Eqs.(22a) and (22b) are extremely difficult to evaluate completely. Fortunately it is not necessary to do so in order to discover the part played by the free modes in the solution to this problem. Since it is to the interpretation of these modes that this paper is primarily devoted, much useful information can be obtained from the aforementioned integrals. In particular, it is clear that the nature of the integrands in Eqs.(22a) and (22b) depends greatly upon the determinant Δ. A substantial digression will therefore be made in order to discuss its properties, and their relationship to the free modes.

According to Eqs.(14), (15), (23) and (24), the values of γ for which Δ vanishes are given by the set of equations

$$\frac{I_1(Z_1)}{Z_1 I_o(Z_1)} = -\lambda \frac{K_1(Z_2)}{Z_2 K_o(Z_2)} \quad , \tag{25a}$$

$$Z_2^2 - Z_1^2 = \omega^2\mu b(\epsilon_1 - \epsilon_2) \quad , \tag{25b}$$

$$-\gamma^2 = Z_2^2 + k_2^2 = Z_1^2 + k_1^2 \quad . \tag{25c}$$

These equations are precisely the ones which determine the <u>free TM modes</u> on the structure, when the fields are axially symmetric. The standard method of searching for free modes leads not only to Eq.(25), but also to a similar equation for TE modes (when the circular-variation index (n) is zero). For $n > 0$, the free modes are mixed TE-TM, and the eigenvalue equation is more complicated (8,9,10,11,12). The results in the latter circumstance are not sufficiently different, however, to warrant detailed consideration here.

Equation (25a) cannot be satisfied when Z_1 and Z_2 are both pure real, because the two sides have opposite algebraic signs (18). Moreover, the <u>left</u> side of that equation remains <u>real</u> whether Z_1 is pure real or pure imaginary. Now, it is easy to show that if Z_2 is pure <u>imaginary</u> ($= j\,\delta_2$), the <u>right side cannot</u> be pure real; for by Eq.(5.8) of the text

$$-\lambda \frac{K_1(j\delta_2)}{j\delta_2 K_o(j\delta_2)} = \lambda \frac{H_1^{*(1)}(\delta_2)}{\delta_2 H_o^{*(1)}(\delta_2)} \quad . \tag{26}$$

In order for the above term to be real, however,

$$\text{Im} \left[\frac{H_1^{(1)}(\delta_2)}{H_o^{(1)}(\delta_2)}\right] = 0 \quad , \tag{27}$$

or

$$\begin{aligned} &\text{Im}\left\{\left[H_1^{(1)}\ (\delta_2)\right]\left[H_o^{*(1)}\ (\delta_2)\right]\right\} \\ &= \text{Im}\left\{\left[J_1(\delta_2) + jN_1(\delta_2)\right]\left[J_o(\delta_2) - jN_o(\delta_2)\right]\right\} \\ &= J_o(\delta_2)\ N_1(\delta_2) - J_1(\delta_2)\ N_o(\delta_2) = 0 \quad . \end{aligned} \tag{28}$$

Since $J_1(\delta_2) = -J_o'(\delta_2)$ and $N_1(\delta_2) = -N_o'(\delta_2)$, Eq.(28) requires that the Wronskian of the two independent solutions (J_o and N_o) must vanish. This is impossible

In view of the fact that γ cannot be complex (as outlined in Section 5.2 of the text), the only remaining possibility is for Z_2 to be real and Z_1 pure imaginary ($= j\delta_1$). Under these conditions, however, Eq.(25b) requires that $\epsilon_1 > \epsilon_2$, while Eq. (25c) demands that $\gamma = j\beta$. Therefore Eq.(25) may be rewritten

$$(a) \quad \frac{J_1(\delta_1)}{\delta_1 J_o(\delta_1)} = -\lambda \frac{K_1(Z_2)}{Z_2 K_o(Z_2)} \quad ,$$

$$(b) \quad \delta_1^2 + Z_2^2 = \omega^2 \mu b^2 (\epsilon_1 - \epsilon_2) \geq 0 \quad , \qquad (29)$$

$$(c) \quad b^2\beta^2 = Z_2^2 + b^2 k_2^2 = b^2 k_1^2 - \delta_1^2 \geq 0 \quad .$$

Clearly Eq.(29c) states that

$$k_1 \geq \beta \geq k_2 \quad . \qquad (30)$$

The general form of the left side $F_1(\delta_1)$ of Eq.(29a) is shown in Figure D-2. The values a_ν and b_ν are given by the roots

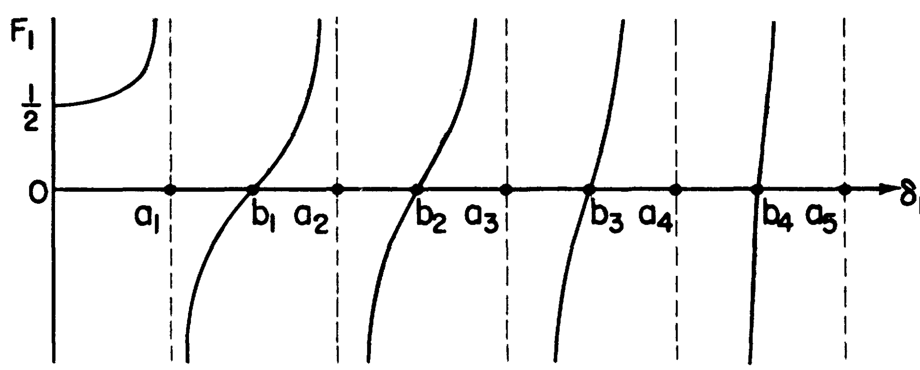

Fig. D-2. Left side of Eq.(29a).

of the Bessel functions (excluding the origin) as follows:

$$(a) \quad J_o(a_\nu) = 0 \quad , \qquad \nu = 1,2,3,\cdots \; ;$$

and (31)

$$(b) \quad J_1(b_\nu) = 0 \quad , \qquad \nu = 1,2,3,\cdots \; .$$

The right side $F_2(Z_2)$ of Eq.(29a), on the other hand, has the behavior illustrated in Figure D-3 below.

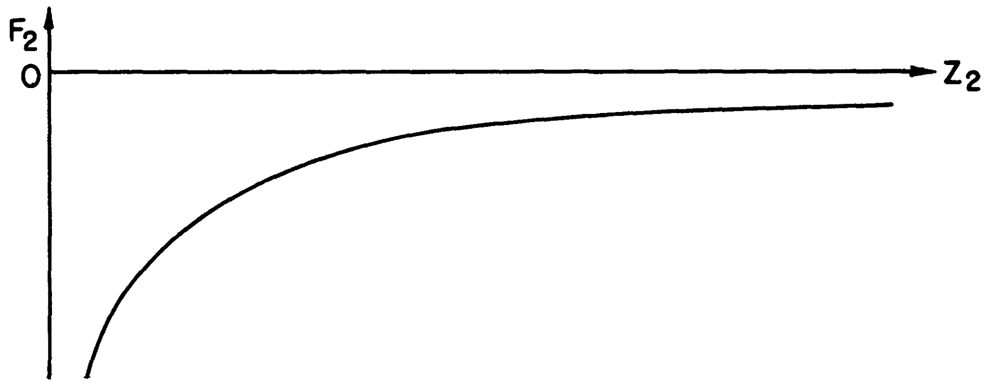

Fig. D-3. Right side of Eq.(29a).

From Figures D-2 and D-3, it may be seen that the equality of F_1 and F_2, required by Eq.(29a), can take place only when

$$a_\nu \leq \delta_1 \leq b_\nu \quad , \qquad \nu = 1,2,3,\cdots. \tag{32}$$

For each value of ν, the values of Z_2 range from zero (at $\delta_1 = a_\nu$) to ∞ (at $\delta_1 = b_\nu$). The values of ν, in fact, designate the modes, and the resulting loci of Z_2 versus δ_1 are shown in Figure D-4. The dotted circle in Figure D-4 represents the rela-

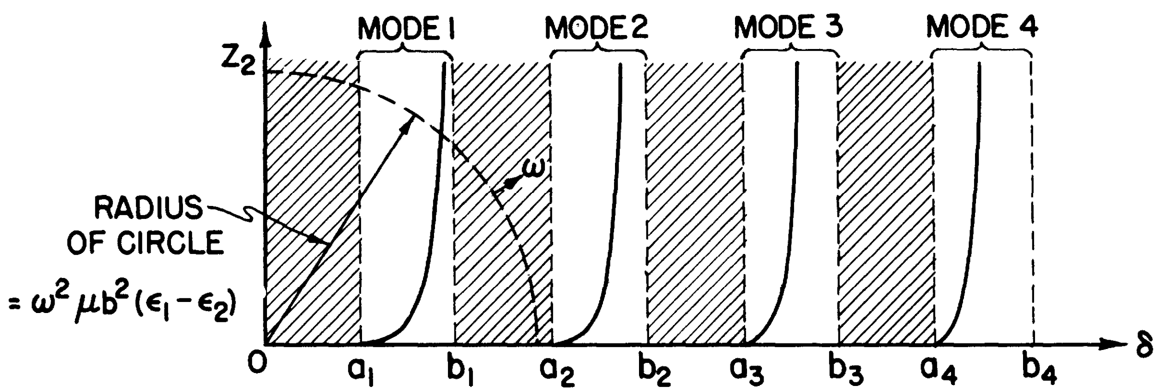

Fig. D-4. Solution of Eqs.(29a) and (29b).

tion (29b), which must be satisfied along with Eq.(29a). The intersection of this circle with each "mode curve" (solid lines) gives the corresponding values of Z_2 and δ_1 in that mode at each frequency. According to Eq.(29c), $\beta = k_2$ when $Z_2 = 0$. From Figure D-4, the frequency at which $Z_{2\nu} = 0$ (in the νth mode) is that for which the circle just touches the corresponding "mode curve". Hence

$$\beta_\nu = k_2 \quad \text{when} \quad \omega_{d\nu}^2 = \frac{a_\nu}{\mu b^2(\epsilon_1 - \epsilon_2)} , \qquad \nu = 1,2,3 \cdots . \tag{33}$$

The $\omega_{d\nu}$ in Eq.(33) are the "divergence frequencies" for each mode "ν". Mode "ν" ceases to exist when $\omega < \omega_{d\nu}$. At very high frequencies, Figure D-4 also shows that $\delta_{1\nu} \rightarrow b_\nu$ (a constant, independent of frequency). Therefore Eq.(29c) leads to the conclusion that $\beta_\nu \rightarrow k_1$ as $\omega \rightarrow$. Since this same equation shows that

$$\beta_\nu^2 - \beta_{\nu+1}^2 = \delta_{1,\nu+1}^2 - \delta_{1,\nu}^2 , \tag{34}$$

and since it is clear from Figure D-4 that

$$\delta_{1,\nu+1} > \delta_{1,\nu} , \tag{35}$$

therefore

$$\beta_\nu > \beta_{\nu+1} . \tag{36}$$

It is not difficult to see from the foregoing considerations that the frequency dependence of the various β_ν will be given by the curves of Figure D-5, in which the lines showing k_1 and k_2 are also included.

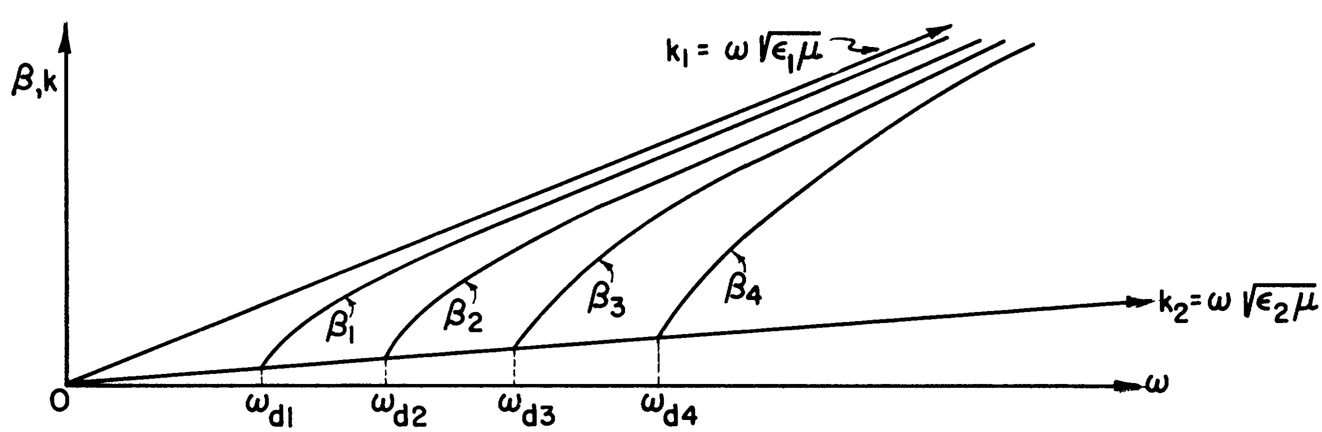

Fig. D-5. β_ν versus ω
for $TM_{o\nu}$ modes on a dielectric rod.

It is a consequence of Figure D-5 that if the frequency is below ω_{d1} (for a given rod), there will be <u>no</u> free $TM_{o\nu}$ modes. Even if the frequency is above this value, there will only be a <u>finite number</u> of these free modes having divergence frequencies below the given frequency. Since the $TE_{o\nu}$ modes are similar, and in fact can be shown to have the same divergence frequencies as the $TM_{o\nu}$ modes, it is therefore clear that these modes cannot be a complete set.

The modes which are not axially symmetric ($n > 0$), differ from the present ones in that there is always <u>one</u> TE-TM mode, for <u>each</u> value of $n > 0$, which persists down to $\omega = 0$. Nevertheless, since it is possible to construct sources whose free-space field requires only a finite number of angular variations n, the TE-TM modes are evidently still not complete. Such a source would be, for example, a dipole at $z = 0$ which is polarized in the transverse plane; for this source excites only modes with $n = 1$.

In any case, there will be no free modes at all if $\epsilon_1 < \epsilon_2$, even though the solution to the dipole problem can always be expressed in an integral form similar to Eqs.(22a) and (22b). It follows that there must be some portions of the integrals which are not representable by free modes. That is, the integrals <u>cannot</u> be entirely represented by a series expansion in the free modes, and therefore the fields due to the dipole source cannot be thus represented. It is interesting to observe that the Fourier integrals in question can furnish a basis for the proof of the completeness of a set of modes, and this technique has been applied (17) to develop the theory of eigenfunction expansions in general.

The foregoing discussion of the significance of Δ (and the connection between its zeros and the free modes) sheds considerable light upon the interpretation of the integrals in Eqs.(22a) and (22b). The simplest integral is that for E_{z2}, which will be taken as an example. With the new notation

$$r = k_2\rho \quad , \qquad B = k_2 b \leq r \quad ,$$

$$Z = k_2 z \quad , \qquad \varkappa = \frac{k_1}{k_2} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \frac{1}{\sqrt{\lambda}} > 1 \quad , \tag{37}$$

the field E_{z2} can be written

$$\frac{E_{z2}}{\varkappa^2 k_2 \alpha} =$$

$$\int_{-\infty}^{\infty} \frac{\left(B\sqrt{W^2-1}\right)\left(B\sqrt{W^2-\varkappa^2}\right) K_o\left(r\sqrt{W^2-1}\right) e^{-jWZ} dW}{\varkappa^2 I_1\left(B\sqrt{W^2-\varkappa^2}\right) K_o\left(B\sqrt{W^2-1}\right) + I_1\left(B\sqrt{W^2-\varkappa^2}\right) K_1\left(B\sqrt{W^2-1}\right)} \quad . \tag{38}$$

The integrand (excluding the exponential) in Eq.(38) has branch points at ∞ and at $W = \pm 1$, because of the functions K_o and K_1. It may also have a finite number of simple poles corresponding to the free modes, as previously discussed. If there are any of these, Eq.(30) shows that they lie symmetrically about the origin in two regions restricted by

$$1 \leq |W| \leq |\varkappa| \quad . \tag{39}$$

Suppose, for example, that $W = \pm W_o$ are the only such simple poles. They lie directly upon the path of integration, since the free modes are undamped.

In this situation, the Fourier integral must be interpreted properly. In order to do so, it is simplest to consider the present lossless rod as the limiting case of a dissipative one, in which k_1 is complex. The propagation constant (γ_o) of the free mode would then be complex, and the particular branches of the Bessel functions which determine γ_o would have been so chosen (on physical grounds) that α_o and β_o would have the same algebraic sign. The free mode would propagate with z-dependence

$$e^{-\gamma_o z} = e^{-(\alpha_o + j\beta_o)z} \quad ;$$

the "incident wave" would have α_o, $\beta_o > 0$, while the corresponding "reflected wave" would have α_o, $\beta_o < 0$. Since the propagation factor in the integral (38) is written instead as e^{-jWZ}, the corresponding value of W_o would be $W_o = \beta_o - j\alpha_o$. The two poles of the integrand in Eq.(38) would thus lie in the second and fourth quandrants of the complex W-plane. The actual lossless problem is now to be interpreted as a limiting form of the dissipative one, when $\alpha_o \rightarrow 0$. It is more convenient, however, to displace the contour of integration where it passes the poles, rather than to move the poles themselves. Therefore the integral (38) is to be interpreted as taken over the path shown in Figure D-6, with the understanding that the radii of the small detours

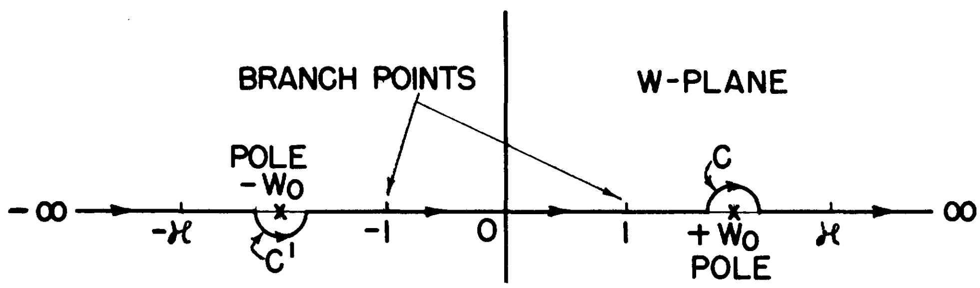

Fig. D-6. Path of integration for Eq.(38)

C and C' will be allowed to approach zero. It is not possible to close the contour on the conventional large semi-circle, because of the branch point at ∞. This is far from unexpected, however, in view of the fact that the possibility of such a closure would mean that the poles furnished the only contribution to the integral. Then the free modes would be the entire solution, and there would be no radiation. It is clear from the physics of the problem, however, that there will be radiation, and the mathematics shows up this fact by requiring contributions to the integral from regions outside the immediate vicinity of the poles.

These statements may be clarified by rewriting the integral in Eq.(38) in the condensed form

$$I(Z,r) = \int_{-\infty}^{\infty} F(W,r) \cos WZ \, dW \quad . \tag{40}$$

By reason of the symmetry of the problem, it is sufficient to consider $Z \geqq 0$. The function $F(W,r)$ is described, qua function of W, by Figure D-6, while its dependence on r is governed primarily by the Bessel function $K_o(r\sqrt{W^2-1})$. The branch points $W = \pm 1$ are zeros of $F(W,r)$, as can be seen from Eq.(38) by using the series expansion of the K-functions near this value of W. The square roots are to be taken in such a way that their angles lie between 0 and $+\frac{\pi}{2}$, because of the physical requirements that the integrand must either represent outgoing waves or remain finite as $r \rightarrow \infty$. Thus $F(W,r)$ is complex when $W < 1$ and real when $W > 1$.

It is therefore convenient to write Eq.(40) in the form

$$I(Z,r) = I_a(Z,r) + I_b(Z,r)$$

$$= \int_o^1 F(W,r) \cos WZ \, dW + \int_1^\infty F(W,r) \cos WZ \, dW \, . \quad (41)$$

According to Figure D-6, the interpretation given to $I_b(Z,r)$ is

$$I_b(Z,r)$$

$$= \lim_{\delta_o \rightarrow 0} \left\{ \int_1^{W_o-\delta_o} F(W,r) \cos WZ \, dW + \int_{W_o+\delta_o}^\infty F(W,r) \cos WZ \, dW \right.$$

$$\left. + \int_{C(-\pi,o)} F(W,r) \cos WZ \, dW \right\}, \quad (42)$$

where $W - W_o = \delta_o e^{j\theta}$ on the contour C. The first two integrals in Eq.(42) define, together, a Cauchy principal value on the real axis of W. The remaining integral about C can be obtained in terms of residues from standard methods of complex-variable theory. Thus

$$I_b(Z,r)$$

$$= P \left\{ \int_1^\infty F(W,r) \cos WZ \, dW \right\} - \pi j \, R(W_o,r) \cos W_o Z \quad , \quad (43)$$

where $R(W_o,r)$ is the residue of $F(W,r)$ in the simple pole at W_o, and P denotes the Cauchy principal value of the first integral. According to the integrand in Eq.(38), the form of $R(W_o r)$ will be

$$R(W_o,r) = g(W_o,B,\mathcal{H})\, K_o(r\sqrt{W_o^2-1}) \quad , \tag{44}$$

where $g(W_o,B,\mathcal{H})$ would be obtained explicitly in the process of finding the residue at $W = W_o$.

In order to evaluate the first integral in Eq.(43), it is convenient to define

$$Q(W,r) = (W-W_o)\, F(W,r) \quad , \tag{45}$$

whence, in particular,

$$Q(W_o,r) = R(W_o,r) \quad . \tag{46}$$

Therefore

$$P\left\{ \int_1^\infty F(W,r) \cos WZ \, dW \right\}$$
$$= P\left\{ \int_1^\infty \frac{Q(W_o,r)}{W-W_o} \cos WZ \, dW \right\}$$
$$+ \int_1^\infty \frac{[Q(W,r) - Q(W_o,r)]}{W-W_o} \cos WZ \, dW \quad . \tag{47}$$

Since the integrand in the second integral on the right side of Eq.(47) remains finite at $W=W_o$, no principal value is required. The first integral can be evaluated directly;

$$P\left\{ \int_1^\infty \frac{Q(W_o,r)}{W-W_o} \cos WZ \, dW \right\}$$
$$= P\left\{ \int_{-(W_o-1)}^\infty \frac{Q(W_o,r)}{\xi} \cos Z(\xi + W_o) \, d\xi \right\}$$
$$= Q(W_o,r)\, P\left\{ \int_{-(W_o-1)Z}^\infty \left(\cos W_o Z \, \frac{\cos \chi}{\chi} - \sin W_o Z \, \frac{\sin \chi}{\chi} \right) d\chi \right\}, \tag{48a}$$

which can be integrated to yield

$$P\left\{\int_1^\infty \frac{Q(W_o,r)}{W-W_o}\cos WZ\, dW\right\}$$

$$= -R(W_o,r)\left\{ Ci\left[(W_o-1)Z\right]\cos W_oZ + \left(\pi + si\left[(W_o-1)Z\right]\right)\sin W_oZ\right\} \quad . \qquad (48b)$$

The Ci and si functions are defined in the standard manner (18).

It is now possible to interpret the integral in Eq.(40) quite effectively. With reference to Eq.(41), $I_a(Z,r)$ is the portion which accounts for the radiation. The function $K_o(r\sqrt{W^2-1})$ becomes a Hankel function when $W<1$, and, for any finite value of Z, this has outgoing-wave character as r becomes large. The integral is thus seen to be a superposition of radiating cylindrical waves, each produced by the Fourier components of the induced charges on the surface of the rod. It is to be observed, however, that when r has any given value ($>B$), the contribution of $I_a(Z,r)$ vanishes at large Z. This circumstance arises from the well-known limit theorem that, if f(x) is sectionally continuous in (a,b), then

$$\lim_{k\to\infty}\int_a^b f(x)\cos kx \; dx = 0 \quad . \qquad (49)$$

The second term in Eq.(41), $I_b(Z,r)$, vanishes exponentially at large radial distances from the rod, on account of the corresponding exponential decay of $K_o(r\sqrt{W^2-1})$ when $W>1$. This integral then represents a combination of the free-mode field and the non-radiating (or "local") field of the charges induced on the rod surface. The separation between the latter portions can be made from Eqs.(43) through (48). From Eqs.(43), (44) and (48), in fact, the free-mode part is seen to be composed of the following terms,

$$\text{Free mode} = -\pi j R(W_o, r) \left[\cos W_o Z - j \sin W_o Z\right]$$

$$= -\pi j g(W_o, B, \varkappa)\, K_o\left(r\sqrt{W_o^2-1}\right) e^{-jW_o Z} \quad , \qquad (50)$$

while the "local" field comprises the terms

$$\text{"Local" field} = \int_1^{\infty} \frac{\left[Q(W,r)-Q(W_o,r)\right]}{W-W_o} \cos WZ\, dW$$

$$-R(W_o, r)\left\{ \mathrm{Ci}\left[(W_o-1)Z\right] \cos W_o Z \right.$$

$$\left. + \mathrm{si}\left[(W_o-1)Z\right] \sin W_o Z \right\} . \quad (51)$$

The justification for the separation made in Eqs.(50) and (51) lies in the fact that all the terms in the latter vanish as $Z \to \infty$ [Eq.(49) and reference (18)], while those in the former persist indefinitely along the rod.

Further numerical work could be given here, in order to examine the field outside the rod in greater detail, but the general picture outlined above appears sufficiently clear for the purposes of this paper.

ACKNOWLEDGMENT

The author wishes to express his gratitude to the Research Laboratory of Electronics, Massachusetts Institute of Technology, for providing the facilities and staff so essential to the conduct of this research. In particular, the untiring assistance of Professor L. J. Chu, supervisor of this work, is gratefully acknowledged. His keen insight has been an indispensible foundation for the investigation. The author desires also to thank Dr. M. V. Cerillo, Dr. P. M. Marcus (lately of this laboratory), Dr. R. Q. Twiss, and Messrs, S. J. Mason and R. J. Harrison for the time they spent in many discussions.

REFERENCES

1a. J. A. Stratton, "Electromagnetic Theory", N.Y., McGraw-Hill Book Co. (1941).

1b. J. Schwinger, Unpublished Lecture Notes, (Hectographed form, Harvard University Circulation).

1c. C. G. Montgomery, R. H. Dicke and E. M. Purcell, "Principles of Microwave Circuits", Rad. Lab. Ser. 8, N.Y., McGraw-Hill Book Co. (1948).

2. W. W. Hansen, Unpublished Lecture Notes, (Hectographed form, R.L.E. Document Room T-2).

3. L. Pincherle, Phys. Rev. 66, 118 (1944).

4. W. C. Hahn, G. E. Rev. 42, 258 (1939).

5. W. C. Hahn, G. E. Rev. 42, 497 (1939).

6. J. R. Pierce, Proc. I.R.E. 35, 111 (1947).

7. L. J. Chu and D. Jackson, R.L.E. Tech. Rep. No. 38, April 28, 1947.

8. A Hondros and P. Debeye, Ann. d. Phys. 32, 465 (1910).

9. J. R. Carson, S. P. Mead and S. A. Schelkunoff, Bell System Tech. Jour. 15, 310 (1936).

10. H. Bondi and M. H. L. Pryce, A.S.E. Rep. (London) M434, August 1942.

11. W. M. Elsasser, "Theory of the Dielectric Rod as a Waveguide", R.C.A. Tech. Rep. Princeton, N.J. (Hectographed form).

12. M. Abele, "Teoria della propagazione di un campo elettromagnetico lungo una guida dielettrica a sezione circolare", (Consiglio Nazionale Delle Richerche), Roma, Tipografia Del Senato (1948).

13. S. W. Harrison, Sylvania Rep. No. 145, December 1, 1947.

14. L. Page, "Introduction to Theoretical Physics", N.Y., D. van Nostrand, 2nd Ed. (1935).

15. J. C. Slater, Rev. Mod. Phys. 18, 441 (1946).

16. G. A. Campbell and R. M. Foster, Bell System Monograph B-584 (1931).

17. E. C. Titchmarsh, "Eigenfunction Expansions", London, Oxford University Press (1946).

18. E. Jahnke and F. Emde, "Tables of Functions", N.Y., Dover (1945).