1 Proof of Farkas’ Lemma

Theorem 1 [Farkas’ Lemma] Either

1. \(Ax = b, \ x \geq 0\) has a solution, or
2. \(A^Ty \geq 0\) and \(y^Tb < 0\) has a solution,

but not both.

The reason that 1 and 2 cannot both occur is that \((y^TA)x = y^Tb\), so if \(y^TA\) is non-negative and \(x\) is non-negative, then \(y^Tb\) can’t be negative.

To prove Farkas’ Lemma we need the Projection Theorem:

Theorem 2 Let \(K\) be a closed, convex and non-empty set in \(\mathbb{R}^n\), and \(b \in \mathbb{R}^n\), \(b \notin K\). Define projection \(p\) of \(b\) onto \(K\) to be \(x \in K\) such that \(||b - x||\) is minimized. Then for all \(z \in K\):

\[(b - p)^T(z - p) \leq 0.\]

Proof of Farkas’ Lemma: Assume \(Ax = b, \ x \geq 0\) is not feasible. Let \(K = \{Ax : x \geq 0\}\). Therefore, \(b \notin K\). Let \(p = A\bar{w}, \bar{w} \geq 0\) be the projection of \(b\) onto \(K\). Then we know that

\[(b - A\bar{w})^T(Ax - A\bar{w}) \leq 0 \text{ for all } x \geq 0 \tag{1}\]

Define \(y = p - b = A\bar{w} - b\). Therefore,

\[(x - \bar{w})^TA^Ty \geq 0 \text{ for all } x \geq 0 \tag{2}\]

Let \(e_i\) be the \(n \times 1\) vector that has 1 in its \(i^{th}\) component and 0 everywhere else. Take \(x = \bar{w} + e_i\). Therefore, \(x - \bar{w} = e_i\), and by (2),

\[e_i^TA^Ty \geq 0 \Rightarrow (A^Ty)_i \geq 0 \text{ for all } i\]

Thus since each element of \(A^Ty\) is non-negative, \(A^Ty \geq 0\).
Now, \( y^T b = y^T (p - y) = y^T p - y^T y \). From (1) if \( x = 0 \),

\[
(b - Aw)^T (Ax - Aw) = (b - Aw)^T (-Aw) = -y^T (-Aw) = y^T Aw = y^T p \leq 0
\]

and

\[
y^T p - y^T y \leq -y^T y < 0
\]

The last inequality comes from the fact that \( y = b - p, b \notin K \), so \( b - p \neq 0 \implies y^T y > 0 \)

\[\Box\]

**Theorem 3 [Another variant of Farkas’ Lemma]** Either

1. \( Ax \leq b \) has a solution, or
2. \( A^T y = 0, b^T y < 0, y \geq 0 \) has a solution,
   
   but not both (for then we would have \( 0 = y^T Ax \leq y^T b < 0 \).)

2 Duality

Consider an LP \( P \) in the standard form (we call this LP the primal). We can write a “dual” LP \( D \) as follows:

**Primal \( P \):**

\[
\begin{align*}
\text{min } & \quad c^T x \\
\text{subject to } & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

**Dual \( D \):**

\[
\begin{align*}
\text{max } & \quad b^T y \\
\text{subject to } & \quad A^T y \leq c
\end{align*}
\]

Weak duality states the following.

**Theorem 4 [Weak Duality]** Let \( x \) be feasible in \( P \), and let \( y \) be feasible in \( D \). Then

\[
c^T x \geq b^T y
\]

Proof of Theorem 4:

\[
c^T x - b^T y = x^T c - x^T A^T y = x^T (c - A^T y) \geq 0,
\]

since \( x \) and \( c - A^T y \) both have nonnegative coordinates.

\[\Box\]

The following three cases are possible for an LP:

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) infeasible ( (z^* = +\infty) )</td>
<td>1') infeasible ( (w^* = -\infty) )</td>
</tr>
<tr>
<td>2) unbounded ( (z^* = -\infty) )</td>
<td>2') unbounded ( (w^* = +\infty) )</td>
</tr>
<tr>
<td>3) finite ( (z^* = \text{finite real number}) )</td>
<td>3') finite ( (w^* = \text{finite real number}) )</td>
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Then $2 \Rightarrow 1'$ because if the dual were feasible, any value $b^Ty$ for the dual would be a lower bound for the primal, which could therefore not be unbounded. Similarly $2' \Rightarrow 1$. Note that we can have 1 and 1' occurring simultaneously.

**Theorem 5 [Strong duality]** If $P$ or $D$ is feasible then $z^* = w^*$.

**Proof of Theorem 2:** It suffices to treat the case when the primal is feasible, because the primal and dual are interchangeable. So assume $P$ is feasible. If $P$ is unbounded then weak duality implies that $D$ is infeasible, and then $z^* = w^* = -\infty$. So from now on assume that the primal is finite.

**Claim 6** There exists a solution of dual of value at least $z^*$, i.e.,

$$\exists y : A^T y \leq c, b^T y \geq z^*$$

**Proof of Claim 3:** We wish to prove that there is a $y$ satisfying

$$\begin{pmatrix} A^T \\ b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}.$$

Assume the claim is wrong. Then the variant of Farkas’ Lemma implies that the LP

$$\begin{pmatrix} A & -b \\ c^T & -z^* \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0$$

has a solution. That is, there exist nonnegative $x, \lambda$ with

$$Ax - b\lambda = 0 \quad c^Tx - z^*\lambda < 0$$

**Case 1:** $\lambda > 0$. Then $A(\frac{x}{\lambda}) = b, \quad c^T(\frac{x}{\lambda}) < z^*$. This contradicts the minimality of $z^*$ for the primal, hence this case cannot occur.

**Case 2:** $\lambda = 0$. Then $Ax = 0, \quad c^Tx < 0$. Take any feasible solution $\hat{x}$ for $P$. Then for every $\mu \geq 0$, $\hat{x} + \mu x$ is feasible for $P$, since

a) $\hat{x} + \mu x \geq 0$ because $\hat{x} \geq 0, x \geq 0, \mu \geq 0$.

b) $A(\hat{x} + \mu x) = A\hat{x} + \mu Ax = b + \mu \cdot 0 = b$.

But $c^T(\hat{x} + \mu x) = c^T\hat{x} + \mu c^Tx \rightarrow -\infty$ as $\mu \rightarrow \infty$. This contradicts the assumption that the primal has finite solution.

The above claim shows that if $P$ or $D$ is finite then the other is too, and the optimums are equal ($z^* \geq w^*$ is weak duality and the claim shows $w^* \geq z^*$.) This concludes the proof of the strong duality theorem. \qed
3 Complementary Slackness

Consider the following primal LP.

\[
\begin{align*}
\min \ & c^T x \\
\text{subject to} \ & Ax = b \\
\ & x \geq 0
\end{align*}
\]

We write the dual as follows:

\[
\begin{align*}
\max \ & b^T y \\
\text{subject to} \ & A^T y + s = c \\
\ & s \geq 0, \quad y \in \mathbb{R}^m, s \in \mathbb{R}^n
\end{align*}
\]

**Theorem 7** Let \( x \) be feasible for the primal, and \( y \) be feasible for the dual. Then \( x \) is optimal for \( P \) and \( y \) is optimal for \( D \) if and only if \( x_j s_j = 0 \) for all \( j \).

**Proof:** We have

\[
\begin{align*}
c^T x - b^T y & = x^T c - x^T A^T y \\
& = x^T (c - A^T y) \\
& = x^T s
\end{align*}
\]

When both \( x \) and \( y \) are optimal, the above difference must be zero, and conversely, if the difference is zero, both must be optimal by weak duality. But since \( x, s \) are nonnegative, \( x^T s \) is zero if and only if \( x_j s_j = 0 \) for all \( j \). \( \square \).

So, to prove that a solution to an LP is optimal, all we need to do is to give an \( x \) and a \((y, s)\) and show that both are feasible and the complementary slackness condition is satisfied.

4 Size of a linear program

Let’s think about how we encode the LP. We can use binary encoding to give the entries of \( A, b, c \), that defines the LP in standard form. For an integer \( k \), it takes \( \text{size}(k) = 1 + \lceil \log_2(|k| + 1) \rceil \) bits to encode \( k \). So,

\[
\text{size}(LP) = \sum_{i,j} \text{size}(a_{ij}) + \sum_j \text{size}(c_j) + \sum_i \text{size}(b_i)
\]

A polynomial-time algorithm for linear programming is an algorithm whose worst-case running time is bounded by a polynomial in the size of the input LP.

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