LOCKING PHENOMENA IN MICROWAVE OSCILLATORS

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TECHNICAL REPORT NO. 63
APRIL 8, 1948

RESEARCH LABORATORY OF ELECTRONICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
The research reported in this document was made possible through support extended the Massachusetts Institute of Technology, Research Laboratory of Electronics, jointly by the Army Signal Corps, the Navy Department (Office of Naval Research), and the Air Force (Air Materiel Command), under the Signal Corps Contract No. W-36-039 sc-32037.
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Abstract

The relationship of the Rieke diagram to the operating conditions of a microwave oscillator is derived. It is shown how this diagram may be used to determine the behavior of the oscillator under the influence of a buffered, externally applied, microwave signal. The oscillator frequency, phase, and power output are determined as functions of the synchronizing signal amplitude, frequency, and phase. These functions are described by contours of constant reflected power or constant synchronizing power on the Rieke diagram. Experimental data on a 707B klystron confirming the theory are presented. This analysis provides a method for determining the locking characteristics of an oscillator by graphical construction on its Rieke diagram. Comparison is made of this graphical analysis with a purely theoretical one.
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Introduction

In many electronic systems using ultra-high frequency or microwave oscillators, it is desirable to synchronize the frequency and phase relative to some invariant master signal. One method of achieving this end is to inject power from the master source directly into the oscillator to be controlled, thereby locking it in frequency and phase. In order to utilize this method in the design of a system, it becomes necessary to know the relation between magnitude of synchronizing signal, locked oscillator power output, and frequency range over which the oscillator may be locked. It is the purpose of this report to present a method of obtaining the necessary information by a graphical computation applied to the load diagram of the oscillator to be locked. For the sake of clarity, some discussion of oscillator operation in general is included. The analysis is intended to present a clear, physical conception of locked oscillator operation. Section 6 of this report contains a theoretical analysis based upon work by J. C. Slater.¹

1. Oscillator Theory

In general, a microwave oscillator consists of a resonant cavity (or its equivalent) excited by some sort of electronic discharge. The load is coupled to this oscillator by means of a transitional transformer such as an iris or an inductive coupling loop. It can be shown that this generator may be represented by the equivalent circuit of Fig. 1a.² The tuned circuits of resonant frequencies \( w_0, w_{01}, w_{02}, \ldots, w_{0v} \) simulate the resonant modes of the microwave cavity. They are excited by the electronic discharge which is characterized by the admittances \( g_0 + jb_0, g_1 + jb_1, \ldots, g_v + jb_v \). These electronic admittances are defined as the ratio of r-f current to voltage at the appropriate terminal pair; that is,

\[
y_n = g_n + jb_n = \frac{(i_{rf,n})}{(v_{rf,n})} \quad n = 0, 1, 2, 3, \ldots
\]

The load admittance \( G + jB \) is coupled to each of the cavity modes by a mutual inductance \( M_n \), and the coupling element has an inductance \( L \). Actually in any practical coupling arrangement, there may be appreciable capacitive and conductive, as well as inductive, coupling. We shall later, however,


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lump all coupling effects into one constant; therefore, the representation of Fig. 1a is adequate.

Fig. 1a. Equivalent circuit representation of a microwave oscillator.

Fig. 1b. Simplified oscillator equivalent circuit.

Oscillator cavities are so designed that the resonant frequencies of the modes are separated sufficiently to allow significant oscillation only in a single mode. The implication of single-mode oscillation is that only the excited mode stores appreciable energy. In the equivalent circuit, therefore, the effect of all except this mode may be neglected. Since the coupling circuit with its associated stray effects act as an impedance transformer, the apparent loading on the principal mode will be $k_c(G+jB)$, where $k_c$ is the impedance transforming characteristic of the coupling system. When these two simplifications are incorporated, the circuit of Fig. 1a becomes that of Fig. 1b.

Since $g+jb$ is the admittance seen at the terminals MN, it may be expressed as:

$$g+jb = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) + k_c(G+jB). \quad (2)$$
Defining
\[ \omega_o = \frac{1}{\sqrt{LC}} \]
and
\[ Q_o = \frac{R}{\omega_o L} = \frac{R}{w_o C}, \]
one may write Eq. (2) as
\[ \frac{g + j b}{\omega_o C} = \frac{1}{Q_o} + j \left( \frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} \right) + \frac{k_c (G + j B)}{\omega_o C} \]
(3)
where \( \omega \) is the angular frequency of oscillation. The coefficient \( k_c/\omega_o C \), of the last term in Eq. (3) may be recognized as defining, as an equivalent \( Q \), the coupling between the resonant cavity and the external circuit. That is, when \( G = 1 \) and \( B = 0 \), \( k_c \) is the conductance loading the cavity, and \( Q \) determined by this external load is
\[ Q_{ext} = \frac{\omega_o C}{k_c}. \]
(4)
The external \( Q \), then, is that determined by the effect of a matched load, exclusive of losses in the cavity itself.

Upon separating the real and imaginary parts of Eq. (3) and substituting Eq. (4), one obtains
\[ \frac{g}{\omega_o C} = \frac{1}{Q_o} + \frac{G}{Q_{ext}} \]
(5)
\[ \frac{b}{\omega_o C} = 2 \left( \frac{\omega - \omega_o}{\omega_o} \right) + \frac{B}{Q_{ext}}. \]
(6)
Equations (5) and (6) express the operational dependence of the oscillator upon the load. These two equations, however, contain the three variables \( g \), \( b \), and \( \omega \), and another relation is needed, therefore, to specify uniquely the operation of the oscillator as a function of the load. This additional constraint may be obtained by relating \( g \) to \( b \) through the characteristics of the electronic discharge. This relation will take the form:
\[
\begin{align*}
g &= f(V_{rf}) \\
b &= F(V_{rf})
\end{align*}
\]
(7)
where \( V_{rf} \) is a parameter. The electronic conductance and susceptance, how-
ever, are also dependent upon the d-c parameters associated with the oscillator. Equation (7) should then be written to include these:

\[
\begin{align*}
  g &= f(V_{rf}, A_{dc}, B_{dc}) \\
  b &= F(V_{rf}, A_{dc}, B_{dc})
\end{align*}
\]  

(8)

The explicit form of Eqs. (8) is determined by the dynamics of the electronic discharge, and the derivation of these equations is, in general, extremely difficult. The approximate expressions however, have been obtained in the case of the reflex klystron.

When d-c variables are fixed therefore, Eqs. (5), (6), and (7) uniquely define the operation of the oscillator into a particular load impedance. That is, when G and B are known, g and b are fixed, and therefore w is specified. The power generated, being the product \( g V_{rf}^2 \), is also determined.

2. The Rieke Diagram

The operating equations of an oscillator, as derived above, may be plotted as contours of constant frequency and constant power output in the reflection coefficient plane. Such a plot is called a Rieke diagram and is the load characteristic of the oscillator for a particular d-c condition.

Consider the normalized load impedance,

\[
z = \frac{Z}{Z_0} = \frac{1}{(G+1)B}
\]  

(9)

It gives rise to a reflection coefficient according to the relation,

\[
r = \frac{z-1}{z+1}
\]  

(10)

where \( r \) = reflection coefficient and is the ratio of the complex voltage magnitudes of the waves incident upon, and reflected from, the impedance \( z \). Equation (10) is a bilinear transformation expressing the relation between the variables \( r \) and \( z \), and having the property of conformality; that is, angles of intersection between contours in the \( z \)-plane are preserved in the \( r \)-plane. Such a transformation also retains the circular shape of a contour mapped from the \( z \)- to the \( r \)-plane. It is well known that contours of constant G and constant B in the complex impedance plane are a set of orthogonal circles. Therefore, they represent a similar set in the complex reflection coefficient plane.

1. In the reflex klystron, these are reflector and accelerator voltage; for the magnetron, current and voltage.
tion coefficient plane and are shown in Fig. 2.

Equation (5), when multiplied by the factor $\omega_0 V_{rf}^2$, becomes

$$gV_{rf}^2 = \frac{V_{rf}^2}{R} + k_0 GV_{rf}^2$$

(11)

or

$$P = P_D + P_o$$

where $P = \text{power generated}$

$P_D = \text{power loss in cavity}$

$P_o = \text{power output to load}$.

Equations (5) and (7) show that for constant $G$, both $g$ and $V_{rf}$ are also constant. The condition of the constant $G$, therefore, is the condition of constant power output. A constant conductance circle in the reflection coefficient plane, then, is a path of constant power output for the oscillator.

Solving Eq. (6) for $\omega$, the frequency of oscillation,

$$\omega = \frac{b}{2o} + \omega_0 - \frac{\omega_0}{2Q_{ext}} B$$

(12)

Fig. 2. The Smith Chart showing contours of constant $B$ and $G$ inside the unit circle of the reflection coefficient plane.
which when \( G \) is a constant is of the form,
\[
\omega = k_0 - k_1 B, \tag{13}
\]
where \( k_0 \) and \( k_1 \) are constants. Therefore, each intersection of a constant frequency locus with a conductance circle corresponds to a unique value of \( B \). If \( k_0 \) and \( k_1 \) did not change with \( G \), then the intersection on any \( G \) circle would correspond to the same value of \( B \), and the lines of constant frequency would coincide with constant \( B \) loci. Since a change in \( G \) causes a change in \( b \) and hence \( k_1 \), the frequency contours are distorted and diverge slightly from the \( B \) lines. The amount of this distortion is dependent on Eqs. (7) and hence, on the mechanics of the electronic discharge.

The Rieke diagram then, is similar in appearance to the chart shown in Fig. 2. Any point on the diagram corresponds to a load admittance, a frequency of oscillation, and a power output. Therefore, it specifies oscillator operation uniquely as a function of the load admittance, provided d-c conditions are fixed.

3. The Locked Oscillator

Consider a microwave oscillator operating into a matched load \((G = 1, B = 0)\). Its output power and frequency are specified by the contours passing through the centers of the Rieke chart where the reflection coefficient is zero. Suppose now, a signal of frequency \( \omega_1 \) is introduced into the oscillator output line. If the signal is of sufficient amplitude, and \( \omega_1 \) is not greatly different from the initial matched-load frequency \( \omega_0 \), the oscillator will change its frequency to \( \omega_1 \). At this new frequency of oscillation, its power output and r-f operating conditions will have changed. Its d-c conditions, however, will remain the same, since they are determined by controls external to the oscillator. As \( \omega_1 \) is changed, the operating frequency of the oscillator will track these changes until the difference \( |\omega_1 - \omega_0| \) becomes too large, at which time the oscillator breaks synchronism. We are concerned here with the changes of operation as the oscillator follows the synchronizing signal frequency.

When the locking signal is applied, the oscillator shifts frequency from \( \omega_0 \) to the frequency \( \omega_1 \) of the locking signal. Since the d-c condition has remained fixed, the point of oscillator operation on the Rieke diagram has been shifted to a new frequency line, where the reflection coefficient has a value greater than zero. This value of reflection coefficient is given by the ratio,
\[
\rho = |\rho| e^{j\theta} = \frac{E_2}{E_1} = \frac{P_2}{P_1} e^{j\theta}, \tag{14}
\]
where

\[ E_s = \text{magnitude of synchronizing voltage}, \]
\[ P_s = \text{magnitude of synchronizing power}, \]
\[ E_i = \text{magnitude of oscillator incident voltage}, \]
\[ P_i = \text{magnitude of oscillator incident power}. \]

This change of reflection coefficient reveals the mechanism of oscillator synchronization; when locking to an external signal, the oscillator assumes a phase and power output for which the resulting reflection coefficient as given by Eq. (14) specifies the frequency \( \omega_1 \). As \( \omega_1 \) is changed, therefore, the reflection coefficient varies in a manner determined by the load characteristic of the oscillator.

In general, therefore, the effect of the locking signal can be interpreted as a change in the load admittance presented to the oscillator. This change causes a frequency shift. Since the Rieke diagram represents the relation between reflection coefficient (load admittance), power output, and frequency, the behavior of the oscillator under locked conditions may be analyzed in terms of this chart. Further, since the characteristics of the electronic discharge are inherent in the Rieke diagram, they do not affect the mechanics of the locking action. As a result, an analysis of this type is applicable to any microwave oscillator.

4. Locked Operation in a Matched Load

It has been seen that under fixed d-c conditions, the frequency and power output of a microwave oscillator are uniquely specified by the reflection coefficient presented to the tube. Further, if the output frequency is to be modified by the application of an external signal, its effect must be a change in the reflection coefficient. The purpose of the following analysis is to determine the variations of this coefficient as a function of the magnitude, frequency, and phase of the externally applied signal. A plot of these variations on the Rieke diagram, with the power in the locking signal as a parameter, will be used to describe the synchronized operation of the oscillator. These curves can be used to determine the frequency, power output, and range of locking for any specified synchronizing signal.

Consider the circuit shown in Fig. 3a. \( O_L \) represents the oscillator to be locked by a signal supplied from the "ideal" injection source \( O_s \). The characteristics of this ideal source are: (1) the synchronizing signal propagates only toward the oscillator to be locked; (2) the injection source has no insertion mismatch; that is, the admittance seen from the main line is zero; and (3) the amplitude, phase, and frequency of the injected signal may be varied independently, and all are independent of the operation of the oscillator.
A system for ideal injection may at present only be approximated in practice, and the design of such components is a problem of microwave circuitry. The effect of an insertion mismatch, however, will be discussed in a later section.

Again in Fig. 3a if ideal injection is assumed, \( O_L \) operates into an equivalent load which approaches \( Z_0 \) as \( E_s \to 0 \). The load, \( Z_o \), and \( O_S \) may, then, be represented by a generator of internal impedance \( Z_0 \) with variable output amplitude, frequency, and phase. This generator supplies the locking signal \( E_s \), which is incident upon \( O_L \). The wave \( E_1 \) is dependent upon \( E_s \); this dependence may alternatively be expressed by (1) representing \( O_L \) as a constant voltage generator whose internal impedance is a function of \( E_s \), or (2) representing \( O_L \) as a generator of internal impedance \( Z_0 \), whose terminal voltage is a function of \( E_s \). The functional dependence of the internal impedance or the terminal voltage on \( E_s \) is expressed by the Rieke diagram of the oscillator. Figure 3b shows the equivalent circuit of the system with the variable voltage representation of \( O_L \) and the fictitious voltage generator \( E' \) generating the signal \( E_s \).

The most important characteristic of this system is that the voltage wave \( E_s \) is dependent only upon the amplitude, phase, and frequency of \( E' \). If the amplitude of \( E' \) is assumed constant, it may be said that...
although its frequency and phase may change, the synchronizing signal contains constant power. At the reference terminals of $O_L$, this power appears identical to that which would be reflected from a mismatched load located at a greater distance from the tube. As the frequency of the locking signal (supplied by $E'$) is varied, the reflection coefficient associated with this fictitious load changes because of variations in amplitude and phase of the generated wave ($E_1$). In the locking region, therefore, $O_L$ will operate into an equivalent load which changes with frequency so as to maintain the power reflected from it constant, independent of the incident power.

Consider the complex reflection coefficient plane with contours of constant power output superimposed upon it. Each point in the plane corresponds to a value of complex reflection coefficient and power output. It may be shown that

$$P_s = P_1 - P_o$$

and

$$P_s = |\rho|^2 P_1$$

where

- $P_s$ = power in reflected wave
- $P_1$ = power in incident wave
- $P_o$ = power output
- $\rho$ = complex reflection coefficient.

Therefore, each point in the plane likewise specifies a reflected power. Contours of constant reflected power, then, may be constructed from the Rieke diagram on the reflection coefficient plane. These contours represent the variation of reflection coefficient required to maintain constant reflected power as the incident power changes with frequency. Lines of constant reflected power, therefore, represent the path of locked-oscillator operation as a function of frequency. The magnitude of reflected power on the contours is the same as that in the synchronizing signal, $E_s$.

If the incident power generated by the oscillator were independent of the reflection coefficient, the CRP (constant reflected power) contours would be circles centered about the origin. Since the incident power depends upon the reflection coefficient, the loci will appear distorted, although retaining a generally circular shape. The amount of distortion increases with reflection coefficient since variations of incident power become more pronounced at large reflection coefficients.

Figure 4a shows the Rieke diagram of a 707B velocity-modulated tube. Figure 4b shows the CRP contours calculated therefrom by means
of Eqs. (15) and (16) and plotted inside the unit circle of the reflection coefficient plane. The loci representing small reflected powers are near the origin since here the reflection coefficient is small. As the contours enlarge, the reflected power increases until, at the unit circle, it becomes equal to the incident power. Since an externally adjusted source actually determines the reflected power, its magnitude may be further increased and
cause the reflection coefficient to become greater than one. The oscillator now absorbs more power than it generates and has, therefore, become a "load". CRP contours which lie outside the unit circle have been observed experimentally when large locking signals were applied. The major interest in the locking problem, however, is in the region where the synchronizing signal is small relative to the oscillator power output. The CRP contours in Fig 4b, therefore, lie well inside the unit circle.
As previously noted, if the magnitude of the synchronizing power is known, the locus of possible oscillator operating points is the corresponding CRP contour. The intersection of this locus with the frequency line corresponding to the locking signal frequency is the point of operation. The oscillator power output is specified by the power output contour passing through this point. If there is no intersection between the CRP and frequency locus, then locking will not occur at that frequency. The range of synchronization with a constant value of locking power, therefore, is defined by the points of tangency between that CRP contour and the frequency loci. Figure 5 shows the CRP and frequency loci with locking boundaries marked.

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**Fig. 5.** CRP contours as used to define locking boundaries.
Since the CRP loci are closed contours there are necessarily two crossing points for each intersecting frequency line. Therefore, there are two possible points of operation for each value of locking signal frequency and amplitude. This ambiguity is unimportant, however, since the oscillator will operate at the more stable load value. Consider, for example, a point as it moves counter-clockwise around the 9-mw CRP contour in Fig. 5, starting from $\Theta = 0$. The constant frequency loci are cut first in a positive, then in a negative sense, while $\Theta$, the angle of the reflection coefficient, is continuously increasing. Figure 6 shows the variation of frequency as a periodic function of $\Theta$ around the 9-mw contour. The boundary between the regions of positive and negative slope are points of tangency between the CRP contour and the frequency loci.

Now
\[ \Theta = \Psi - \Phi \]  
(17)

where
\[ \Psi = \text{phase of synchronizing signal}, \]
\[ \Phi = \text{phase of generated wave}. \]

Differentiating,
\[ \frac{d\Theta}{dt} = - \frac{d\Phi}{dt} = - \Delta \omega \]
(18)

since $\Psi$ is a constant.

Fig. 6. Variation of frequency as a periodic function of the angle of the reflection coefficient on the 9-mw CRP contour. The slope of such curves defines stable operating points for the locked oscillator.
In Fig. 6, lines of constant frequency intersect at two possible operating points 0₁ and 0₂. A random positive dynamic rate of change in \( \varphi \) results in a positive frequency change \( \Delta \omega \) and a negative change, \( \frac{d\varphi}{dt} \). The point 0₁, under these conditions, moves in the direction of the arrow in a cumulative manner and is, therefore, unstable. At point 0₂, the negative \( \frac{d\varphi}{dt} \) produces a negative \( \Delta \omega \), which conflicts with the condition (18). Therefore, no motion of 0₂ is possible, and it may be concluded that points in the region of positive derivative are stable. In Fig. 5, then, the loci of stable, locked, operating points are arcs which are concave upward.

The CRP contours, then uniquely specify the locked operation of an oscillator as a function of the amplitude and frequency of the synchronizing signal. Since the apparent reflection coefficient is fixed so long as these two parameters are constant, the operation does not depend upon the phase of the applied signal.

5. Experimental Results

An experimental circuit to reproduce the CRP contours may be devised. The block diagram of such a circuit is shown in Fig. 7. Here, the ideal injection source is simulated by the system to the right of the attenuator. Approximately 50 watts of microwave power, supplied by the tunable magnetron, are dissipated in the matched load after passing through the power divider. A few milliwatts, however, provide a locking signal to the 707B after transmission through the directional coupler and attenuator. The magnetron is, therefore, isolated from the rest of the system and supplies a small, buffered signal, variable in amplitude and frequency.
These parameters are measurable by means of the directional coupler and wavemeter. A matched load is provided for the 707B by the high-loss attenuator. In order to observe a CRP contour, the power divider is set to supply a constant amplitude signal; as the magnetron is tuned through the locking range of the klystron, values of reflection coefficient are recorded at the slotted section. These values, plotted in the reflection coefficient plane, give the CRP contour corresponding to the signal amplitude supplied through the attenuator. Contours taken in this manner are shown as solid curves in Fig. 8. The dotted curves are contours calculated from the 707B
Rieke diagram. Limits of the locking range are given at the end of each experimental contour and should be compared with that indicated in Fig. 5.

6. Effects of Mismatched Load

In a practical system, it is not possible at the present time to devise an ideal injection source. Injection circuits may be designed which satisfy the requirements of directed propagation of the locking signal and production of a buffered, variable-frequency wave; however, such a system will invariably have an insertion mismatch. Also, in some cases, the actual load impedance may present a reflection coefficient other than unity. The effect of such mismatches may be determined.

Consider a system similar to that of Fig. 3a, with the load impedance no longer specified as \( Z_o \), and with \( Z_s \) having a finite insertion mismatch. Such a system may be represented as shown in Fig. 9, where \( Z' \) is an equivalent impedance lumping the load and insertion impedances; \( E' \) is still a function of \( E_s \), although \( E_s \) now consists of the two components \( E'_s \) and \( E''_s \) where \( E'_s \) is the wave reflected from \( Z' \), and \( E''_s \) is the synchronizing signal supplied by \( E'' \). Vectorially then

\[
E_s = E'_s + E''_s
\]  

(19)

and the reflection coefficient \( \rho \) presented to the locked tube is

\[
\rho = \frac{E'_s}{E'_i} = \frac{E'_s + E''_s}{E'_i} = \rho' + \frac{E''_s}{E'_i}
\]

(20)

where \( \rho' = \) reflection coefficient due to \( Z' \).

\[\text{Fig. 9. Equivalent circuit of synchronizing system with mismatched load.}\]

Now, as a function of propagated powers

\[
P_s = P''_s + P'_s
\]

(21)
where
\[ P = \text{total power associated with the wave } E_s \]
\[ P_s'' = \text{power in synchronizing signal} \]
\[ P_s' = \text{power reflected from } Z'. \]

Then
\[ P_s = P_s'' + \left| \rho' \right|^2 P_1 = P_s'' + \frac{\left| \rho' \right|^2}{\left| \rho \right|^2} P_s \]

or
\[ P_s = \frac{P_s''}{1 - \left| \rho' \right|^2 \left| \rho \right|^2} \quad (22) \]

The CRP contours previously derived are contours of constant \( P_s'. \) In the system of Fig. 9, \( P_s'' \) is constant, while \( P_s \) is related to the variable \( \rho'. \)
Therefore, the CRP contours are no longer the locus of operating points for a constant amplitude locking signal. For each value of \( \left| \rho' \right|^2 \) and \( P_s'' \), however, there exists a locus on the Rieke diagram which satisfies Eq. (22). This is the locus of operating points when the synchronizing power is \( P_s'' \) and the mismatched load \( Z' \) presents a reflection coefficient \( \rho' \). It is seen that \( P_s \) becomes identical with \( P_s'' \) when there is no mismatch (\( \rho' = 0 \)).

Note that the CRP contours are fundamentally a characteristic of the oscillator itself. Just as the Rieke diagram is determined by the electronics of the oscillator, so the locking characteristics, as expressed by the CRP loci, are inherently specified by the tube design. Conversely, the loci expressed by Eq. (22) are a function of the load impedance \( Z' \). They, therefore, are dynamic loci characterized by the system external to the oscillator. For this reason, they may be referred to as dynamic-load contours, or simply DL contours.

Since the DL contours represent the locus of operating points for the oscillator, the criterion for locking range and stability of the operating points is the same as for CRP contours. The point of oscillator operation is likewise determined by the intersection of a DL line and the frequency line specified by the signal \( E_s'' \).

7. Theoretical Analysis

It is interesting to compare the graphical analysis just presented with the results of a theoretical analysis. If a system identical to that in Fig. 3b is assumed, current and voltage of the microwave oscillator may be denoted by \( V e^{j(\omega t+\phi)} \) and \( i e^{j(\omega t+\psi)} \), respectively. The synchronizing signal may be represented by a voltage and current, \( V e^{j(\omega_1 t+\psi)} \) and \( i e^{j(\omega_1 t+\psi)} \), respectively. Then the equivalent load admittance seen at a
plane of reference in the r-f line is

\[ Y_L = \frac{\frac{1}{2} e^{J(\omega_L t + \phi)} + e^{J(\omega t + \phi)}}{V_1 e^{J(\omega_L t + \phi)} + V_0 e^{J(\omega t + \phi)}} \]

or

\[ Y_L = \frac{1}{V} \left[ \frac{1 + \frac{1}{2} e^{J[(\omega_L - \omega) t + \phi - \varphi]}}{1 + \frac{1}{2} e^{J[(\omega_L - \omega) t + \phi - \varphi]}} \right]. \quad (23) \]

If the indicated division is carried out,

\[ Y_L = Y_0 \left\{ 1 + \frac{1}{2} \frac{e^{J(\omega_L - \omega) t + \theta}}{1 - \frac{1}{2} e^{J(\omega_L - \omega) t + \theta}} \right\} \quad (24) \]

where \( \theta = \varphi - \varphi \).

If the synchronizing signal is small, all but the first two terms of this expansion may be neglected. Normalizing with respect to \( Y_0 \),

\[ Y_e \equiv 1 + \frac{1}{2} \frac{e^{J(\omega_L - \omega) t + \theta}}{1 - \frac{1}{2} e^{J(\omega_L - \omega) t + \theta}} \quad (25) \]

\[ \equiv 1 - 2|p|e^{J(\omega_L - \omega) t} \]

where \( p \) is the reflection coefficient of the apparent load. If the oscillator is locked in frequency to the external signal, then \( \omega = \omega_L \) and Eq. (25) becomes

\[ Y_e \equiv 1 - 2|p|e^{J\theta}. \quad (26) \]

Now \( \theta \) is the phase angle between the wave generated by the oscillator and the locking signal at the reference plane. It is, therefore the angle associated with the reflection coefficient \( p \). Equation (25) may now be substituted into the oscillator-operating equation (3). Assuming that \( G = 1 \) and \( B = 0 \) (matched load), Eq. (3) becomes

\[ \frac{\omega + b}{\omega_0} = j\left(\frac{\omega_0 - \omega}{\omega_0}\right) + \frac{1}{Q_0} + \frac{1 - 2|p|e^{J\theta}}{Q_{\text{ext}}} \quad (27) \]

or after separating the real and imaginary parts,

\[ \frac{a}{\omega_0} = \frac{1}{Q_0} + \frac{1}{Q_{\text{ext}}} \left\{ 2|p| \cos \theta + \frac{2|p|e^{J\theta}}{Q_{\text{ext}}} \right\} \quad \text{(28a)} \]

\[ \frac{b}{\omega_0} = \frac{2(\omega - \omega_0)}{\omega_0} - \frac{2|p| \cos \theta}{Q_{\text{ext}}} \quad \text{(28b)} \]
The effect of the locking signal, then, is to add the last term on the right in Eqs. (28). These terms represent a load conductance and susceptance introduced by the locking signal. These cause a modification of $\rho$, $b$, and $V_{rf}$; hence the output power and frequency are likewise modified.

Now,

$$\rho = \sqrt{\frac{P_s}{P_i}} e^{j\theta} = \sqrt{\frac{P_s}{k_V V_{rf}}} e^{j\theta} \quad (29)$$

Equation (29) shows that $\rho$ is a function of $V_{rf}$; therefore, the equivalent oscillator load is a function of its own operating conditions. When locking to an external signal then, the oscillator assumes such an r-f voltage and phase that the apparent load specifies the frequency of the locking signal. Note that the synchronizing action is independent of the phase of the locking signal. Only the difference in phase, $\theta$, enters into the analysis.

Let us see how the CRP contours are related to this analysis.

Upon solving Eq. (28),

$$\frac{|\rho| \sin \omega}{Q_{ext}} = \frac{w-w_o}{w_o} - \frac{b}{2w_o \sigma} \quad (30)$$

By letting $|\rho| = 0$ in this expression, the frequency $w'$ of the tube operating into a matched load is found to be

$$w' = w_o + \frac{b}{2\sigma} \quad (31)$$

From (31) and (30),

$$\frac{|\rho| \sin \theta}{Q_{ext}} = \frac{w-w'}{w_o} \quad (32)$$

As $\sin \theta$ passes through values of 1 to -1, $|\rho|$ also changes, as may be seen from Eqs. (28) and (29). That is, if $\theta$ changes, $V_{rf}$ changes, and a new value of $\rho$ results. However, if the synchronizing signal is small, as we have assumed, the changes of $|\rho|$ are small. Therefore, it may be stated that the maximum value of $|w-w'|$ occurs when $\sin \theta = \pm 1$. Therefore, the boundaries of locking are given by the expression

$$+ (w-w') = \frac{w_o |\rho|}{Q_{ext}} \quad (33)$$

It has been assumed that the magnitude of the reflection coefficient is constant, while the phase changes through $180^\circ$. On the reflection coefficient plane, paths of constant $|\rho|$ are circles centered about the origin. In this theoretical treatment, therefore, the CRP contours are approximated by circles of constant reflection coefficient. These sets of loci merge as they approach the region where $|\rho|$ is small.