A Design Study of Radial Inflow Turbines with Splitter Blades in Three-Dimensional Flow

by

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submitted to the department of
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ABSTRACT

An inverse design technique to design turbomachinery blading with splitter blades in three-dimensional flow is presented. Using the Clebsch formulation, the velocity field is decomposed into a potential part and a rotational part; the rotational part can be expressed in terms of the mean swirl schedule (product of radius and tangential velocity) and blade geometry which includes that of the main blade as well as the splitter blade. This leads to an inverse design approach in which the main and splitter blade geometry are determined from a specification of the swirl schedule. In this study, the division of the swirl at the leading edge between the main and splitter blades is based on simple linear proportionality that involves the length of the splitter blade. At the trailing edge, the swirl is assigned a zero value. Along the hub and the shroud, polynomials are used to interpolate the values of the swirl that result in the continuity of the second derivative of the swirl schedule. Within the main and splitter blade regions the mean swirl schedule is generated through the use of the Biharmonic Equation. The numerical implementation of the inverse design method is based on the use of the finite-element method on the meridional plane and Fourier collocation in the circumferential direction. The discrete governing equations are solved iteratively until the difference in the blade camber for the main blade as well as the splitter blade between two successive iterations has become sufficiently small. This technique is applied to the design study of a radial inflow turbine with splitter blades; the splitter blades are proposed as a means for reducing the blade lean angle and for possibly improving the aerodynamics as indicated by the reduction of inviscid reversed flow region on the pressure side of the blades. The results indicate that the use of splitter blades is an effective means for making the blade filament at an axial location more radial; appropriate use of splitter blades can eliminate any inviscid reversed flow region that may exists on the pressure side of the blades. In addition, the length of splitter blades has an influence on the extent of the reversed flow region. In agreement with earlier results, stacking position should be near the location of maximum blade loading for a good design. A number referred to as the Wrap Factor has been introduced; it is defined in terms of the specified swirl along the hub and the shroud and is shown to correlate with the blade lean angle. For instance, by increasing the Wrap Factor appropriately the blade lean angle near the trailing edge has been reduced to 4.7° as compared to 56.3° for the situation with no splitter blades. At the same time the inviscid reversed flow region has been reduced also. By relaxing the condition of zero incidence angle at the leading edge, the blade lean angle there can be substantially reduced. In general the present work indicates that there is a trade-off between the structural and the aerodynamic constraints since an increase in the Wrap Factor is generally accompanied by an increase in the Wake Number; the latter is a measure of the extent of the formation of secondary flow in the blade passage.

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Nomenclature

Roman Letters

\( B \) \hspace{1cm} \text{number of blades} \\
\( b \) \hspace{1cm} \text{blockage, Eq.}(2.35) \\
\( f \) \hspace{1cm} \text{blade shape, Eq.}(2.5) \\
\( H \) \hspace{1cm} \text{enthalpy} \\
\( N \) \hspace{1cm} \text{shape function, Eq.}(3.5) \\
\( p \) \hspace{1cm} \text{pressure} \\
\( S \) \hspace{1cm} \text{entropy} \\
\( S(\alpha) \) \hspace{1cm} \text{sawtooth function, Eq.}(2.12) \\
\( T \) \hspace{1cm} \text{temperature} \\
\( t_n \) \hspace{1cm} \text{blade normal thickness, Eq.}(2.37) \\
\( t_\theta \) \hspace{1cm} \text{blade tangential thickness, Eq.}(2.36) \\
\( V \) \hspace{1cm} \text{absolute velocity} \\
\( W \) \hspace{1cm} \text{relative velocity} \\

Greek Letters

\( \alpha \) \hspace{1cm} \text{blade surface, Eq.}(2.5) \\
\( \gamma \) \hspace{1cm} \text{ratio of specific heats} \\
\( \delta_p(\alpha) \) \hspace{1cm} \text{periodic delta function, Eq.}(2.6) \\
\( \rho \) \hspace{1cm} \text{density} \\
\( \rho_a \) \hspace{1cm} \text{artificial density, Eq.}(2.18) \\
\( \sigma \) \hspace{1cm} \text{relative angle between main and splitter blades as fraction of local pitch (Table II) and Lanczos factor (App. B)}
\( \psi \)  \quad \text{Stokes stream function, Eq.(2.20,21,22)} \\
\( \omega \)  \quad \text{impeller rotational speed} \\
\( \Phi \)  \quad \text{periodic potential function, Eq. (2.15)} \\
\( \phi \)  \quad \text{harmonic mode of periodic potential function, Eq. (2.25)} \\
\( \Omega \)  \quad \text{vorticity} \\

\textbf{Subscripts} \\
bl  \quad \text{" at " the blade} \\
j, k \quad \text{denotes the main blade } (j = 1) \text{ or the splitter blade } (j = 2) \\
r, \theta, z \quad \text{the } r-, \theta-, z- \text{ component} \\
LE \quad \text{blades leading edge} \\
TE \quad \text{blades trailing edge} \\
t \quad \text{total, stagnation} \\

\textbf{Superscripts} \\
+/− \quad \text{the blade pressure/suction surface} \\
* \quad \text{rotary} \\

\textbf{Others} \\
(\cdot) \quad \text{vector quantity} \\
(\overline{\cdot}) \quad \text{tangential mean, Eq.(2.8)} \\
(\widetilde{\cdot}) \quad \text{periodic part} \\
(\cdot) \quad \text{column matrix} \\
[\cdot] \quad \text{matrix} \\

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Chapter 1

Introduction

In the design of turbomachinery blading, there are two problems that are of interest. One is the direct approach and the other is the "inverse" approach. In the direct approach the designer specifies the geometrical configuration and computes the performance at design or off-design operating point (e.g. possibly through the use of an Euler or Navier-Stokes code). In the inverse method, the blade geometry is determined to yield a specified performance characteristic. In the next section, a review of previous work on inverse design of turbomachinery blading will be presented. The review has intentionally been focused on inverse design of turbomachinery blading in three-dimensional flow. Since the direct approaches based on Euler solver and Navier-Stokes solver are quite commonly known, no attempt will be made to review these works. Following the section on review, the technical objective of the present work is stated and the approach taken to accomplish the stated objective is described. Finally we present the organization of this thesis.

1.1 Review of Previous Work

In contrast to the direct problem, research activities pertaining to the inverse design problem has not been extensive; this is especially so in three-dimensional flow. One of the earliest work is found in Ref. 4, in which Tan at al. presented an inverse design technique for highly loaded blades in annular cascades. The flow equations are solved by spectral methods in which the velocity is represented by a Fourier Series in the tangential direction and a Bessel Series in the radial direction. The blade wrap angle is represented by a Chebyshev Series in the axial direction and a Cosine Series in the radial direction. Borges (Ref. 22) applied this inverse design technique for the design of a
low-speed radial inflow turbine. The flow was assumed to be incompressible and a finite
difference scheme was used. He designed and constructed two different impellers, one of
which was designed using the inverse approach described in Ref. 4. His work shows that
the inverse designed impeller had a peak total-to-static efficiency 1.5 points better than
the one designed using the conventional approach based on throughflow calculation
 technique. Ghaly (Ref. 3) extended the inverse design theory to compressible flow
regime and used finite element method to numerically solve the equations. Zangeneh
(Ref. 8) extended the work of Borges to compressible subsonic flow. He noted that the
aerodynamically superior blade shape designed by the inverse approach showed 2.5%
improvement compared to the impeller designed using the conventional approach based
on throughflow procedure.

More recently, Yang applied the inverse technique to a design study of radial inflow
turbine wheel in three-dimensional flow (Ref. 1); the design specifications for the
radial inflow turbine are summarized in Table I. It can be seen that this is a rather
highly-loaded radial inflow turbine. Yang developed a rational technique for generating
the swirl distribution (see chapter 4 for more details) from which the blading is then
determined using the 3-D inverse design procedure developed in Ref. 4. His work
has been focused on determining a blade geometry that will yield good aerodynamics
without any due consideration to the structural aspect. In all his design calculations, he
found that for a wide variety of swirl distributions there always exists a region of inviscid
reversed flow on the pressure surface of the blade. Furthermore the resulting computed
blade camber distribution is such that the blade filament is highly non-radial; this is
particularly so in the trailing edge region. The fact that there exists an inviscid reversed
flow region on the pressure surface implies the presence of strong adverse reduced static
pressure gradient which may result in flow separation in real flows. Thus this aspect
of his design calculations may be viewed to be inadequate in aerodynamic terms. The
highly non-radial blade filaments in his design is also unsatisfactory from strutural
considerations.

A set of typical results from his work is presented in Fig. 1.1 to 1.6. Fig. 1.1 and 1.2 show the $r\overline{V}_\theta$ distribution along the hub and the shroud, respectively. The values have been non-dimensionalized by the value of $r\overline{V}_\theta$ at the leading edge. The maximum derivatives of $r\overline{V}_\theta$, denoted by $(\frac{\partial r\overline{V}_\theta}{\partial \varphi})_{\max}$, along the hub and shroud are the same. Fig. 1.3 shows the $r\overline{V}_\theta$ distribution within the blade region based on the rational technique Yang (Ref. 1) developed for generating $r\overline{V}_\theta$ distribution. The resulting blade shape is shown in Fig. 1.4. It is not too difficult to deduce that the blade filament is highly non-radial. The relative Mach number contours on the pressure side is shown in Fig. 1.5 indicating a region with reversed flow. This region of inviscid reversed flow covers almost half of the blade region. The blade cross section at constant z section is shown in Fig. 1.6; note that the value of lean angle can get as high as 56.3° at the trailing edge. Near the leading edge (to be more precise, on cross section at $z = 0.05 \cdot z_{T.E}$), the lean angle is 6.8°.

Thus there is a need to find means of relieving these two inadequacies in Yang’s design calculations. The need to eliminate or to reduce the inviscid reversed flow region on the pressure surface as well as to make the blade filament as radial as possible motivate the work presented in this thesis.

1.2 Technical Approach

As noted by Yang, a way of reducing the extent of the reversed flow region on the pressure side is to increase the number of blades (Ref. 1, section 6.6). However, this approach does not make the blade filament more radial. Instead of adding more "full-length" blades, we can choose the alternative of adding splitter blades, which are "partial-length" blades. The splitter blades will extend from the leading edge to a distance which is approximately 2/3 chord of the main blades. The splitter blades will
carry some of the loading which is mostly concentrated near the leading edge. Near the trailing edge, where the loading is usually small, the splitter blades are not needed. This will also avoid choking the flow near the outlet. An obvious advantage of using splitter blades is that it will result in less surface; another advantage is the availability of an additional degree of freedom in specifying an \( r \vec{V}_\phi \) distribution on the splitter blade that can be different from that on the main blade.

The addition of splitter blades will modify the governing equations for the inverse design procedure (Chapter 2). The corresponding inverse design code as developed by Yang has to be modified accordingly. With the additional degree of freedom, more parameters need to be specified before using the design code to compute the blade shape. Since the total \( r \vec{V}_\phi \) will be divided between the main and splitter blades, one of the immediate consequences is that this will reduce the loading on the main blades; this in turn might reduce the region of reversed flow somewhat. Likewise the use of splitter blade will introduce a flexibility into the choice of \( r \vec{V}_\phi \) distribution that can potentially make the blade filament more radial.

Attempt is also made to relieve some of the constraints in Yang’s original specification of \( r \vec{V}_\phi \) distribution; these include the impositions of zero incidence angle (which requires \( \frac{\partial r \vec{V}_\phi}{\partial \phi} = 0 \) at the leading edge) and the condition that requires \( \frac{\partial^2 r \vec{V}_\phi}{\partial \phi^2} = 0 \) at the leading and trailing edges.

1.3 Thesis Organization

This thesis is organized as follows:

The governing equations for the inverse design technique that include splitter blades will be presented in chapter 2, along with the various boundary conditions.

The numerical techniques to solve those governing equations are presented in chapter 3. Basically the numerical techniques are similar to those used by Yang in his
original work (Ref. 1).

The method of specifying the $r\overline{V}_\theta$ distribution on the blades is discussed in chapter 4.

Some of the remarks in the use of the inverse design code are presented in chapter 5. These include the computational domain, the division of $r\overline{V}_\theta$, stacking conditions, and the iteration process.

In chapter 6, the influence of the various parameters on the blade shape and on the extent of the reversed flow region is discussed. Then the influence of the blade shape and the boundary conditions of $r\overline{V}_\theta$ on the aerodynamic performance is presented. An attempt to analyze one of the resulting blade shapes using a viscous code is made at the end of this chapter. The code is described and the results are presented.

The conclusions of this study and some suggestions for future work are presented in chapter 7.
Chapter 2

Theoretical Formulation

In this chapter, the governing equations of the inverse design theory will be presented. It will be shown that these will reduce to a set of coupled partial differential equations involving the stream function $\psi$, the periodic velocity potential $\Phi$, the blade camber $f_1$ and the splitter camber $f_2$. The boundary conditions associated with these equations are described. A brief discussion on the use of an axisymmetric blockage distribution to account for the effects of the blade thickness is also given.

2.1 Governing Equations

The flow through the turbine wheel is assumed to be steady, irrotational, inviscid, and non-heat conducting. It follows then that the flow is both homenthalpic and homentropic, or that the rothalpy ($H^*$) and the entropy ($S$) are constant throughout the flow field. The governing equations (i.e., the conservation of mass, momentum, and energy, and the equation of state) then assume the forms:

$$\nabla \cdot (\rho \vec{W}) = 0$$  \hspace{1cm} (2.1)

$$\vec{W} \times \vec{\Omega} = 0$$  \hspace{1cm} (2.2)

$$H^* = H + \frac{1}{2} W^2 - \frac{1}{2} \omega^2 r^2 = H^* + \frac{1}{2} \omega^2 V^2 - \omega \tau \vec{V}_\theta = constant$$  \hspace{1cm} (2.3)

$$\left(\frac{p}{p_T}\right) = \left(\frac{\rho}{\rho_T}\right)^\gamma = \left(\frac{T}{T_T}\right)^\frac{\gamma - 1}{\gamma}$$  \hspace{1cm} (2.4)

where $p$, $\rho$, $T$ denote pressure, density and temperature, respectively. $\vec{V}$ is the absolute velocity, and $\vec{W}$ is the relative velocity, i.e., the velocity in the rotating frame ($\vec{W} = \vec{V} - \omega \vec{r}_\theta$). $H^*_T$, $H^*$, $H$ are the rothalpy (rotary total enthalpy), rotary static enthalpy and static enthalpy, respectively. $\gamma$ denotes the ratio of specific heats.
In cylindrical coordinate system, blade surfaces may be represented by

\[
\alpha_j(r, \theta, z) = \theta - f_j(r, z) = \pm \frac{n2\pi}{B} for n = 0, 1, 2, 3, \ldots, (B - 1) \tag{2.5}
\]

where \((r, \theta, z)\) is the right-handed coordinate system, \(f\) is the angular coordinate of a point on the blade surface (also called the Wrap angle), \(B\) is the number of blades, and \(n\) is a nonnegative integer. The subscript \(j\) can assume a value of 1 or 2; the value of 1 pertains to the main blade while the value 2 pertains to the splitter blade. Thus the main blade surfaces correspond to \(\alpha_1\) assuming a value of \(\pm \frac{n2\pi}{B}\); likewise \(\alpha_2 = \pm \frac{n2\pi}{B}\) are the splitter blade surfaces. Hence the blade surfaces are at \(\theta = f_j(r, z) \pm \frac{n2\pi}{B}\) or \(\alpha_j = \frac{n2\pi}{B}\) for \(n = 0, 1, 2, 3, \ldots, (B - 1)\).

The only vorticity in the flow field is located at the blade surfaces, and under these assumptions they can therefore be conveniently represented in terms of generalized function \(\delta_p\) as follows:

\[
\vec{\Omega} = \sum_{j=1}^{2} \vec{\Omega}_j \delta_p(\alpha_j) \tag{2.6}
\]

where \(\delta_p(\alpha_j)\) is the periodic delta function (Ref.2), given as

\[
\delta_p(\alpha_j) = \frac{2\pi}{B} \sum_{n=-\infty}^{\infty} \delta(\alpha_j - \frac{n2\pi}{B}) = \sum_{n=-\infty}^{\infty} e^{inB\alpha_j} \tag{2.7}
\]

and the over bar "\(" defines a tangential mean:

\[
\overline{A}(r, z) = \frac{B}{2\pi} \int_{0}^{2\pi} A(r, \theta, z)d\theta \tag{2.8}
\]

Using Clebsch formulation (Ref.5), the velocity vector may be written as the sum of a potential and a rotational part:

\[
\vec{V} = \nabla \Phi + \lambda \nabla \mu \tag{2.9}
\]

where \(\Phi(r, \theta, z), \lambda(r, \theta, z)\) and \(\mu(r, \theta, z)\) are scalars (also called Clebsch variables). Taking the curl of the above equation, we obtain

\[
\vec{\Omega} = \nabla \times \vec{V} = \nabla \lambda \times \nabla \mu \tag{2.10}
\]
Using the expression for the mean tangential vorticity component $\overline{\Omega}_\theta$ from Eq. (2.10) and noting that $\overline{\Omega}_\theta = \frac{\partial V_r}{\partial \theta} - \frac{\partial V_\theta}{\partial r}$, one can readily identify the Clebsch Variables to be

$$\lambda = r\overline{V}_\theta$$

$$\mu = \alpha$$

(see Ref.1). Hence all the vorticity in the flowfield is given by

$$\mathbf{\Omega} = \sum_{j=1}^{2} (\nabla r \overline{V}_{\theta j} \times \nabla \alpha_j) \delta_p(\alpha_j)$$

(2.11)

Upon integrating the above equation, we obtain

$$\overline{V} = \nabla \Phi + \sum_{j=1}^{2} [r\overline{V}_{\theta j} \nabla \alpha_j - S(\alpha_j) \nabla r \overline{V}_{\theta j}]$$

(2.12)

where $S(\alpha_j)$ is the classical periodic sawtooth function (Ref.2), given as

$$S(\alpha_j) = \sum_{n=-\infty}^{\infty} \frac{e^{inB\alpha_j}}{inB}$$

(2.13)

We can decompose the velocity as a sum of a mean part $\overline{V}$ and a periodic part $\tilde{V}$ as

$$\overline{V} = \nabla \Phi + \sum_{j=1}^{2} r\overline{V}_{\theta j} \nabla \alpha_j$$

(2.14)

$$\tilde{V} = \nabla \tilde{\Phi} - \sum_{j=1}^{2} S(\alpha_j) \nabla r \overline{V}_{\theta j}$$

(2.15)

The continuity equation, Eq.(2.1), can be rewritten as

$$\nabla \cdot \vec{W} = -\vec{W} \cdot \nabla \ln \rho$$

(2.16)

Taking the pitch average of the above equation, we obtain

$$\nabla \cdot \overline{W} = -\overline{W} \cdot \nabla \ln \rho$$

(2.17)
It has been found convenient to introduce an artificial density \( \rho_a(r,z) \) to account for compressibility effects (Ghaly, see Ref.3); the artificial density is defined so as to satisfy

\[
\nabla \cdot \rho_a \vec{W} = 0
\]

(2.18)

Use of Eq.(2.17) in Eq.(2.18) yields

\[
\vec{W} \cdot \nabla \ln \rho_a = \vec{W} \cdot \nabla \ln \rho
\]

(2.19)

The artificial density has the physical meaning of fluid density in the limit of \( B \to \infty \) (which is referred to as Bladed Actuator Duct flow model, Tan et al., Ref.4).

We use Stokes stream function, \( \psi \), to describe the mean flow (as well as the bladed actuator duct flow). Thus the velocity components in the mean flow (also in actuator duct flow) are given as:

\[
\rho_a \vec{W}_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}
\]

(2.20)

\[
\vec{W}_\theta = \sum_{j=1}^{2} \frac{r \vec{V}_{\theta j}}{r} - r \omega
\]

(2.21)

\[
\rho_a \vec{W}_z = \frac{1}{r} \frac{\partial \psi}{\partial r}
\]

(2.22)

Substituting for \( \vec{V}_r (= \vec{W}_r) \) and \( \vec{V}_z (= \vec{W}_z) \) in the equation for the \( \theta- \) component of the mean vorticity \( \vec{\Omega}_\theta = \frac{\partial \vec{V}_z}{\partial z} - \frac{\partial \vec{V}_r}{\partial r} \), we arrive at a governing equation for \( \psi \):

\[
\left[ \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{\partial (\ln \rho_a)}{\partial r} \frac{\partial}{\partial z} - \frac{\partial (\ln \rho_a)}{\partial z} \frac{\partial}{\partial r} \right] \psi = \rho_a \sum_{j=1}^{2} \left[ \frac{\partial r \vec{V}_{\theta j}}{\partial z} \frac{\partial f_j}{\partial r} - \frac{\partial r \vec{V}_{\theta j}}{\partial r} \frac{\partial f_j}{\partial z} \right]
\]

(2.23)

Upon subtracting Eq.(2.17) from Eq.(2.16) we obtain a governing equation for the periodic velocity \( \vec{V} \) as:

\[
\nabla \cdot \vec{V} = -\vec{W} \cdot \nabla \ln \rho + \vec{W} \cdot \nabla \ln \rho
\]

(2.24)

It has been found convenient to use Fourier Series for representing the \( \theta- \) dependence of the periodic velocity potential \( \Phi \) (since the flow has inherent blade-to-blade
periodicity); doing so, we have

$$\tilde{\Phi} = \sum_{n=-\infty, \neq 0}^{+\infty} \tilde{\phi}_n(r, z)e^{inB\theta}$$  \hspace{1cm} (2.25)

Substituting the above equation and Eq.(2.15) into Eq.(2.24), the result is

$$\nabla^2_{2D} \tilde{\phi}_n = \sum_{j=1}^{2} \left[ \frac{e^{-inBf_j}}{inB} \nabla^2 r\nabla_{\theta_j} - e^{-inBf_j} \left( \frac{\partial f_j}{\partial r} \frac{\partial r \nabla_{\theta_j}}{\partial r} + \frac{\partial f_j}{\partial z} \frac{\partial r \nabla_{\theta_j}}{\partial z} \right) \right] +$$

$$\left[ -\tilde{W} \cdot \nabla \ln \rho + \tilde{W} \cdot \nabla \ln \rho \right]_{FT(n)}$$  \hspace{1cm} (2.26)

where $\nabla^2_{2D} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} - \frac{n^2 \pi^2}{r^2}$ and $FT(n)$ denotes the Fourier coefficients of the $n^{th}$ mode.

The condition of no normal velocity on the blade surface yields

$$\tilde{W}_{bl,j} \cdot \nabla \alpha_j = 0$$  \hspace{1cm} (2.27)

where $\tilde{W}_{bl} = \frac{1}{2}(\tilde{W}^+ + \tilde{W}^-)$ is the velocity at the blade (superscript $\pm$ denotes flow variables pertaining to the blade pressure/suction surface). Upon expanding the above equation in cylindrical coordinates, we have

$$\overline{W}_r \frac{\partial f_j}{\partial r} + \overline{W}_z \frac{\partial f_j}{\partial z} = \sum_{k=1}^{2} \left[ \frac{r \nabla_{\theta_k}}{r^2} \right] - \omega + (\tilde{V})_{bl,j} \cdot \nabla \alpha_j$$  \hspace{1cm} (2.28)

In the absence of splitter blades, all the above equations reduce to those presented in Chapter 2 of Ref.1, i.e., the equations for the situation with no splitter blades.

### 2.2 Boundary Conditions

The governing equation for stream function, Eq.(2.23), is a Poisson’s Equation, so that the boundary conditions are:

- along the inflow section: $\psi$ is given
- along the outflow section: $\frac{\partial \psi}{\partial n} = 0$ (parallel flow)
• along the hub and shroud: $\psi$ is constant (no flow normal to the wall)

The boundary conditions for Eq.(2.26) for the (periodic) velocity potential are:

• along the inflow section: $\frac{\partial \tilde{\psi}}{\partial n} = 0$

• along the outflow section: $\frac{\partial \tilde{\psi}}{\partial n} = 0$ (but $\tilde{\psi}_n$ at a point (e.g. at the hub) is taken to be zero so that the solution becomes unique)

• along the hub and shroud: $\frac{\partial \tilde{\psi}_n}{\partial \phi} = 0$ (no flow normal to the wall)

The governing equation for artificial density is Eq.(2.19), and is a first order convective equation so that an appropriate initial condition is:

• along the inflow section: $\rho_a = \bar{\rho}$ is specified.

The initial condition for the blade camber distribution $f$, which is governed by Eq.(2.28), is

• $f_j = f_{st,j}(r_{st}, z_{st})$ where $f_{st,j}$ must be a line from hub to shroud (not coinciding with any of the streamlines)

This initial condition is referred to as the blade stacking condition in Ref. 1.

Besides those boundary conditions which arise from the mathematical nature of the governing equations, we have other conditions which arises from the physical nature of the problem. One of this condition is the Kutta condition, which requires smooth flow at a cusp trailing edge. As a consequence, in a subsonic flow, the pressure must be continuous there. In a homenthalpic flow, this translates to the requirement that $H^+ = H^-$ at the trailing edge. Uniformity in rothalpy (see Eq.(2.3)) enables us to write

$$H^+ - H^- = \frac{1}{2}(W^-)^2 - \frac{1}{2}(W^+)^2 = \frac{1}{2}(\tilde{W}^- + \tilde{W}^+) \times (\tilde{W}^- - \tilde{W}^+) = -\tilde{W}_{st} \cdot \Delta \tilde{W} \quad (2.29)$$

24
The velocity jump $\Delta \vec{W}$ across the blade itself further can be written in the form (see Ref. 1, Appendix A)

$$\Delta \vec{W} = \frac{2\pi \vec{\Omega} \times \nabla \alpha}{B} = \frac{2\pi}{B} \left( \frac{\nabla \alpha \cdot \nabla r V_\theta}{|\nabla \alpha|^2} \nabla \alpha - \nabla r V_\theta \right)$$  \hspace{1cm} (2.30)

Substituting for $\Delta \vec{W}$ in Eq. (2.29) from Eq. (2.30), we obtain

$$H^+ - H^- = \frac{2\pi}{B} [\vec{W}_n \cdot \nabla r V_\theta]$$  \hspace{1cm} (2.31)

where we have used the condition of vanishing normal velocity at the blade surface (i.e., $\vec{W}_n \cdot \nabla \alpha = 0$). The above condition has to be satisfied on both the main and splitter blades so that at the trailing edges we have

$$\vec{W}_{bl,i} \cdot \nabla r V_{\theta j} = 0$$  \hspace{1cm} (2.32)

This equation serves as a useful supplement to the Kutta condition in finite wing theory.

To ensure that Eq. (2.32) is satisfied, we can simply impose $\frac{\partial r V_{\theta i}}{\partial r} = 0$ at the trailing edge since $\frac{\partial r V_{\theta i}}{\partial r}$ is zero at the leading and trailing edge in a free vortex design (i.e., $r V_\theta$ is constant along the leading edge and trailing edge).

Because of the assumptions adopted in Section 2.1, the requirement of no normal velocity on the hub and the shroud surfaces imposes an additional constraint on the $r V_\theta$ distribution:

$$\frac{\partial (r V_\theta)}{\partial n} = 0$$  \hspace{1cm} (2.33)

along the hub and shroud (see Ref. 4, Ref. 12, Ref. 13, and Ref. 14). This results in an additional boundary condition for the blade shape, as noted by Yang (Ref. 1, chap. 4), which is

$$\frac{\partial f}{\partial n} = 0$$  \hspace{1cm} (2.34)

at the intersection of the blade with the walls.
2.3 Blockage Effects

So far we have assumed that the blades are infinitely thin; this may be a fairly good assumption in the shroud region, but not so in the hub region because of stress considerations. We can take into account the effect of blade thickness by introducing a blockage distribution \( b(r, z) \) into the mean flow. Its effect on the periodic flow is neglected (see Ref.6 and Ref.7). Introducing \( b(r, z) \) into the mean flow, the mean flow continuity equation becomes

\[
\nabla \cdot \overline{b \rho \overrightarrow{W}} = 0 \tag{2.35}
\]

where

\[
b = 1 - \frac{\sum_{j=1}^{2} t_{\theta j}}{2 \pi r} \tag{2.36}
\]

with \( t_{\theta} \) denoting the tangential thickness, \( r \) denoting the radius and \( B \) denoting the number of blades.

However, usually it is the normal thickness \( t_n \) that is specified. The tangential and normal thickness can be related (see Ref.8) through the equation

\[
t_{\theta j} = t_{n,j} \sqrt{1 + r^2 \left( \frac{\partial f_i}{\partial r} \right)^2 + r^2 \left( \frac{\partial f_i}{\partial z} \right)^2} \tag{2.37}
\]

The artificial density will be redefined through the equation

\[
\nabla \cdot \rho_a \overline{b \overrightarrow{W}} = 0 \tag{2.38}
\]

to incorporate blockage effects due to finite blade thickness.

Accordingly, the governing equation for the artificial density becomes

\[
\overline{W} \cdot \nabla \ln \rho_a = \overline{W} \cdot \nabla \ln \overline{\rho} + \frac{\nabla \cdot (b \overline{\rho \overrightarrow{W}})}{b \overline{\rho}} \tag{2.39}
\]

Eq.(2.38) implies the following expressions for the Stokes stream function \( \psi \):

\[
b \rho_a \overline{W}_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} \tag{2.40}
\]
The governing equation for \( \psi \) now becomes

\[
\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + 2 \frac{\partial}{\partial z} - \frac{\partial (\ln \rho)}{\partial r} \frac{\partial}{\partial r} - \frac{\partial (\ln \rho)}{\partial z} \frac{\partial}{\partial z} \psi = r \rho_a b \sum_{j=1}^{2} \left( \frac{\partial (r V_\psi)}{\partial z} \frac{\partial f_j}{\partial r} - \frac{\partial (r V_\psi)}{\partial r} \frac{\partial f_j}{\partial z} \right)
\]

(2.43)

The blockage does not change the boundary conditions discussed in Section 2.2.
Chapter 3

The Numerical Technique

This chapter discusses the numerical technique to solve all the governing equations subjected to the appropriate boundary conditions described in chapter 2. A more comprehensive discussion can be found on the original work by Yang (see Ref.1, Chap.5).

3.1 Grid Generation

Although the problem under consideration is a three-dimensional flow, the periodic nature of the flow has made it possible to use Fourier Series to represent the flow variables in the circumferential direction. As a result, we have to generate only a two-dimensional mesh in the meridional plane. Given the profile of the hub and shroud, we can generate a mesh by two means.

The first method is the Algebraic method which is the simpler one. We can cast the expression for \((r, z)\) coordinates in terms of the mapping coordinates \((\xi, \eta)\) as

\[
\begin{align*}
  r(\xi, \eta) &= \frac{1}{2}[r(I_{MAX}, J)\xi + r(I, J)(1 - \xi)] + \frac{1}{2}[r(I, J_{MAX})\eta + r(I, 1)(1 - \eta)] \\
  z(\xi, \eta) &= \frac{1}{2}[z(I_{MAX}, J)\xi + z(I, J)(1 - \xi)] + \frac{1}{2}[z(I, J_{MAX})\eta + z(I, 1)(1 - \eta)]
\end{align*}
\]

and the mapping coordinates \((\xi, \eta)\) are given by

\[
\begin{align*}
  \xi &= \frac{I - 1}{I_{MAX} - 1} \\
  \eta &= \frac{J - 1}{J_{MAX} - 1}
\end{align*}
\]

Because the equations are linear, no numerical iteration is needed.

The second one is the Elliptic method. We solve two elliptic equations for mapping coordinates \(\xi\) and \(\eta\), which are

\[
\begin{align*}
  \xi_{rr} + \eta_{zz} &= P(\xi, \eta) \\
  \eta_{rr} + \xi_{zz} &= Q(\xi, \eta)
\end{align*}
\]
where $P$ and $Q$ are terms that control the point spacing on the interior of the domain. If we transform the above equation by interchanging the roles of the dependent and independent variables, we get

$$\begin{align*}
\alpha r_{\xi\xi} - 2\beta r_{\xi\eta} + \gamma r_{\eta\eta} &= -|J|^2 (Pr_\xi + Qr_\eta) \\
\alpha z_{\xi\xi} - 2\beta z_{\xi\eta} + \gamma z_{\eta\eta} &= -|J|^2 (Pz_\xi + Qz_\eta)
\end{align*}$$

(3.4)

where

$$\begin{align*}
\alpha &= r_n^2 + z_n^2 \\
\beta &= r_\xi r_\eta + z_\xi z_\eta \\
\gamma &= r_\xi^2 + z_\xi^2 \\
|J| &= \frac{\partial(x,y)}{\partial(\xi,\eta)} = r_\xi z_\eta - r_\eta z_\xi
\end{align*}$$

The above equations are partial differential equations of elliptic type and they can be solved iteratively.

3.2 The Elliptic Solver

Eq.(2.23) for the stream function and Eq.(2.26) for the periodic velocity potential are non-linear partial differential equations. The Finite Element Method is used to solve them. The procedure used is that all the nonlinear terms are transposed to the right hand sides of the equations to become the "forcing functions" of the equations. Since the left hand sides are now linear, they can be solved by a finite element method. The RHS is then recomputed based on newly obtained solutions; such a procedure is repeated until the solution converges.

Here, we use a 4-node isoparametric element to interpolate the variables. The coordinates $r$ and $z$ are expressed in terms of the coordinates of the four corner nodes by

$$\begin{align*}
\begin{bmatrix} r \\ z \end{bmatrix} &= \sum_{i=1}^{4} (1 + \xi_i \xi)(1 + \eta_i \eta) \begin{bmatrix} r_i \\ z_i \end{bmatrix} = N^T \begin{bmatrix} r_i \\ z_i \end{bmatrix}
\end{align*}$$

(3.5)
where $N^T$ is the transpose of the shape function vector. The Jacobian, in terms of the coordinates of $r_i$ and $z_i$, is given as

$$[J] = \begin{bmatrix}
\frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta} \\
\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta}
\end{bmatrix} \quad (3.6)$$

The derivative of variables in the physical plane can be expressed in terms of those in the computational plane as follows:

$$\begin{bmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial z}
\end{bmatrix} = [J]^{-1} \begin{bmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\
I_{21} & I_{22}
\end{bmatrix} \begin{bmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{bmatrix} \quad (3.7)$$

If we use the Galerkin method, the variational form of equation (2.23) becomes

$$I = \frac{1}{2} \iint \psi^T \left[ 1 \left( \frac{\partial N^T}{\partial r} \right) \left( \frac{\partial N}{\partial r} \right) + 1 \left( \frac{\partial N^T}{\partial z} \right) \left( \frac{\partial N}{\partial z} \right) \right] \psi dA + \iint \psi^T N^T N q dA$$

$$= \frac{1}{2} \psi^T [k] \psi + \psi^T q \quad (3.8)$$

where $q$ is the nodal value of $[1 \left( \frac{\partial \ln a}{\partial r} \right) \left( \frac{\partial \psi}{\partial r} \right) + 1 \left( \frac{\partial \ln a}{\partial z} \right) \left( \frac{\partial \psi}{\partial z} \right) + \sum_{j=1}^{2} \rho_a \left( \frac{\partial \psi}{\partial x} \right) \frac{\partial f_j}{\partial r} - \rho_a \left( \frac{\partial \psi}{\partial r} \right) \frac{\partial f_j}{\partial z}]$. The evaluation of $[k]$ and $q$ is done by Gaussian Quadrature. For stationary value of $I$, i.e., $\frac{\delta I}{\delta \psi} = 0$, we obtain the Euler equation

$$[k] \psi = -q \quad (3.9)$$

with the corresponding boundary conditions. This is done locally from element to element. To assemble the contribution from each element, we apply the direct stiffness technique to obtain the global matrix over the whole domain as

$$[K] \psi = -Q \quad (3.10)$$

The above procedure is also applied for the periodic velocity potential in Eq.(2.26). The above equation can be solved directly by a method such as LU decomposition, or iteratively, such as by conjugate gradient method. As there is no need to get the exact solution for the stream function and the periodic velocity potential before the blade
camber and the artificial density converge, the use of an iterative procedure using the conjugate gradient method should be more efficient and it is chosen over direct solution approach.

3.8 The Convective Equation Solver

The governing equation for the artificial density, Eq.(2.19), and the governing equation for the blade camber, Eq.(2.28), are two first order equations of hyperbolic type. It can be solved by using the method of characteristics, or alternatively, it can be solved by using direct numerical differentiation and integration using the finite element method.

The RHS of these equations are obtained from the previous iteration or from initial guess. The discrete approximations to operator $\bar{W}_r \frac{\partial f}{\partial r} + \bar{W}_z \frac{\partial f}{\partial z}$ can be separated into two parts. The parts that contain the given initial nodal points will be transposed to the RHS, and the other parts that contain the unknown nodal points are retained on the LHS. As an illustration, Eq.(2.28) can be written as

$$\bar{W}_r \frac{\partial f}{\partial r} + \bar{W}_z \frac{\partial f}{\partial z} = q$$

(3.11)

where $q$ equals $[\sum_{i=1}^{4} \bar{V} \theta_k - \omega + \bar{V} \mu \cdot \nabla \alpha]$.

Now if we use an element with four sides with $f_1$ and $f_4$ given, then the unknowns $f_2$ and $f_3$ are solved for by satisfying the above equation at nodes 1 and 4 (or alternatively, at two Gaussian points, which are preferable, since they give a better weighted accuracy). Using Eq.(3.7) we get

$$\begin{cases} [\bar{W}_{r_1}(I_{11}N_i,\xi f_i + I_{12}N_i,\eta f_i)]_{\xi=1,\eta=1} + [\bar{W}_{z_1}(I_{21}N_i,\xi f_i + I_{22}N_i,\eta f_i)]_{\xi=1,\eta=-1} = q_4 \\ [\bar{W}_{r_1}(I_{11}N_i,\xi f_i + I_{12}N_i,\eta f_i)]_{\xi=1,\eta=1} + [\bar{W}_{z_1}(I_{21}N_i,\xi f_i + I_{22}N_i,\eta f_i)]_{\xi=1,\eta=-1} = q_1 \end{cases}$$

(3.12)
where $N_i,j$ denoting $\frac{\partial N_i}{\partial \xi}$ and the free index $i$ represents the sum from 1 to 4. If the known values $f_1$ and $f_4$ are moved to the RHS, Eq.(3.12) can be rewritten as

$$[k] \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} = q$$

(3.13)

The above equations are solved for the unknowns $f_2$ and $f_3$ on each element. Between the two adjacent elements, an average value is used for $f$. This procedure is done to solve all the values of $f$ along the span; then it marches to the next streamwise location using the previous solutions as the new initial conditions, either forward or backward, depending on the location of the stacking axis. The artificial density is solved in the same way.

The additional boundary condition on the blade shape, i.e., $\frac{\partial f}{\partial n} = 0$ at the hub and shroud (see Eq.(2.34)), is imposed explicitly. The blade camber is determined, and then the value of $f$ along the hub and shroud is modified to satisfy this condition. This is done by setting $q = 0$ and by replacing $\overline{W_r}$ and $\overline{W_s}$ with $-\cos \nu$ and $\sin \nu$, where $\nu$ is the angle normal to the boundary; $f_2$ and $f_3$ are then solved to satisfy the condition that $\frac{\partial f}{\partial n} = 0$.

### 3.4 The Biharmonic Solver

The distribution of the mean swirl schedule is generated through the solution of the biharmonic equation

$$\nabla^4 (r\overline{V}_s) = R(r, z)$$

(3.14)

; this will be discussed further in the next chapter. Equation (3.14) can in principle be solved by the usual finite element method as in structural mechanics. However, this method was found to be rather impractical (Ref.1, pg.58). Thus the same method as in the original work by Yang is used; this technique is based on the idea of reformulating the above equation as two coupled Poisson equations (Ref.15, 16, 17, 18) in the
following:
\[
\nabla^2 r \bar{V}_\theta^{(k)} = \omega^{(k-1)} \quad \text{in } G
\]
\[
r \bar{V}_\theta^{(k)} = r \bar{V}_{\text{specified}} \quad \text{on } \partial G
\]
\[
r \bar{V}_\theta^{(k)} = (1 - \epsilon) r \bar{V}_\theta^{(k-1)} + \epsilon r \bar{V}_\theta^{(k)}
\]
\[
\nabla^2 \omega^{(k)} = R \quad \text{in } G
\]
\[
\omega^{(k)} = \nabla^2 r \bar{V}_\theta^{(k)} - c[\frac{\partial r \bar{V}_\theta^{(k)}}{\partial n} - g] \quad \text{on } \partial G
\]
\[
\omega^{(k)} = (1 - \delta) \omega^{(k-1)} + \delta \omega^{(k)}
\]

where \( G \) is the domain of the blade region, \( \partial G \) is the boundary of the blade region, \( c \) is an arbitrary nonzero constant, \( \epsilon \) and \( \delta \) are two relaxation constants and \( g \) is set to be zero for vanishing normal gradient of \( r \bar{V}_\theta \) and nonzero otherwise.

The two Poisson equations are solved by the same method as has been described in Sec.3.2. It has been noted by Yang that if under-relaxation is used, small numbers of conjugate gradient iterations (usually 10 - 15) are needed on each local iteration. With \( c = 1, \epsilon = \delta = 0.15 \), in general the solution requires \( k = 250 \) global iterations to achieve machine accuracy.

3.5 The Computational Scheme

The same computer programs that were developed by Yang were modified to include the option of incorporating the inverse design of the splitter blades. The procedure is basically similar:

Step 1. Grid Generation

The same hub and shroud profiles as in the original work by Yang are used. The algebraic method is used to generate the grid. For the design calculations presented in this thesis, the number of gridpoints in the streamwise direction is taken to be 97 and that in the spanwise direction to be 33. However, in the tangential direction,
8 complex harmonic modes (which corresponds to 17 Fourier collocation points from blade to blade) are used. The inlet and outlet boundaries are taken to have distances of a quarter of chord (averaged meridional distances of the hub and the shroud) from the leading edge and the trailing edge, respectively.

Step 2. Mean Swirl Schedule

The value of $rV_\phi$ is specified along the hub and shroud of the main blade and the splitter blade. The value of $rV_\phi$ within the blade region and the splitter blade region is obtained through the use of biharmonic equation solver. These aspects of $rV_\phi$ generation will be discussed in the next chapter.

Step 3. Inverse Design

As in the original work by Yang, all the flow variables have been made dimensionless based on tip radius, meridional velocity and rotary stagnation quantities at the leading edge (see Appendix A). Before implementing the determination of blade geometry using the inverse design code, the followings need to be specified:

- Velocity triangle and Mach number at the leading edge
- Ratio of specific heats, $\gamma$
- Stacking condition
- Mean swirl schedule, $rV_\phi(r, z)$
- Normal thickness distribution of blade, $t_n(r, z)$

The iteration procedure proceeds as follows: The code begins with an initial guess of the blade camber. It first solves for the stream function $\psi$ from Eq.(2.23) and $\phi_n$ for each $n$ from Eq.(2.26). Then the velocity field is calculated to update the blade.
camber \( f \) (Eq.(2.28)), artificial density \( \rho_a \) (Eq.(2.19)) and density field. If the difference in the blade camber between the current iteration and the previous one is large, then the stream function and potential harmonic modes will be calculated again. When the difference in the blade camber for the main blade as well as the splitter blade between two successive iterations has become sufficiently small (this difference is typically \( 10^{-5} \) radian in the present study), the solution from the inverse design procedure is taken to have converged.

The outputs from the solution are:

- Blade shape \( f_1 \) and the splitter blade shape \( f_2 \)
- Velocity field (except \( V_e \) which has been specified), pressure field and etc.

In the computational procedure, the gradient of the density (or \( \ln \rho \)) is evaluated in the Fourier space by taking its Fourier Transform \( (FT) \) first. The physical values are obtained from the Inverse Fourier Transform \( (IFT) \). If the number of harmonic modes is an exponent of 2, then a fast Fourier Transform (Ref. 19) is available to evaluate the \( FT \) and \( IFT \) of \( \tilde{W} \) and \( \ln \rho \). As the periodic velocity field has a sawtooth-like behavior (see Eq.(2.15)), Gibbs phenomenon involving overshoots and undershoots near the discontinuity is expected; this is a consequence of representing a piecewise continuous function in terms of Fourier Series. To overcome this problem, Lanczos filter (Ref. 20) is used with slight modification (App.B).
Chapter 4

Design Criteria and Specification of Swirl Distribution

In this chapter the specification of the swirl ($rV_\theta$) and the criteria used for assessing the aerodynamic goodness of results from design calculation for the main and splitter blades are discussed.

4.1 Design Criteria

As has been noted in the Introduction (Chapter 1), the work by Yang has resulted in guidelines for choosing an optimum $rV_\theta$ distribution based on two criteria:

- the maximum adverse reduced static pressure gradient on the suction surface of the blade, which can influence boundary layer separation in real flow

- the wake number, which gives a measure of the extent of secondary flow development

In Ref.1, the parametric influence of loading distribution, stacking position, lean in stacking angle, slip factor, number of blades, and hub and shroud profile geometry on the blade shape and the resulting flow field has been explored. However the results in Ref.1 show that for a rather wide choices of $rV_\theta$ distribution, the resulting blade shapes are such that the blade filaments at constant z section are highly nonradial; this aspect of the blade shape is usually not acceptable from structural/stress consideration. Furthermore there also exists a region of inviscid reversed flow on the pressure surface which may imply flow separation for real fluid.

In the present investigation, the inclusion of splitter blades in the design is proposed as a mean to eliminate or reduce the reversed flow region and to arrive at designs in
which the blade filaments at constant z section are more radial. In view of these, we have adopted the following criteria for assessing the goodness of the resulting design calculation:

- the curvature of the blade shapes along constant z section
- the extent of the region of reversed flow

However, the extent of the reversed flow region can be considered as secondary as the reversed flow region does not necessarily result in an unacceptable level of losses (Ref.1, Chap.7).

Upon examining the governing equations in Chapter 2, we deduce that by including the splitter blades we obtain several additional degrees of freedom. These are:

- the length of the splitter blades
- $r \bar{V}_s$ distribution on the splitter blades
- stacking position of the splitter blades
- lean in stacking angle of the splitter blades
- the relative angular distance between the main blades and the splitter blades at the stacking position

However, they are not really arbitrary; for instance we have the following constraint:

- the relative angular distance between the main and splitter blades has to be within the values of 0 and $\frac{2\pi}{B}$

This implies that the resulting splitter blade shape can not cross the main blade. This constraint clearly arises because of physical consideration, although it did not necessarily arise in the mathematical formulation. In practice, to avoid choking the
flow, it is desirable to have relative angular distance that has a value between 0.4 and 0.6 when expressed as a fraction of the local pitch (which is equal to the angle between main blades, i.e., $2\pi/B$).

4.2 Boundary Conditions on $r\overline{V}_\theta$ Distribution

As has been noted by Yang, the swirl distribution $r\overline{V}_\theta$ can not be specified arbitrarily; it has to satisfy several boundary conditions which involve its value and its normal derivative along the boundary of the blade region. The boundary conditions needed to be specified for the generation of $r\overline{V}_\theta$ distribution are:

- at the leading edge : $r\overline{V}_\theta$ is a constant which is given; this is needed for a free vortex design.

- along the hub and shroud : $r\overline{V}_\theta$ is prescribed as a mean to control the loading distribution; this will be discussed in the next section.

- at the trailing edge : $r\overline{V}_\theta$ is a constant which is specified; this is again needed for the free vortex design.

The boundary conditions which involve the normal derivative of $r\overline{V}_\theta$ are:

- at the leading edge: $\frac{\partial r\overline{V}_\theta}{\partial s}$ is given; if the condition of zero incidence is assumed, then $\frac{\partial r\overline{V}_\theta}{\partial s} = 0$.

- along the hub and shroud : the kinematic constraint requires that $\frac{\partial r\overline{V}_\theta}{\partial n} = 0$ (see Ref.1, Eq.(4.4); this is needed so that there are no shed vorticities due to non-zero normal gradient of the swirl at the wall).

- at the trailing edge : $\frac{\partial r\overline{V}_\theta}{\partial s} = 0$; this is needed to satisfy the Kutta-Joukowski condition.
Because we have two conditions along each boundary so that the generation of a smooth $r\vec{V}_\theta$ distribution from second order equation (such as Poisson equation) will not work, we need to have at least a fourth order equation so as to satisfy all the boundary conditions; a natural choice would be to use the biharmonic equation

$$\nabla^4 (r\vec{V}_\theta) = R(r, z) \tag{4.1}$$

where

$$\nabla^4 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)^2$$

The forcing function $R(r, z)$ can be chosen appropriately and it can actually be used as a means to control the swirl distribution. As in the original work by Yang, however, $R(r, z)$ is taken to be identically zero here for simplicity. The swirl distribution is controlled by the specification of the value of $r\vec{V}_\theta$ along the hub and the shroud. In this way, the problem is reduced from specifying the swirl in a two dimensional region to a one dimensional region, i.e., along the boundary only.

### 4.3 Method of Specifying $r\vec{V}_\theta$ along the Hub and the Shroud

In general, we only need to specify $r\vec{V}_\theta$ along the leading edge, the hub, the shroud and the trailing edge. As in the original work by Yang, the hub or the shroud region is divided into three sections (see Fig. 4.1): from the leading edge ($L.E.$) to a point called $A$, from point $A$ to a second point called $B$, and from point $B$ to the trailing edge ($T.E.$). We will specify $r\vec{V}_\theta$ as polynomials in those three sections and their degrees in each section is determined to ensure certain continuity condition in $r\vec{V}_\theta$ distribution across the regions.

Because the biharmonic and the governing equations require the evaluation of $\nabla^2 r\vec{V}_\theta$ (see Eq.(2.26) and Eq.(4.1)), we need to enforce the continuity of the second derivative of $r\vec{V}_\theta$ along the hub and shroud. This implies that the polynomials will
have degrees of at least two. Furthermore the maximum value of \( \frac{\partial r\vec{V}_\theta}{\partial s} \) is specified at point A. To control the shape of the loading, we specify the magnitude of \( \frac{\partial r\vec{V}_\theta}{\partial s} \) along the second section (from point A to point B). The Kutta-Joukowski condition requires that \( \frac{\partial r\vec{V}_\theta}{\partial s} = 0 \) at the trailing edge, and \( \frac{\partial r\vec{V}_\theta}{\partial s} \) at the leading edge is specified, which is zero for zero incidence angle. A typical \( r\vec{V}_\theta \) distribution along the hub of the main and splitter blades is shown in Fig. 4.1, where lowercase a and b denote the points for the splitter blade and the uppercase A and B for the main blade.

From the above discussion, we conclude that we need to have at least polynomials of the following degrees:

- **Section 1**: \( r\vec{V}_\theta(s) \) is cubic in \( s \)
  
  specify \( r\vec{V}_\theta = (r\vec{V}_\theta)_{L.E.}, \ (r\vec{V}_\theta)' = (r\vec{V}_\theta)'_{L.E.} \) at \( L.E. \)

  specify \( (r\vec{V}_\theta)' = (r\vec{V}_\theta)'_{max}, \ (r\vec{V}_\theta)'' = (r\vec{V}_\theta)''_{A-B} \) at \( A \)

- **Section 2**: \( r\vec{V}_\theta(s) \) is quadratic in \( s \)
  
  obtain from section 1, \( r\vec{V}_\theta = (r\vec{V}_\theta)_A, \ (r\vec{V}_\theta)' = (r\vec{V}_\theta)'_A, \ (r\vec{V}_\theta)'' = (r\vec{V}_\theta)''_A \) at \( A \)

- **Section 3**: \( r\vec{V}_\theta(s) \) is quartic in \( s \)
  
  obtain from section 2, \( r\vec{V}_\theta = (r\vec{V}_\theta)_B, \ (r\vec{V}_\theta)' = (r\vec{V}_\theta)'_B, \ (r\vec{V}_\theta)'' = (r\vec{V}_\theta)''_B \) at \( B \)

  specify \( r\vec{V}_\theta = (r\vec{V}_\theta)_{T.E.}, \ (r\vec{V}_\theta)' = 0 \) at \( T.E. \).

The specifications above will satisfy all the constraints. If, in addition, the conditions of \( (r\vec{V}_\theta)'' = 0 \) at the leading edge and the trailing edge and continuity in \( D^2 r\vec{V}_\theta \) are imposed, higher degree polynomials for Section 1 and 3 are needed as indicated below:

- **Section 1**: \( r\vec{V}_\theta(s) \) is quartic in \( s \)
specify \( r\bar{V}_\theta = (r\bar{V}_\theta)_{L.E.}, \) \( (r\bar{V}_\theta)' = (r\bar{V}_\theta)'_{L.E.}, \) \( (r\bar{V}_\theta)'' = 0 \) at \( L.E. \)

specify \( (r\bar{V}_\theta)' = (r\bar{V}_\theta)'_{\text{max}}, \) \( (r\bar{V}_\theta)'' = (r\bar{V}_\theta)''_{A-B} \) at \( A \)

- Section 2 : \( r\bar{V}_\theta(s) \) is quadratic in \( s \)

obtain from section 1, \( r\bar{V}_\theta = (r\bar{V}_\theta)_A, \) \( (r\bar{V}_\theta)' = (r\bar{V}_\theta)'_A, \) \( (r\bar{V}_\theta)'' = (r\bar{V}_\theta)''_A \) at \( A \)

- Section 3 : \( r\bar{V}_\theta(s) \) is quintic in \( s \)

obtain from section 2, \( r\bar{V}_\theta = (r\bar{V}_\theta)_B, \) \( (r\bar{V}_\theta)' = (r\bar{V}_\theta)'_B, \) \( (r\bar{V}_\theta)'' = (r\bar{V}_\theta)''_B \) at \( B \)

specify \( r\bar{V}_\theta = (r\bar{V}_\theta)_{T.E.}, \) \( (r\bar{V}_\theta)' = 0, \) \( (r\bar{V}_\theta)'' = 0 \) at \( T.E. \)

In general, it is the latter that is used in the present study unless otherwise stated. Also, in the next chapter, except for one case, \( (r\bar{V}_\theta)' \) at \( L.E. \) is set to zero. This leave us with the choice of the following variables: the location of point \( A \) (the maximum loading position), the location of point \( B \) (the extent of finite loading), \( (r\bar{V}_\theta)'_{\text{max}} \) (the maximum loading), and \( (r\bar{V}_\theta)''_{A-B} \) (the shape of the loading). They can be used to produce a variety of swirl distribution.
Chapter 5
Additional Remarks on Computational Procedure for Design

In utilizing the inverse design code, there are several points to be stated. These are the grid used, the division of the total value of $r\overline{V}_{\theta}$ at the leading edge between the main and splitter blades, the stacking conditions of the main and splitter blades, and the iteration process.

5.1 The Computational Domain

The turbine design conditions for the present study are summarized in Table I. The velocity triangle at the leading edge is shown in Fig. 5.1. The absolute Mach number at the inlet is computed to be equal to 0.98. The grid used for study has 97 nodes in the streamwise direction and 33 nodes in the spanwise direction. The number of harmonic modes is taken to be 8, which corresponds to 17 Fourier collocation points.

5.2 The Division of $r\overline{V}_{\theta}$ between the Main and Splitter Blades

The total value of $r\overline{V}_{\theta}$ is to be divided between the main blade and the splitter blade. Especially, we have to decide what fraction of $r\overline{V}_{\theta}$ at the leading edge is to be assigned to the splitter. That division can be arbitrary. However, in the present study, that division is based on the length of the splitter blade (the average of the meridional distances along the hub and the shroud). For instance, if we have a splitter blade which is half the length of the main blade, then the value of $r\overline{V}_{\theta}$ at the leading edge will be $0.5/(1 + 0.5) = 1/3$ of the total value of $r\overline{V}_{\theta}$ at the leading edge. This linear division
implies that in the absence of the splitter blade \( r\bar{V}_\theta \) is zero for the splitter, and when the splitter is of the same length as the main blade (i.e., we have twice the number of blades) \( r\bar{V}_\theta \) for the splitter is identical to that for the main blade.

There is another reason for choosing the above procedure for the specification of the value of \( r\bar{V}_\theta \) at the splitter leading edge. This is described in the following. Suppose the swirl to be removed varies linearly from leading edge to trailing edge; because of this, the main and the splitter blades will have exactly the same value of \( \frac{\partial r\bar{V}_\theta}{\partial z} \) along the hub and the shroud. A similar value in the first derivative of \( r\bar{V}_\theta \) is desirable, as it is related to the loading on the blades. However, in the present study, the specified swirl schedule is such that it is not removed linearly; accordingly the above procedure for assigning the swirl at the leading edges is expected to result in the specification of loading distribution which is approximately the same for the main and the splitter blades.

5.3 The Stacking Conditions

As explained in chapter 2, the blade shapes are obtained iteratively from Eq.(2.28), which is again given below:

\[
\frac{\partial f_i}{\partial r} + \frac{\partial f_i}{\partial z} = \sum_{k=1}^{2} \left[ \frac{r\bar{V}_{\theta k}}{r^2} \right] - \omega + (\bar{V})_{bl,i} \cdot \nabla \alpha_j
\] (5.1)

Now upon neglecting the last term on the RHS, we deduce that both the main blade and the splitter blade would have the same shape. This is so because the mean velocity is used on the LHS while the sum of \( r\bar{V}_\theta \) (not the \( r\bar{V}_\theta \) on each blade) is used on the RHS. Thus if the periodic velocity term is indeed small compared to the other terms in the equation, the blades will have approximately the same camber distributions for the same stacking conditions. It is for this reason that the same stacking conditions for the main and the splitter blades are chosen.
The relative angular distance between the blades at the stacking positions need not be necessarily set to \( \pi/B \) (i.e., the splitter blade is exactly halfway between the main blades at the stacking position) so that it can be viewed as an additional degree of freedom. The choice of this additional degree of freedom at the stacking position can influence the relative mass flow between the two different passages: one the passage between the suction side of the main blade and the pressure side of the splitter blade and the other the passage between the suction side of the splitter blade and the pressure side of the main blade. This implies that the relative angular distance can influence the velocity field. In this study, the relative angle is expressed as a fraction of the angle between the main blades (i.e., \( 2\pi/B \)) and is denoted by \( \sigma \). For \( \sigma \) close to zero the splitter blade is close to the suction side of the main blade while for \( \sigma \) close to unity the splitter blade is close to the pressure side of the main blade.

5.4 The Iteration Process

As Eq.(5.1) implies, the amount of wrap of the blade shapes is determined by the mean velocity on the left hand side. Thus if the right hand side is finite and at some point the mean velocity approaches zero, the blades can have a large change in the camber angle; this is undesirable from the design point of view. For this reason, the minimum mean velocity is monitored in the computational procedure and if it falls below a certain threshold, the \( r\overline{V}_\theta \) distribution is then altered accordingly in an attempt to increase it. Usually, it is done by reducing the value of \( \frac{\partial r\overline{V}_\theta}{\partial z} \) at the point of minimum mean velocity, which can readily be done if the point is close to the hub or shroud. The parameter \( \sigma \) can also be varied in an attempt to increase the mean velocity.
Chapter 6

Results from Design Calculations

We will now proceed to present the results from the design calculations carried out in this thesis. Specifically we will focus our effort on examining the influences of \( rV_e \) distribution, length of the splitter blades, stacking position, \( \frac{\partial V}{\partial r} \), non-vanishing first derivative of \( r\hat{V}_t \) at the leading edge (i.e., finite incidence angle), blockage, and slip factor on the blade shape as well as on the extent of inviscid reversed flow region on the pressure surface of the blades.

6.1 Length of Splitter Blades

Three splitter blade lengths are examined; these are 0.53, 0.61, and 0.74, and will be referred to as case a, b, and c, respectively. The numbers represent fractions of splitter blade lengths with respect to the main blade length, which is kept constant. The length itself is calculated as the average of the meridional distances along the hub and the shroud. Using the simple procedure for assigning the \( r\hat{V}_t \) distribution between the main and splitter blades described in Chapter 5, the corresponding values of \( r\hat{V}_t \) at the leading edges of the main blades are 0.65, 0.62, and 0.57, respectively, of the net \( r\hat{V}_t \) at the inlet.

The meshes used are shown in Fig. 6.1a, 6.1b, and 6.1c where the locations of the leading and trailing edges are indicated as well. The grids are set so that the leading and trailing edges coincide with the quasi-orthogonal grid line extending from the hub to the shroud. The splitter trailing edge for case b has been made more radial for assessing the change in the flow as we increase the length from case a to case c.

The \( r\hat{V}_t \) distribution for each case on the hub is shown in Fig. 6.2a, 6.2b, and 6.2c and on the shroud in Fig. 6.3a, 6.3b, and 6.3c, respectively. They have been
all normalized by the value of $r\bar{V}_\phi$ at the leading edges of the main blades. The distances along the hub and the shroud have been normalized by the diameter. Along the shroud, a maximum value of 2 for $\frac{\partial r\bar{V}_\phi}{\partial s}$ has been selected for all cases; the region for which $\left(\frac{\partial r\bar{V}_\phi}{\partial s}\right)_{\text{max}} = -2.0$ is chosen to extend as far downstream as possible so long as it does not result in reversed loading (i.e., an "overshoot" in the curve of $\frac{\partial r\bar{V}_\phi}{\partial s}$). Along the hub, the loading distribution has a triangular shape with $\left(\frac{\partial^2 r\bar{V}_\phi}{\partial s^2}\right)_{A-B}$ set to positive values for case a and c (see Table III and IV). For case b it is set to zero.

The resulting main blade cross-sections at constant $z$ are shown in Fig. 6.4a, 6.4b, and 6.4c. In these cases the blades have been stacked at the leading edge. The computed results indicate that as the splitter blade length is increased, the reduction in the lean angle at the trailing edge of the main blade is rather insignificant. In fact, case b has a larger wrap angle at the trailing edge (more than -2.4 radians) compared to that of case a and c. This may be attributed to the fact that the region over which the loading is maximum for case b has been extended for a finite length.

The corresponding reversed flow regions on the pressure sides are shown in Fig. 6.5a, 6.5b, 6.5c for the main blades and in Fig. 6.6a, 6.6b, and 6.6c for the splitter blades. We observe that the splitter blades with lengths of 0.53 and 0.61 have eliminated reversed flow regions on the pressure side of the main blade entirely. However, when the length is increased to 0.65 (case c), small reversed flow regions are observed to appear on the pressure sides of both the main and splitter blades. Hence we may deduce that increasing the length of splitter blades does not necessarily reduce or eliminate the inviscid reversed flow region. Moreover, the increasing length of the splitter blades will result in additional profile loss in real flow. The elimination of the reversed flow region can be attributed to the nature of the specified loading, and not the extent of the splitter blade. This gives a good indication that we are taking a step in the right direction: we add splitter blades to allow us the flexibility of tailoring the loading on the main blades appropriately.
We note from these results that the length of the splitter blades appear not to have any significant effect on the blade shape. From structural consideration, however, it is desirable to have splitter blades with blade filament being more radial at the trailing edge such as that in case b. For this reason, the splitter blade with length of 0.61 will be examined further for assessing the parametric influence of other key design parameters.

6.2 Stacking Position

The effects of stacking position on the flow field have been studied by Yang (Ref.1, Sec. 6.2), and in general, to avoid unacceptable low value of $W_{bl}$ (which causes the solution to fail to converge to the final blade shape), stacking position is chosen to be near where the loading is maximum and preferably slightly downstream of it. In this section the effect of stacking position on the extent of reversed flow region will be examined in the context of the use of splitter blade in radial inflow turbine wheel.

A new distribution of $rV_\phi$ is specified for the configuration with splitter of length 0.61, of which the grid is shown in Fig. 6.1b. This will be referred to as case d. The $rV_\phi$ along the hub is shown in Fig. 6.7 and the $rV_\phi$ along the shroud is shown in Fig. 6.8. Instead of having an equal value of -2 for $(\frac{\partial rV_\phi}{\partial s})_{max}$ on the hub and the shroud, the maximum value of $\frac{\partial rV_\phi}{\partial s}$ on the hub is decreased to -1.75 and the maximum value of $\frac{\partial rV_\phi}{\partial s}$ on the shroud is increased to -3. The location of maximum $\frac{\partial rV_\phi}{\partial s}$ is moved further away from the leading edge. This new $rV_\phi$ distribution along the hub and shroud is rather similar to that used by Zangeneh (Ref. 8) which has yielded a blade shape with a lean angle of no more than 10°. The corresponding swirl distributions on the main and splitter blades are shown in Fig. 6.9 and 6.10, respectively.

When stacked at the leading edge, this new $rV_\phi$ distribution fails to give convergent blade shape. So the stacking axis is moved to a location 1/3 chord ($J=40$) downstream from the leading edge; this corresponds to the location of the maximum loading. The
radial distribution of resulting main blade camber along constant \( z \) section is shown in Fig. 6.11. Indeed this new \( r\bar{V}_\theta \) distribution gives significant improvement in the blade camber distribution. The lean angle along constant \( z \) section has been reduced to 25.8° from a value of 56.3° in Yang’s original design calculation (see Fig. 1.6). These results appear to indicate that for designing blade with more radial filament along constant \( z \) section, one should specify the maximum loading along the shroud to have a value more negative than that along the hub. However, as will be shown in the following section, there is a better procedure to accomplish this (i.e., through the use of the Wrap Factor, to be defined below).

The corresponding reversed flow regions are shown in Fig. 6.12a and 6.13a. Apparently the large spanwise gradient of \( r\bar{V}_\theta \) that results from the new distribution has caused the reversed flow region to reappear; this is in contrast to the absence of reversed flow region using the \( r\bar{V}_\theta \) distribution in case b, which is more typical of Yang’s distribution with similar maximum loading along the hub and shroud. This is to be expected, since Yang attempted to optimize the aerodynamics without any due consideration for the structural constraints; however, we are here attempting to strike a balance between aerodynamics and structural constraints.

Nevertheless, we can still attempt to utilize Yang’s guideline to move the stacking axis further downstream of the position of maximum loading. The results from this design calculation will be referred to as case e. The resulting reversed flow regions with the stacking position set at \( J=45 \) (slightly further downstream than that in case d) are shown in Fig. 6.12b and 6.13b. The reversed flow region on the main blade has been reduced, and the reversed flow region on the splitter blade is now confined to a small extent near the shroud. It appears that a minor alteration in the stacking condition does not influence the extent of the region of reversed flow significantly. The extent of the reversed flow region is far more affected by the \( r\bar{V}_\theta \) distribution.
6.3 Distribution of $\overline{W}_r$ and the Wrap Factor

The camber distribution of the main blade for case d is shown in Fig. 6.14 and the distribution of $\overline{W}_r$ for the same case is shown in Fig. 6.15. It is seen that the contours of $\overline{W}_r$ near the stacking axis are oriented in approximately the same direction as that of the stacking axis. However, toward the trailing edge, the contour lines of $\overline{W}_r$ become more radial (invariant with r-coordinates) while the contours of the blade camber have approximately the same orientation as that at the stacking axis. For blade with more radial blade filament along constant z section, it is desirable to have blade contour distribution similar to the $\overline{W}_r$ distribution.

In the iteration process, the blade shape is updated in each iteration through Eq.(5.1) which is rewritten here for convenience:

$$\overline{W}_r \frac{\partial f_i}{\partial r} + \overline{W}_z \frac{\partial f_i}{\partial z} = \frac{\overline{W}_\theta}{r} + (\overline{V})_{u,j} \cdot \nabla \alpha_j \tag{6.1}$$

where $\overline{W}_r = \sum_{k=1}^{2} [r \overline{V}_r] - \omega$. The $\overline{W}_r$ term is fixed once we specify a certain $r \overline{V}_\theta$ distribution; the $(\overline{V})_{u,j} \cdot \nabla \alpha_j$ term, however, is updated in each iteration. Nevertheless, if we assume that the $(\overline{V})_{u,j} \cdot \nabla \alpha_j$ term is much smaller than the $\overline{W}_r$ term, then this suggests that the key in obtaining more radial blade shape lies in the specification of $\overline{W}_r$.

It is more difficult to examine the $\overline{W}_r$ distribution on the whole blade region than to examine its distribution along the hub and shroud only. Moreover, the distribution of $r \overline{V}_\theta$ (and $\overline{W}_r$) along the hub and shroud is obtained through polynomials, and not any differential equation, so it is far easier to control.

Comparing the results in Fig. 6.14 and Fig. 6.15, we observe that downstream of the stacking axis, the values of $\overline{W}_r$ along the hub and along the shroud both decrease approximately from a value of 0.0 to -3.6. The blade wrap angle along the hub and shroud can be obtained approximately from the value of $\overline{W}_r$ along the hub and shroud.
Because the distance along the hub is greater than that along the shroud, the wrap angle at the trailing edge is thus more negative on the hub than on the shroud. To have radial blade shape, the wrap angles at the trailing edge on the hub and the shroud should approximately be the same. This suggests that we have to specify an \( r\mathbf{V}_\theta \) distribution such that the value of \( \frac{\mathbf{W}_s}{r} \) decreases more on the shroud than on the hub from the stacking axis to the trailing edge. This observation leads to the development of an indicator which will be referred to as the "Wrap Factor" in the following.

Neglecting the last term in Eq.(6.1), the blade boundary condition reduces to:

\[
\frac{\partial f_i}{\partial s} = \frac{1}{V_s} \frac{\mathbf{W}_\theta}{r}
\]  

(6.2)

where \( s \) denotes the meridional distance along the blades (Ref.8, Eq.(5.3)). This suggests that if we integrate the value of \( \frac{\mathbf{W}_s}{r} \) with respect to the distance along the hub and the shroud from the stacking position to the trailing edge and take the difference between them, we will get some guidance on how to make the blade filament at constant \( z \) more radial. This integrated value will be referred to as the Wrap Factor which is given below:

\[
\text{Wrap Factor} = \int_{\text{stacking axis}}^{\text{trailing edge}} \left( \frac{\mathbf{W}_\theta}{r} \right)_{\text{hub}} ds - \int_{\text{stacking axis}}^{\text{trailing edge}} \left( \frac{\mathbf{W}_\theta}{r} \right)_{\text{shroud}} ds
\]

(6.3)

The Wrap Factor has little to do with the real flow field, nor does it take into account the mean meridional velocity \( V_s \) in Eq.(6.2), which is not known a priori before a converged solution of the blade shape is obtained. On the contrary, the values of \( \frac{\mathbf{W}_s}{r} \) and the distances along the hub and the shroud are precisely known once we specify the geometry and the \( r\mathbf{V}_\theta \) distributions along the hub and shroud. As will be shown below, it is possible to use the Wrap Factor as a measure of the degree of curvature in the blade filament; this can be done a priori without having to implement the design calculation.

If the values of the mean meridional velocity on the hub and the shroud are the same at the same axial location and if the last term in Eq.(6.1) is indeed small, then we
can expect to have the Wrap Factor close to zero for highly radial blade cross sections at constant z. If this is not the case, then as shown below, the value of the Wrap Factor is optimum at a value different from zero.

Two additional $r\tilde{\nabla}_s$ distributions are considered for evaluating the usefulness of the Wrap Factor; the results from one of these are shown in Fig. 6.16a and 6.17a (this will be referred to as case f) while those from the other are shown in Fig. 6.16b and 6.17b (case g). For case f, the maximum loading along the hub is taken to be the same as that for case d (-1.75); however, the maximum loading along the shroud is made more negative than that of case d. It is taken to be -3.5. This increase in the value of $(\frac{\partial r\tilde{\nabla}_s}{\partial s})_{\text{max}}$ along the shroud is made to obtain a larger Wrap Factor. However, we cannot increase the maximum loading along the shroud further nor can we decrease the maximum loading along the hub to obtain an even larger Wrap Factor as this will result in negative velocity on the blade which causes the inverse design procedure to fail to converge to a final blade shape. For case g, the technique used is to move the position of point A (the point of maximum loading) so that a larger value of Wrap Factor is obtained.

The values of the Wrap Factor for case d, f, and g are shown below in Table 6.1 with the corresponding wrap angle differences (or lean angles) at the trailing edge, which is denoted by $\Delta f_{T.E.}$, and at $z = 0.05z_{T.E.}$, which is denoted by $\Delta f_{0.05}$. Yang’s results as described in Chapter 1 are also included.

<table>
<thead>
<tr>
<th>Case</th>
<th>Wrap Factor</th>
<th>$\Delta f_{T.E.}(\text{deg})$</th>
<th>$\Delta f_{0.05}(\text{deg})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yang</td>
<td>-0.75</td>
<td>56.3</td>
<td>6.8</td>
</tr>
<tr>
<td>d</td>
<td>-0.03</td>
<td>25.8</td>
<td>11.0</td>
</tr>
<tr>
<td>f</td>
<td>0.11</td>
<td>16.7</td>
<td>10.0</td>
</tr>
<tr>
<td>g</td>
<td>0.35</td>
<td>4.7</td>
<td>10.7</td>
</tr>
</tbody>
</table>
The resulting main blade shapes are shown in Fig. 6.18a for case f and in Fig. 6.18b for case g. From the results in Table 6.1, there appears to be a correlation between the Wrap Factor and the lean angle at the trailing edge. Thus the Wrap Factor may be used as an indicator of the curvature of the blade filaments.

The distribution of $\frac{W_x}{r}$ along the hub and shroud for case d, f, and g are shown in Fig. 6.19a, 6.19b and 6.19c. The values are not plotted against the distances along the hub and the shroud since they are different, but rather they are plotted against the node number in the streamwise direction so that they coincide at the leading edge and the trailing edge.

There are several observations that can be made. In the first place, we see from the above table and from Fig. 1.6, 6.11, 6.18a, and 6.18b that as the Wrap Factor is increased, the blade cross sections at constant $z$ get more and more radial, and it is optimal at Wrap Factor of approximately 0.35. This implies that the values of $V_s$ are not the same along the hub and the shroud (see Eq.(6.2) above).

Secondly, the wrap angle difference near $z = 0.05z_{T,E}$ does not get smaller with an increase in the Wrap Factor. This is to be expected since the Wrap Factor pertains to a region downstream of the stacking axis. One could also construct a second wrap factor; however, as will be shown in the following section, there is an alternative way that provides a simpler means of controlling the wrap angle near the leading edge.

Thirdly, the resulting splitter blade shapes at constant $z$ section are shown in Fig. 6.20a, 6.20b, and 6.20c for cases d, f, and g, respectively. These figures show that the splitter blades themselves do not get more radial with an increase in the Wrap Factor. The main blade and the splitter blade should have approximately the same camber distribution if the term $(\tilde{V})_{H,j} \cdot \nabla \alpha_j$ in Eq.(6.1) is small; however the computed results imply that this is not the case. The influence of $(\tilde{V})_{H,j} \cdot \nabla \alpha_j$ term can also be seen in the relative angular distance of the splitter blade $\sigma$, of which the maximum and minimum values are tabulated in Table II as a fraction of the local pitch (the angular
distance between two successive main blades, $2\pi/B$) for all the cases considered in this study. Ideally, without this term, the main and splitter blades would have exactly the same shapes, and the splitter blades will be exactly halfway between the main blades ($\sigma = 50\%$). Because of the influence of this term, the splitter blade shapes deviate from the middle of the passage between blade to blade and can reach a value of $\sigma$ as high as 78.9\% (close to the pressure side of the main blade) and as low as 30.3\% (close to the suction side of the main blade) as in case g.

The corresponding reversed flow regions for case f and g are shown in Fig. 6.21a and 6.21b, respectively, for the main blades and in Fig. 6.22a and 6.22b for the splitter blades. There are regions of reversed flow on the pressure surface because we have used a class of $rV_\phi$ distribution with a fairly large spanwise gradients.

We may now deduce that one has to specify $rV_\phi$ on the hub and the shroud that results in an increase of the Wrap Factor in order to make the blade filament at constant $z$ section of the main blades more radial. A way is to increase the value of $(\frac{\partial rV_\phi}{\partial z})_{max}$ on the shroud and reduce it on the hub. The value of $(\frac{\partial rV_\phi}{\partial z})_{max}$ on the shroud for case g is -3.6 and it is much higher than the value of $(\frac{\partial rV_\phi}{\partial z})_{max}$ in Yang's original design calculations, a case of which has a maximum value of -2.5. However, since the plotted $rV_\phi$ distributions are all normalized by the value of $rV_\phi$ at the leading edge of the main blade, the equivalent value of $(\frac{\partial rV_\phi}{\partial z})_{max}$ for case g is $-2.2 = 0.62 \times (-3.6)$; this is smaller than the value used by Yang. Hence, the advantage of using splitter blades is that it gives more room for manipulating the $rV_\phi$ distribution so as to alter the Wrap Factor without specifying too high a loading $(\frac{\partial rV_\phi}{\partial z})$ on either the main blade or the splitter blade; this would be difficult to do if we have main blades only.

6.4 Non-zero Incidence Angle at the Leading Edge

In all of Yang's design calculations, the values of $\frac{\partial rV_\phi}{\partial z}$ at the leading edge are always
set to zero. This corresponds to the condition of zero incidence angle at the leading edge. In this section, a swirl distribution with non-vanishing $\frac{\partial r\bar{V}_\theta}{\partial s}$ at the leading edge (i.e., non-zero incident angle) is examined, and this case will be referred to as case h.

The $r\bar{V}_\theta$ distributions along the hub and the shroud are shown in Fig. 6.23 and 6.24. The maximum loading along the shroud for the main blade is the same as that of case d (-3), but on the splitter blade it is reduced to -2.5 because the distance along the shroud for the splitter is shorter than that for the main blade. Along the hub of the main blade, a region of finite loading is used to avoid the occurrence of small mean velocity.

The value of $\frac{\partial r\bar{V}_\theta}{\partial s}$ at the leading edge is taken to be -1.25 (normalized by the value of $r\bar{V}_\theta$ at the leading edge of the main blade). This number is chosen so that the resulting $\bar{W}_r$ along the hub and shroud near the leading edge is close to zero (see Fig. 6.25). As expected, this vanishing value of $\bar{W}_r$ near the leading edge results in highly radial blade shape at location $z = 0.05z_{T.E.}$, a location near the leading edge (see Fig. 6.26). This is desirable from structural consideration.

The Wrap Factor is calculated to be 0.27 for this case. It results in lean angle of 6.0° at the trailing edge. This again shows the correlation between the Wrap Factor and the lean angle and confirms to the trend shown in Table 6.1 above.

6.5 Non-vanishing $\frac{\partial^2 r\bar{V}_\theta}{\partial s^2}$ at the Leading Edge and the Trailing Edge

The condition of $\frac{\partial^2 r\bar{V}_\theta}{\partial s^2} = 0$ at the leading edge and trailing edge has been imposed so that $\nabla^2 r\bar{V}_\theta$ is $C^2$ continuous at those locations. However, it is not required by any physical or mathematical constraint. Hence, in this section this condition is relaxed and the degrees of polynomials at Section 1 and Section 3 are reduced by one (see Sec 4.3). This will be referred to as case i.

The $r\bar{V}_\theta$ distributions along the hub and shroud are shown in Fig. 6.27 and 6.28,
respectively. The distributions are similar to those of case d. The two have exactly
the same locations of point A and B, but case i has a smaller value of \((\partial r_V/\partial s)_{max}\). The
value of \((\partial r_V/\partial s)_{max}\) has been reduced to avoid reversed loading (i.e., an overshoot in the
curve of \(\partial r_V/\partial s\)) near the trailing edge. As a result the Wrap Factor here is larger than
that of case d (see Table 6.1 above) and the blade shape tends to be more radial, as
shown in Fig 6.29. The Wrap Factor is computed to be 0.08 and again, we see a strong
correlation between the Wrap Factor and the difference in wrap angle at the trailing
edge. The value of \(\Delta f_{T,E}\) is 17.1°, which again is consistent with the trend shown in
Table 6.1.

The reversed flow regions, shown in Fig. 6.30 and 6.31, are reduced in comparison
to those of case d (see Fig. 6.12a and 6.13a). This can be attributed to a smaller value
of \((\partial r_V/\partial s)_{max}\), which has been made possible because the loading \((\partial r_V/\partial s)\) is distributed
more evenly from the leading edge to the trailing edge. With the condition \(\partial^2 r_V/\partial s^2 = 0\)
at the leading edge and the trailing edge is specified, the loading at the blade near
the leading and trailing edges is relatively small. Hence for the particular example
examined here, the removal of the constraint that \(\partial^2 r_V/\partial s^2 = 0\) at the leading and trailing
edges has worked in favor of reducing the reversed flow region.

6.6 Blockage Effects

So far the blades have been assumed to be infinitely thin in the design calculations.
However, a blade thickness distribution can be specified and accounted for partially
through the use of the blockage distribution (see Eq.(2.35)).

The normal thickness distributions \(t_n\) on the main and splitter blades are shown
in Fig. 6.32 and 6.33. The tangential thickness distributions \(t_t\) that result from the
design calculation are shown in Fig. 6.34 and 6.35.

The \(rV_{\theta}\) distribution that is used in this case, which is referred to as case j, is...
exactly the same as that of case d, and hence the two have exactly the same Wrap Factor. The effect of blockage is an increase in the mean level of velocity because of the reduction in the flow passage area. Since $\bar{W}_\theta$ is the same with or without blockage, we would thus expect the blades to have smaller wrap angles. However, upon examining the resulting blade shape in Fig. 6.36 and upon comparing it to Fig. 6.11, we observe only a minor reduction in the lean angle at the trailing edge. We can only conclude that the $(\bar{V})_{W_\theta}$ factor in Eq.(6.1) has changed in such a way that the blockage does not improve the blade shapes significantly.

The reversed flow regions, however, are significantly reduced, as can be seen in Fig. 6.37 and 6.38. On the pressure side of the splitter blade it is eliminated entirely. We conclude that the velocity on the blade surface has increased to eliminate the low momentum fluid region which causes the flow to reverse its direction.

6.7 Slip Factor

The work done through a turbomachinery $\dot{W}$ is given by the Euler Turbine Equation which can be written as

$$\dot{W} = \dot{m}\omega(r\bar{V}_o - r\bar{V}_i)$$

(6.4)

where $\dot{m}$ denotes the mass flow, $\omega$ denotes the wheel rotating speed, and subscripts 2 and 1 pertain to the outlet and inlet, respectively. We see that the same amount of work done with the same mass flow can be achieved by increasing the wheel rotating speed and reducing the incoming swirl or vice versa.

Slip factor is defined as the ratio of the tangential velocity over the blade rotating speed. So far, the slip factor is taken to be unity. Here an attempt will be made to assess the consequence of slip factor assuming a value less than unity. This case will be referred to as case k. A slip factor with value less than unity can be obtained by reducing the incoming swirl (tangential velocity) and increasing the rotating speed.
while maintaining the work output to be constant. The decrease in the loading is expected to give a more radial blade filaments at constant $z$ section.

With a slip factor of 0.858, the new blade rotating speed is 69,100 rpm and the new net non-dimensional $r\overline{V}_\theta$ at the leading edge is 0.99 times that with a slip factor of 1. The velocity triangle at the leading edge is shown in Fig. 6.39. The $r\overline{V}_\theta$ distribution used is similar to that of case d. The resulting $\overline{W}_r$ distribution along the hub and shroud is shown is Fig. 6.39. Comparing Fig. 6.19a and Fig. 6.40, we see that the effect of slip factor less than unity is a shift of the overall curve of $\overline{W}_r$ along the hub and shroud in the negative direction and this results in higher wrap angle near the trailing edge as shown in Fig. 6.41. However, since it is the lean angle that is important for structural consideration, a design with slip factor less than unity does not introduce any additional structural disadvantage.

The corresponding Wrap Factor is calculated to be -0.13. As the Wrap Factor is less than that of case d, the wrap angle difference at the trailing edge $\Delta f_{T.E.}$ is also larger. But as the value of $\overline{W}_r$ on the shroud near the leading edge is more negative, the blade is more radial near the leading edge.

Since the decrease in non-dimensional $r\overline{V}_\theta$ at the leading edge is only 1%, there is only a minor reduction of reversed flow region on the main blade as well as on the splitter blade.

6.8 Aerodynamics Consideration

In all the previous sections, the effects of various parameters and boundary conditions on blade shapes and the extent of the reversed flow regions have been examined. We will next examine the effects on the two aerodynamics parameter: the maximum adverse reduced static pressure gradient and the wake number.
6.8.1 Effects of $r\nabla \phi$ Distribution

The effects of various parameters (location of $(\frac{\partial r\nabla \phi}{\partial s})_{max}$, regions of vanishing $\frac{\partial r\nabla \phi}{\partial n}$, the maximum loading $(\frac{\partial r\nabla \phi}{\partial x})_{max}$, stacking position, lean in stacking line, slip factor, blockage effects, number of blades, exit swirl, and modified hub and shroud profiles) on the two aerodynamics parameters have been examined by Yang in his original work (Ref.1). Here we will examine cases that have not been included, i.e., a case with non-vanishing $\frac{\partial r\nabla \phi}{\partial s}$ at the leading edge and a case with non-vanishing $\frac{\partial^2 r\nabla \phi}{\partial x^2}$ at the leading edge and the trailing edge, which will be represented by case h and case i, respectively, and we will use case d for comparison because it has similar $r\nabla \phi$ distributions along the hub and the shroud.

The effects on the maximum adverse static pressure gradients along the hub, mid-span, and shroud, of the main blade and on the wake number on the hub surface, suction side of main blade, and shroud surface are summarized in the table below.

<table>
<thead>
<tr>
<th>Table 6.2 Effect of Different Boundary Conditions on the Maximum Adverse Pressure Gradient and Wake Number on the Main Blade</th>
</tr>
</thead>
<tbody>
<tr>
<td>case</td>
</tr>
<tr>
<td>d</td>
</tr>
<tr>
<td>h</td>
</tr>
<tr>
<td>i</td>
</tr>
</tbody>
</table>

The effects on the maximum adverse static pressure gradients along the hub, midspan and shroud of the splitter blade and on the wake number on the suction surface of the splitter blade are summarized below.
Table 6.3 Effect of Different Boundary Conditions on the Maximum Adverse Pressure Gradient and Wake Number on the Splitter Blade

<table>
<thead>
<tr>
<th>case</th>
<th>$(\frac{\partial P_{rad}}{\partial z})_h$</th>
<th>$(\frac{\partial P_{rad}}{\partial z})_m$</th>
<th>$(\frac{\partial P_{rad}}{\partial z})_s$</th>
<th>(Wake no)$_{s,s.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>1.100</td>
<td>1.260</td>
<td>1.142</td>
<td>2.510</td>
</tr>
<tr>
<td>h</td>
<td>1.001</td>
<td>1.057</td>
<td>0.946</td>
<td>3.167</td>
</tr>
<tr>
<td>i</td>
<td>1.051</td>
<td>1.120</td>
<td>0.797</td>
<td>2.546</td>
</tr>
</tbody>
</table>

In general, the condition of non-vanishing $\frac{\partial P_{rad}}{\partial z}$ at the leading edge increases the maximum adverse reduced static pressure gradient on the main blade. The increase is especially large along the shroud. The wake numbers also increase. On the splitter blade, however, the maximum adverse pressure gradients are reduced. The values of maximum adverse pressure gradients on the splitter blade are larger than those on the main blade for case d; thus the condition of non-zero incidence at the leading edge has reduced the overall maximum adverse pressure gradient. In view of this, it is only the increase in the wake number that has to be considered against the benefit of having more radial blade near the leading edge.

On the contrary, the removal of the condition that $\frac{\partial P_{rad}}{\partial z} = 0$ at the leading edge and trailing edge has reduced all the maximum adverse pressure gradients and all the wake numbers, except on the suction surfaces, where the wake numbers increase only slightly. It is concluded that by distributing the loading more evenly on the overall blade region, a better design measured in terms of aerodynamics as well as blade that has more radial blade filaments appears possible.

6.8.2 Effects of the Wrap Factor

The main objective of the present study has been to design blade shapes that have more radial filaments at constant z section; this has been achieved by increasing
the Wrap Factor. In this section we will examine the influence of Wrap Factor on aerodynamics in terms of wake number and reduced static pressure distribution.

For the aerodynamics considerations, cases b, d, f, and g are taken as representatives because they have more or less decreasing order of blade curvature (compare Fig. 6.4b, 6.11, 6.18a, and 6.18b) and correspondingly, increasing Wrap Factor. The maximum advered reduced static pressure gradients along the hub, mid-span, and shroud of the main blade and the wake numbers are summarized in Table 6.4 below together with the Wrap Factor (W.F.), where Yang's case has also been included:

Table 6.4 Effect of Wrap Factor on the Maximum Adverse Pressure Gradient and Wake Number on the Main Blade

<table>
<thead>
<tr>
<th>case</th>
<th>W.F.</th>
<th>$(\frac{\partial \text{Pres}}{\partial s})_h$</th>
<th>$(\frac{\partial \text{Pres}}{\partial s})_m$</th>
<th>$(\frac{\partial \text{Pres}}{\partial s})_s$</th>
<th>$(\text{Wake no})_h$</th>
<th>$(\text{Wake no})_m$</th>
<th>$(\text{Wake no})_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yang</td>
<td>-0.75</td>
<td>1.242</td>
<td>1.298</td>
<td>1.309</td>
<td>8.395</td>
<td>2.942</td>
<td>6.597</td>
</tr>
<tr>
<td>b</td>
<td>-0.90</td>
<td>0.575</td>
<td>0.455</td>
<td>0.410</td>
<td>3.970</td>
<td>3.544</td>
<td>2.455</td>
</tr>
<tr>
<td>d</td>
<td>-0.03</td>
<td>1.021</td>
<td>0.916</td>
<td>0.550</td>
<td>10.976</td>
<td>4.510</td>
<td>2.953</td>
</tr>
<tr>
<td>f</td>
<td>0.11</td>
<td>1.276</td>
<td>1.482</td>
<td>0.740</td>
<td>150.4</td>
<td>4.461</td>
<td>56.01</td>
</tr>
<tr>
<td>g</td>
<td>0.35</td>
<td>1.209</td>
<td>2.717</td>
<td>1.654</td>
<td>20.74</td>
<td>4.573</td>
<td>37.88</td>
</tr>
</tbody>
</table>

The maximum adverse pressure gradients along the hub, mid-span, and shroud and the wake number on the suction side of the splitter blade are summarized in Table 6.5 below:

Table 6.5 Effect of Wrap Factor on the Maximum Adverse Pressure Gradient and Wake Number on the Splitter Blade

<table>
<thead>
<tr>
<th>case</th>
<th>$(\frac{\partial \text{Pres}}{\partial s})_h$</th>
<th>$(\frac{\partial \text{Pres}}{\partial s})_m$</th>
<th>$(\frac{\partial \text{Pres}}{\partial s})_s$</th>
<th>$(\text{Wake no})_{s.s.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>0.689</td>
<td>0.657</td>
<td>0.438</td>
<td>2.206</td>
</tr>
<tr>
<td>d</td>
<td>1.100</td>
<td>1.260</td>
<td>1.142</td>
<td>2.510</td>
</tr>
<tr>
<td>f</td>
<td>1.140</td>
<td>1.763</td>
<td>2.389</td>
<td>2.299</td>
</tr>
<tr>
<td>g</td>
<td>1.035</td>
<td>1.045</td>
<td>1.299</td>
<td>2.861</td>
</tr>
</tbody>
</table>
The general overall effect of increasing the Wrap Factor has been an increase in the maximum adverse reduced static pressure gradient. The maximum gradient, however, is not a monotonically increasing function of the Wrap Factor. This is to be expected since the same value of Wrap Factor can be obtained through an infinitely many specifications of $r\overline{V}_\theta$ along the hub and shroud. This suggests that there are ways to increase the Wrap Factor by keeping the increase in adverse pressure gradient to a minimum.

The computed results indicate that an increase in wake number is associated with an increase in Wrap Factor. The results in Ref. 1 show that for an optimal value of wake number, the $r\overline{V}_\theta$ distribution along the hub should be similar to that along the shroud, i.e., the spanwise of gradient of $r\overline{V}_\theta$ should be minimized. However, as the distance along the hub is longer than that along the shroud, a different $r\overline{V}_\theta$ distribution along the shroud should be specified in order to increase the Wrap Factor. Thus it appears that some trade-offs between the aerodynamic and structural considerations may not be avoided.

6.9 Viscous Code Analysis

So far all the results presented have been made under the assumption that the flow through the turbine wheel is both homenthalpic and homentropic. In this section the flow through the resulting blade passage from the inverse design calculation will be analyzed using a viscous code to assess the aerodynamic goodness of the design. The main and splitter blade shape from case j will be taken as representative since it has a finite thickness distribution and hence the computed flow field is expected to give a good approximation to the real flow in a real turbine.

6.9.1 Description of the Code
The code used is "BTOB3DSP" program which is developed by W. N. Dawes from Whittle Laboratory. This code is similar to "BTOB3D" except that it has been modified to include splitters. This code uses structured grid and all flow quantities are cell-centered. It solves the Reynolds averaged Navier-Stokes equation with a mixing length turbulence model in a rotating cylindrical coordinate system. The mixing length turbulence model used is that due to Baldwin and Lomax (Ref. 23). The shear stress on the hub and casing is obtained from log law. The code also assumes that the flow is spatially periodic from blade passage to blade passage.

The code uses control volume approach in solving the discretized conservation laws. Fluxes through cell faces are found by linear interpolation between cell centers which results in second order formal spatial accuracy on smoothly varying meshes and ensured global conservation. The code uses an artificial viscosity which consists of an adaptive-combination of a second difference and fourth difference which is blended through shock waves to eliminate wiggles and give sharp shock capture. This adaptive artificial viscosity is used to control odd-even point solution decoupling and to suppress oscillation in regions with strong pressure gradients. The time marching scheme of the code uses Beam-Warming implicit algorithm (Ref. 24) with two step evaluation of the residue analogous to the Brailovskaya scheme (Ref. 25).

At the upstream boundary, the values of absolute stagnation pressure and temperature are specified along with the absolute swirl angle and the meridional pitch angle of the incoming flow. The tangential and meridional velocities are used as initial guesses only. At the downstream boundary, the static pressure at the hub is specified and its radial variation is obtained from simple radial equilibrium. The static pressure at the hub is adjusted until the right amount of mass flow is obtained.

6.9.2 Results from the Code

The main and splitter blade shapes for the viscous code calculation are shown in
The blade thickness is not shown in these 3-D perspectives. The code is run at the design point of the turbine, which is specified in Table I. The Reynolds number based on hub axial chord and exit meridional velocity is computed to be $8.8 \times 10^5$.

For the viscous calculation the number of grid points from hub to shroud is taken to be 17. The number of grid points from inlet to outlet is taken to be 107. The outlet section has been expanded to ensure that the influence of the blade on the flow field at the downstream boundary is minimum. The meridional mesh is shown in Fig. 6.43. The number of grid points from blade to blade is taken to be 25. However, as the numerical scheme uses 4 cells to represent the splitter blade thickness, there are effectively only 20 computational cells with half of them on each side of the splitter blade. The quasi stream-surface projection of the grid on the hub is shown in Fig. 6.44.

The reduced static pressure distributions from the inverse design calculation on the suction surface, pressure surface, hub surface, and shroud surface are shown in Fig. 6.45 to Fig. 6.48. This pressure field does not seem to have any strong adverse gradient. However, the numerical solution shows that there appears to be a breakdown in the flow field. The velocity fields along the quasi stream-surface after 400 iterations on the hub, mid-span, and shroud surfaces are shown in Fig. 6.49, 6.50 and 6.51. There is a flow separation from both the suction and pressure surfaces near the trailing edge. The separation occurs earlier near the hub surface than on the mid-span. Near the shroud surface it does not happen.

This separation in the flow field causes some difficulty in the code for later iterations. It produces a lot of swirl leaving the trailing edge of the blade which induces backward flow at the outlet. The velocity field near the outlet for quasi stream-surface near the hub and shroud after 4500 iterations are shown in Fig. 6.52 and 6.53. In Fig 6.53 we see that there is an incoming flow at the downstream boundary. Since there is no specification whatsoever on the flow at the downstream boundary except the static
pressure at the hub, we have an undeterministic system that results in the breakdown of the whole flow field.

In summary the results from the use of viscous calculation cannot be taken to be conclusive. Further work needs to be implemented.
Chapter 7
Conclusions and Suggestions for Future Work

7.1 Summary and Conclusions

The inverse design theory and the accompanying computational procedure have been extended to include the design of splitter blade. In particular, the present study has concentrated on blade design that attempts to make the blade filament along constant z section more radial. The extent of region of reversed flow on the pressure side has also been examined.

In addition, the present study also includes two situations that have not been considered by Yang. They are the condition of non-vanishing $\frac{\partial rV_\theta}{\partial z}$ at the leading edge (non-zero incidence) and the condition of non-vanishing $\frac{\partial^2 rV_\theta}{\partial z^2}$ at the leading and trailing edge.

The present study examines the influence of various parameters on blade design; these include the length of splitter blades, stacking position, $rV_\theta$ distribution, non-zero incidence angle, blockage, and slip factor. Based on the results from these parametric design calculations, we deduce the following conclusions:

1. The use of splitter blades gives more flexibility for tailoring the loading on the main blades and this has resulted in the reduction or elimination of inviscid reversed flow region on the pressure side of the blades. Its use has also resulted in blades with nearly radial blade filament at an axial location.

2. The extent of the reversed flow region is more influenced by the specified loading than by the stacking position or the slip factor.
A number referred to as the Wrap Factor can be constructed to give an indication of the curvature of the blade shape at constant z section. The use of this number can reduce the number of design calculations aimed at obtaining more radial blade shape.

In general, the main blade filament near the trailing edge can be made more radial by increasing the Wrap Factor while that near the leading edge can be made more radial by imposing non-zero incidence angle.

The blockage has more effect on the reduction of the inviscid reversed flow region than on the improvement on the blade filament curvature.

An increase in the Wrap Factor is generally accompanied by an increase in the Wake Number, but it does not always result in an increase in the maximum adverse reduced static pressure gradient.

7.2 Suggestions for Future Work

Based on the parametric study implemented here and in Ref. 1, some suggestions for future work should include:

1. Experimental investigations of flow in an inverse-designed radial inflow turbine wheel for assessing the results arrived at in the present work as well as those in Ref. 1.

2. Further developmental work to extend the design theory to include flow rotationalities upstream of the blade row as well as non-free-vortex design; some preliminary work has been done (Ref. 26 and 27).

3. A unified approach for optimizing the blade design in terms of the structure and aerodynamic performance of the turbine.
Reference


Table I Design Specifications for Radial Inflow Turbine

<table>
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<tr>
<th>Specification</th>
<th>Value</th>
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<td>Inlet Total Temperature, °K</td>
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<td>Inlet Total Pressure, N/m²</td>
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Table II Stacking Positions $J_{st}$ and Relative Angle $\sigma$

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Note:

$\sigma_{st}$

relative angular distance between main and splitter blades at the stacking axis

$\sigma_{lower}$

the minimum relative angular distance between the splitter blade and the suction side of the main blade

$1 - \sigma_{upper}$

the minimum relative angular distance between the splitter blade and the pressure side of the main blade
Table III Specifications of $r\overline{V}_\theta$ on the Main Blade

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<th>Case</th>
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<th>$(\frac{\partial^2 r\overline{V}<em>\theta}{\partial \theta^2})</em>{hb}$</th>
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<th>$B_{sh}$</th>
<th>$(\frac{\partial r\overline{V}<em>\theta}{\partial \theta})</em>{sh}$</th>
<th>$(\frac{\partial^2 r\overline{V}<em>\theta}{\partial \theta^2})</em>{sh}$</th>
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Table IV Specifications of \( r\dot{V} \) on the Splitter Blade

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<th>( A_{sh} )</th>
<th>( B_{sh} )</th>
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Fig. 1.1  The distribution of $rV_{\theta}$, $\frac{W_r}{r}$, and $\frac{\partial rV_{\theta}}{\partial s}$ along the hub for Yang's case.

Fig. 1.2  The distribution of $rV_{\theta}$, $\frac{W_r}{r}$, and $\frac{\partial rV_{\theta}}{\partial s}$ along the shroud for Yang's case.
Fig. 1.3  The distribution of $r\overline{V}_\phi$ for Yang's case

Fig. 1.4  The blade shape for Yang's case
Fig. 1.5  The $M_{rel.}$ on the pressure side for Yang's case

Fig. 1.6  Radial distribution of blade camber along constant z-section for Yang's case
Schematic plot of $rV_\theta$ distribution along the hub and shroud of the main and splitter blades.

Velocity triangle at leading edge
Fig. 6.1a  Grid used for calculation for case a (length 0.53)

Fig. 6.1b  Grid used for calculation for case b (length 0.61)
Fig. 6.1c  Grid used for calculation for case c (length 0.74)
Fig. 6.2a The distribution of $r\overline{V}_\theta$, $\overline{W}_r$, and $\frac{\partial r\overline{V}_\theta}{\partial \theta}$ along the hub for case a (length 0.53)

Fig. 6.3a The distribution of $r\overline{V}_\theta$, $\overline{W}_r$, and $\frac{\partial r\overline{V}_\theta}{\partial \theta}$ along the shroud for case a (length 0.53)
Fig. 6.2b  The distribution of $r \overline{V}_{\theta}$, $\overline{W}_{\theta}$, and $\frac{\partial r \overline{V}_s}{\partial \theta}$ along the hub for case b (length 0.61)

Fig. 6.3b  The distribution of $r \overline{V}_{\theta}$, $\overline{W}_{\theta}$, and $\frac{\partial r \overline{V}_s}{\partial \theta}$ along the shroud for case b (length 0.61)
Fig. 6.2c  The distribution of $rV_\theta$, \(\overline{W_r}\), and \(\frac{\partial rV_\theta}{\partial s}\) along the hub for case c (length 0.74)

Fig. 6.3c  The distribution of $rV_\theta$, \(\overline{W_r}\), and \(\frac{\partial rV_\theta}{\partial s}\) along the shroud for case c (length 0.74)
Fig. 6.4a  Radial distribution of main blade camber along constant z-section for case a (length 0.53)

Fig. 6.4b  Radial distribution of main blade camber along constant z-section for case b (length 0.61)
Fig. 6.4c  Radial distribution of main blade camber along constant z-section for case c (length 0.74)
Fig. 6.5a  The $M_{rel}$ on the pressure side of main blade for case a (length 0.53)

Fig. 6.6a  The $M_{rel}$ on the pressure side of splitter blade for case a (length 0.53)
Fig. 6.5b  The $M_{rel}$ on the pressure side of main blade for case b (length 0.61)

Fig. 6.6b  The $M_{rel}$ on the pressure side of splitter blade for case b (length 0.61)
Fig. 6.5c  The $M_{rel.}$ on the pressure side of main blade for case c (length 0.74)

Fig. 6.6c  The $M_{rel.}$ on the pressure side of splitter blade for case c (length 0.74)
Fig. 6.7  The distribution of $rV_\theta$, $\frac{W_s}{r}$, and $\frac{\partial rV_s}{\partial s}$ along the hub for case d (stacking at 1/3 chord)

Fig. 6.8  The distribution of $rV_\theta$, $\frac{W_s}{r}$, and $\frac{\partial rV_s}{\partial s}$ along the shroud for case d (stacking at 1/3 chord)
Fig. 6.9  The distribution of $r\overline{V}_\theta$ on main blade for case d (stacking at 1/3 chord)

Fig. 6.10  The distribution of $r\overline{V}_\theta$ on splitter blade for case d (stacking at 1/3 chord)
Fig. 6.11  Radial distribution of main blade camber along constant z-section for case d (stacking at 1/3 chord)
Fig. 6.12a  The $M_{rel}$ on the pressure side of main blade for case d (stacking at 1/3 chord)

Fig. 6.12b  The $M_{rel}$ on the pressure side of main blade for case e (stacking further downstream)
Fig. 6.13a  The $M_{rel.}$ on the pressure side of splitter blade for case d (stacking at 1/3 chord)

Fig. 6.13b  The $M_{rel.}$ on the pressure side of splitter blade for case e (stacking further downstream)
Fig. 6.14  The contour of main blade for case d (stacking at 1/3 chord)

Fig. 6.15  The distribution of $\frac{W}{r}$ for case d (stacking at 1/3 chord)
Fig. 6.16a The distribution of $r\bar{V}_\theta$, $\bar{W}_s$, and $\frac{\partial \bar{V}_s}{\partial \theta}$ along the hub for case f (large Wrap Factor)

Fig. 6.17a The distribution of $r\bar{V}_\theta$, $\bar{W}_s$, and $\frac{\partial \bar{V}_s}{\partial \theta}$ along the shroud for case f (large Wrap Factor)
Fig. 6.16b The distribution of $rV_\theta$, $\frac{W_\theta}{r}$, and $\frac{\partial rV_\theta}{\partial s}$ along the hub for case g (larger Wrap Factor)

Fig. 6.17b The distribution of $rV_\theta$, $\frac{W_\theta}{r}$, and $\frac{\partial rV_\theta}{\partial s}$ along the shroud for case g (larger Wrap Factor)
Fig. 6.18a Radial distribution of main blade camber along constant z-section for case f (large Wrap Factor)

Fig. 6.18b Radial distribution of main blade camber along constant z-section for case g (larger Wrap Factor)
Fig. 6.19a  The distribution of $\bar{W}_r$ along the hub and shroud for case d (small Wrap Factor)

Fig. 6.19b  The distribution of $\bar{W}_r$ along the hub and shroud for case f (large Wrap Factor)
Fig. 6.19c  The distribution of $\frac{W_f}{r}$ along the hub and shroud for case g (larger Wrap Factor)

Fig. 6.20a  Radial distribution of splitter blade camber along constant z-section for case d (small Wrap Factor)
**Fig. 6.20b** Radial distribution of splitter blade camber along constant z-section for case f (large Wrap Factor)

**Fig. 6.20c** Radial distribution of splitter blade camber along constant z-section for case g (larger Wrap Factor)
Fig. 6.21a  The $M_{rel.}$ on the pressure side of main blade for case f (large Wrap Factor)

Fig. 6.21b  The $M_{rel.}$ on the pressure side of main blade for case g (larger Wrap Factor)
Fig. 6.22a The $M_{rel.}$ on the pressure side of splitter blade for case f (large Wrap Factor)

Fig. 6.22b The $M_{rel.}$ on the pressure side of splitter blade for case g (larger Wrap Factor)
Fig. 6.23  The distribution of $r\frac{\partial V_s}{\partial r}$, $\frac{W_z}{r}$, and $\frac{\partial rV_s}{\partial s}$ along the hub for case h (nonzero $\frac{\partial V_s}{\partial s}$ at L.E.)

Fig. 6.24  The distribution of $r\frac{V_s}{s}$, $\frac{W_z}{r}$, and $\frac{\partial rV_s}{\partial s}$ along the shroud for case h (nonzero $\frac{\partial V_s}{\partial s}$ at L.E.)
Fig. 6.25  The distribution of $\frac{W}{r}$ along the hub and shroud for case h (nonzero $\frac{\partial r V_z}{\partial s}$ at L.E.)

Fig. 6.26  Radial distribution of main blade camber along constant $z$-section for case h (nonzero $\frac{\partial r V_z}{\partial s}$ at L.E.)
Fig. 6.27 The distribution of $rV_{\theta}$, $\frac{\partial V_{r}}{\partial s}$, and $\frac{\partial^{2} V_{r}}{\partial s^{2}}$ along the hub for case i (nonzero $\frac{\partial^{2} V_{r}}{\partial s^{2}}$ at L.E. and T.E.)

Fig. 6.28 The distribution of $rV_{\theta}$, $\frac{\partial V_{r}}{\partial s}$, and $\frac{\partial^{2} V_{r}}{\partial s^{2}}$ along the shroud for case i (nonzero $\frac{\partial^{2} V_{r}}{\partial s^{2}}$ at L.E. and T.E.)
Fig. 6.29  Radial distribution of main blade camber along constant z-section for case i (nonzero $\partial^2 \gamma / \partial s^2$ at L.E. and T.E.)
Fig. 6.30  The $M_{rel.}$ on the pressure side of main blade for case i (nonzero $\frac{\partial^2 V}{\partial \zeta^2}$ at L.E. and T.E.)

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Fig. 6.53  Quasi stream-surface velocity field near the shroud after 4500 viscous iterations
Appendix A

Nondimensional Form of Variables

I. Normalized Form of Variables in the Design Code.

In the design code, the length is normalized by the tip radius. The velocity is normalized by the radial velocity at the leading edge. The thermodynamic quantities are normalized by the rotary total quantities at the leading edges. They are summarized below:

\[ L_{\text{ref}} = r_{\text{tip}} \]
\[ V_{\text{ref}} = W_r \]
\[ T_{\text{ref}} = T_T \]
\[ \rho_{\text{ref}} = \rho_T \]
\[ P_{\text{ref}} = P_T^* \]

II. Normalized Form of Variables in the Output

In the presentation of the result in this thesis, the length is normalized by the turbine diameter, and all other variables are normalized by the upstream total pressure, temperature, and density. They are summarized below:

\[ L_{\text{ref}} = \text{Diameter} \]
\[ V_{\text{ref}} = \sqrt{\frac{P_T}{\rho_T}} \]
\[ T_{\text{ref}} = T_T \]
\[ \rho_{\text{ref}} = \rho_T \]
\[ P_{\text{ref}} = P_T \]
For example, the reduced static pressure is nondimensionalized by $P_{ref}$, i.e.,

$$
\frac{P_{red}}{P_{ref}} = \frac{P - \frac{1}{2} \rho \omega^2 r^2}{P_T} = \frac{P}{P_T} - \frac{1}{2} \rho_T \omega^2 r^2 = \overline{P} - \frac{1}{2} \overline{p} (\omega r)^2
$$
Lanczos Filter is used in the inverse design program to suppress the amplitudes of high harmonics which cause some overshoot near point of discontinuity. This filter is particularly needed in the program because the periodic velocity field behaves like a sawtooth function (see Eq.(2.15)). The usual application of Lanczos Filter utilizes the following factor:

\[
\sigma_k = \frac{\sin\left(k\pi/(m+1)\right)}{k\pi/(m+1)}
\] (B.1)

which is applied to every harmonic of mode \( k \) and \( m \) is the total number of harmonics (Ref. 21).

However, suppose we have a turbine of \( N \) main blades with \( m \) harmonic modes used in circumferential direction. If we have full-length splitter blades, so that basically we have twice the number of blades, we want to use twice the number of harmonics so that the design result is exactly the same as a turbine of \( 2N \) blades with \( 2m \) harmonics. Then the filter above will not give the same result unless it is modified as

\[
\sigma_k = \frac{\sin\left(k\pi/(m+j)\right)}{k\pi/(m+j)}
\] (B.2)

where \( j = 1 \) for a design without splitter blades and \( j = 2 \) for a design with splitter blades. This filter is what is used in the inverse design code.