DESIGN OF A CIRCUIT TO APPROXIMATE A PRESCRIBED AMPLITUDE AND PHASE

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Abstract
In discussing the system function of a network one may make the transformation
\[ w = \tan \frac{\phi}{2}, \]
so that the frequency range \( 0 \leq w < \infty \) becomes \( 0 \leq \phi < \pi \). If the amplitude
\[ A(\phi) = \sum a_n \cos n\phi, \]
then Wiener gives the corresponding phase \( -B(\phi) = -\sum a_n \sin n\phi, \)
equality of coefficients representing the condition for physical realizability. Here we
approximate arbitrary functions \( a(\phi) \) and \( b(\phi) \) simultaneously by functions \( A(\phi) \) and \( B(\phi) \)
satisfying the above condition. Minimizing the sum of the mean square errors, one finds that
the \( a_n \) are the arithmetic means of the Fourier coefficients, for approximation over an
infinite frequency range; they are solutions of a linear system for a finite range; and
the solution involves a linear system and an algebraic equation for the finite range with
\( A(\phi), B(\phi) \) subject to a condition of boundedness. Linear equations are also obtained for
minimizing the mean error, subject to exact fit at one point. The conditions are proved
sufficient to insure an absolute minimum. It is shown that the error cannot generally be
made arbitrarily small over the infinite range, but always can over a finite range; and
that approximation by partial sums of an infinite series is generally impossible, as is
approximation by polynomials which remain uniformly bounded. We include graphs of specific
functions with their approximating polynomials.
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1. Introduction

The system function of a network is a complex function, and hence has a real part, an imaginary part, an amplitude, and a phase. If any one of these four "parts" of the function (as we may call them) is specified over the whole frequency range, then the other three parts are automatically determined. Or again if any two parts are specified over a finite frequency range, no matter how small, then all four parts are determined over the whole frequency range. These and similar results follow from the fact that the system function is an analytic function of the complex frequency.

Suppose, then, that we are given two arbitrary curves, to be matched respectively by two parts of the system function. If we succeed in matching one of these prescribed curves over the whole frequency range, we cannot hope, in general, to come anywhere near the other; for no freedom remains to make adjustments of any kind. Alternatively, if we succeed in matching both curves over some small frequency interval, we cannot expect to match them, even approximately, outside of that range; for again no freedom of adjustment remains.

In practical work, nevertheless, one often requires a network for which the real and imaginary parts, or the amplitude and phase, do conform to prescribed functions. Examples of problems leading to such requirements are the broadbanding problem, and certain problems connected with feedback circuits in servos or amplifiers. It is natural, therefore, to inquire whether two parts of the system function can be prescribed at least approximately over a frequency range. The chief problems are to find some systematic method of minimizing the error, and to estimate the minimum error so obtained. Such is the purpose of the present note. The subject and its applications were suggested by E. O. Guillemin, who also supplied all results taken from circuit theory.

By the transformation \( w = \tan \theta / 2 \) the frequency domain \( w \) is transformed to a corresponding \( \theta \) domain, the range \( 0 \leq w \leq \infty \) going into \( 0 \leq \theta < \pi \). In the \( \theta \) domain we may confine our attention to this range of \( \theta \), and hence all functions may be assumed even or odd at pleasure, and may be supposed without loss of generality to have period \( 2\pi \). The amplitude \( A(\theta) \) may then be written as a Fourier cosine series,

\[ A(\theta) = a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \ldots \]  

(1)

As noted above, the phase \( -B(\theta) \) is completely determined by such a specification of amplitude. With the present choice of variable, which is used by N. Wiener, the specification of phase has a particularly simple form, namely,

\[ B(\theta) = a_1 \sin \theta + a_2 \sin 2\theta + \ldots \]  

(2)

with the coefficients \( a_k \) the same as those in Eq. (1). This equality of coefficients
represents the Wiener condition for physical realizability of the two functions $A(\phi)$ and $B(\phi)$. The phase is taken as $-B(\phi)$ and below as $-b(\phi)$ rather than as $B(\phi)$ or $b(\phi)$, incidently, to give the theory a more symmetric form.

The functions $A(\phi)$ and $B(\phi)$ represent the actual output of the network, while $a(\phi)$ and $b(\phi)$ represent the desired output. Our task, then, is to investigate the approximation of two arbitrary functions $a(\phi)$, $b(\phi)$ by Fourier cosine and sine series, respectively, the same coefficients being used in both cases. The results to be described are obtained with this formulation of the problem as starting point, and the approximation is always specified in the mean-square sense; that is, the coefficients are so determined that the integral

$$I = \int \left| a(\phi) + ib(\phi) - \sum_{k} a_{k} e^{ik\phi} \right|^2 d\phi$$

is minimized. Our attention will be confined to the coefficients $a_{k}$; once these are known, design of the circuit can be carried out by methods standard in circuit theory.

2. Procedure

2.1 Approximation Over Whole Frequency Range Without Constraint. To approximate $a(\phi)$ and $b(\phi)$ over the whole frequency range, which corresponds to limits of integration $0-n$ in Eq. (3), one should determine the coefficients $a_{k}$ by the equations

$$a_{k} = \frac{1}{n} \int_{0}^{n} [a(\phi) \cos k\phi + b(\phi) \sin k\phi] d\phi.$$  

(4)

The Fourier cosine coefficients of $a(\phi)$ would be the optimum values for $a_{k}$ if we were interested in approximating $a(\phi)$ alone, while the Fourier sine coefficients of $b(\phi)$ would be optimum for $b(\phi)$ alone. This is well known from the theory of Fourier series. The above result, then, says that the optimum coefficients for simultaneous approximation are the arithmetic means of the optimum coefficients for individual approximation.

2.2 Approximation Over Finite Frequency Range Without Constraint. It may be shown (Sec. 4.1) that the error in the above calculation will not usually tend to zero as $n$ tends to infinity; in other words an arbitrarily good approximation over the whole frequency range cannot be obtained, no matter how many terms are used in the approximating series.

If on the other hand we attempt to obtain a good approximation over only part of the frequency range, with no restriction on the behaviour of the approximating functions outside of that range, then the error can always be made arbitrarily small. Because of this property, which is proved in Sec. 5.1, the question of approximation over a finite interval is worth considering in detail.

The optimum coefficients are determined as follows, for a range $0-s$ in the $\phi$ domain.
with \( s \leq \pi \). First compute a set of numbers \( P_{jk} \) given by

\[
P_{jk} = \frac{\sin((j-k)s)}{(j-k)(k-j)}; \quad P_{jj} = s.
\] (5)

These depend only on \( s \), the range of approximation, and not on \( n \) or the functions \( a(\phi), b(\phi) \) being approximated. For routine calculations, therefore, they would be computed once for all, and would be available in the form of a table.

Next compute a set of constants \( A_k \) given by the formula

\[
A_k = \frac{1}{s} \left[ a(\phi) \cos k\phi + b(\phi) \sin k\phi \right] d\phi
\] (6)

which depends on the functions being approximated. The optimum coefficients are obtained by solving the following set of linear equations:

\[
\sum_{j=0}^{n} P_{jk} a_j = A_k \quad (j=0, 1, 2, \ldots, n).
\] (7)

Graphical examples of the results obtained by this procedure are given in Figs. 1-3.
Figure 2. Mean-square approximation, n=4, s=n/2.

Figure 3. Mean-square approximation, n=7, s=n/2.
2.3 Approximation Over Finite Frequency Range With Constraint. Although the error tends to zero over the range $0 \leq \phi \leq s$, so that the approximating functions are certainly bounded in this range, it may be shown that they will not in general be bounded for $s \leq \phi \leq \pi$ (see Sec. 5.4). An example of the behavior to be expected is given in Fig. 4.

![Figure 4](image)

Figure 4. Mean-square approximation, $n=7$, $s=\pi/2$, showing behavior in interval $\pi/2-\pi$.

Physically, the meaning is that the loss in gain increases without limit as the error decreases to zero. If $M'$ is the maximum of the function $\sum_{k=0}^{n} a_k \cos k\phi$ in $0-\pi$, while $M''$ is the maximum of $a(\phi)$ in $0-s$, then $M''/M'$ represents the sacrifice in gain necessary to achieve the given approximation. We are thus led to consider the above problems, subject to a condition of boundedness on the approximating polynomials.

It turns out that this condition is most conveniently specified, not by the inequality $\sum_{k=0}^{n} a_k \cos k\phi \leq M'$, but by

$$\sum_{k=0}^{n} a_k^2 \leq M'. \quad (8)$$

Since the value of the prescribed loss of gain is rather arbitrary in ordinary practice, we can generally pass sufficiently accurately from the one condition to the other by assuming $M'$ proportional to $M$. Physically, $M$ represents the average of $a^2(\phi)$ or $b^2(\phi)$ over a period.

The procedure is as follows. First carry out the calculations for the unrestrained case, as described in Sec. 2.2 above, and plot the approximating polynomial for $a(\phi)$ to
estimate its maximum, \( M' \). If \( M' \) does not exceed the prescribed maximum, which we may take as \( M'' \), then the polynomial so obtained is the correct one. If \( M' \) does exceed \( M'' \), compute the sum of the squares of the approximating polynomial, and reduce this sum by the factor \( (M'/M'')^2 \) to obtain the value of \( M \) in Eq. (8).

The optimum coefficients are given by the equations

\[
\begin{align*}
\sum_{j=0}^{n} P_{jk} a_j + \xi_k a_k &= A_k \quad (k = 0, 1, 2, \ldots n) \\
\sum_{j=0}^{n} a_j^2 &= M
\end{align*}
\]

where the \( A_k \) and the \( P_{jk} \) are as previously defined [Eqs. (5), (6)]. The unknowns are \( \xi \) and \( a_j \). To solve the system (9), (10), we note that the first set of equations (9) is linear in the \( a_j \) if \( \xi \) is regarded as a constant, and hence one may use standard methods to find the \( a_j \) in terms of \( \xi \). When this has been done, substitution in (10) will give an algebraic equation in \( \xi \).

2.4 Fixed Points. In the foregoing methods it is only the average error that is considered, and the approximating polynomials may not give an exact fit at any point. An alternative procedure which has been suggested is to match a number of points on the two curves \( a(\phi) \), \( b(\phi) \), using as many unknowns \( a_j \) as there are points to be fitted. Direct use of this procedure is not satisfactory because of its unreliability; that is, one can sometimes match a large number of points without obtaining a good fit at all. An example of the sort of thing that can happen is given by the simple case

\[
a(\phi) = 1 \cos 16\phi, \quad b(\phi) = 0 \sin 16\phi
\]

in which there is a perfect fit at the points 0, \( \pi/8 \), \( 2\pi/8 \),... even though the curve as a whole is not well approximated. A graphical comparison with the mean-square method is given in Figs. 5 and 6.

Nevertheless there are cases of practical importance in which the value at or near one point may be quite critical. In other words, it is sometimes desirable to specify that the two curves shall be in some sense close, as above, and to specify in addition that there shall be an exact fit at certain prescribed points. We show in Sec. 5.2 that such an added restriction will not prevent the mean-square error from going to zero.

Prescribing that there be an exact fit of amplitude \( a(\phi) \) at \( q \) fixed points \( \phi_k \), and of phase \( -b(\phi) \) at \( r \) points \( \phi_k \), amounts to a set of linear relations

\[
\begin{align*}
\sum_{j=0}^{n} a_j \cos j\phi_k &= a(\phi_k) \quad (k = 1, 2, \ldots q) \\
\sum_{j=0}^{n} a_j \sin j\phi_k &= b(\phi_k) \quad (k = 1, 2, \ldots r)
\end{align*}
\]
Figure 5. Mean-square approximation, \( a(\phi) = 2 - \cos \phi \), \( b(\phi) = \sin \phi \); \( n = 4 \), \( s = m/2 \).

Figure 6. Same functions and range as in Fig. 5, points matched at 0°, 45°, 90°.
The proper procedure is to minimize the integral (3) subject to these conditions. For simplicity we shall assume that only one point is prescribed, though the theory is probably not much different for the general case. If

\[ \sum p_j a_j = \mathbf{p} \]  

(13)

is the linear relation analogous to (12), then the optimum coefficients \( a_k \) are obtained from (13) and the equations

\[ \sum_{j=0}^{n} (P_{jk} - P_{jm} P_k/P_m) = A_k - A_n P_k/P_m \quad (k = 0, 1, \ldots n) \]  

(14)

which are also linear, and may be solved in the same way as the system (7). The coefficient \( P_m \) is any coefficient in (13) which is not zero; the \( P_{jk} \) and \( A_k \) have the usual meaning (Eqs. (5), (6)).

3. Derivations Without Constraint

In this work the functions \( a(\phi), b(\phi) \) are usually given in the form of a table or a graph, or at any rate they are specified only approximately. Hence one may assume much in the way of boundedness and smoothness without impairing the utility of the results. The class of continuous functions gives ample generality, for example, though the following derivations are actually valid for functions of class \( \mathcal{L}^2 \) on \( 0--\pi \).

3.1 Approximation Over Whole Frequency Range. First we derive Eq. (4), which gives a convenient introduction to the questions to be considered later. The expression (3) is a polynomial in the \( a_k \)'s and hence certainly differentiable. A necessary condition for a minimum, therefore, is that the derivative with respect to each \( a_k \) be zero. In other words the \( a_k \) must be so determined that

\[ \frac{1}{d} \frac{d}{d} a_j = \int_0^\pi 2 \left[ a(\phi) - \frac{n}{2} a_k \cos k\phi \right] \left[ - \cos j\phi \right] d\phi \\
+ \int_0^\pi 2 \left[ b(\phi) - \frac{n}{2} a_k \sin k\phi \right] \left[ - \sin j\phi \right] d\phi \\
= -2 \int_0^\pi \left[ a(\phi) \cos j\phi + b(\phi) \sin j\phi \right] d\phi + 2a_j \left[ \pi/2 + \pi/2 \right] = 0 \]  

(15)

which is the condition given in the text.

To complete the derivation we must show that the condition is sufficient to insure a minimum, as well as necessary, and the reasoning on this point, which will be frequently used in the sequel, is as follows. The expression (3) is non-negative for all choices of the \( a_k \)'s and it is a continuous function of them. Hence it has a non-negative lower bound which it attains for some set of values \( a_k \). At the lower bound the necessary condition (4) must be satisfied, and since the solution of these equations is unique, the sufficiency is proved. We have shown that the minimum so obtained is an absolute minimum, in the sense that (3) does not take a smaller value for any values of the \( a_k \)'s.
3.2 Approximation Over Finite Frequency Range. To derive the system (7) we proceed exactly as in (15), collecting terms in the same way by taking corresponding parts of the two integrals together. The result follows immediately if we note that

\[
P_{jk} = \int_0^s (\cos j\phi \cos k\phi + \sin j\phi \sin k\phi) d\phi \tag{16}
\]

From the derivation one observes that approximation over an interval \(s'--s\), say, instead of \(0--s\), requires only that we change the definition of the \(A_k\)'s in Eq.(6), taking the integral from \(s'\) to \(s\) rather than from \(0\) to \(s\). All the expressions for partial intervals here considered can be similarly generalized.

To prove sufficiency of the condition it is enough, as above, to show that the system has a unique solution. This will be true when and only when the coefficient determinant \(\|P_{ij}\|\) is different from zero for the particular values of \(s\) and \(n\) under consideration. We already know, then, that the solution will give the absolute minimum if the procedure of the text can be carried out at all; for the procedure breaks down automatically whenever \(\|P_{ij}\| = 0\).

Actually, however, much more is true; viz., the procedure of the text is always operable, for every value of \(n\) and \(s\), and the values of \(a_k\) so obtained always lead to the absolute minimum. To prove this it suffices to show that \(\|P_{ij}\|\) vanishes only when \(s = 0\), and we proceed to this question forthwith.

In the first place it is easy to see that the determinant cannot vanish for \(s\) a rational multiple of \(\pi\); for in this case the non-diagonal entries will be algebraic numbers. Hence if \(\|P_{ij}\| = 0\) we have \(s\) satisfying an equation with algebraic coefficients, which is impossible. This result is already sufficient for every application, since it is no real restriction in a practical problem to assume \(s/\pi\) rational.

To give a proof in all generality one must proceed otherwise. The following argument, which is presented in the general form on account of its interest, is taken almost word for word from an article by M. Kac and R. P. Boas in the Duke Mathematical Journal for 1945. The author's attention was called to this reference by R. Salem.

To begin with we note that

\[
\left| \sum_{k=1}^n x_k e^{ib_k t} \right|^2 = \sum_{k=1}^n \sum_{l=1}^n x_k x_l e^{i(b_k - b_l)t} \geq 0 \tag{17}
\]

for all values of \(t\). Hence if \(g(t)\) is any non-negative function belonging to \(L\) on \((-\infty, \infty)\), then

\[
\int_{-\infty}^{\infty} g(t) \sum \sum x_k x_l e^{i(b_k - b_l)t} dt \geq 0 \tag{18}
\]

or, defining \(f(x)\) as the Fourier transform of \(g(t)\),

\[
f(x) = \int_{-\infty}^{\infty} e^{ixt} g(t) dt, \tag{19}
\]
we find from (18)
\[ \sum \sum f(b_k - b_L) x_k x_L \geq 0. \] (20)

Since the quadratic form (20) is positive definite, its determinant is non-negative, so that
\[ ||f(b_k - b_L)|| \geq 0. \] (21)

This is the result in question; viz., if \( f(x) \) is the Fourier transform of a non-negative function, all determinants of the form (21) will be non-negative. The article cited proved only that \( ||f(b(k-t))|| \geq 0 \) for any \( b \), but the method is identical.

Before we can apply this to the case at hand we must replace \( \geq \) by \( > \), a procedure which requires only a few trivial modifications. First, for any given values of \( x_k \) not all zero we note that the set in \( t \) for which (17) is zero is a set of zero measure. This follows most easily from the fact that the expression is an analytic function of \( t \), regarded as a complex variable, and is not identically zero.

From this we see that if we have
\[ \int_{-\infty}^{\infty} g(t) \, dt > 0 \] (22)
as well as \( g(t) \geq 0 \), then the inequality (20) will be a strict inequality. This in turn implies that all the characteristic values of the determinant are \( > 0 \), since if one were equal to zero we could make the whole expression zero, after reduction to diagonal form, by using a non-zero value for the corresponding \( x_k \) and zero for the others. Hence the determinant, which equals the product of the characteristic values, is also \( > 0 \).

To show that the required result \( ||P_{jk}|| \neq 0 \) is but a special case of this one, let us divide each element by \( s \) to obtain
\[ ||P_{jk}|| = s^{-n} \left| \frac{\sin((j-k)s)}{(j-k)s} \right|. \] (23)

The general term is of the form \( f(js - ks) \) with \( f(x) = \sin x/x \), which is the Fourier transform of a non-negative function meeting the condition (22). Thus the result is established, and with it the sufficiency of the conditions (7).

4. Derivations With Constraint

4.1 Approximation Over Whole Range. The question of minimizing the error over the whole range subject to a constraint (8) was omitted from Sec. 2 because it has little practical importance. We give a brief discussion here, however, since it leads to a few rather interesting theoretical results, and forms a convenient introduction to the corresponding problem for the finite interval.

Replacing (8) by an equality, solving for \( a_k \), and differentiating, we find
\[ \frac{\partial a_n}{\partial a_k} = -a_k/a_n \quad (k = 0, 1, \ldots, n-1) \] (24)
where we assume for convenience that $a_n \neq 0$. If we now proceed as in (15), taking account of this relation, we obtain $a_k / A_k = a_n / A_n$ or $a_k = x A_k$ with a proportionality constant $x$ independent of $k$. This may be substituted in (10) to give

$$M = \sum_{k} x^2 A_k^2 .$$

(25)

The optimum coefficients $a_k$ are thus equal to

$$a_k = x A_k = \sqrt{M/\sum_{0}^{n} A_k^2} .$$

(26)

These are proportional to the values for the unrestrained case, the proportionality constant being so adjusted that (10) is satisfied. It is easy to verify that the two solutions are the same if $M$ is assigned the value $1/4 \sum_{0}^{n} A_k^2$ that it would have in the case without constraint.

Having evaluated the coefficients, which we know by uniqueness to be truly optimum when (8) is an equality, we proceed to the question of the minimum error. If we abbreviate the Fourier coefficients by

$$c_k = (1/n) \int_{0}^{\pi} a(\phi) \cos k\phi d\phi, \quad s_k = (1/n) \int_{0}^{\pi} b(\phi) \sin k\phi d\phi ,$$

(27)

we find, upon substituting (26) into (3),

$$I_{\text{min}} = \left[ a^2(\phi) + s^2(\phi) \right] d\phi - 2x(1-x) \sum_{0}^{n} A_k^2 .$$

(28)

This gives the minimum value of the sum of the mean-square errors for all choices of the $a_k$. By using the Parseval equality for the first member and regrouping the terms, we find

$$I_{\text{min}} = \sum_{n+1}^{\infty} \left[ s_k^2 + c_k^2 \right] + (1/2) \sum_{0}^{n} (s_k - c_k)^2 + (1/2)(2x-1)^2 \sum_{0}^{n} (s_k + c_k)^2$$

(29)

which is a sum of three non-negative expressions, each representing a different type of error. The first expression, I, exhibits the error due to use of only a finite number of terms in the approximating series. By the Bessel inequality we know that this term of the error will tend to zero as $n$ becomes infinite.

The second term II represents the error due to the fact that the two functions $a(\phi)$ and $b(\phi)$ are not conjugate—the condition that they be conjugate is precisely that $c_k = s_k$ for all $k$. The presence of this term shows that the error will not generally approach zero for arbitrary functions, no matter how large $n$ becomes.

The third term exhibits the error due to the constraint. The optimum value of $M$ leads to a proportionality $x = \frac{1}{2}$, as is evident from (29) itself or may be proved directly from (4). If $M$ has this optimum value the third term drops out and hence the first two, as was tacitly assumed above, represent the error for the unrestrained case.

So far we have assumed (8) to be an equality. That this procedure is permissible is shown by the behaviour of III in (29). Thus, the error increases steadily as $M$ deviates from its optimum value. Hence, if the prescribed $M$ is already at least
as large as the value \( M \) would have without constraint, the unrestrained solution should be used; but if the prescribed \( M \) is too small, the condition of minimum error will automatically make (8) an equality. Thus the above procedure is completely justified.

The foregoing minimizes the expression (3); that is, it minimizes the sum of the two mean-square errors for \( a(\phi) \) and \( b(\phi) \). One naturally inquires whether a procedure which minimizes the larger of these two errors, rather than their sum, would be very different. To answer this question, we note that the new condition will be fulfilled if matters are so adjusted that the two errors are equal to each other. By a method similar to that used for (29) we find the difference of the errors

\[
\int_0^\pi \left[ a(\phi) - \sum_0^n a_k \cos k\phi \right]^2 \, d\phi - \int_0^\pi \left[ b(\phi) - \sum_0^n a_k \sin k\phi \right]^2 \, d\phi = \sum_{n+1}^\infty (c_k^2 - s_k^2) + (1-2x)\sum_0^n (c_k^2 - s_k^2)
\]

so that the individual errors, though not generally equal, tend to the same limit as \( n \to \infty \) in the unrestrained case (\( x = \frac{1}{2} \)). Hence in this particular case minimizing Eq. (3) is practically the same thing as minimizing the maximum error, for large \( n \). The question is less significant for the finite interval, since both errors tend to zero.

4.2 Finite Frequency Range. Proceeding as usual, and noting Eq. (24) we find

\[
-A_k + \sum_0^n P_{nk} + \delta k = \frac{a_k}{a_n} \sum_{n=0} P_{nk} a_n = 0 \quad (k=0, \ldots, n-1)
\]

to be the equations for determining the coefficients subject to equality in (8). As they stand, these equations represent a set of simultaneous equations of the second degree, and no systematic method of solution is apparent. We therefore introduce a new variable \( \xi \) defined by

\[
\xi = \frac{a_n}{a_n} - \left( \frac{1}{a_n} \right) \sum_0^n P_{nk} a_n
\]

so that \( \xi \), though it involves each \( a_k \), is independent of \( k \). Upon substituting into (31) we obtain the equations (9) of Sec. 2.

To complete the derivation we must show that the condition (8) may be replaced by an equality. The procedure for this question will be similar to that used in Sec. 2.3 above, except that matters are complicated by the fact that the error cannot be exhibited in simple form. The desired condition, viz., that the error increase steadily as \( M \) deviates from the optimum, is equivalent to saying that the derivative of the error with respect to \( M \) equals zero for just one value of \( M \). With this new formulation of the condition the proof becomes relatively simple.

For the value of the error we have

\[
I = \int_0^a \left[ a^2(\phi) + b^2(\phi) \right] \, d\phi - 2 \sum_0^n \frac{a_k}{a_n} A_k \int_0^a \left[ (\frac{2}{a_k} \cos k\phi)^2 + (\frac{2}{a_k} \sin k\phi)^2 \right] \, d\phi
\]
which becomes

\[ I = \int_0^a \left[ a^2(\phi) + b^2(\phi) \right] d\phi - 2 \sum_{k=0}^n a_k A_k + \sum_{k=0}^n a_j a_k P_{jk} \]  

(34)

if we replace the squares in the last integral by double sums and use Eq. (16). For convenience we write (9) again:

\[ \sum_{j=0}^n P_{jk} a_j + \xi, a_k = A_k \]  

(35)

This may be multiplied by \( a_k \) and summed on \( k \) to give

\[ \sum_{j=0}^n \sum_{k=0}^n P_{jk} a_j a_k + \xi, \sum_{k=0}^n a_k^2 = \sum_{k=0}^n a_k A_k \]  

(36)

which leads to

\[ I_{\text{min}} = \int_0^a \left[ a^2(\phi) + b^2(\phi) \right] d\phi - \sum_{k=0}^n \sum_{j=0}^n a_j a_k P_{jk} - 2 \xi, \sum_{k=0}^n a_k^2 \]  

(37)

upon substitution in (34). This expression for the error is suitable for the purpose in view. Differentiating it we obtain

\[ \frac{dI_{\text{min}}}{dM} = -2 \sum_{j=0}^n \sum_{k=0}^n P_{jk} a_j a_k' - 2 \xi, (\sum_{k=0}^n a_k^2)' \]  

(38)

On the other hand we know, by differentiating (35), that

\[ \sum_{j=0}^n P_{jk} a_j' + (\xi, a_k)' = 0 \]  

(39)

where in both cases the prime stands for differentiation with respect to \( M \). Equation (39) may be multiplied by \( a_k \) and summed to give

\[ \sum_{j=0}^n \sum_{k=0}^n P_{jk} a_j a_k' + \sum_{k=0}^n (\xi, a_k)' = 0 \]  

(40)

which leads to

\[ \frac{dI_{\text{min}}}{dM} = -\xi, (a_k^2)' = -\xi, M' = -\xi \]  

(41)

by virtue of (38). Thus, the unknown \( \xi \), originally introduced just to make the system linear, actually has a definite physical significance; viz., it is minus the derivative of the error with respect to \( M \).

The original task, to prove that the error increases as \( M \) deviates from the optimum, requires us to show that \( \xi = 0 \) is possible for just one value of \( M \). Let us suppose that in fact \( \xi = 0 \). Then the system (9) reduces to the system (7), that is, to the system for the unrestrained case. But we know that this system has a unique solution. Therefore \( \xi = 0 \) is possible for just one set of \( a_k \), hence \textbf{a fortiori} for just one value of \( M = \sum a_k^2 \). Thus the proof is complete.
Only necessity has been established so far, which is the most important aspect of the problem in practical work. For sufficiency, we see from (9) that the $a_k$ will have a unique solution in terms of $\zeta$ if and only if $\zeta$ is not a characteristic value of the matrix $(P_{jk})$. In this case substitution of the unique set of values in (10) will lead to a finite number of solutions only, and hence one of these will certainly be sufficient.

Unfortunately the interesting case practically is that in which the prescribed $M$ is less than the optimum $M$. We know then by (41) and the above discussion that $\zeta$ will be positive. It has been proved that the characteristic values are also positive (Sec. 2.2) and hence the undesirable case of equality could possibly arise in practice.

5. Existence Theorems

5.1 Existence of Polynomial Approximation. In Sec. 2.3 we remarked that the two functions $a(\phi)$ and $b(\phi)$ cannot be approximated arbitrarily well over the whole frequency range; in other words the integral (2) with $s = \pi$ will not generally tend to 0 as $n$ tends to infinity, for any choices of the $a_k$. If on the other hand $s < \pi$, the two functions can always be approximated arbitrarily well, and in fact uniformly if they are continuous. This theorem and its proof are due to R. Salem.

To begin with we assume $a(\phi)$ even, $b(\phi)$ odd as heretofore, and approximate $a(\phi)$ by a polynomial containing only even powers, $b(\phi)$ by one containing only odd powers:

$$ a(\phi) \approx \sum_n a_n \phi^n, \quad b(\phi) \approx \sum_n b_n \phi^n. \quad (42) $$

That this is possible may be shown by the Weierstrass approximation theorem, or it may be proved by writing down the two Fourier series, taking the Fejéer sums, and replacing the cosine and sine terms by their power series expansions in $\phi$. In Eq. (42) we write $(\log z)/i$ for $\phi$, multiply the second relation by $i$, and add to obtain

$$ a(\log z)/i + ib(\log z)/i \approx \sum_n \chi_n (\log z)^n \quad (43) $$

where the constants $\chi_n$, by the condition on the approximating polynomials, will all be real. Next we approximate $\log z$ in the second member by a polynomial in $z$, which is possible in any simply connected region throughout which the function is regular. In particular it is possible over any portion of the unit circle (though not over the whole circle). If finally we let $z = e^{i\phi}$ we obtain

$$ a(\phi) + ib(\phi) \approx \sum_n e^{in\phi} \quad (\text{B}_n \text{ real}) \quad (44) $$

which gives the desired result after separation into real and imaginary parts. This argument assumes continuity but gives uniform approximation; and since continuous functions are dense in $L^2$, the corresponding theorem for approximation in the mean is also obtained.

5.2 Approximation With Fixed Point. We wish next to show that the error can still be made arbitrarily small, even if an exact fit is specified at one point, as in Sec. 2.4.

To this end we observe that for any fixed point $\phi > 0$ at least one of the functions $\sin \phi$, $\cos \phi$,
sin 2 \phi, \ldots \sin n \phi, \text{ for sufficiently large } n, \text{ will be greater than 0.1, say. We therefore take a polynomial which will approximate within an error } \epsilon/20 \text{ at all points. If we change the coefficient of the above-mentioned sine or cosine term enough to get an exact fit, the added error introduced will be no worse than } 10 \epsilon/20 \text{ so that the approximation is still within } \epsilon. \text{ Since } \epsilon \text{ may be arbitrarily small the result is established for continuous functions which pass through the assigned point. The case of the cosine series is even easier, since one may simply change the constant term enough to get an exact fit, after approximating within } \epsilon/2. \n
5.3 Non-existence of Series Approximation. Having shown that an arbitrarily good approximation may be obtained by polynomials, one is naturally led to inquire whether an infinite series might be used. The difference between the two cases is that for polynomials all the coefficients may be changed as } n \text{ increases to } n+1, \text{ while for series we simply add a coefficient to those we already have.}

That no series exists for which the error will approach zero as we take larger and larger partial sums has also been shown by R. Salem, as follows. Suppose we take the special case in which } a(0) \text{ and } b(0) \text{ are both constant in some interval } (0--s). \n
If

\[ a(\phi) = \sum_{0}^{\infty} a_n \cos n\phi, \quad b(\phi) = \sum_{0}^{\infty} a_n \sin n\phi, \quad (45) \]

then multiplying the second by } i \text{ and adding gives

\[ a(\phi) + i b(\phi) = \sum_{0}^{n} a_n e^{in\phi} \quad (46) \]

so that the exponential series on the right is constant on } (0--s). \text{ But this series is also the boundary value of an analytic function; viz., it is

\[ \lim_{r \to 1} \sum_{0}^{n} a_n r e^{in\phi} = \lim_{z \to 1} \sum_{0}^{n} a_n z^n. \quad (47) \]

Hence the function must be constant everywhere, so that all coefficients except } a_0 \text{ must be zero. Since } a_0 \text{ must be real, Eq. (46) is therefore impossible in general, and so is our assumption that the functions could be expanded simultaneously in infinite series.}

5.4 Non-existence of Bounded Approximation. It was observed in Sec. 2.3 that the approximating polynomials would generally become larger and larger, as } n \text{ increases, for the interval } s < \phi \leq \pi \text{ over which they are not specified. The material for proving this statement is now at hand. Suppose in fact that one can approximate two arbitrary continuous functions by polynomials of the form (1) and (2) which remain bounded over } 0-\pi \text{ as the error goes to zero. If we pick } a(\phi) \text{ and } b(\phi) \text{ to have continuous derivatives, in particular, then these derivatives can be approximated in the specified manner, so that we have

\[ a'(\phi) \cong a_n \sum_{0}^{n} a_k \cos k\phi = P_n'(\phi), \quad b'(\phi) \cong a_n \sum_{0}^{n} a_k \sin k\phi = Q_n'(\phi) \quad (48) \]
with the trigonometric polynomials $P_n(\phi)$ and $Q_n(\phi)$ bounded by some constant $M$ in $\Omega - \pi$.

Integrating these equations we find

$$a(\phi) \geq P_n(\phi) \leq M\pi, \quad b(\phi) \geq Q_n(\phi) \leq M\pi$$

in absolute value (49)

where the error still goes to zero as $n$ increases indefinitely. Thus, if the functions $a(\phi)$ and $b(\phi)$ have continuous derivatives, then they can be approximated by polynomials which are not only uniformly bounded themselves, but which have uniformly bounded derivatives. By the mean value theorem and the family of polynomials so obtained will be uniformly equicontinuous on the closed interval $0--\pi$. Hence one may use Ascoli's theorem to select a subsequence $P_{n_1}(\phi)$ which converges to a limit $P(\phi)$ on $0--\pi$. From the set $Q_{n_1}(\phi)$ we can similarly select a convergent subsequence $Q_{n_1}(\phi)$. We have therefore $P_{n_1}(\phi) \to P(\phi)$, $Q_{n_1}(\phi) \to Q(\phi)$ with $|P|$ and $|Q| \leq M$ on $0--\pi$, since $|P_{n_1}|$ and $|Q_{n_1}|$ are. By hypothesis the coefficients are equal for each polynomial of the sequence, so that

$$\int_0^\pi P_{n_1}(\phi) \cos k\phi \, d\phi = \int_0^\pi Q_{n_1}(\phi) \sin k\phi \, d\phi$$

and hence by the Lebesgue theorem on bounded convergence

$$\int_0^\pi P(\phi) \cos k\phi \, d\phi = \int_0^\pi Q(\phi) \sin k\phi \, d\phi.$$ (51)

We have thus represented the functions $a(\phi)$ and $b(\phi)$ by infinite Fourier series with equality of coefficients. In the preceding paragraphs, however, this was proved impossible; and the counter-example there given requires no more generality in the functions $a(\phi)$ and $b(\phi)$ than we assume here. Thus the proof is complete. The same methods show that the approximating polynomials will not generally remain bounded, no matter how many derivatives of $a(\phi)$ and $b(\phi)$ are assumed continuous.

5.5 Other Methods. In conclusion we note a few alterations that could have been made in the general procedure hitherto followed. Instead of minimizing the expression (2), for example, one could treat the corresponding expression with the second integral multiplied by some constant $\lambda$. In this way the relative accuracy of the two approximations for $a(\phi)$ and $b(\phi)$ could be changed. A large value of $\lambda$ will tend to favor $b(\phi)$, so that it is accurately approximated, while a small value will weight the error in $a(\phi)$ the more heavily. It is probable that this generalization of the problem would lead to no new complications, though we shall not carry out the calculations here.

A second more radical departure from our procedure would be to work in the impedance plane rather than in the $\phi$-plane. The condition for physical realizability would then be essentially that the Fourier transforms of the two functions be equal, and we should be led to a minimizing problem subject to this as a constraint.

Finally, for the partial interval, which is perhaps the most interesting case in practice, one could begin by constructing a set of polynomials
orthogonal over the fundamental interval $0--\pi$ rather than over $0--\pi$. The treatment of the error would then be much simpler, and would proceed much as in the $0--\pi$ case treated above. But this increased simplicity would be compensated by an increased complexity of the condition for physical realizability: instead of having mere equality of the coefficients, one would have a set of linear equations.