Prediction of Microcracking Distributions in Composite Laminates Using a Monte Carlo Simulation Method

by

Yuki Christopher Michii

B.S., Aerospace Engineering (1994)

University of Maryland at College Park

Submitted to the Department of Aeronautics and Astronautics in Partial Fulfillment of the Requirements for the Degree of Master of Science in Aeronautics and Astronautics

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Signature of Author ......

Department of Aeronautics and Astronautics
January 17, 1997

Certified by ....................

Hugh L. McManus
Class of 1943 Career Development Assistant Professor of Aeronautics and Astronautics
Thesis Supervisor

Accepted by ....................

Professor Jaime Peraire
Chair, Graduate Office
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ABSTRACT

Space structures are subject to thermal and mechanical loads. Matrix cracks can form in composite components, which results in a change in their thermal and elastic properties. The objective of this study is to develop a method to predict transverse microcracking in general composite laminates subject to thermal and mechanical loads. The approach combines probabilistic and analytical components in an incremental damage method. The probabilistic components include a distribution of flaws characterized by a Weibull probability function, seeding of flaws at random locations, inspection of flaws in random order for crack initiation, and inspection of cracks in random order for extension. The analytical components include fracture mechanics based energy criteria that uses a shear lag derivation of the stress and displacement fields. Degradation of material properties, temperature-dependent material properties, and a material variations model are incorporated into the method. This method is implemented through a computer program that predicts crack densities, crack distributions, and degraded laminate properties as functions of an arbitrary thermomechanical load profile. Parametric analyses are used to understand the behaviors predicted by the method and their sensitivities to model parameters. Predictions are compared to previously collected data and observations for different laminate configurations and material systems. For both thermal and mechanical loads, crack density predictions capture general trends and agree with much of the data. The method shows improvements on the current state of the art in several areas. The effective flaw model predicts the initiation and gradual accumulation of cracks. The material variations model allows the method to emulate the intrinsic variability of the crack data. The method predicts crack distributions and their evolution as cracking progresses. This evolution can include the formation of different crack types and patterns. The effect of ply thickness on this evolution is correctly predicted. The success of the method shows its superior as a tool for predicting cracking. By replicating complex observed behavior using a relatively simple method, the work supports the physical soundness of the method and increases our understanding of the mechanisms of microcracking in composite laminates.

Thesis Supervisor:  Hugh L. McManus
Title:  Class of 1943 Career Development Assistant Professor of Aeronautics and Astronautics
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NOMENCLATURE

A  Maximum amplitude of the material variation in the material variations model
A  Laminate stiffness matrix
A_{11}  Component of the laminate stiffness matrix
ac  Thickness of cracking ply group
ao  Total thickness of the laminate
aq  Thickness of shear transfer region
ar  Thickness of the rest of the laminate
d  Reference flaw size
d^{\alpha}  Weibull probability function scale parameter
Dt  Variable of convenience used to simplify ply stiffness expressions
Ec  Axial stiffness of the cracking ply group
E_{ii}  Longitudinal ply stiffness of the ith ply
E_o  Effective axial stiffness of laminate
E_r  Axial stiffness of the rest of the laminate
E_{ii}  Transverse ply stiffness of the ith ply
F_{\delta}  Cumulative probability of the flaw having a size smaller than \delta
G^{\alpha}  Effective shear modulus of the shear transfer region
Gi  Shear ply stiffness of the ith ply
G_{lc}  Critical strain energy release rate, fracture toughness
G_{lc}^{y'}  Critical strain energy release rate in the y'-direction
G_{lc}^{z'}  Critical strain energy release rate in the z'-direction
G_{lc}^{i}  Fracture toughness with uniform material conditions
G_{lc}^{i}(x', y')  Fracture toughness as a function of two spatial variables
h  Crack spacing
i  Direction of interest (y' or z')
$K$  Stiffness constant in the shear lag analysis which relates the shear stress and the displacements of the cracked ply group and the rest of the laminate

$m$  Weibull probability function shape parameter

$N$  Laminate load vector in the $xyz$ coordinate system

$N'$  Laminate load vector in the $x'y'z'$ coordinate system

$N_{x'}$  Laminate load in the $x'$-direction, a component of $N'$

$q$  Shear stress

$Q$  Ply stiffness matrix

$\bar{Q}_i$  Rotated reduced ply stiffnesses of the $i$th ply group in the $x'y'z'$ coordinate system

$Q_{11}$  Component of the ply stiffness matrix

$Q_{22}$  Component of the ply stiffness matrix

$Q_{66}$  Component of the ply stiffness matrix

$Q_{12}$  Component of the ply stiffness matrix

$S_1(x',y')$  Function of two spatial variables representing the local fiber volume fractions, fiber-packing, and other aspects affecting local stresses

$S_2(x',y')$  Function of two spatial variables representing the local material inhomogeneities such as voids, flaws, non-uniformity of the matrix material, etc. that affect the local material toughness within the laminate

$S_3(x',y')$  Arbitrary function of two spatial variables used to represent the combination of $S_1$ and $S_2$

$T$  Transformation matrix

$t_i$  Thickness of the $i$th ply

$T_0$  Stress-free temperature

$U$  Internal strain energy of a body

$u_c$  Axial displacement in the cracking ply group

$U_q$  Internal strain energy contribution of shear stresses

$u_r$  Axial displacement in the rest of the laminate

$U_\sigma$  Internal strain energy contribution of normal stresses

$W$  Work done by an external force

$x$  Global coordinate system axis
\( x' \)  
Axis in local coordinate system of the cracking ply group

\( y \)  
Global coordinate system axis

\( y' \)  
Axis in local coordinate system of the cracking ply group

\( z \)  
Global coordinate system axis

\( z' \)  
Axis in local coordinate system of the cracking ply group

\( \alpha_c \)  
Axial coefficient of thermal expansion of the cracking ply group

\( \alpha_i \)  
Coefficient of thermal expansion vector of the \( i \)th ply group

\( \alpha_{ii} \)  
Longitudinal ply CTE of the \( i \)th ply

\( \alpha_r \)  
Smeared coefficient of thermal expansion vector of the rest of the laminate

\( \alpha_r' \)  
Axial coefficient of thermal expansion of the rest of the laminate

\( \alpha_{ii} \)  
Transverse ply CTE of the \( i \)th ply

\( \overline{\alpha}_i \)  
Rotated coefficient of thermal expansion vector of \( i \)th ply group

\( \delta \)  
Effective flaw size

\( \Delta G \)  
Change in strain energy release rate

\( \Delta G_{i'}' \)  
Change in strain energy release rate of a fully-formed incremental extension of a partial crack

\( \Delta G_{i'}^{*} \)  
Change in strain energy release rate of an effective flaw

\( \Delta G'_{\omega} \)  
Change in strain energy release rate with uniform material conditions

\( \Delta G'(x', y') \)  
Change in strain energy release rate as a function of two spatial variables

\( \Delta T \)  
Change in temperature relative to a stress-free temperature

\( \Delta U \)  
Change in internal strain energy of a body

\( \Delta W \)  
Change in work done by an external force

\( \Delta y' \)  
Length of an incremental, fully-formed extension of a partial crack in the \( y' \)-direction

\( \varepsilon_a \)  
Laminate axial strain

\( \varepsilon_c \)  
Axial strain in the cracking ply group

\( \varepsilon_r \)  
Axial strain in the rest of the laminate
\( \Gamma \) Variable of convenience used in simplifying the shear lag analysis
\( \gamma \) Scaling factor between the critical strain energy release rates in the \( y' \)-direction and the \( z' \)-direction
\( \varphi_x \) Phase shift in the \( x' \)-direction in the material variations model
\( \varphi_y \) Phase shift in the \( y' \)-direction in the material variations model
\( \kappa \) Material property degradation knockdown factor
\( \Lambda \) Compliance of the cracking ply group equal to \( 2/E_c \)
\( \lambda \) Variable of convenience used to simplify the shear lag analysis
\( \lambda_x \) Length parameter in the \( x' \)-direction over which \( A \) varies in the material variations model
\( \lambda_y \) Length parameter in the \( y' \)-direction over which \( A \) varies in the material variations model
\( \nu_i \) Major Poisson's ratio of the \( i \)th ply
\( \phi_c \) Angle between the global \( x \)-axis and the cracking ply group \( x' \)-axis
\( \rho \) Crack density
\( \rho_{\text{apparent}} \) Observed edge crack density
\( \rho_{\text{true}} \) Calculated true crack density
\( \sigma_a \) Applied mechanical stress
\( \sigma_c \) Longitudinal stress in the cracking ply group
\( \sigma_{\infty} \) Far-field stress in the cracking ply group
\( \sigma_r \) Longitudinal stress in the rest of the laminate
\( \theta_c \) Angle between the fibers of the cracking ply group to the global \( x \)-axis
\( \theta_i \) Angle between the global \( x \)-axis and the ply \( x_i \)-axis; ply angle of the \( i \)th ply
\( \theta'_i \) Angle between the ply \( x_i \)-axis and the cracking ply group \( x' \)-axis
\( \xi \) Shear lag parameter
CHAPTER 1

INTRODUCTION

Material properties including high specific stiffness, low coefficient of thermal expansion (CTE), and high specific strength make composite materials attractive for use in a variety of areas. For example, dimensionally critical components of space structures, such as antenna booms, precision reflectors for space telescopes, and space truss tubes, are constructed of composite materials. These components take advantage of composite materials' unique qualities in order to achieve often demanding material performance goals such as very high stiffness or near-zero CTE.

1.1 ORBITAL ENVIRONMENT

Space structures orbiting the Earth must be designed to operate in a number of different environmental conditions. Among these conditions are temperature extremes, the result of passing in and out of Earth's shadow, which can reach values of ±250°F [121°C, -157°C] in geosynchronous orbit [1]. In addition to thermal loads, a space structure may be required to handle applied mechanical loads. For example, one early design of the multi-national space station performed by NASA required an axial load capacity of ±1200 lbs. (±5.3 kN) for composite truss tubes [2].

Typically, transverse microcracks, which form in the matrix of composites, are the first observed damage caused by these loads. A transverse microcrack is defined as a crack that runs parallel to the direction of the fibers and whose plane is perpendicular to that of the laminate. Microcracks are observed in many different lay-up configurations and different materials.
Transverse microcracks degrade the thermal and mechanical properties of composites. This degradation can affect the integrity and response of a structure. Also, transverse microcracks can instigate other types of damage, such as delamination. Transverse microcrack damage or damage instigated by transverse microcracks may cause a structure to respond outside of its design specifications. Additionally, transverse microcracks may be the first sign of a process leading to premature failure. This damage mode is very important to dimensionally critical structures that must adhere to strict design criteria on stiffness and CTE.

Under thermal loads, a mismatch of CTE's between plies of different orientations causes transverse microcracks in composite laminates. In Figure 1.1a, individual, unidirectional plies exposed to a change in temperature expand or contract according to their CTE's depending on their orientation. However, when these plies are used together in the form of a laminate, as shown in Figure 1.1b, each ply constrains the adjacent, neighbouring plies. This constraint and the CTE mismatch between plies of different orientations produce ply stresses. If these stresses are sufficiently high, ply matrices can fail in the form of transverse microcracks, as shown in Figure 1.1b.

Mechanical loads can also cause the ply matrices to fail in the form of transverse microcracks due to the combination of loading and the constraint of adjacent plies. Depicted in Figure 1.1c, a mechanical load is applied to the laminate in a direction parallel to the fibers of some of the plies; other plies are loaded in the weaker matrix-dominated direction. For a sufficiently high load, transverse microcracks will form in the matrix-strength dominated plies.

Considerable effort has been devoted to the investigation of the microcracking phenomenon. Experimental studies have shown that microcracks cause material properties to degrade. Also, experimentation has identified material type, laminate geometry, and ply thickness as variables affecting the damage state. Despite the contributions of experimental methods to the investigation of transverse microcracking,
Figure 1.1  (a) Unconstrained plies subject to thermal load - No crack formation. (b) Constrained laminate subject to thermal load - Crack formation due to internal stresses. (c) Constrained laminate subject to mechanical load - Crack formation in transverse ply.
performing experiments to determine transverse microcrack damage for every material, laminate geometry, and loading situation would prove inefficient as well as costly. To avoid this level of experimentation, a better understanding of this damage mode is necessary. Greater knowledge of the mechanisms of transverse microcrack damage can be achieved through the development of analytical methods. These methods will facilitate the quantification of the damage as well as provide a more efficient means of analyzing a particular situation.

1.2 PRESENT WORK

The present study works toward developing an analytical method to predict transverse microcracking in composite laminates. In this study, analytical and probabilistic methods are combined to predict transverse microcrack initiation and growth in three dimensions while accounting for the effects of local variations of the material. The analytical aspect is a synthesis of a shear lag solution of stress and displacement fields near a given crack and energy-based cracking criteria founded on fracture mechanics to determine the energetic favorability of cracking. The probabilistic aspects include the use of probability functions to characterize flaw distributions and a simulation scheme. Additionally, local variations of the material are modeled to account for the inhomogeneities of a given material and the influence of these inhomogeneities on the cracks within the volume of a laminate.

An incremental damage method combines the probabilistic and analytical components to predict transverse microcrack damage in composite laminates at discrete load increments of a given load profile. The method simulates random factors that affect transverse microcracking including random flaw locations and distributions of flaw sizes. Energy-based cracking criteria determine whether microcracks will grow from initial flaws to span the thickness of a layer and, subsequently, whether these microcracks will extend across the width of the laminate. Also, the method incorporates a material
variations model to examine their effect on cracking. Temperature dependence of material properties and material softening effects due to progressive cracking can also be taken into account.

A computer program implements the incremental damage method. The results produced by the encoded method are then compared to previously gathered experimental data for verification. Crack distribution histories are also illustrated for select load increments to provide insight into crack initiation and development.

1.3 OVERVIEW

Chapter 2 will cover the background of the microcracking problem. Chapter 3 summarizes the present study through a problem statement and a description of the approach used. Chapter 4 describes the methods used to investigate microcracking. Chapter 5 describes the implementation of these methods. Chapter 6 presents the results of the method and parametric studies. Chapter 7 discusses the results presented in Chapter 6. Chapter 8 presents conclusions and recommendations for future work on the problem.
CHAPTER 2

BACKGROUND

For over two decades, transverse microcrack damage has been intensively studied, as evidenced by the many papers that have been published on the subject. For monotonic and cyclic loads, the effects of transverse microcracks on composite laminates have been investigated. Also, predictive methods have been developed for both of these load types. Cross-ply laminates have received most of the attention because damage in these laminates is more easily observed and modeled in comparison to general angle ply laminates. Most of the studies that predict transverse microcrack damage concentrate on mechanical loads; however, some limited work has been done to develop predictive methods for thermal loads. Some studies incorporate probabilistic aspects; these include studies using distributions of flaws and studies on strength distributions. Some studies have been published that investigate crack growth; a modest amount of work has been conducted to experimentally observe and model transverse microcrack growth for both static and fatigue type mechanical loads. Studies outside of microcracking that deal with material variations within composites have implications for transverse microcrack damage; included in these studies are some experimental observations of local variations within a given material and some preliminary work modeling the effects of local variations in the material.

In this Chapter, the background relevant to the present work is reviewed in five sections. Section 2.1 reviews research in transverse microcrack damage. Section 2.2 reviews studies with probabilistic components. Section 2.3 reviews papers that deal with crack growth mechanisms. Section 2.4 reviews studies on material variations. Section 2.5
reviews recent research directly related to the present work. Section 2.6 summaries the background studies reviewed in Sections 2.1 - 2.5. Section 2.7 summarizes and discusses the material properties and analytical parameters used in the current model.

2.1 STUDIES OF TRANSVERSE MICROCRACK DAMAGE

In this section, studies of transverse microcrack damage are organized into four separate categories: early work, mechanical loads, thermal loads, and general angle ply laminates. The first category concentrates on early work on this damage type. These studies illustrated the importance of this mode of damage and the need to gain a better understanding of it as well as helping to initially characterize the problem. The second category concentrates on mechanical loads. These studies have segregated into two general groups: strength-based methods and fracture mechanics-based methods [3]. Additionally, researchers have published work on the effects of transverse microcrack damage for this load type. The third category concentrates on thermal loads. Similar to mechanical loads, strength-based and fracture mechanics-based studies represent the two groups of studies for thermal loads. Also, researchers have studied the effects of transverse microcracks under thermal loads; most of the research focuses on the effects rather than on developing predictive methods for thermal loads. The fourth category concentrates on studies dealing with general angle ply lay-ups. Most of the work on transverse microcrack damage use cross-ply laminates rather than more complex configurations.

2.1.1 Early Work

The early experimental studies of Camahort et al. [4] illustrated the relevance of transverse microcrack damage to composites, especially for dimensionally critical space structural components. By measuring the CTE of experimental specimens made of different GFRP materials subjected to thermal cyclic loads, transverse microcracks are
shown to cause changes in the values of the CTE. This effect of transverse microcracks demonstrates that laminates are sensitive to thermal loads. In a similar manner, residual strains in experimental specimens demonstrate the effects of transverse microcracks caused by monotonic mechanical loads. Additionally, the type of material is shown to be a factor in microcracking.

The work of Garrett and Bailey [5] is an early example of an analytical method developed to predict transverse microcracking in cross-ply laminates. In their study, they use a one-dimensional shear lag solution to calculate the transfer of stresses between plies and they derive expressions for degraded laminate properties as functions of the applied load and uncracked properties. Comparison between predictions and data shows some correlation. For example, predictions capture the general shape of some crack spacing data. The impact of their work is that it demonstrated the potential of predictive methods to model transverse microcrack damage.

2.1.2 Mechanical Loads

Transverse microcrack studies involving mechanical loads have segregated into two basic types: strength-based methods and fracture mechanics-based methods [3]. Common to both methods is the use of a stress analysis [3]. Stresses have been determined using a variety of methods including shear lag solutions, variational methods, continuum damage mechanics approaches, classical laminated plate theory (CLPT), and finite element methods.

Some of the simpler strength-based models use CLPT to describe the stresses [6-9]. Shear lag solutions [10-14] have also been used to obtain stresses. Cracking in the strength-based methods is determined by comparing the derived stress-state with a material failure property. In situ transverse strength is used most often in this capacity. However, using the in situ transverse strength of the laminate as a material property has been shown to be the weakness of many of the strength-based approaches. As discovered
by Flaggs and Kural [9], \textit{in situ} transverse strength is a property of the laminate that depends on laminate geometry, stacking sequence, and thickness. The use of \textit{in situ} transverse strength as a material property in many of the strength-based studies is a limiting factor in the effectiveness of these methods to predict transverse microcracks.

The other approach involves the use of fracture mechanics-based methods. Fracture mechanics-based methods use energy to determine cracking; derived stresses are employed in calculating the energy. Shear lag stress solutions for cross-ply laminates have been used in many studies [15-24]. An example of one of these studies is a progressive model by Laws and Dvorak [24] that predicts initiation and accumulation of transverse microcracks. Variational methods [25-27], which minimize complementary energy, have been employed to obtain stresses and strains in cross-ply laminates. Finite element models [28-30] and elasticity models [31,32] have also been used to determine stress distributions. Continuum damage mechanics approaches [33-35] use the internal state variable concept to determine the stress fields in cracked laminates. Damage in the form of transverse microcracks is represented by second-order tensors in these approaches.

Although most of the mechanical load studies have concentrated on monotonic loads, some attention has been given to mechanical fatigue loads [36-43]. However, transverse microcracks are not the main focus of these fatigue studies. Instead, these studies use transverse microcracks as a measure of the amount of incurred damage. Those that do incorporate an analysis of transverse microcracks employ methods developed by others [33,44-47].

2.1.3 Thermal Loads

Overall, in comparison to the mechanical load prediction literature, only a few papers exist for thermal loads. Many mechanical load studies include thermal loads
through constant residual thermal stresses [11,12,13,19,20,22,24,25,27-29,48-51]; these residual stresses are assumed to be due to manufacture.

Some predictive methods have been developed for thermal loads using both strength-based and fracture mechanics-based approaches. Examples of strength-based approaches include CLPT [52-55] and ply discount methods [56]. However, these approaches suffer from the same limitations revealed by Flaggs and Kural as their mechanical load counterpart studies. Fracture mechanics-based approaches to predict transverse microcrack damage under thermal loads have also been developed. These studies will be reviewed in Section 2.5.

Thermal load studies concentrate mostly on areas outside of transverse microcrack prediction methods such as on the effects of microcracks. A number of studies by Bowles [52-54,56] and Tompkins [57-61] have investigated the effects of transverse microcracks on laminate material properties. Transverse microcrack suppression by examining the effects of reduced thicknesses have been studied by Bowles and Shen [54] and Manders and Maas [62]. Thermal fatigue is another area that has been the subject of several studies [53,56-64]. Techniques including finite elements and continuum damage mechanics methods have been used in the study of this area.

2.1.4 General Angle Ply Lay-ups

The most common lay-up configuration used in studying transverse microcracking is the cross-ply laminate. Most of the methods developed to predict microcracking deal with monotonic mechanical loads. The cross-ply configuration is popular because microcracks are more easily observed and modeled in this configuration. Only a few papers exist that investigate transverse microcrack damage in general angle ply configurations [15,21,30,48].
2.2 STUDIES WITH PROBABILISTIC COMPONENTS

Studies have used probability functions to characterize strength distributions and fatigue life. Distributions of effective flaws that are assumed to exist within the volume of composite laminates have also been characterized through the use of probability functions.

A set of papers authored by Peters [11-14] deals with strength predictions of cross-ply laminate specimens. At the heart of the method is the assumption that a two-parameter Weibull function characterizes cross-ply strength distributions. The two parameters are the shape parameter and the scaling factor; the scaling factor is referred to as the "characteristic strength." The analysis assumes that the fracture of the 90° plies is the result of statistically distributed defects of various size. Each defect occupies a single geometric element. The size of these elements is determined using a shear lag stress solution; the distance over which the stress returns to the far-field value is used in sizing the elements. The laminate is modeled as being comprised of these single flaw elements. Experiments are conducted to obtain the probability function parameters that describe this distribution of defects. Total number of cracks as a function of applied load data are collected from the experiments. Using this data, the shape parameter and the "characteristic strength" can be extracted. In essence, a Weibull distribution is employed to characterize the number of cracks or failures in the 90° plies as a function of strain or applied load.

Statistical functions have been used to characterize the distribution of residual strength and fatigue life of composite laminates subjected to fatigue loads. In a set of papers by Yang [40,41], a Weibull function is assumed to describe the "ultimate", or initial, strength of the laminate. An expression for the number of cycles to failure is developed which is founded upon the ultimate strength. Since the ultimate strength is assumed to be characterized by a Weibull distribution, the number of cycles to failure can also be described by a Weibull distribution. Similar to Peters' studies, testing is conducted
to obtain data for the probability function parameters. Ultimate strength data obtained from experimental fatigue specimens is used to extract the necessary parameters.

A similar method to Yang's is called the Strength-Life Equal Rank Assumption (SLERA) [42,43]. SLERA also assumes that a Weibull function adequately characterizes the fatigue life and static strength of the laminates. The basis of the model is that specimen strength directly correlates with fatigue life expectancy. The model implies that life expectancy is a function of applied load. For example, the lower the applied stress, the longer the expected fatigue life. Fatigue experiments are performed to obtain strength data from experimental specimens. This data is used to extract probability function parameters for the Weibull distribution that characterizes this relationship.

Flaws within the volume of composite laminates have been characterized by statistical functions [49,65-68]. These flaws serve as sources of transverse microcracks as described by Wang's "effective" flaw hypothesis. According to the hypothesis, certain types of microflaws within the laminate can be simulated by considering effective flaws which act like cracks of a given size and serve as initiation sites for transverse microcracks. In a set of studies by Wang [49,65], a probability function is employed to describe the distributions of effective flaw sizes and spacing.. A study by Xu [66] uses Rayleigh and Weibull functions to describe the distribution of flaw sizes as part of a predictive methodology. Fiber-bridging studies have also used Weibull functions to describe the distribution of effective flaws [67,68].

2.3 STUDIES RELEVANT TO CRACK GROWTH

Few papers in the published literature discuss or deal with crack growth. Most transverse microcracking models assume that initiation and growth occur instantaneously. However, some research has been done to experimentally observe and analytically model the growth of transverse microcracks subject to both static and fatigue loads.
A common assumption used in transverse microcrack damage models is that microcracks initiate and propagate instantaneously. Although difficult to verify experimentally, this assumption allows the strain energy release rate to be modeled as independent of crack length for the case of static loads [27]. Based on experimental observations by various sources, microcracks often initiate at fiber-matrix debonds [5,27,51,64] and areas of high fiber volume fraction, $V_f$. Initiating at these debonds, flaws grow quickly across the thickness of a given layer to form microcracks; subsequently, these microcracks then quickly grow across the width [27]. Another experimental investigation of cross-ply laminates observes that unstable growth of a flaw into a microcrack spanning the thickness of the ply occurs when a flaw attains a critical size [69].

In terms of modeling, Wang and Crossman [70] model flaw growth using two curves: the available energy release rate ($G_F$) curve and the fracture resistance ($G_R$) curve. $G_F$ as a function of flaw length in the thickness direction is calculated through an elasticity approach. $G_R$ as a function of flaw length in the thickness direction, also known as the R-curve in material testing, is obtained experimentally. By superposing the $G_F$ and the $G_R$ curves, the regions of stable crack growth and unstable crack growth can be determined for a given value of mechanical applied load. The value of crack length at which the $G_F$ and $G_R$ curve intersect represents the largest stable flaw length. The region below this largest flaw length represents stable growth; the region above this largest flaw length represents unstable growth.

Boniface et al. [71] model transverse microcrack growth for a specific case. Their study juxtaposes energy approaches and stress intensity approaches. Their model consists of a transverse microcrack growing across the width of the laminate half way between two microcracks positioned on either side of the growing microcrack. The two outside microcracks are assumed to be fully-formed through-cracks before the introduction of the growing microcrack. In the energy approach portion, an estimate of the strain energy
release rate as a function of \( a/W \) (\( a \) is the crack length of the growing crack, \( W \) is the width of the laminate) and the laminate stiffnesses is developed. By comparing values of strain energy release rates calculated from this model with those from two other models, they conclude that the dominant parameter is crack spacing, although crack length does have some influence.

In the stress-intensity approach portion, the stress intensity factor, \( K \), at the tip of the crack is found to depend on ply thickness and crack spacing. \( K \) is not found to have any explicit dependence on the crack length. The primary conclusion that is drawn on the basis of both portions of their study is that crack spacing is the dominant parameter in modeling transverse microcracks for this situation and possibly in a more general sense.

Dvorak and Laws [50,51] develop energy based cracking criteria for both the thickness and width directions for transverse microcracks. Energy release rate expressions that are applied separately for thickness and width directions are derived. Also, the difference and relationship between the critical strain energy release rates, or fracture toughness, in the thickness direction and the width direction are discussed.

Wang et al. [49,65] develop a stochastic method to model transverse microcrack growth from flaws in the thickness direction under quasi-static and fatigue conditions. Normal distributions determine the flaw sizes and spacings in a layer of a laminate. Both size and spacing are variables in their crack criteria. The flaws are described by the effective flaw hypothesis. Distributions of flaws that serve as initiation sites for transverse microcrack growth are used by a handful of others [66-68] in a similar manner but only flaw size distributions are described by probability functions.

Studies dealing with fatigue loads have also yielded interesting results. For example, a group of studies [3,69,72,73] details the effects of using different magnitudes of loads on test specimens. From experimental observation, if the maximum fatigue stress is below that of the maximum static cracking threshold stress then the resulting transverse microcracks grow slowly across the laminate. In contrast, if the maximum fatigue stress is
above that of the maximum static cracking threshold stress then the resulting transverse microcracks grow instantaneously.

Lafarie-Frenot and Henaff-Gardin [45,46] provide both experimental observations about fatigue loads and some analytical work modeling cracking. From their experimental specimens, four types of cracks were observed and classified. Based upon experimental observations, they model crack growth through a cracking law which provides the cracked surface growth rate as a function of the strain energy release rate. This cracking law is similar to a Paris law. A conventional Paris law to model crack growth in cross-ply composite laminates for fatigue loads is developed by Boniface and Ogin [47]. A stress-intensity approach is taken in developing their Paris law.

2.4 STUDIES RELEVANT TO MATERIAL VARIATIONS

Because transverse microcracks are due to the fracture of the matrix of composite materials [74], inhomogeneities in the material have implications for the cracking behavior of a given laminate. The effects of material variations can be seen at different levels. Scatter in the data of experimental values of fracture toughness for a given material partly reflects the effects of inhomeogeneities. Defects and variations in the microstructure of the material, which have been observed experimentally, can be modeled to demonstrate the effects of these inhomogeneities.

Material variations can be reflected through the scatter in experimental fracture toughness values. For epoxy matrix, an early study reports a toughness range between 86 and 200 J/m$^2$ [75]. More recently, O'Brien and Martin have measured fracture toughness values for different material systems [76]. For example, fracture toughness results obtained for AS4/3501-6 material system range from 80 to 350 J/m$^2$, approximately. Some of the scatter in the experimental results is due to gathering results from different laboratories even though the same material and test is used by all. Additionally, scatter in experimental data can be affected by data reduction methods. Different methods of data
reduction affect the resulting values of "effective" fracture toughness [77]. For a given test, the effective fracture toughness value depends upon the specific data reduction method used to obtain the final number. A combination of different laboratories, different tests, different data reduction methods, and material variations results in differences in fracture toughness values that are obtained for a given material. Experimental scatter reflects the impact of these intertwined factors. Although this data hints at significant material variations, they cannot be isolated from the other factors. Thus, no quantitative measure of material variations can be reduced from this data.

Experimental observations of transverse microcracks can be helpful in understanding material variations seen in composites. As reported in one study, fiber volume fractions, \( V_f \), vary widely within a given layer [55]. Regions of higher \( V_f \) usually contain a higher concentration of defects, or flaws, that serve as initiation sites for transverse microcracks [5,51]. For some material systems, such as GFRP's and CFRP's, flaws often form around the outside of a fiber. As they grow, these flaws jump across the matrix such that they can follow a path along the outside of neighboring fibers. These flaws continue to grow by following fiber-to-fiber paths and avoiding resin-rich areas [64]. The combination of regions of higher \( V_f \)'s and the presence of flaws create regions that are favorable for transverse microcracks.

In terms of modeling the effects of material variations, some analytical work has been conducted by a handful of researchers on modeling the effects of inhomogeneities on cracks [78-80]. Pijaudier-Cabot and Bazant [78] investigate the interaction between cracks and between cracks and voids. They find that apparent fracture toughness varies with fiber volume fraction, distribution of fibers and voids, and the elastic properties of the constituents.

Axelson and Pyrz [79] use a two-dimensional stress analysis to look at the varying interaction between fibers and matrix for the case of randomly dispersed fibers. The effect of a fiber upon its surroundings is examined by modeling the stresses around the fiber. In
a qualitative discussion, the authors hypothesize the stress-intensity factor (SIF) alters the fracture toughness of the material surrounding the fiber; SIF is a function of the stress field around the fiber. Depending upon the location of the tip of a crack around the fiber, the material is either effectively toughened or effectively softened depending upon the stress at that location. Although the work considers only a single fiber in an infinite medium, the authors note that this analysis can be extended to include other cases such as fibers in a composite ply. A small step in this direction is a brief discussion about the effect of a cluster of fibers on the SIF of a crack. They find that as the crack moves toward the cluster the SIF decreases; in contrast, the SIF increases when the crack is alongside the cluster of fibers.

In another paper by Pyrz [80], the effect of modeling fiber distribution is considered. This study shows that modeling fiber distributions in a regularly patterned manner results in overestimation of strength and fracture properties. From the analysis, cracks tend to avoid regularly patterned fiber distributions and favor disordered areas for propagation. Thus, geometrical disorder has a significant role in the fracture and microcracking behavior of composites.

Support for the preceding studies is found in a set of papers by Bowles [53,81-83]. In regions of higher $V_f$, the fibers are more closely spaced. As a result, the radial stresses around individual fibers increases [53,81,82]. A concurrent view is found in the work of Hiemstra and Sottos [83]. They concluded that decreases in fiber-spacing leads to increases in local stresses. Regions of higher $V_f$ are areas of higher local stresses as compared to regions of lower $V_f$.

Another type of material variation is examined by Paluch [84]. Paluch models the amount of fiber "waviness" of three different material systems in three-dimensions; fiber "waviness" is the term used to describe fiber undulations and fiber misalignments. Each material system uses the same matrix material but different types of fibers. Experimental specimens are used to determine individual fiber paths by using regularly spaced cut
sections. Using the experimental results, three-dimensional fiber networks of the experimental specimens can be reconstructed for a model volume. Through this method, the amount of fiber waviness can be determined for the specimens of each material system. For each of the three material systems, results of the study demonstrate that the amount of fiber waviness differs according to fiber type. Results show a broad range from fairly straight to wavy. The motivation behind the work is to investigate the effects of this form of material variation upon compressive strengths. However, the results obtained through this research have implications for transverse microcrack damage. For example, fiber waviness demonstrates the variations of a material in the direction parallel to the fibers. Also, fiber waviness can influence the propagation of cracks. Fiber waviness is a random phenomenon whose degree of influence depends on the combination of types of fiber and matrix.

2.5 RECENT WORK

Those studies that have directly influenced the development of the transverse microcrack damage model in the present work are described in this section. The works reviewed include analytical and experimental research by Xu, studies that employ effective flaw distributions, progressive incremental damage models using fracture mechanics-based approaches with shear lag solutions to obtain stresses, and energy-based cracking criteria for two directions.

Xu [66] develops a methodology to predict transverse microcracks in cross-ply laminates. This method is referred to as a "probabilistic-analytic" method. In other words, probabilistic and analytic aspects are combined in a single transverse microcrack damage model that predicts crack density. The probabilistic aspect is incorporated through the use of probability functions to characterize the distribution of effective flaws; these flaws conform to Wang's effective flaw hypothesis. The analytical aspect is based on a shear lag solution of the stress fields.
From experimentation, Xu notes that certain types of defects appear to serve as the source of microcracks while others do not. "Globe"-shaped voids appear to serve as initiation sites for microcracks while "oblate"-shaped voids do not. Transverse microcracks are observed to initiate generally from the edges of the experimental specimens. X-ray inspection of test specimens reveal cracks that fall within the different crack types classified by Lafarie-Frenot and Henaff-Gardin [45,46]. Finally, qualitative mention is noted of behavior that can be described by an apparent fracture toughness.

Wang's effective flaw hypothesis has been used by other researchers to model the effects of defects in composite laminates [49,65-67]. As in the case of Xu, all of these studies use probability functions to describe the distribution of effective flaws.

Wang, Chou, and Lei [49] also develop a method that combines probabilistic and analytical components to predict transverse microcrack damage. The probabilistic aspect is embodied by normal distributions of flaw sizes and spacing and a Monte Carlo simulation scheme to simulate cracking. Briefly, Monte Carlo methods simulate processes involving elements of chance; random numbers are often used to emulate these probabilistic mechanisms because they are easy to generate [85]. The analytical aspect is the use of an energy-based cracking criteria. Both flaw size and spacing are variables in the cracking criteria.

Laws and Dvorak develop a progressive incremental damage method [24]. This work serves as the basis for a group of studies. Their model, which is a fracture mechanics-based approach with a shear lag solution of the stresses, predicts the initiation and accumulation of transverse microcracks in cross-ply laminates subject to mechanical loads. Accumulation refers to the increase in the number of transverse microcracks forming in the matrix of a layer in a laminate after cracking has initiated in the layer. Similarly, McManus et al. [86] develop a predictive methodology to predict initiation and accumulation of transverse microcracks in cross-ply laminates subject to thermal loads. Crack densities and degraded laminate material properties are calculated as functions of
progressively decreasing temperature; thermal-cycling is modeled as well. Also, a knockdown factor approach is developed to determine degraded laminate material properties.

Building on the work of McManus et al, Park [87] develops an incremental damage method to predict transverse microcracks in general angle-ply laminates subject to decreasing temperatures and thermal fatigue loads. Incorporated into the model is the ability to use temperature-dependent material properties. The incremental damage method is implemented through a computer program originally developed by McManus et al.

Further expanding on this incremental damage method, Maddocks [88] modifies Park's method to incorporate mechanical in addition to thermal loads. Crack densities and degraded laminate properties are predicted for general angle-ply laminates subject to progressively increasing mechanical loads and/or decreasing temperatures. Shear lag parameters and fracture toughness values necessary to perform the analysis are determined experimentally. Comparisons with experimental results show that predictions capture crack densities trends reasonably well after microcrack initiation. Initiation of transverse microcracking is not as well predicted as accumulation. However, a rather useful general predictive tool is developed.

All of the above models are one-dimensional. Dvorak and Laws [51] develop cracking criteria in two dimensions. Separate energy expressions for the thickness and width directions, respectively, are obtained. This study is unique in that it models crack growth in two directions while other studies are confined to only one direction.

2.6 SUMMARY OF BACKGROUND STUDIES

Background research is categorized into five topics. These topics include studies concerning transverse microcrack damage, studies with probabilistic components, studies dealing with crack growth mechanisms, studies dealing with material variations, and
studies that have directly contributed to development of the transverse microcrack damage model in the present work.

Early experimental studies demonstrated the effects of transverse microcracks on the material properties of composites. Early predictive methods illustrated the potential of analytical approaches to model transverse microcrack damage. Transverse microcrack studies for mechanical and thermal loads have segregated into two groups: strength-based methods and fracture mechanics-based methods. Thermal load transverse microcrack studies have concentrated mostly on the effects of microcracking while little attention has been directed toward developing predictive methodologies.

Studies with probabilistic elements use probability functions to characterize distributions. These functions have been used to characterize distributions of strength and fatigue life. Also, in cracking studies, these functions have been used to characterize distributions of flaws that serve as sources of cracks. The most common function used to characterize the various distributions is the Weibull function.

Studies that discuss or deal with crack growth mechanisms in connection with transverse microcracks have been conducted for both static and fatigue loads. These studies include experimental observations and analytical approaches. Analytical methods for static loads include energy and stress-intensity approaches. Modeling crack growth for fatigue loads has been accomplished using growth laws.

Studies outside of microcracking dealing with material variations are both experimental and analytical. Experimental data and observations demonstrate the effects of local material variations within laminates. Scatter in fracture toughness values partly shows the effects of inhomogeneities. Also, experimental observations of transverse microcracks help in gaining a better understanding of material variations. Analytical methods model the effect of material variations on local conditions within the matrix material. Local conditions can appear to affect the local fracture toughness of a material resulting in a locally varying apparent fracture toughness.
Various works have directly impacted on the development of the transverse microcrack damage model in the present study. Transverse microcrack studies that use a combination of probabilistic and analytic components have been published. Fracture mechanics-based methods with shear lag approaches have been developed to predict transverse microcracking in cross-ply laminates for thermal and mechanical loads. These methods have been expanded to include general angle ply laminates. One transverse microcrack damage study develops energy-based cracking criteria for two directions.

2.7 DAMAGE MODEL PARAMETERS

This section summarizes the material properties and analytical parameters necessary for the current microcrack damage model. An understanding of these parameters is critical to the development of the model. Incorporating a probability function to describe a distribution of effective flaws, a fracture mechanics-based method with a shear lag approach, and a model of material variations in a transverse microcrack damage model requires three sets of parameters. These are probability function parameters, shear lag stress solution and fracture toughness parameters, and material variation parameters.

A two parameter Weibull probability function requires two parameters, the shape factor and the scaling factor, to characterize a given distribution. Researchers who have used probability functions, such as in strength distributions, have adjusted the parameters such that they reflect the desired characteristics. Examples of these adjustments can be seen in strength-based studies for fatigue loads \([11-14,40-43]\). From experimental data, Weibull distributions of fatigue specimens can be characterized by extracting values for the two parameters.

Distributions of effective flaws are determined in a similar method. In a study published by Xu \([66]\), parameters for probability functions are selected such that they reflect the cracking traits exhibited by experimental specimens. A collection of papers by
Wang et al. [49,65] select values to describe distributions of effective flaw sizes and spacing.

Shear lag approaches usually require a variable called a shear lag parameter, which is difficult to measure directly. Depending on the particular method, shear lag parameters can differ because no standard definition exists and, hence, they depend on the derivation of each particular solution. Some assume that this parameter is laminate dependent [10,24] while others assume it is laminate independent [12,23,86-88].

Fracture toughness of a given material is measured by a critical strain energy release rate. Intralaminar fracture toughness that governs transverse microcracking is currently impossible to measure. Therefore, as an approximation, the interlaminar fracture toughness, which governs delamination, is measured by a variety of tests such as the Double Cantilever Beam Test.

Toughness values often exhibit a considerable amount of scatter [75,76]. Within a given laminate, there exist local variations in toughness, an apparent fracture toughness, depending upon geometric location. Reasons for apparent fracture toughness variations within composite materials are attributable to inhomogeneities that include non-uniformity of the matrix, various defects, and variability of fiber volume fractions. Refinement of processing techniques have probably shifted the onus towards the latter two, at least in the case of prepregs. Experimental observation of local changes in $V_f$ has been recorded [64]. A few studies have been conducted on the effects of $V_f$ variations on toughness variations [53,78-84] but more work appears to be necessary to gain a quantitative understanding.

In comparison to crack growth in the longitudinal direction, crack growth in the thickness direction has received much less attention. In the developing a transverse microcrack prediction model, Dvorak and Laws investigate crack growth in the thickness direction [51]. Expressions for strain energy release rate in the thickness and longitudinal directions are derived. Fracture toughness in the thickness direction of composite
laminate is measured proportionally to the longitudinal direction by a scaling factor. However, these scaling factors are essentially non-existent in the published literature. Only ranges of values for these scaling factors can be estimated.
CHAPTER 3

APPROACH

3.1 PROBLEM STATEMENT

The purpose of this study is to enhance our understanding of transverse microcracking in the matrix of composite laminates. Motivating this study is a continuing lack of knowledge of this damage type despite the considerable volume of literature. Development of a tool that predicts crack densities, degraded laminate properties, and crack distributions assuming prior knowledge of laminate geometry, material properties, load histories, parameters characterizing the probability function that describes a distribution of flaws, and parameters describing material variations addresses this problem. Material softening effects and temperature dependence of material properties will also be taken into account.

The general approach used to resolve the problem has four stages. First, a predictive methodology is developed. Second, key parameters in the method are investigated to determine their influence on microcrack predictions. Third, results from the method are compared to previously collected experimental data to evaluate the validity of the method. Fourth, some insights into the mechanisms by which microcracks occur will be gleaned from the results.

3.2 MODELING APPROACH

The modeling approach combines probabilistic and analytic components. The probabilistic component is divided into two sub-sets. One sub-set is a distribution of effective flaws, which is characterized by a Weibull probability function that describes
the variation in flaw sizes. Effective flaws are described by Wang's effective flaw hypothesis [49,65]. The other sub-set is comprised of the random seeding of flaw locations for each ply group, random inspection of flaws and cracks in each layer, and two random length parameters in the material variations model. The analytical component is a synthesis of a shear lag solution of the stress and displacement fields near a given transverse microcrack and energy-based cracking criteria founded on fracture mechanics.

Prior to performing the microcracking damage calculations, each layer of the laminate is randomly seeded with effective flaws. Then, given a specified load profile, the incremental microcrack damage method is applied to simulate transverse microcracking within the laminate; material softening and temperature dependence of material properties can be taken into account. At every load increment, each layer, in turn, is inspected individually while the properties of the other layers are smeared. Two distinct energy based cracking criteria that model the formation and growth of transverse microcracks are utilized. The first criterion, which must be satisfied before application of the second, determines whether a transverse microcrack will form from an effective flaw. If it is energetically favorable to do so, a transverse microcrack that spans the thickness of the layer will form instantaneously from an effective flaw. The second criterion is applied incrementally across the width of the laminate to model the growth of a transverse microcrack. If it is energetically favorable to do so, a given microcrack will extend an incremental distance in the width direction. For a given load increment, this crack will continue to progress across the width an increment at a time as long as conditions are favorable. The general process is repeated for the entire load profile.

The method also incorporates local material variations within each layer of the laminate through an apparent fracture toughness model. The effects of inhomogeneities in the material on the development of microcracks throughout the load profile are investigated by modeling the fracture toughness as a function of spatial variables. The
inhomogeneities include the effects of local material and geometric variations on the local stresses and non-uniformities affecting the toughness of the matrix material.

The incremental microcrack damage simulation method is encoded in a computer program. The computer code predicts crack densities, crack distributions, and effective laminate properties at discrete load increments for a specified load profile. Investigation of various parameters and their effects on the development of transverse microcracks within the volume of the laminate is then studied with the aid of this tool.
CHAPTER 4

THEORY AND ANALYSIS

The probabilistic-analytical transverse microcracking model is described in this chapter. The key components of the model include: laminate geometry, the Weibull function that will be used to describe the distribution of effective flaw sizes, shear lag solutions of the stress and displacement fields, energy expressions used in the cracking criteria, the material variations model, the material property degradation knockdown factor, effective ply and laminate properties, and a discussion of the difference between true and apparent crack density. Implementation of the model is described in the next chapter.

4.1 MODEL GEOMETRY

A given section of the laminate, aligned with a global \( xyz \) coordinate system, is modeled as shown in Figure 4.1. The laminate is comprised of unidirectional plies. Contiguous, stacked plies oriented at the same ply angle are referred to as ply groups, which are assumed to act as a single thick ply. Loading is applied on the \( x \) edges; the \( y \) edges are assumed to be free.

Each ply group is assumed to contain effective flaws either on the free edges or within the interior. These effective flaws vary in size according to a Weibull function and are located randomly in each ply group. "Starter" cracks are assumed to initiate from these flaws. Starter cracks are partial cracks that span the thickness of a ply group, have initial lengths that extend a short distance in the \( y'-\)direction, and whose plane is aligned parallel with the \( y'-z' \) plane. "Partial" cracks are cracks that have extended from starter
Figure 4.1 (a) Laminate model geometry. (b) Different crack types.
cracks in the fiber direction (y'-direction) but do not extend from one side of the laminate to the other. "Through" cracks are cracks that extend from one side of the laminate to the other side in the fiber direction.

To model the behavior of a selected flaw or crack, the laminate is modeled as two separate components, as illustrated in Figure 4.2: the ply group containing the flaw or crack, referred to as the "cracking" ply group, and the "rest" of the laminate, the properties of which are smeared. A local x'y'z' coordinate system is defined for a fully-formed through crack as shown in Figure 4.1. The y'-axis is aligned parallel to the fiber-direction of the ply group containing the flaw or crack. Only fully-formed through cracks are depicted in Figure 4.1. The origin of the local coordinate system is located at the center of a given flaw, partial crack, or through crack.

4.2 FLAW MODEL

The material of a laminate is assumed to contain a variety of inherent microflaws. These microflaws are described by Wang's effective flaw hypothesis. According to the hypothesis, a distribution of effective flaw sizes represents the gross effects of various inherent microflaws that exist within a material. The distribution is described through the use of a probability function. The effective flaws serve as initiation sites for microcracks. In this study, the effective flaws are modeled as flat and circular. The plane of the flaws is aligned with the \( y'-z' \) plane of the particular ply group. A two-parameter Weibull function will be used for the effective flaw size distribution. The two parameters that describe any Weibull distribution are the shape parameter and the scaling parameter, respectively.

A modified version of the Weibull function used by Spearing and Zok [68] to describe the cumulative probability distribution of effective flaw sizes in their fiber-bridging study will be used in a similar capacity in this transverse microcrack damage study. In this work, the cumulative distribution function describing the effective flaw sizes is given by
Figure 4.2  Two separate components used in modeling the laminate - "Cracking" ply group and the "Rest of the Laminate"
\[ F_\delta = \exp \left[ - \left( \frac{d}{\delta} \right)^m \right] \]  

(4.1)

where \( F_\delta \) is the cumulative probability of the flaw having a size smaller than \( \delta \), \( \delta \) is the effective flaw half-size as shown in Figure 4.3, \( d \) is the characteristic effective flaw size, and \( m \) is the shape parameter. The scaling parameter is equal to \( d^m \). Figure 4.4 is an example of a cumulative probability distribution given by Equation (4.1).

4.3 SHEAR LAG SOLUTION OF STRESSES AND DISPLACEMENTS

The shear lag solution of the stress and displacement fields used in this study is borrowed from Maddocks [88]. Some of the basic details of the solution are reproduced here for clarity and understanding. A complete derivation can be found in the original work of Maddocks.

A one-dimensional shear lag model is used to determine the stress and displacement fields near a given crack. The model is aligned with the \( x'y'z' \) local coordinate system for each ply group in the laminate as defined in Figure 4.1. Assumptions contained within the shear lag model are uniform through-thickness displacement and normal stresses in every ply group. Also, shear stresses are assumed to exist only within the shear transfer region between ply groups and are assumed to be uniform through the thickness of the region. The shear transfer region has a thickness of \( a_q \).

Stiffnesses, CTE's, and displacements in the \( x' \)-direction are denoted by the variables \( E, \alpha, \) and \( u \), respectively. Normal stresses in the \( x' \)-direction are denoted by \( \sigma \). The shear stresses between the uncracked and cracked ply groups in the \( x'y' \)-plane are denoted by \( q \). Thicknesses, which are measured in the \( z \)-direction, are denoted by the variable \( a \). Subscripts denoted by \( c, r, q, \) and \( o \) symbolize the cracking ply group, the rest
Effective flaw size, $\delta$. Flaws are situated along the center-line of the cracking ply group.

Figure 4.3
Example Weibull cumulative probability distribution describing the distribution of effective flaw sizes.

Figure 4.4
of the laminate, the shear transfer region, and the entire laminate, respectively. Many of these variables are depicted in Figure 4.5.

The laminate is subjected to both a thermal load and an applied mechanical load. The thermal load, \( \Delta T \), is the difference between the laminate temperature, which is assumed constant, and a defined stress-free temperature. The applied mechanical load expressed as an applied stress is given by

\[
\sigma_a = \frac{N_x}{a_o}
\]  

(4.2)

where \( N_x \) is the laminate load in the \( x' \)-direction. \( N_x \) is a component of the laminate load vector, \( N' \); \( N' \) is obtained by transforming \( N \), the laminate load in the \( xyz \) coordinate system, to the \( x'y'z' \) coordinate system using the CLPT transformation matrix [89].

The system of equations that must be solved to obtain the stress and displacement fields includes equilibrium, stress-strain, and shear stress equations. All three sets sum to a total of seven equations, given in Equations (4.3) - (4.9). Figure 4.6 is the shear lag model used to obtain the system of equations.

Laminate equilibrium gives

\[
\sigma_a a_o = \sigma_r a_r + \sigma_c a_c
\]  

(4.3)

Equilibrium of the cracked ply group gives

\[
q = \frac{a_c}{2} \frac{\partial \sigma_c}{\partial x'}
\]  

(4.4)

Equilibrium of the rest of the laminate gives

\[
q = -\frac{a_r}{2} \frac{\partial \sigma_r}{\partial x'}
\]  

(4.5)

Combining the stress-strain and strain-displacement equations for the cracked ply group and the rest of the laminate results in Equation (4.6) and Equation (4.7), respectively. For the cracked ply group,
Illustration of the laminate model aligned with the $x'y'z'$ coordinate system and many of the variables used in the shear lag stress solution.
Figure 4.6  Shear lag model used to obtain the system of equations necessary to obtain the stress and displacement fields.
For the rest of the laminate,

\[ \frac{\sigma_r}{E_r} = \frac{du_r}{dx'} - \alpha_r \Delta T \]  

(4.6)

The shear stress between the cracked ply group and the rest of the laminate is represented through a stress-strain type equation of the form

\[ q = K(u_c - u_r) \]  

(4.8)

where \( K \) is a stiffness constant, \( u_c \) is the displacement in the cracking ply group, and \( u_r \) is the displacement in the rest of the laminate. \( K \) relates the shear stress and the displacements of the cracked ply group and the rest of the laminate. \( K \) is defined as

\[ K = \frac{G_{\text{eff}}}{a_q} \]  

(4.8)

where \( G_{\text{eff}} \) is the effective shear modulus of the shear transfer region.

By combining the previous seven equations, a nonhomogeneous, second order, linear differential equation as a function of the stress in the cracking ply group, \( \sigma_c \), can be obtained. Most of the basic details of the derivation of the stress and displacements are provided in Appendix A. Derivation of the effective constants used in the equations in this section are supplied in Section 4.7. The differential equation is

\[ \frac{\partial^2}{(\partial x')^2} (\sigma_c) - \frac{4\xi^2}{a_c^2} \sigma_c = -\lambda \]  

(4.9)

where the shear lag parameter is

\[ \xi = \sqrt{\frac{Ka_c a_c E_c}{2a_r E_r E_c}} \]  

(4.10)

\[ \lambda, \text{ a variable of convenience, is} \]
\[
\lambda = \frac{2K}{a_c} \left[ \frac{a_c \sigma_a}{a_r E_r} - (\alpha_c - \alpha_r) \Delta T \right] 
\]  

(4.12)

The far-field stress in the cracking ply group is

\[
\sigma_{\text{far}} = \left[ \frac{E_c}{E_o} \sigma_a + E_c (\alpha_o - \alpha_c) \Delta T \right] 
\]  

(4.13)

Using standard methods [90] with the boundary conditions given by \( \sigma (x' = \pm h) = 0 \), the solution of Equation (4.10) is

\[
\begin{align*}
\sigma_c (x') &= \left[ 1 - \frac{\cosh \left( \frac{2\xi x'}{a_c} \right)}{\cosh \left( \frac{2\xi h}{a_c} \right)} \right] \sigma_{\text{far}} \\
&= \left[ 1 - \frac{\cosh \left( \frac{2\xi x'}{a_c} \right)}{\cosh \left( \frac{2\xi h}{a_c} \right)} \right] \sigma_{\text{far}} 
\end{align*}
\]  

(4.14)

Physically, upon inspection of Equation (4.14), the shear lag parameter scales the distance from the crack over which the stress in the cracking ply group rises to the far-field stress. The shear lag parameter is easier to determine than separate values for \( K \), \( G^\text{eff} \), and \( a_q \). In practice, the shear lag parameter is approximately equal to 1.

Average stress in the rest of the laminate, \( \sigma_n \), is

\[
\sigma_n (x') = \frac{\sigma_o a_n}{a_r} \frac{a_c}{a_r} \left[ 1 - \frac{\cosh \left( \frac{2\xi x'}{a_c} \right)}{\cosh \left( \frac{2\xi h}{a_c} \right)} \right] \sigma_{\text{far}} 
\]  

(4.15)

The displacement in the cracking ply group, \( u_c \), is
\[ u_c(x') = \frac{a_r E_r x'}{a_o E_o} \left( \frac{a_o \sigma_r}{a_r E_r} + \alpha_r \Delta T \right) \left[ 1 - \frac{a_c}{2 \xi x'} \sinh \left( \frac{2 \xi x'}{a_c} \right) \right] \]

\[ + \alpha_c \Delta T x' \left[ 1 - \frac{a_r E_r}{a_o E_o} \left( 1 - \frac{a_c}{2 \xi x'} \sinh \left( \frac{2 \xi x'}{a_c} \right) \right) \right] \]

The displacement in the rest of the laminate, \( u_r \), is

\[ u_r(x') = \frac{\sigma_r x'}{a_r E_r E_o} \left[ a_o E_o - a_c E_r \left( 1 - \frac{a_c}{2 \xi x'} \sinh \left( \frac{2 \xi x'}{a_c} \right) \right) \right] \]

\[ + \frac{\Delta T x'}{a_o E_o} a_o E_o \alpha_r + a_c E_r (\alpha_r - \alpha_c) \left[ 1 - \frac{a_c}{2 \xi x'} \sinh \left( \frac{2 \xi x'}{a_c} \right) \right] \]

\[ (4.17) \]

4.4 ENERGY EXPRESSIONS

For either a "starter" crack to initiate from an effective flaw or a "partial" crack to extend an incremental distance, conditions must be energetically favourable for them to do so. A starter crack is a partial transverse microcrack that has initiated from an effective flaw. A partial crack is a transverse microcrack that has extended in the \( y' \)-direction from a starter crack but has not completely extended across the width of the laminate to form a through-crack. The general criteria used to measure favourability for both formation and extension of a crack is given by

\[ \Delta G \geq G_{lc} \]

\[ (4.18) \]
where $\Delta G$ is the change in strain energy release rate and $G_{fc}$ is the critical strain energy release rate (fracture toughness); $G_{fc}$ is considered to be a material property. Separate versions of Equation (4.18) are used to determine growth of an effective flaw to a starter crack and extension of a partial crack.

In this study, only Mode I cracking is considered because Mode II cracking is taken to be a secondary effect. In the Section 4.3, Mode I stresses (in the $x'$-direction) are calculated through the shear lag model. In most laminates of practical interest, these Mode I stresses are greater than or equal to Mode II (shear) stresses in the $x'y'$-plane. Also, the Mode II fracture toughness, $G_{IIc}$, is generally greater than the Mode I fracture toughness, $G_{Ic}$.

### 4.4.1 Effective Flaw-to-Starter Crack Growth

Calculating the strain energy release rate for a given effective flaw is accomplished through the use of an energy expression developed by Dvorak and Laws [50,51]. In their study, the change in strain energy release rate for a Mode I crack, which determines the growth of an effective flaw to a starter crack, is derived. The change in strain energy release rate for a given effective flaw in an orthotropic medium aligned with the $y'-z'$ plane for a constant load is given by

\[ \Delta G^f_i = \frac{1}{2} \pi \delta A \sigma_c^2 \]  

(4.19)

where $\delta$ is the effective flaw size shown in Figure 4.3, $A$ is a compliance, and $\sigma_c$, the stress in the cracking ply group at the location of the effective flaw, is given by Equation (4.14). The compliance, $A$, is taken to be equal to $2/E$. In calculating $\sigma_c$, the stress field between two existing cracks is assumed to be unaffected by the effective flaw in question as a first approximation. Figure 4.7 illustrates the geometry used in calculating the stress at the location of an effective flaw.
Figure 4.7 Illustration of the model geometry used in calculating the stresses in the cracking ply group, $\sigma_c$. 
The effective flaws are modeled as flat, circular cracks. The planes of the seeded effective flaws are aligned with the $y'-z'$ plane of the respective ply groups. The flaws are assumed to be aligned along the center-line of each of the ply groups.

By substituting Equation (4.14) into Equation (4.19), the strain energy release rate for an effective flaw is

$$\Delta G' = \frac{1}{2} \pi \delta \Lambda \sigma^2 \left[ 1 - \frac{\cosh \left( \frac{2 \xi x'}{a_c} \right)}{\cosh \left( \frac{2 \xi h}{a_c} \right)} \right]^2$$

By substituting the expression for the far-field stress in the cracking ply group, $\sigma_{cw}$, Equation (4.20) can also be expressed as

$$\Delta G' = \frac{1}{2} \pi \delta \Lambda \left[ \frac{E_c}{E_o} \sigma_a + E_c \left( \alpha_o - \alpha_c \right) \Delta T \right] \left[ 1 - \frac{\cosh \left( \frac{2 \xi x'}{a_c} \right)}{\cosh \left( \frac{2 \xi h}{a_c} \right)} \right]^2$$

The fracture toughness used to determine the growth of an effective flaw to a starter crack, $G_{lc}'$, is different from that used to determine the extension of a partial crack, $G_{lc}$. The differences between the fracture toughnesses are mostly due to the different geometric conditions in each of the cracking situations. However, according to Dvorak and Laws [51], $G_{lc}$ can be related to $G_{lc}'$ through a scaling factor. The scaling factor accounts for the differences between the separate cracking conditions. The relationship between the fracture toughness for the two situations is given by

$$G_{lc} = \gamma G_{lc}'$$

where $\gamma$ is the scaling factor. Unfortunately, values for $\gamma$ are confined to a couple of obscure sources. Because of such limited data, concrete conclusions about the exact magnitudes cannot be made [51]. Only a range can be estimated for $\gamma$; the range is
considered to be between 0.6 and 1 according to Dvorak and Laws. For the present work, \( \gamma \) will be considered to be equal to 1.

### 4.4.2 Partial Crack Extension

The change in strain energy release rate in Equation (4.18) for the extension of a partial crack is modeled as independent of crack length, \( l \), which is a common assumption made by many researchers as discussed in Section 2.3. Upon attaining a critical length, \( l_c \), the change in strain energy release rate for a given crack is independent of \( l \) due to the restraints of adjacent ply groups [87,88] as illustrated in Figure 4.8. In calculating the change in strain energy release rate, the starter crack that forms from an effective flaw upon compliance with the effective flaw-to-starter crack energy criterion is assumed to have a length that is greater than or equal to \( l_c \).

The strain energy release rate of an incremental extension of a partial crack is a slightly modified version of the one derived by Maddocks [88]. Change in strain energy release rate for an incremental extension of a partial crack in the \( y' \)-direction is the change in total energy from an uncracked state to a cracked state for a self-similar extension of length \( \Delta y' \). This is given by

\[
\Delta G_i = \frac{\Delta W - \Delta U}{a_c \Delta y'}
\]  

(4.23)

where \( \Delta G_i \) is the strain energy release rate for an incremental extension of a partial crack, \( \Delta W \) is the change in external work, and \( \Delta U \) is the change in internal energy.

Derivation of the change in strain energy release rate is same as the one-dimensional expression developed by Maddocks [88] except that the position of a hypothetical crack extension is not assumed to be at a location precisely halfway between pre-existing, neighbouring cracks, as illustrated in Figure 4.9. This follows the work of Laws and Dvorak [24]. The change in external work due to an incremental extension of a partial crack is
Figure 4.8 Change in strain energy release rate as a function of length, $l$. 
Figure 4.9  Illustration of an incremental extension of a partial transverse microcrack in the cracking ply group.
\[ \Delta W = \left( W_{1h} + W_{2h} \right) - W_{2h} \]  
(4.24)

where \( W_{1h} \) is the work done by the applied load before the extension of the crack and \( \left( W_{1h} + W_{2h} \right) \) is the work done by the applied load after the extension of the crack. \( W_{2h} \) is given by

\[ W_{2h} = 2a_o \sigma_a \Delta y' u_r \left( x' = h \right) \]  
(4.25)

where \( u_r(x' = h) \) is found using Equation (4.17). The quantities \( W_{i h} \), where \( i = 1, 2 \), is given by

\[ W_{ih} = 2a_o \sigma_a \Delta y' u_r \left( x' = \frac{h_i}{2} \right) \]  
(4.26)

The displacement, \( u_r(x' = h_i/2) \), can be obtained by re-solving Equation (4.10) using the boundary conditions: \( \sigma_r(x' = \pm h_i/2) = 0 \). This calculation of the change in external energy uses an approximation to simplify the calculations. Details are given in Appendix B.

The change in internal energy for an incremental extension of a partial crack has two contributing components: the change in internal strain energy due to the normal stresses and the change in internal strain energy due to the shear stresses residing in the shear transfer region. This is given by

\[ \Delta U = \Delta U_\sigma + \Delta U_q \]  
(4.27)

where \( \Delta U \) is the total change in internal strain energy due to the incremental extension of a partial crack, \( \Delta U_\sigma \) is the change in internal strain energy due to the normal stresses, and \( \Delta U_q \) is the change in internal strain energy due to the shear stresses located in the shear transfer region. \( \Delta U_\sigma \) and \( \Delta U_q \), respectively, are given by

\[ \Delta U_\sigma = \left( U_\sigma \bigg|_{h_i} + U_\sigma \bigg|_{h_2} \right) - U_\sigma \bigg|_{2h} \]  
(4.28)

and

\[ \Delta U_q = \left( U_q \bigg|_{h_i} + U_q \bigg|_{h_2} \right) - U_q \bigg|_{2h} \]  
(4.29)
The internal strain energy from the normal stresses, $U_\sigma$, and the shear stresses, $U_q$, respectively, are given by

$$U_\sigma = \frac{1}{2} \int \sigma^2 \, dV$$

(4.30)

and

$$U_q = \frac{1}{2} \int \frac{q^2}{G_{eff}} \, dV$$

(4.31)

The volume in Equation (4.30) is integrated from the location of the partial crack extension to neighbouring cracks in the $x'$-direction, over the length of the incremental extension, $Ay'$, in the $y'$-direction, and over the combined thicknesses of the cracking ply group and the rest of the laminate in the $z$-direction. Equation (4.31) is integrated over the same lengths in the $x'$ and $y'$ directions as Equation (4.30) but over the shear transfer regions between the cracking ply group and the rest of the laminate in the $z$-direction. The thicknesses of the shear transfer regions sum to $2a_q$ since shear stresses are transferred at the top and bottom of the cracking ply group.

The components of Equation (4.28) can be expressed in terms of Equation (4.30). The internal strain energy of the volume of the partial crack due to normal stresses before extending, $U_\sigma|_{2h}$, can be expressed as

$$U_\sigma|_{2h} = 2 \left( \frac{1}{2} a, Ay' \int_0^h \frac{\sigma_f^2}{E_r} \, dx' + \frac{1}{2} a_c, Ay' \int_0^h \frac{\sigma_c^2}{E_c} \, dx' \right)$$

(4.32)

Due to symmetry, Equation (4.32) can be integrated from 0 to $h$ and the integrals can be multiplied by 2. This gives the same result as integrating from $-h$ to $h$. The quantities $U_\sigma|_{h}$ where $i = 1, 2$, the internal strain energy of the volumes of the partial crack due to normal stresses after an incremental extension, can be expressed as

$$U_\sigma|_{h} = 2 \left( \frac{1}{2} a, Ay' \int_0^{h/2} \frac{\sigma_f^2}{E_r} \, dx' + \frac{1}{2} a_c, Ay' \int_0^{h/2} \frac{\sigma_c^2}{E_c} \, dx' \right)$$

(4.33)
The same symmetry considerations applied to Equation (4.32) can be applied to Equation (4.33). The normal stresses, $\sigma_r$ and $\sigma_o$, used in Equation (4.33) are obtained by solving Equation (4.10) using the boundary conditions $\sigma_c (x' = \pm h/2) = 0$.

Likewise, the components of Equation (4.29) can be expressed in terms of Equation (4.31). The internal strain energy of the volume of the partial crack due to shear stresses before extending, $U_q|_{2h}$, can be expressed as

$$ U_q|_{2h} = \frac{2}{K} \Delta y' \int_0^{h} q^2 dx' $$

(4.34)

The above form is obtained by substituting Equation (4.9) into Equation (4.31), integrating over the incremental crack extension length, $\Delta y'$, in the $y'$-direction, and integrating over the thicknesses of the shear transfer regions in the $z$-direction. Due to symmetry, Equation (4.34) can be integrated from 0 to $h$ and the integrals can be multiplied by 2. This gives the same result as integrating from $-h$ to $h$. An additional multiplicative factor of 2 is included in Equation (4.34) as well as Equation (4.35) because, as noted previously, the integrated volumes include both the top and bottom of the cracking ply group which have a combined thickness of $2a_q$. The internal strain energy of the volumes of the partial crack due to shear stresses after extending, $U_q|_{h}$ where $i = 1, 2$, can be expressed as

$$ U_q|_{h} = \frac{2}{K} \Delta y' \int_0^{h/2} q^2 dx' $$

(4.35)

The same symmetry considerations applied to Equation (4.34) can be applied to Equation (4.35). The shear stress, $q$, used in Equation (4.35) is obtained by solving Equation (4.10) using the boundary conditions $\sigma_c (x' = \pm h/2) = 0$.

More details of the derivation in this Section are supplied in Appendix C. Performing the appropriate integrations and combining Equations (4.23) - (4.35), the change in strain energy release rate for an incremental extension of a partial crack in the $y'$ direction can be expressed as
\[ \Delta G_I = \frac{a_c a_0 E_o}{2 \xi a_r E_r E_c} \sigma^2 \left[ \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) - \tanh \left( \frac{2 \xi h}{a_c} \right) \right] \]  

Substituting Equation (4.13) into Equation (4.36) and manipulating, Equation (4.36) can be re-arranged into the following form

\[ \Delta G_I = \frac{a_c E_c}{2 \xi a_r E_r a_0 E_o} \left[ a_o \sigma_a - a_r (\alpha_c - \alpha_r) \Delta T \right]^2 \left[ \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) - \tanh \left( \frac{2 \xi h}{a_c} \right) \right] \]  

(4.37)

4.5 MATERIAL VARIATIONS MODEL

Local inhomogeneities in the volume of the laminate can impact the behaviour of transverse microcracks. These local material variations can affect the change in strain energy release rate for both the formation of a microcrack from a flaw and the extension of microcrack into the laminate. Random distributions of fibers is a primary cause. Random spatial distributions of fibers create regions with different fiber volume fractions, \( V_f \). Dissimilarities in local \( V_f \) can occur in multiple dimensions. In the \( x' \)-direction, clustering can be observed experimentally along the edges of a specimen. Into the width (\( y' \)-direction) of the laminate, fiber clustering is more difficult to observe. Among the physical causes of these inhomogeneities are fiber waviness, tow twisting, and slight differences in fiber diameter along the length of a fiber. The effect of these inhomogeneities is that fiber clustering patterns vary as a function of depth into the width of the laminate. Overall, the local \( V_f \) patterns are a function of both the \( x' \) and \( y' \).

Researchers have observed these phenomena and preliminary attempts to model the effects of these inhomogeneities have shown that variations in local \( V_f \) affect the local stresses surrounding fibers. These local stress variations, which depend on location and the attendant local conditions at that location, can affect the change in strain energy.
release rate for a given flaw or microcrack. A first-order model of these effects on changes in strain energy release rates is

$$\Delta G(x', y') = \Delta G_o S_1(x', y')$$  \hspace{1cm} (4.38)$$

where $\Delta G(x', y')$ is the change in strain energy release rate as a function of two spatial variables, $\Delta G_o$ is the change in strain energy release rate with uniform material conditions, and $S_1(x', y')$ is function of two spatial variables representing the local fiber volume fractions, fiber-packing, and other aspects affecting local stresses. Equation (4.38) applies to both effective flaw-to-starter crack growth and partial crack extension.

Material variations can also be reflected in the fracture toughness of a material system. Despite refinements and improvements in the manufacturing techniques of composite materials, non-uniformities are still inherent within composites. Also, the degree of the variation can differ between production batches; in other words, some batches may have greater consistency than others. As a consequence, the matrix material within a particular laminate will not be entirely uniform throughout the entire volume. An argument can be made that an apparent local fracture toughness exists that is a function of spatial position. Additionally, defects, such as certain types of voids, in the matrix can appear to affect the fracture toughness of the material depending upon location of the defect relative to a crack. Fracture toughness as a function of spatial variables can be represented as

$$G_{fc}(x', y') = G_{fco} S_2(x', y')$$  \hspace{1cm} (4.39)$$

where $G_{fc}(x', y')$ is the fracture toughness as a function of two spatial variables, $G_{fco}$ is the fracture toughness with uniform material conditions, and $S_2(x', y')$ is function of two spatial variables representing the local material inhomogeneities such as voids, certain types of flaws, non-uniformity of the matrix material, etc. that affect the local material toughness within the laminate. Equation (4.39) applies to both effective flaw-to-starter crack growth and partial crack extension.
Equations (4.38) and (4.39) can be substituted into Equation (4.18) resulting in a spatially dependent energy criteria

\[ \Delta G(x', y') \geq G_{lc}(x', y') \] (4.40)

\[ \Delta G_{o}S_1(x', y') \geq G_{lo}S_2(x', y') \] (4.41)

Placing both spatial functions, \( S_1 \) and \( S_2 \), on the right side of the inequality gives

\[ \Delta G_{o} \geq G_{lo} \frac{S_2(x', y')}{S_1(x', y')} \] (4.42)

A single function, \( S_3 \), will arbitrarily be used to represent the combination of \( S_1 \) and \( S_2 \) since the form of these two functions is unknown. Equation (4.42) becomes

\[ \Delta G_{o} \geq G_{lo}S_3(x', y') \] (4.43)

where

\[ S_3(x', y') = \frac{S_2(x', y')}{S_1(x', y')} \] (4.44)

The form of \( S_3 \) is unknown. For parametric explorations, a sinusoidal function will be used to represent local material variations:

\[ S_3(x', y') = 1 + A \sin \left( \frac{2\pi}{\lambda_x} x' + \varphi_x \right) \sin \left( \frac{2\pi}{\lambda_y} y' + \varphi_y \right) \] (4.45)

where \( A \) is the maximum amplitude of the variation, the \( \lambda \)'s are length parameters (i.e. wavelengths), and the \( \varphi \)'s are phase shifts. The apparent fracture toughness model is obtained by substituting Equation (4.45) into Equation (4.43). The result of the substitution is

\[ \Delta G_{o} \geq G_{lo} \left[ 1 + A \sin \left( \frac{2\pi}{\lambda_x} x' + \varphi_x \right) \sin \left( \frac{2\pi}{\lambda_y} y' + \varphi_y \right) \right] \] (4.46)
The values of $A$ and the $\lambda$'s are assumed to be the same for all ply groups, but the $\varphi$'s are selected randomly for each. Equation (4.46) applies to both effective flaw-to-starter crack growth and partial crack extension.

The apparent fracture toughness model given by Equation (4.46) possesses a regularity in the variation that is not characteristic of real materials. Actual variations in apparent fracture toughness are erratic and random in nature. However, the simplistic model in Equation (4.46) does have magnitude and length parameters, which captures two key features of any real distribution. An illustration of the apparent fracture toughness is shown in Figure 4.10.

### 4.6 DEGRADATION OF LAMINATE PROPERTIES

The effective properties of the laminate are reduced due to transverse microcrack damage. The effects of microcracks in one ply group on the formation and extension of microcracks in other ply groups are included through expressions of reduced material properties that are functions of crack density.

Microcracks are assumed to degrade ply group properties as a function of crack density. Effective laminate properties are obtained from the degraded ply properties using CLPT. Stiffness loss in a cracked laminate is derived in a study by Laws and Dvorak [24]. Average strain of a segment between two microcracks in an uncracked portion of a mechanically loaded laminate can be shown to be

$$
\varepsilon_a = \frac{\sigma_a}{E_o} \left[ 1 + \frac{a_c^2 E_c}{2h \xi a_c E_r} \tan \left( \frac{2h \xi}{a_c} \right) \right]
$$

Equation (4.47) is valid for any two microcracks separated by a distance of $2h$. The average crack density will be expressed as

$$
\rho = \frac{1}{2h}
$$
Figure 4.10 Illustration of the apparent fracture toughness of the cracking ply group as calculated through the material variations model. (a) 90° ply group. (b) 60° ply group.
Substituting Equation (4.48) into Equation (4.47) and re-arranging gives the effective stress-strain relation for the cracked laminate, which is given by

$$\sigma_u = E_0(\rho)\epsilon_u$$  \hspace{1cm} (4.49)

where

$$E_0(\rho) = \frac{E_o}{1 + \frac{\rho a_c^2 E_c}{\xi a_r E_r} \tanh \left( \frac{\xi}{\rho a_c} \right)}$$  \hspace{1cm} (4.50)

$E_0(\rho)$ is the new laminate stiffness as a function of crack density.

Expanding on this work, McManus et al. [86] derive the reduction of all laminate properties due to microcracks. Considering that the reduction in stiffness to be caused entirely by a reduction of the effective stiffness of the cracking ply group, a knockdown factor, $\kappa$, due to microcracks is defined as

$$E_c(\rho) = \kappa E_c$$  \hspace{1cm} (4.51)

Solving Equation (A.14) for $E_o$ gives

$$E_o = \frac{a_r E_r + a_c E_c}{a_o}$$  \hspace{1cm} (4.52)

Substituting the above expressions of $E_o(\rho)$ and $E_c(\rho)$ into $E_o$ and $E_c$, respectively, in Equation (4.52) gives

$$E_o(\rho) = \frac{a_r E_r + a_c \kappa E_c}{a_o}$$  \hspace{1cm} (4.53)

Substituting Equations (4.52) and (4.53) into Equation (4.50) and solving for $\kappa$ gives

$$\kappa = \frac{a_r E_r \left[ 1 - \frac{a_c \rho}{\xi} \tanh \left( \frac{\xi}{a_c \rho} \right) \right]}{a_r E_r + a_c E_c \frac{a_c \rho}{\xi} \tanh \left( \frac{\xi}{a_c \rho} \right)}$$  \hspace{1cm} (4.54)
\( \kappa \) is used to calculate the degraded laminate properties due to transverse microcracks. Details of the implementation of the knockdown factor are elucidated in Section 4.7.

### 4.7 DERIVATION OF EFFECTIVE LAMINATE AND PLY PROPERTIES

The stiffness constants used in the analysis, originally developed by Park [87], are derived using CLPT [89]. The equivalent stiffnesses \((E_o, E_r, E_c)\) are needed in many of the equations. Each ply \(i\) possesses the standard material properties: \(E_{li}\) (longitudinal stiffness), \(E_{ri}\) (transverse stiffness), \(v_i\) (major Poisson's ratio), \(G_i\) (shear stiffness), \(\alpha_{li}\) (longitudinal CTE), and \(\alpha_{ri}\) (transverse CTE). Ply \(i\) has a thickness \(t_i\). If available, appropriate temperature-dependent material properties can be taken into account.

The angles used in the analysis, illustrated in Figure 4.11, are defined in the following manner. The fibers of each ply are aligned at an angle of \(\theta_i\) to the \(x\)-axis of the global coordinate system. The coordinate system of a given ply has the \(x_i\)-axis aligned parallel to the fibers and the \(y_i\)-axis aligned perpendicular to the fibers. \(\theta_i\) is the angle between the global \(x\)-axis and the ply \(x_i\)-axis. The cracking ply group has its fibers aligned at an angle of \(\Theta_c\) to the global coordinate system. The local \(x'y'z'\) coordinate system of the cracking ply group is aligned such that the \(x'\)-axis is perpendicular to the fiber direction and the \(y'\)-axis is aligned parallel to the fibers. In the \(x'y'z'\) coordinate system, the ply angles are defined:

\[
\theta' = \theta + \phi_c
\]

where

\[
\phi_c = 90^\circ - \Theta_c
\]

\(\phi_c\) is the angle between the global \(x\)-axis and the cracking ply group \(x'\)-axis. \(\theta'\) is the angle between the ply \(x_i\)-axis and the cracking ply group \(x'\)-axis.

The laminate properties necessary to perform the analysis are derived using CLPT. Laminate stiffness in the \(x'y'z'\) coordinate system is given by
Figure 4.11 Illustration of the angles used in the calculations. Ply $c$ is the cracking ply group. Ply $i$ is the ply group under consideration.
\[ A = \sum_{i=1}^{n} \overline{Q}_i t_i \]  

(4.57)

\[ \overline{Q}_i, \text{ the rotated reduced ply stiffnesses in the } x’y’z’ \text{ coordinate system, is given by} \]

\[ \overline{Q}_i = T_i^{-1} Q_i T_i^{-\top} \]  

(4.58)

where

\[ Q_i = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \]  

(4.59)

and

\[ T_i = \begin{bmatrix} \cos^2 \theta_i' & \sin^2 \theta_i' & 2 \sin \theta_i' \cos \theta_i' \\ \sin^2 \theta_i' & \cos^2 \theta_i' & -2 \sin \theta_i' \cos \theta_i' \\ -\sin \theta_i' \cos \theta_i' & \sin \theta_i' \cos \theta_i' & \cos^2 \theta_i' - \sin^2 \theta_i' \end{bmatrix} \]  

(4.60)

The reduced ply stiffnesses in Equation (4.59) are given by

\[ Q_{11i} = \frac{E_{ii}}{D_i} \quad Q_{12i} = \kappa_i \frac{\nu_i E_{ii}}{D_i} \]

\[ Q_{22i} = \kappa_i \frac{E_{ii}}{D_i} \quad Q_{66i} = \kappa_i G_{ii} \]  

(4.61a-e)

\[ D_i = 1 - \nu_i^2 \frac{E_{ii}}{E_{uu}} \]

\( \kappa_i \) is the knockdown factor for ply \( i \) as defined by Equation (4.54); the same value of \( \kappa_i \) is applied to all matrix-dominated properties. \( \kappa_i = 1 \) until cracking initiates in ply \( i \).

The CTE of each ply is given by

\[ \alpha_i = \begin{bmatrix} \alpha_{ii} \\ \alpha_{ii} \\ 0 \end{bmatrix} \]  

(4.62)

In the \( x’y’z’ \) coordinate system, the rotated ply CTE’s are given by
\[ \bar{\alpha}_i = T_i^T \alpha_i \quad (4.63) \]

The laminate constants needed to perform the analysis are derived below. The total laminate constants are given by

\[ E_o = \frac{A_{11}}{a_o} \quad a_o = \sum_{i=1}^{n} t_i \quad (4.64a-b) \]

The cracking ply group constants are given by

\[ E_c = Q_{22} = \frac{E_{\text{inv}}}{D_c} \quad \alpha_c = \alpha_{tc} \quad a_c = t_c \quad (4.65a-c) \]

The smeared properties and constants for the rest of the laminate are given by

\[ E_r = \frac{A_{11} - E_c a_c}{a_r} \quad \alpha_r = \alpha_1 \quad a_r = a_o - a_c \quad (4.66a-c) \]

\( \alpha_1 \) is the first element in the smeared CTE of the rest of the laminate, which is given by

\[ \alpha_r = \left[ A - \overline{Q}_c t_c \right]^{-1} \left( \sum_{i=1}^{n} \overline{Q}_i \bar{\alpha}_i t_i - \overline{Q}_c \bar{\alpha}_c t_c \right) \quad (4.67) \]

After performing the cracking analysis for all the ply groups for a given load increment, the degraded laminate properties in the global x-direction are calculated. The degraded laminate properties are calculated using

\[ E^{\text{eff}} = \frac{1}{A_{11} \cdot a_o} \quad A^{\text{inv}} = A^{-1} \quad \alpha^{\text{eff}} = \alpha_1 \quad (4.68a-c) \]

\( \alpha_1 \) is the first element of

\[ \alpha^r = A^{-1} \sum_{i=1}^{n} \overline{Q}_i \bar{\alpha}_i t_i \quad (4.69) \]

In Equations (4.68) and (4.69), \( A, \overline{Q}, \) and \( \bar{\alpha} \) are calculated from Equations (4.57) - (4.63) with \( \theta' = \theta \).

Bending is not incorporated into this model so that its use is restricted to symmetric laminates. The above equations are used in the calculations at each load increment in a defined load profile. Each load increment incorporates the knocked-down
Figure 4.12 True crack density versus apparent crack density.
properties of all the plies from the previous load increment and temperature-dependent properties for the conditions of the current increment.

4.8 TRUE AND APPARENT CRACK DENSITY

True crack densities and apparent crack densities differ when dealing with angle ply laminates. True crack densities calculated in the analysis represent the perpendicular distance between cracks in the local $x' y' z'$ coordinate system of each of the ply groups. Apparent crack densities represent the distance measured between cracks in the global $x y z$ coordinate system. The apparent crack density is the one observed during edge inspections of ply groups of experimental specimens. Thus, depending on the angle of a particular ply group, the apparent crack spacing differs as a function of the ply angle. This geometric effect is illustrated in Figure 4.12. To account for this geometric effect, the true crack density for a particular ply group is multiplied by a geometric factor. Multiplication of the true crack density by the geometric factor for a particular ply group results in the apparent crack density as shown in Equation (4.70).

$$\rho_{\text{apparent}} = \rho_{\text{true}} \sin(\theta_i) \quad (4.70)$$
CHAPTER 5

IMPLEMENTATION

The components developed in Chapter 4 are combined into an incremental transverse microcrack damage model. This model will predict transverse microcrack damage in a symmetric laminate for a given load profile. The damage model is encoded into a computer program.

The implementation consists of four general parts, as illustrated in Figure 5.1. Each of the parts in Figure 5.1a corresponds to a section in this chapter. Section 5.1 contains the inputs necessary to perform the analysis. Section 5.2 represents the calculations and steps prior to executing the analysis. Section 5.3 represents the execution of the incremental damage analysis. Section 5.4 represents the results from the analysis.

5.1 INPUTS

The inputs needed to perform the analysis can be divided into two classes: physical input parameters and numerical input parameters. The physical input parameters represent parameters that reflect physical characteristics, material properties, etc. Numerical input parameters are values that facilitate the execution of the analysis. Almost all of the input parameters are specified by the user except for those determined through the random number generator.

The physical input parameters include the material properties, laminate geometry, load profiles, the fracture toughness scaling factor, Weibull function parameters describing the distribution of effective flaw sizes, the effective flaw density, parameters for the material variations model, and laminate dimensions. Material properties are
Program Inputs

Pre-Analysis Calculations and Steps

Load Increment

Calculate Material Properties at Temperature

Set to Local Coord. Sys. of CPG

Calculate Stiffness Constants

Inspection for Crack Formation and Crack Extension*

Record Crack Density and Crack Distribution Information

Calculate Laminate Properties

Update Knockdown Factor

Results of Analysis

Figure 5.1a Flow chart of the implementation of the incremental damage model.
Figure 5.1b  Flow chart of the inspection algorithm of effective flaws and partial cracks.
standard information or can be obtained through experimental testing. Laminate geometry includes the lay-up configuration and the ply group thicknesses. Load profiles include the amount and sequence of mechanical and/or thermal loads applied to the laminate. The fracture toughness scaling factor, $\gamma$, described in Equation (4.22) relates the different fracture toughnesses used to determine effective flaw-to-starter crack growth and partial crack extension. The Weibull parameters include the shape parameter, $m$, and characteristic effective flaw size, $d$, in Equation (4.1). These parameters characterize the distribution of effective flaw sizes used in the analysis. The density of effective flaws, $EFD$, is the number of effective flaws per unit length that are seeded in each of the ply groups of the laminate. The material variations model described by Equation (4.46) requires three parameters. These parameters include the amplitude of the material variations, $A$, and the lengths over which the amplitude varies in the $x'$-direction, $\lambda_{x'}$, and the $y'$-direction, $\lambda_{y'}$. The laminate dimensions needed to perform the analysis include the length and the width of the laminate over which the analysis is conducted.

Numerical parameters that facilitate the execution of the analysis include the incremental crack extension length of partial cracks, values used to discretize the Weibull function, and an initialization value for the random number generator. The incremental crack extension length of a partial crack, $\Delta y'$, is the length that a partial crack extends when conditions are energetically favorable. Discretization of the Weibull function that characterizes the distribution of effective flaw sizes requires three parameters. These include the lower flaw size bound of the Weibull function, $\delta_{\text{low}}$, the upper flaw size bound, $\delta_{\text{up}}$, and the number of discrete flaw size increments between the lower and upper flaw size bounds, $N_{\text{DFS}}$. The random number generator requires an initialization value. The random number generator serves in a few different capacities including the selection of phase shift values for the material variations model for each of the ply groups in the laminate, the selection of seeding locations of individual effective flaws in each of the ply
groups of the laminate, and the determination of the inspection order of effective flaws or partial cracks within each ply group at each load increment.

5.2 PRE-ANALYSIS CALCULATIONS

Prior to performing the damage analysis, the distribution of effective flaw sizes as characterized by the Weibull function given by Equation (4.1) is discretized. The same Weibull function is used for all the ply groups in the laminate. To discretize the cumulative distribution function (CDF), three variables \(N_{DFS}, \delta_{low},\) and \(\delta_{up}\) must be selected by the user.

The upper flaw size bound, \(\delta_{up}\), is typically selected using the ply group that is most likely to initiate cracking first as predicted by the one dimensional incremental damage analysis [88]. This value should be selected such that it considerably larger than any likely effective flaw in the Weibull distribution. The lower flaw size bound, \(\delta_{low}\), is selected such that the smallest flaws will not cause cracking for the particular load profile.

Once the three parameters are selected, the assigned flaw sizes and the total number of flaws corresponding to that flaw size are calculated in the following manner. The CDF is discretized into \(N_{DFS}\) increments between \(\delta_{low}\) and \(\delta_{up}\). For each increment along the abscissa in Figure 5.2, such as between \(\delta_{i+1}\) and \(\delta_{i}\), the assigned effective flaw size for that increment is the average between the upper value and the lower value of the increment, \((\delta_{i+1}+\delta_{i})/2\). Corresponding to this effective flaw size increment is a discrete number of effective flaws. This number is determined by taking the difference between the upper probability value, \(CP_{i+1}\), and the lower probability value, \(CP_{i}\), for this increment then multiplying by the flaw density and either the length over which the analysis is conducted for edge seeding or the area for volume seeding, i.e.

\[
NF_{i,(i+1)} = (CP_{i+1} - CP_i) \cdot EFD \cdot AL
\]  

(5.1)
Discretized Weibull CDF

Discretized Weibull cumulative probability function (CDF) of effective flaw sizes.

Figure 5.2
where \( NF_{(i+1)} \) is the total number of flaws for the assigned flaw size of the increment, \( EFD \) is the effective flaw density per unit length or area and \( AL \) is the analysis length or area. Probability values, \( CP \), are calculated using Equation (4.1) for a specific effective flaw size; for example, \( CP \), is the value of \( F \) calculated from Equation (4.1) using \( \delta \).

After the effective flaw sizes and the corresponding numbers of effective flaws for each increment of effective flaw sizes are determined, flaws are seeded in each of the ply groups within the laminate. The user has two options in seeding the effective flaws; this option is selected prior to executing the cracking analysis. The first option seeds the effective flaws random locations along the edges of each ply group. The second option seeds the effective flaws random locations in the \( x'-y' \) plane in each of the ply groups.

5.3 INCREMENTAL DAMAGE ANALYSIS

Referring to Figure 5.1a, the analysis begins at load-free conditions, which usually correspond to the stress-free temperature and zero applied mechanical load. Then, the analysis proceeds to the next load increment.

At each load increment, material properties are obtained from temperature-dependent material property data, if such information is available. Degraded properties at each increment reflect the damage incurred in all ply groups at all previous load increments. These properties are calculated by applying the knockdown factor, \( \kappa \), for the cracking state resulting from the previous load increment.

In each of the ply groups at a given load increment, cracking calculations are performed. A given ply group is designated as the cracking ply group and the analysis to determine cracking is performed in the local \( x'y'z' \) coordinate system of this ply group. The properties of the rest of the laminate are smeared. Each of the effective flaws in the cracking ply group is inspected to determine whether conditions are energetically favorable for the formation of a starter crack. The effective flaws are inspected in random
order. Energetic favorability is determined through the use of Equations (4.18), (4.21), and (4.22). Combining these Equations gives

$$\frac{1}{2\pi} \delta \left[ \frac{E_c}{E_o} s_o + E_e (\alpha_o - \alpha_e) \Delta T \right]^2 \left[ \cosh \left( \frac{2 \xi x}{a_e} \right) \right]^2 1 - \frac{cosh \left( \frac{2 \xi h}{a_e} \right)}{cosh \left( \frac{2 \xi h}{a_e} \right)} \geq \frac{G_{lc}}{\gamma}$$ (5.2)

When Equation (5.2) is satisfied, a starter crack forms instantaneously from an effective flaw. Starter crack formation is illustrated in Figure 5.3. The assumed length of the starter crack depends on whether the effective flaw is seeded along the edges or within the interior of the cracking ply group.

When all of the effective flaws are inspected, each of the partial cracks are inspected in random order to determine whether conditions favor a crack to extend an incremental distance across the laminate. As is the case with the formation of starter cracks from effective flaws, partial cracks will only extend if conditions are energetically favorable. Energetic favorability is determined through the use of Equations (4.18) and (4.37). Combining these Equations gives

$$\frac{a_c E_c}{2 \xi a_c E_a E_o} \left[ a_o \sigma - a_c E e (\alpha_c - \alpha_e) \Delta T \right]^2 \left[ \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) - \tanh \left( \frac{2 \xi h}{a_e} \right) \right] \geq G_{lc}$$ (5.3)

Incremental extension of a partial crack across the laminate is illustrated in Figure 5.4; incremental extension of partial cracks growing from the edge and in the volume of the cracking ply group is shown in Figure 5.4a and Figure 5.4b, respectively. If conditions defined by Equation (5.3) are satisfied, then a partial crack will extend an incremental length of $\Delta y'$. The calculation is then repeated. Partial cracks continue to extend, at an increment at a time, as long as conditions are energetically favorable.
Figure 5.3  Formation of starter cracks from effective flaws seeded at the edge and within the volume of the cracking ply group.
Existing Cracks
Extending Partial Crack

Figure 5.4a  Inspection of a partial crack extending from the edge of the cracking ply group. The partial crack is inspected an increment at a time in the \( y' \)-direction (a-c) until it is no longer energetically favorable to do so (d).
Figure 5.4b Inspection of a partial crack extending within the volume of the cracking ply group. The partial crack is inspected an increment at a time in the $y'$-direction (a-c) until it is no longer energetically favorable to do so (d) or until it reaches an edge.
When conditions for extension of all of the partial cracks in a ply group become energetically unfavorable, the inspection process is concluded and these cracking calculations are performed for the next ply group in the laminate for the particular load increment. The inspection algorithm for both effective flaws and partial cracks are illustrated in Figure 5.1b. The inspection process is repeated for all the ply groups at the given load increment as illustrated in Figure 5.1a. The order of inspection for both effective flaws and partial cracks is different at each load increment.

Fracture toughness to the right of the inequality in both Equations (5.2) and (5.3) is calculated by the apparent fracture toughness at a specific location in the \( x'-y' \) plane. Spatially-dependent apparent fracture toughness is calculated through the material variations model given by Equation (4.46).

Once the inspections have been performed for all of the ply groups, the crack density and crack distribution information for each of the ply groups is recorded. Also, new knockdown factors for each ply group and new effective laminate properties are calculated through Equations (4.54) and CLPT (Section 4.7) to reflect the damage at this load increment.

The analysis is performed for the entire load history as illustrated in Figure 5.1a. Residual thermal stresses incurred during manufacture are taken into account by performing the analysis from the load-free conditions to the first user-specified load conditions. Then, execution of the analysis is continued through the user-specified load profile until completion of the entire load profile.

\section{RESULTS}

Upon completion of the analysis, results at each load increment of the specified load profile can be examined. Included in the results are crack density, crack distribution data, and effective laminate properties. Results are presented and discussed in the following Chapters.
CHAPTER 6

RESULTS

This chapter contains six sections. Section 6.1 presents the parameters necessary to execute the incremental damage method. Section 6.2 presents comparisons between crack density predictions from the deterministic method and from models using various aspects of the current method. Section 6.3 presents parametric studies. Section 6.4 presents predictions of crack densities and crack distribution histories for three different laminate configurations composed of AS4/3501-6 plies. Crack densities predictions are compared to previously gathered experimental data. Section 6.5 presents predictions for a laminate composed of P75/934 plies subject to thermal loads. These predictions are compared with previously gathered experimental data. Section 6.6 qualitatively compares experimental observations on the development of crack distributions under fatigue loads with predictions using the current method.

6.1 INCREMENTAL DAMAGE MODEL PARAMETERS

Parameters needed to execute the incremental damage model in this chapter are presented in the following sub-sections. There are two general categories of parameters as presented in Chapter 5: physical and numerical. The physical parameters include laminate configurations and material properties, parameters for the Weibull probability functions that characterize the effective flaw size distributions, the shear lag parameter and fracture toughness parameters, and parameters for the material variations model. Physical parameters are presented in sub-sections 6.1.1 - 6.1.4. Numerical parameters include several different variables and are presented in sub-section 6.1.5.
6.1.1 Laminate Configurations and Material Properties

Two material systems are examined in this chapter: AS4/3501-6 and P75/934. For AS4/3501-6, three laminate configurations are investigated. These are [0/2/45/90/2/-45], [0/45/90/4/-45], and [0/±60]. Material properties for AS4/3501-6 are supplied in Table 6.1. For P75/934, one laminate configuration is investigated, which is [0/45/90/-45]. Material properties for P75/934 are supplied in Table 6.2. For both AS4/3501-6 and P75/934, material properties are assumed to be temperature-independent.

6.1.2 Effective Flaw Distribution Parameters

Three parameters are required by the Weibull probability function that describes the distribution of effective flaw sizes for each ply group in the laminate. The parameters for the Weibull function described by Equation (4.1) are obtained through a heuristic method. First, the effective flaw density, \( EFD \), is chosen. Using previously gathered experimental data as a guide, the current method is executed with selected values of the shape parameter and the characteristic effective flaw size. In this study, the parameters were obtained using the data for various laminates subject to mechanical loads. Initial values for the two parameters are selected and the method is executed. The two parameters are adjusted and the method is re-executed. The process of adjusting the two parameters and re-executing the method is continued until the predictions reflect the trend of the experimental data.

For this study, the shape and the characteristic effective flaw size, respectively, for [0/2/45/90/2/-45], [0/45/90/4/-45], and [0/±60], laminates composed of AS4/6501-6 are provided in Table 6.3. The shape and the characteristic effective flaw size for the [0/45/90/-45] laminate composed of P75/934 is provided in Table 6.4.
Table 6.1 AS4/3501-6 Material Properties

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_t$ (GPa)</td>
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<tr>
<td>$E_t$ (GPa)</td>
<td>9.81</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td>$G_{lt}$ (GPa)</td>
<td>6.0</td>
</tr>
<tr>
<td>$\alpha_0$ (µε/°C)</td>
<td>-0.36</td>
</tr>
<tr>
<td>$\alpha_t$ (µε/°C)</td>
<td>28.8</td>
</tr>
<tr>
<td>$T_o$ (°C)</td>
<td>177</td>
</tr>
<tr>
<td>$t_{ply}$ (mm)</td>
<td>0.134</td>
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</table>

Table 6.2 P75/934 Material Properties

<table>
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<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>$E_t$ (GPa)</td>
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</tr>
<tr>
<td>$E_t$ (GPa)</td>
<td>6.21</td>
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<tr>
<td>$\nu$</td>
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<tr>
<td>$G_{lt}$ (GPa)</td>
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</tr>
<tr>
<td>$\alpha_0$ (µε/°C)</td>
<td>-1.22</td>
</tr>
<tr>
<td>$\alpha_t$ (µε/°C)</td>
<td>28.8</td>
</tr>
<tr>
<td>$T_o$ (°C)</td>
<td>177</td>
</tr>
<tr>
<td>$t_{ply}$ (mm)</td>
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Table 6.3 Effective Flaw Distribution Parameters for AS4/6501-6

<table>
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<tbody>
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<td>$m$</td>
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<td>10</td>
<td>10</td>
</tr>
<tr>
<td>$d$ (cm)</td>
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<td>0.00650</td>
<td>0.00325</td>
</tr>
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</table>

Table 6.4 Effective Flaw Distribution Parameters for P75/934

<table>
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<th>$m$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$d$ (cm)</td>
<td>0.003175</td>
</tr>
</tbody>
</table>
6.1.3 Shear Lag Parameter and Fracture Toughnesses

The shear lag parameter, $\xi$, for AS4/3501-6 used in this study is taken from Maddocks [88]. Maddocks acquired a value for $\xi$ through the use of a data fitting scheme and experimental data. Briefly, $\xi$ was procured by minimizing the mean square error between predictions and the data from all of his experimental specimens; details of this method are available in the original work [88]. The shear lag parameter for P75/934 is taken from Park [87].

Fracture toughness, $G_{lc}$, for AS4/3501-6 is also taken from Maddocks. The same data fitting scheme used to obtain a value of $\xi$ is also employed to extract fracture toughness values. Fracture toughness for P75/934 is taken from Park [87].

For the fracture toughness used to determine effective flaw-to-starter crack growth, $G_{lc}'$, a scaling factor, $\gamma$, described by Dvorak and Laws [51] is necessary. As seen in Equation 4.22, $G_{lc}$ is proportional to $G_{lc}'$ through $\gamma$. The range of these values is generally be considered to be less than or equal to unity. For the present work, $\gamma$ will be assumed to be equal to 1.

The shear lag parameter, $\xi$, fracture toughness used to determine the extension of a partial crack, $G_{lc}$, and the scaling parameter for the fracture toughness determine effective flaw-to-starter crack growth, $\gamma$, are supplied in Tables 6.5 and 6.6 for AS4/3501-6 and P75/934, respectively.

6.1.4 Material Variations Parameters

Parameters needed for the material variations model include the amplitude of the local apparent fracture toughness variations, $A$, and the length scale over which the toughness values change in two dimensions, $\lambda_x$ and $\lambda_y$, respectively. Values for these parameters are difficult to obtain. Rough estimates for $A$ can be extracted from a few sources in the published literature. Length scale values must be obtained from experimental observations.
Table 6.5  Shear Lag, Fracture Toughness in the \( y' \)-Direction, and Fracture Toughness Scaling Parameter in the \( z' \)-Direction for AS4/6501-6

| \( \xi \) | 1.0 |
| \( G_{fc} \) (J/m\(^2\)) | 141.0 |
| \( \gamma \) | 1.0 |

Table 6.6  Shear Lag, Fracture Toughness in the \( y' \)-Direction, and Fracture Toughness Scaling Parameter in the \( z' \)-Direction for P75/934

| \( \xi \) | .65 |
| \( G_{fc} \) (J/m\(^2\)) | 40 |
| \( \gamma \) | 1.0 |
For the amplitude of the variations, $A$, a few sources are available from which values can be inferred for fiber-reinforced composites [78,79,81]. Variations in amplitude are assumed to be the result of variations in local maximum stress which, in turn, are caused by variations in local $V_f$. These changes in stresses create regions where the potential for cracking is increased or decreased. Bowles and Griffen [81] mention that the relationship between volume fraction, $V_f$, and stress is non-linear. They add that as a result of this relationship a change in stress amplitude of 30%, approximately, can exist from a local low volume fraction to a local high volume fraction. From the work of Pijaudier-Cabot and Bazant [78], an effective fracture toughness amplitude variation of 30%, approximately, can exist depending upon a crack's position relative to a fiber or cluster of fibers. In extreme situations, variations can be as much as 100% according to Axelson and Pyrz [79]. However, well-made composites would, more reasonably, lean toward the lower end of the scale. More extreme values could be argued to be as high as 20% to 30%. In the present study, the amplitude of the variation in effective fracture toughness is assumed to be 20%.

Indirect support for the effective fracture toughness is evident in interlaminar fracture toughness data. Large amounts of scatter exhibited by the data suggest that the effective fracture toughness may, in fact, vary. However, the effect of the material variations is inextricable from other causes of the scatter, such as test methods, data reduction techniques, etc. so scatter in fracture toughness data was not used in this study.

The physical length over which the amplitudes vary must be estimated using informly gathered experimental observations. Information of this kind is absent in the published literature. For the present work, the length parameter in the $x'$-direction, $\lambda_{x'}$, can be determined from previously tested experimental specimens. The length parameter in the $y'$-direction, $\lambda_{y'}$, can be estimated from unpublished records of previous experimental observations.
For the direction perpendicular to the fibers in the $x'$-direction, the method used to determine the length scale over which the amplitude changes, $\lambda_x$, involves subjective visual observation. Experimental specimens constructed and subjected to mechanical loads by Maddocks [88] were examined using an optical microscope. The pattern of microcrack "bunching" was surveyed and the number of "bunches" was counted over a pre-determined length of 2.54 cm., or one inch. Microcrack bunches refers to regions along the edge of a ply group where microcracks appear to be more closely spaced. The frequency of the microcrack bunches from 8 specimens was observed to be 2 cm.$^{-1}$, approximately, which corresponds to a length of 0.5 cm. Additionally, in previously recorded experimental observations collected by Park, frequency values ranged from approximately 1 cm.$^{-1}$ to 2 cm.$^{-1}$, corresponding to lengths of 1 cm. and 0.5 cm., respectively.

Into the depth of a laminate in the $y'$-direction, the length parameter, $\lambda_y$, is more difficult to estimate. Examination of rough figures of changes in crack distributions versus depth into the width of a laminate specimen, recorded by Park in unpublished laboratory records, provides a means of obtaining initial estimates. Comparing crack distributions with the depth into the width of the laminate provides a subjective means of estimating a value of $\lambda_y$. Using this information, an initial estimate for the $y'$-direction of 2 cm.$^{-1}$, corresponding to a length of 0.5 cm., will be employed.

The values of apparent fracture toughness amplitude, $A$, and the length scales over which the amplitudes change in the $x'$ and $y'$ directions, $\lambda_x$ and $\lambda_y$, respectively, are summarized in Table 6.7.

### 6.1.5 Numerical Parameters

Numerical parameters described in Chapter 5 include the incremental crack extension length of partial cracks, values used to discretize the Weibull function, and an
Table 6.7  Apparent Fracture Toughness Amplitude and Length Scale of Amplitude Variation in the $x'$ and $y'$ directions

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ (% of $G_c$)</td>
<td>20</td>
</tr>
<tr>
<td>$\lambda_x$ (cm)</td>
<td>0.5</td>
</tr>
<tr>
<td>$\lambda_y$ (cm)</td>
<td>0.5</td>
</tr>
</tbody>
</table>
initialization value for the random number generator. Table 6.8 lists these parameters for both AS4/3501-6 and P75/934.

The incremental crack extension length, \( \Delta y' \), is selected as a fraction of the material variation length parameter in the \( y' \)-direction, \( \lambda_y \). The value of the extension length is determined through a convergence study. As a result of this study, \( \Delta y' \) is selected smaller than \( \lambda_y/4 \). \( \Delta y' \) is selected sufficiently large to be computationally efficient but sufficiently small that the predictions are not significantly affected.

Three discretizing parameters for the Weibull distribution include \( N_{DFS}, \delta_{own}, \) and \( \delta_{up} \). The method of selection for these three parameters is discussed in Section 5.2. In this study, \( \delta_{own} \) and \( \delta_{up} \) are selected such that for the given effective flaw density, the number of effective flaws corresponding to the first and last assigned discrete effective flaw sizes is equal to 0.

The random number generator requires an initialization value as a starting point for number generation. The restrictions on this value is that it must be a negative integer.

### 6.2 COMPARISON BETWEEN METHODS

In this section, some brief descriptions of Maddocks' deterministic method and the current method will be presented first. Comparisons of crack density predictions from the deterministic method and components of the current method will be presented second. There are two components: the flaw model and the material model. The flaws from which crack initiate and extend are modeled in two ways. The first assumes that the flaws already span the thickness of the ply group and are randomly positioned. The second uses the effective flaw distribution. Similarly, the material is modeled in two ways. The first assumes uniform material. The second uses the material variations model. Combinations of each of the flaw components with each of the material components demonstrates how each of these combinations affect predictions of crack density and crack distribution. There are four combinations: random crack locations with uniform material, random
Table 6.8 Numerical Parameters for AS4/3501-6 and P75/934

<table>
<thead>
<tr>
<th>Seeding Assumption</th>
<th>EFD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edge (1/cm)*</td>
<td>250</td>
</tr>
<tr>
<td>Volume (1/cm²)*</td>
<td>500</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>* AS4/3501-6</td>
<td></td>
</tr>
<tr>
<td>** P75/934</td>
<td></td>
</tr>
</tbody>
</table>

|   |  
|---|---|
| $\Delta y^*$ (cm.) | .0125 |
| $N_{DFS}$ | 85 |
| $\delta_{low}$ (cm.) | .00055 |
| $\delta_{up}$ (cm.)  | .0065 |

<table>
<thead>
<tr>
<th>RNG Seed</th>
<th>Negative Integer</th>
</tr>
</thead>
</table>

* AS4/3501-6
** P75/934

| $N_{DFS}$ | 85 |
| $\delta_{low}$ (cm.) | .00055 |
| $\delta_{up}$ (cm.)  | .0065 |

** [0/45/90/-45]s

| $N_{DFS}$ | 85 |
| $\delta_{low}$ (cm.) | .000635 |
| $\delta_{up}$ (cm.)  | .0089 |
crack locations with material variations, effective flaw distribution with uniform material, and effective flaw distribution with material variations. In this section, only mechanical loads are studied. Features of the results are illustrated using a select group of figures. A complete set of results are included in Appendix D.

In all of the crack density figures, hereafter, predictions produced by the deterministic method are presented by a solid line. Predictions produced the current method are presented by "×"s and/or "+"s. These represent discrete crack states for a given run at a given load.

The deterministic method was developed by Maddocks [88]. In this method, uniform crack spacing is assumed at each load increment for each ply group. Assuming that existing cracks are uniformly spaced at a distance of \(2h\), hypothetical new cracks are assumed to form midway between these existing cracks when conditions are energetically favourable. A uniform crack spacing below \(2h\) will render conditions energetically unfavourable and new cracks will not form. The crack density calculated by the deterministic method corresponds to a uniform crack spacing between existing cracks of \(2h\) which gives the minimum crack density of \(1/2h\).

In the current incremental damage method, cracks are randomly located in each ply group of the laminate. The random crack positions more accurately represent reality. The crack spacing between randomly positioned cracks will result in a crack density above the minimum crack density of \(1/2h\) calculated by the deterministic method.

The first set of predictions from the current method includes only random crack locations in uniform material. This illustrates the geometric effects of random locations on the crack density predictions. In this situation, crack initiation for the current incremental damage model is assumed to depend only on the energetic favourability for a self-similar incremental extension to form for a particular load increment. Thus, the incremental crack extension criteria determines both initiation and extension of cracks. Additionally, the material is assumed to be uniform throughout the volume of the
laminate, which means that the amplitude of the material variations model is equal to 0. For each set of crack density predictions, four separate simulation runs by the current method are presented. Each simulation run produces crack density predictions for both edges of each ply group. Symmetric laminates have two ply groups of each angle except the one in the center; both are shown in the figures.

Comparisons of crack density predictions for the 45₂ ply group of the [0₂/45₂/90₂/-45₂]ₙ laminate subject to mechanical loads are presented in Figure 6.1. Crack initiation in Figure 6.1 is predicted to occur at the same load by both methods. However, crack accumulation predictions from the current method are higher than the minimum crack density that is predicted by the deterministic method. Also, the separate simulation runs by the current methods produce a "swath" of crack density predictions. The "swath" refers to the scatter of crack densities from one run to another.

Predictions of crack distribution at select loads for the above case are presented in Figure 6.2. In each of the ply groups, only through cracks are present. A through crack is a crack that extends from one edge to the other.

The second set of predictions from the current method includes random crack locations and material variations. Figure 6.3 shows crack density predictions in the 45₂ ply group assuming flaw seeding along the edges of the ply groups. Parameters for the material variations model are listed in Table 6.7. In Figure 6.3, crack initiation is predicted earlier compared to the predictions shown in Figure 6.1. Another notable feature of Figure 6.3 is the broader swath of crack density predictions that results from the material variations.

Predictions of crack distributions at selected loads for the above case are presented in Figure 6.4. In each of the ply groups, two different types of partial cracks can be seen at the different loads; in contrast to a through crack, a partial crack is a crack that does not extend from one edge to the other. One type of partial crack is designated as a type P partial crack. A type P partial crack is a crack that appears to have stopped
Crack density predictions that depend only on geometry with uniform material for the $45_2$ ply group. Flaws are assumed to be seeded along the edges.

Figure 6.1
Crack distribution predictions at select loads that include random crack locations in uniform material. Flaws are assumed to be seeded along the edges.
Crack density predictions that random crack locations and material variations for the 45_2 ply group. Flaws are assumed to be seeded along the edges.

Figure 6.3
Crack distribution predictions at select loads that include random crack locations and material variations. Flaws are assumed to be seeded along the edges.

Figure 6.4
extending because of the propinquity of the ends of other partial cracks which has rendered conditions energetically unfavourable for continued extension of the partial crack in question. Visually, the end of a type P partial crack appears to encounter and sense, in the direction generally perpendicular to its own extension, the presence of the ends of counterpart type P partial cracks that are extending in the opposite direction. The other type of partial crack is designated as a type E partial crack. A type E partial crack is a partial crack that appears to have stopped extending because material variations have rendered conditions energetically unfavourable. Visually, in contrast to type P partial cracks, type E partial cracks appear to have stopped extending in the absence of the ends of counterpart type P partial cracks extending in the opposite direction. At the lower loads, many type E partial cracks are visible. At the higher loads, type P partial cracks are in the majority.

The third set of predictions from the current method uses an effective flaw distribution with uniform material. Figure 6.5 shows crack density predictions assuming flaw seeding along the edges of the ply groups for the 45₂ ply group. Crack initiation is predicted to occur at an earlier load than the deterministic method. The predictions also show a more gradual accumulation of cracks early in the load history in comparison to the abrupt initiation seen in the deterministic predictions. Predictions assuming flaw seeding throughout the volume of the ply groups are shown in Figure 6.6. The general trends of the predictions are the same as in Figure 6.5 except that the predictions are shifted toward the higher loads.

Predictions of crack distributions at selected loads for the above case are presented in Figure 6.7. In the -45₄ ply group, only through cracks can be seen. In the 45₂ and 90₂ ply groups, starter cracks form first. Starter cracks are partial cracks that span the thickness of a ply group and extend only a short distance. The formation of starter cracks contrasts with the crack predictions in Figure 6.2 in which only long type P partial cracks and through cracks can be seen at any of the loads. As load increases, through cracks and
Crack density predictions using a Weibull distribution of effective flaws and assuming uniform material for the 45₂ ply group. Flaws are assumed to be seeded along the edges.
Crack density predictions using a Weibull distribution of effective flaws and assuming uniform material for the $45_2$ ply group. Flaws are assumed to be seeded within the volume.
Crack distribution predictions at select loads using a Weibull distribution of effective flaws and assuming uniform material. Flaws are assumed to be seeded along the edges.
type P partial cracks can be seen. The type P cracks are relatively long in length. Predictions assuming flaw seeding throughout the volume of the ply groups is shown in Figure 6.8. Similar to Figure 6.7, starter cracks form in the 45° and 90° ply groups. However, these starter cracks are concealed from sight at the edges. This affects the crack density predictions seen in Figure 6.6.

The fourth set of predictions from the current method uses an effective flaw distribution and assumes material variations. Figure 6.9 shows crack density predictions assuming flaw seeding along the edges of the ply groups for the 45° ply group. The general trend of the predictions is the same as those in Figure 6.5 except that the swath of the predictions is wider. The wider swath is attributable to the material variations as shown in Figure 6.3. Parallel trends can be seen for flaw seeding within the volume of the ply groups in Figure 6.10. As in Figure 6.6, the predictions are shifted toward higher loads.

Predictions of crack distributions at selected loads for the above case are presented in Figure 6.11. In the -454 ply group, only through cracks can be seen. In the 45° and 90° ply groups, starter cracks and type E partial cracks can be seen when cracking initiates in each of the ply groups. As load increases, through cracks and type P partial cracks are also present. In the 90° ply group, there are only a few type E partial cracks at the higher loads. Figure 6.12, which assumes flaw seeding throughout the volume of the ply group, shows the same cracking patterns as Figure 6.11. Thus at lower loads, starter and short type E partial cracks are present but concealed from view at the edges. Compared with Figures 6.7 and 6.8, cracking in Figures 6.11 and 6.12 is more gradual.

6.3 PARAMETRIC STUDIES

In this section, some of the different parameters of the current incremental damage method will be investigated. These include investigating the effects of adjusting the amplitude parameter in the material variations model, the effects of adjusting the length
Crack distribution predictions at select loads using a Weibull distribution of effective flaws and assuming uniform material. Flaws are assumed to be seeded within the volume.
Crack density predictions using a Weibull distribution of effective flaws and assuming material variations for the 45₂ ply group. Flaws are assumed to be seeded along the edges.
Figure 6.10  Crack density predictions using a Weibull distribution of effective flaws and assuming material variations for the 45$_2$ ply group. Flaws are assumed to be seeded within the volume.
Figure 6.11 Crack distribution predictions at select loads using a Weibull distribution of effective flaws and assuming material variations. Flaws are assumed to be seeded along the edges.
Figure 6.12 Crack distribution predictions at select loads using a Weibull distribution of effective flaws and assuming material variations. Flaws are assumed to be seeded within the volume.
parameters in the material variations model, the effects of assuming the material variations model depends on only one length parameter, the effects of adjusting the two Weibull parameters (i.e. the shape and characteristic effective flaw size), and the effects of flaw density. All predictions presented in this section are archived in Appendix E.

For the parametric studies presented in this section, a single laminate configuration and loading type will be considered. For these studies, the \([0_2/45_2/90_2/-45_2]_s\) laminate composed of AS4/3501-6 plies subject to mechanical loads will be employed.

Investigation of the amplitude parameter, \(A\), is accomplished through the use of three different amplitude values. The Weibull parameters are listed in Table 6.3 and the length parameters are the same as those listed in Table 6.7. The amplitudes for the material variations model for each of the three cases are listed in Table 6.9. Comparison of the crack density predictions for each of the three cases will be presented first. Comparison between crack distribution histories will be presented second.

Figure 6.13 shows crack density predictions for Cases 1 - 3 for the 45\(_2\) ply group assuming edge flaw seeding. The crack density predictions for the different amplitudes show a gradually widening swath with increasing amplitude which is especially evident at the lower loads prior to the initiation point predicted by the deterministic method. The general trend of the predictions remains unaffected by the increasing values of amplitude from Case 1 through Case 3.

Predictions of crack distributions at select loads for the 90\(_2\) ply group for each of the three cases are shown in Figures 6.14 and 6.15. Figure 6.14 assumes flaw seeding along the edges of the ply groups, Figure 6.15 assumes flaw seeding throughout the volume of the ply groups. At the low load for Case 1, for both flaw seeding assumptions, starter cracks can be seen. At the higher loads, type P partial cracks and through cracks are the only cracks present. However, as the amplitude increases in Cases 2 and 3, for both seeding assumptions, at the low loads, starter cracks and type E partial cracks can be seen. As load increases, a greater variety of cracks are present at different loads. This can
Table 6.9  Apparent Fracture Toughness Parameters Used in Parametric Study

<table>
<thead>
<tr>
<th>Case</th>
<th>A (% of $G_c$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
</tr>
</tbody>
</table>
Material variations model amplitude parametric study. Predictions of crack density vs. applied load for the 452 ply group. Flaws are assumed to be seeded along the edges.
Figure 6.14  Crack distributions for the $90_2$ ply group at select applied loads for the material variations amplitude parametric study. Flaws are assumed to be seeded along the edges. Case 1 - 0%. Case 2 - 10%. Case 3 - 20%.
Figure 6.15 Crack distribution for the $90_2$ at select applied loads for the material variations amplitude parametric study. Flaws are assumed to be seeded within the volume. Case 1 - 0%. Case 2 - 10%. Case 3 - 20%.
be seen especially in Figure 6.15. As amplitude increases, the variety of different cracks at different loads becomes greater. Additionally, increasing the amplitude appears to increase the amount of disorder in the cracking patterns.

Investigating the effect of the length parameters in the material variations model is accomplished using five different values listed in Table 6.10; the same value is used in both the $x'$-direction and $y'$-direction. First, the crack density predictions for the $45_2$ ply group for length parameters will be presented; the base-line is shown in Figure 6.9 and Cases 1 - 4 are plotted together in Figure 6.16.

Crack density predictions remain unaffected when changing the value of the length parameters in the material variations model. For each of the four cases and base-line case, the crack density predictions are very similar for all. The general trend of the predictions, including crack initiation and accumulation are approximately the same for any value of the length parameters.

Although crack density predictions are unaffected, crack distribution histories for each of the cases and base-line case demonstrate the effect of the various length parameters on the crack predictions. The cracking pattern in the $45_2$ ply group for last two loads (25.2 kN and 29.3 kN) appears to change with the length parameters, as shown in Figure 6.17. The different loads are used to more clearly illustrate the length parameter effects on the crack distributions. These select distributions show that the cracks congregate in regions of weaker material forming bands. The spacing of the bands naturally reflects the value of the length parameter. For example, increasing the length parameter increases the spacing between each of the bands as well as increasing the width of each of the bands.

The effect of considering the material variations model as a function of only one direction is investigated. The effect on predictions of crack density and crack distribution is investigated for both the $x'$-direction and $y'$-direction. The values of the amplitude and length parameters for each condition are listed in Table 6.11.
Table 6.10  Length Parameters Used in Parametric Study

<table>
<thead>
<tr>
<th>Case</th>
<th>$\lambda_x$ (cm)</th>
<th>$\lambda_y$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Base-line</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.667</td>
<td>0.667</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Figure 6.16 Material variations model length parameter study. Predictions of crack density vs. applied load for the 452 ply group. Flaws are assumed to be seeded along the edges.
Figure 6.17  Length parameter study crack distributions for the $45_2$ ply group at select applied loads. Flaws are assumed to be seeded along the edges.
Table 6.11  Length and Phase Shift Parameters Used in Parametric Study of Material Variations as a Function of One Direction

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
 & $x'$ & $y'$ \\
\hline\hline
$\lambda_x'$ (cm) & 0.5 & $\infty$ \\
\hline
$\lambda_y'$ (cm) & $\infty$ & 0.5 \\
\hline
$\phi_x'$ & Random & $\pi/2$ \\
\hline
$\phi_y'$ & $\pi/2$ & Random \\
\hline
\end{tabular}
\end{table}
Figure 6.18 shows the combined crack density predictions of the material variations model as a function of the $x'$-direction assuming flaw seeding along the edges, as a function of the $x'$-direction assuming flaw seeding within the volume, as a function of the $y'$-direction assuming flaw seeding along the edges, and as a function of the $y'$-direction assuming flaw seeding within the volume.

The general trends of the crack density predictions for the $x'$-direction and the $y'$-direction variations are the same except for the crack density predictions in the $y'$-direction assuming flaw seeding throughout the volume. This exception possesses a pronounced dip downward in the general trend of the predictions whereas the others possess a more gradual trend in comparison. The cause of this exception is evident upon examination of the corresponding distribution as shown in Figure 6.19. Material variations combined with flaw seeding within the volume results in bands of cracks that are prevented from extending to the edges of the ply group. Thus, their presence is not reflected in the crack density predictions along the edges.

Parametric studies of the shape parameter and characteristic effective flaw size are performed by comparing crack density predictions using various values for each parameter. The base-line case is the set of crack density predictions that use the Weibull parameters in Table 6.3 and assumes edge flaw seeding as well as uniform material. The parameters are investigated in the following manner. First, the characteristic effective flaw size is changed while the shape parameter remains fixed. Then, the shape parameter is changed while the characteristic effective flaw size remains fixed. For each of the two parameters, two different values are investigated: one lower than the original and one higher than the original. Thus, a total of four cases will be investigated. Weibull parameters for each case including the base-line are listed in Table 6.12. Crack density predictions for the $45_2$ ply group for the original specified conditions are shown in Figure 6.5. Cases 1 and 2 are shown in Figure 6.20. Cases 3 and 4 are shown in Figure 6.21.
Figure 6.18 Study of the material variations model as a function of one direction. Predictions of crack density vs. applied load for the 45\_2 ply group. Flaws are assumed to be seeded along the edges and within the volume.
Figure 6.19  Study of the material variations model is a function of one direction. Crack distributions for the 45_2 ply group at 29.3 kN. (a) Edge seeding, x'-dependent. (b) Volume seeding, x'-dependent. (c) Edge seeding, y'-dependent. (d) Volume seeding, y'-dependent.
Table 6.12  Effective Flaw Distribution Parameters Used in Parametric Study

<table>
<thead>
<tr>
<th>Case</th>
<th>$m$</th>
<th>$d$ (cm.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base-line</td>
<td>10</td>
<td>0.00325</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>0.00225</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>0.00425</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>0.00325</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>0.00325</td>
</tr>
</tbody>
</table>
Figure 6.20 Predictions of crack density vs. applied load for the $45_2$ ply group for the Weibull parameter study. Flaws are assumed to be seeded along the edges. Case 1: $d = 0.00225$. Case 2: $d = 0.00425$. 
Figure 6.21 Predictions of crack density vs. applied load for the 45₂ ply group for the Weibull parameter study. Flaws are assumed to be seeded along the edges. Case 3: \( m = 5 \). Case 4: \( m = 15 \).
Decreasing the characteristic effective flaw size as in Case 1 and increasing the characteristic effective flaw size as in Case 2 shifts the crack density predictions along the abscissa while the general trends are not significantly affected. Decreasing the characteristic effective flaw size scales down the effective flaw sizes of the base-line case while retaining the shape of the cumulative distribution of sizes. Increasing the characteristic effective flaw size scales up the effective flaw sizes.

Decreasing the shape parameter as in Case 3 and increasing the shape parameter as in Case 4 shifts the crack density predictions along the abscissa and alters the general trends of the predictions. Decreasing the shape parameter increases the range of effective flaw sizes by reducing the slope of the cumulative distribution function. The effect on the crack density predictions is that the predictions are shifted toward the lower loads and the general trend of the predictions is flatter. Increasing the shape parameter decreases the range of effective flaw sizes by increasing the slope of the cumulative distribution function. In terms of the effect on the crack density predictions, crack initiation is delayed until a higher load such that it coincides approximately with the deterministic prediction. Crack accumulation predictions rise to match those of the base-line. The general trend of the predictions possesses a higher overall slope than the base-line case.

The effect of flaw density on crack density predictions will be investigated by selecting a lower and higher value than original; values are listed in Table 6.13. Figure 6.5 shows the base-line, and Figure 6.22 shows Cases 1 and 2. By decreasing the flaw density as in Case 1 the crack density predictions shifts toward the higher loads compared to the base-line. Increasing the flaw density as in Case 2, the crack density predictions shift toward the lower loads compared to the base-line case.

The numerical parameters used in discretizing the distribution of effective flaw sizes \((N_{DFS}, \delta_{low}, \delta_{up})\) do not affect the crack density predictions as long as the parameters are selected using the same methods. For \(N_{DFS}\), as long as values are reasonable, crack density predictions remain unaffected. For \(\delta_{low}\), as long as the value is small enough such
Table 6.13  Effective Flaw Density Parameters Used in Parametric Study

<table>
<thead>
<tr>
<th>Case</th>
<th>EFD (cm.$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>20</td>
</tr>
<tr>
<td>Base-line</td>
<td>500</td>
</tr>
<tr>
<td>2</td>
<td>1000</td>
</tr>
</tbody>
</table>
Figure 6.22  Predictions of crack density vs. applied load for the 45_2 ply group for the effective flaw density parameter study. Flaws are assumed to be seeded along the edges. Case 1: $EFD = 20$ cm.$^{-1}$. Case 2: $EFD = 1000$ cm.$^{-1}$. 
that crack initiation will not occur for effective flaws at this size, crack density predictions remain unaffected. Similarly, for $\delta_w$, as long as the value is large enough such that there are no flaws present for the assigned discrete flaw size, crack density predictions remain unaffected.

6.4 COMPARISON TO EXPERIMENTAL DATA

In this section, predictions of crack densities and crack distribution histories from the current method are presented. Crack density predictions are compared to previously gathered experimental data; crack density data were collected by Maddocks [88]. Predictions for mechanical loads are presented in sub-Section 6.4.1 and predictions for thermal loads are presented in Section 6.4.2. In each sub-section, select results from the different laminate configurations are presented to demonstrate the general trends of the predictions.

In all of the crack density figures presented in this section, experimental data is represented by the symbol "o" and error bars. Predictions produced by the current method are represented by the symbol "x". Prediction of the minimum crack density produced by the deterministic method is represented by a solid line. All results presented in this section are archived in Appendix F.

6.4.1 Mechanical Load Predictions

A few select crack density predictions from each of the three laminates are presented to demonstrate the general trends. Figure 6.23 shows the 45$_2$ ply group from the [0$_2$/45$_2$/90$_2$/-45$_2$]s laminate assuming edge flaw seeding. Figure 6.24 shows the 90$_2$ ply group from the [0$_2$/45$_2$/90$_2$/-45$_2$]s laminate assuming edge flaw seeding. Figure 6.25 shows the -45$_4$ ply group from the [0$_4$/45$_4$/90$_4$/-45$_4$]s laminate assuming volume flaw seeding. Figure 6.26 shows the -45$_8$ ply group from the [0$_4$/45$_4$/90$_4$/-45$_4$]s laminate assuming volume flaw seeding. Figure 6.27 shows the -60$_2$ ply group from the [0$_2$/±60$_2$],
laminate assuming edge flaw seeding. A complete set for each of the laminates for both flaw seeding assumptions is compiled in Appendix F.

Comparison between crack density predictions from the current method and experimental data for the 452 ply group of the [02/452/902/-452]s laminate is shown in Figure 6.23. For the 452 ply group, predictions from the current methods match the experimental data. The swath of predicted densities falls within both error bars for the experimental data points from initiation through crack accumulation. Agreement is also good for the 454 ply group of the [04/454/904/-454]s laminate whose predictions resemble those in Figure 6.23.

Agreement between the crack density predictions for the 902 ply group of the [02/452/902/-452]s laminate and the experimental data, shown in Figure 6.24, is less successful. Prediction of crack initiation by the current method falls within the error bars of the experimental data points. Prediction of crack accumulation for loads above 22 kN, approximately, also falls within the error bars of the experimental data. For the loads between 17 kN and 22 kN, crack density is under-predicted. Similar trends are seen in the 904 ply group of the [04/454/904/-454]s laminate. Predictions underpredict the data for a region of lower loads although the predictions are closer than for the 902 ply group shown in Figure 6.24.

Figure 6.25 shows crack density predictions and experimental data for the -454 ply group of the [02/452/902/-452]s laminate. Although the current method predicts crack initiation at a higher load, agreement between predictions and experimental data is good above 18 kN, approximately. Above this load, the rate of crack accumulation increases. Predicted initiation occurs at about the same load where the rate of crack accumulation increases.

Figure 6.26 shows crack density predictions and experimental data for the -458 ply group of the [02/452/902/-452]s laminate. Predicted initiation occurs at approximately the same load at which the rate of crack accumulation increases. The general trend of the
Figure 6.23 Experimental data and predictions of crack density vs. applied load for the 45₂ ply group. Flaws are assumed to be seeded along the edges.
Figure 6.24 Experimental data and predictions of crack density vs. applied load for the 90₂ ply group. Flaws are assumed to be seeded along the edges.
Figure 6.25  Experimental data and predictions of crack density vs. applied load for the -454 ply group. Flaws are assumed to be seeded within the volume.
Figure 6.26 Experimental data and predictions of crack density vs. applied load for the \(-45_8\) ply group. Flaws are assumed to be seeded along the edges.
predictions above 30 kN, approximately, emulates that of the data. The error bars for a couple of data points falls just within the predicted crack density swath. Overall, crack density is over-predicted.

Comparison between crack density predictions from the current method and experimental data for the 60\textsubscript{2} ply group of the [0\textsubscript{2}/±60\textsubscript{2}]\textsubscript{s} laminate is shown in Figure 6.27. Predictions and experimental data agree below 19 kN, approximately. Prediction of crack initiation and crack accumulation below 19 kN coincides with the experimental data. Above this load, crack accumulation is overpredicted. This pattern is also seen in the -60\textsubscript{4} ply group.

6.4.2 Thermal Load Predictions

A few select predictions of laminate response to decreasing temperature are presented here to demonstrate general trends. Figure 6.28 shows the 90\textsubscript{2} ply group from the [0\textsubscript{2}/45\textsubscript{2}/90\textsubscript{2}/-45\textsubscript{2}]\textsubscript{s} laminate assuming edge flaw seeding. Figure 6.29 shows the -45\textsubscript{4} ply group from the [0\textsubscript{2}/45\textsubscript{2}/90\textsubscript{2}/-45\textsubscript{2}]\textsubscript{s} laminate assuming volume flaw seeding. Figure 6.30 shows the 45\textsubscript{4} ply group from the [0\textsubscript{d}/45\textsubscript{d}/90\textsubscript{d}/-45\textsubscript{d}]\textsubscript{s} laminate assuming edge flaw seeding edges. Figure 6.31 shows the 90\textsubscript{4} ply group from the [0\textsubscript{d}/45\textsubscript{d}/90\textsubscript{d}/-45\textsubscript{d}]\textsubscript{s} laminate assuming edge flaw seeding. Predictions of crack distributions at selects loads are also presented. Figure 6.32 shows the crack distribution for the the [0\textsubscript{2}/45\textsubscript{2}/90\textsubscript{2}/-45\textsubscript{2}]\textsubscript{s} laminate assuming edge flaw seeding. Figure 6.33 shows the crack distribution for the [0\textsubscript{d}/45\textsubscript{d}/90\textsubscript{d}/-45\textsubscript{d}]\textsubscript{s} laminate assuming volume flaw seeding. In the [0\textsubscript{2}/±60\textsubscript{2}]\textsubscript{s}, no cracks were observed in the experimental specimens which yields no crack density data with which to compare predictions. A complete set for each of the laminates for both flaw seeding assumptions is compiled in Appendix F.

Comparison between crack density predictions from the current method and experimental data for the 90\textsubscript{2} ply group of the [0\textsubscript{2}/45\textsubscript{2}/90\textsubscript{2}/-45\textsubscript{2}]\textsubscript{s} laminate are shown in Figure 6.28. Experimental data for this laminate is limited. Only crack initiation can be
Figure 6.27  Experimental data and predictions of crack density vs. applied load for the 602 ply group. Flaws are assumed to be seeded along the edges.
Figure 6.28 Experimental data and predictions of crack density vs. temperature for the 90₂ ply group. Flaws are assumed to be seeded along the edges.
seen from the data. Crack accumulation occurs at temperatures below the range of the experimental investigations. For the limited amount of recorded data, the predicted swath falls within both error bars. Similar results can be seen for 452 ply group. Figure 6.29 shows crack density predictions and experimental data for the -454 ply group of the [02/452/902/-452]s laminate. For the limited amount of data, the predictions of initiation and accumulation agree with the experimental data. The general trends are similar for the -458 ply group in the [04/454/904/-454]s laminate where prediction and data also agree well.

Figure 6.30 shows crack density predictions and experimental data for the 454 ply group of the [04/454/904/-454]s laminate. Crack initiation predictions correspond with experimental data. However, crack accumulation is over-predicted for temperatures below -100 °C, approximately.

Figure 6.31 shows crack density predictions and experimental data for the 904 ply group of the [04/454/904/-454]s laminate. Crack initiation is predicted for a lower temperature than the experimental data. At the predicted initiation temperature, crack accumulation exists. Below -100 °C, approximately, crack density is over-predicted.

Predictions of crack distributions for the [02/452/902/-452]s laminate assuming edge flaw seeding are shown in Figure 6.32. Crack distributions for the ply groups are illustrated for three select temperatures. These temperatures are -153 °C, -183 °C, and -213 °C.

At the first temperature of -153 °C, the 452 and 902 ply groups possess cracks. Only starter cracks have formed in each of these ply groups.

At -183 °C, all the ply groups possess cracks. In the 452 and 902 ply groups, more starter cracks have formed and type E partial cracks are present in each. In the -454 ply group, only through cracks are present.

At the last temperature of -213 °C, in the 452 and 902 ply groups, both type E and type P partial cracks can be seen as well as through cracks. Some of the partial cracks in
Figure 6.29  Experimental data and predictions of crack density vs. temperature for the -45₄ ply group. Flaws are assumed to be seeded within the volume.
Figure 6.30  Experimental data and predictions of crack density vs. temperature for the 454 ply group. Flaws are assumed to be seeded along the edges.
Figure 6.31  Experimental data and predictions of crack density vs. temperature for the 904 ply group. Flaws are assumed to be seeded along the edges.
Figure 6.32 Crack distribution history for the [0\_/45\_/90\_/-45\_], at select temperatures. Flaws are assumed to be seeded along the edges.
each of the ply groups have extended since the last temperature. More starter crack have formed in both ply groups. In the -45, ply group, the number of through cracks has increased.

Predictions of crack distributions for the [0/45/-90/45]s laminate assuming volume flaw seeding are shown in Figures 6.33. Crack distributions for the ply groups are illustrated for three select temperatures. These temperatures are -153 °C, -183 °C, and -213 °C.

At the first temperature of -153 °C, only the -45, ply group shows cracking. Only through cracks are present.

At -183 °C, all the ply groups possess cracks. In the 45 and 90, ply groups, starter cracks and type E partial cracks are present. In the -45, ply group, more through cracks are present.

At the last temperature of -213 °C, in the 45 and 90, ply groups, both type E and type P partial cracks can be seen as well as through cracks. Some of the partial cracks in each of the ply groups have extended since the last temperature. A few new starter cracks have formed in both ply groups. In the -45, ply group, the number of through cracks has increased.

Visual comparison of the crack distributions in Figures 6.32 and 6.33 reveal obvious differences in the cracking patterns assuming edge flaw seeding and volume flaw seeding. In Figure 6.32, since the flaws are seeded along the edges, cracks extend toward the middle of the ply groups. As a result, the number of cracks is higher along the edges of the ply groups than near the center. In Figure 6.33, since flaws are seeded at various locations throughout the volume, the number of cracks at various locations through the depth of the ply groups is more even.
Figure 6.33  Crack distribution history for the \([0_2/45_2/90_2/-45_2]\), at select temperatures. Flaws are assumed to be seeded within the volume.
6.5 COMPARISON WITH P75/934 THERMAL EXPERIMENTAL DATA

In this section, crack density predictions from the current method are compared to experimental data collected by Park [87]. Crack densities throughout the volume away from the edges were found by sanding away the specimen edges. At various increments into the depth of the specimens, densities were recorded. Crack density predictions as a function of temperature and as a function of depth into the width at a constant temperature are compared to data for a [0/45/90/-45]ₗ laminate composed of P75/934 material. Also, prediction of crack distributions will be presented. Material properties for the P75/934 material system are listed in Table 6.2. Effective flaw distribution parameters are listed in Table 6.4. Shear lag and fracture toughness parameters are listed in Table 6.6. Material variations parameters are listed in Table 6.7. Results presented in this section are archived in Appendix G.

Crack density predictions as functions of temperature assuming volume flaw seeding are shown in Figures 6.34 - 6.36. For the 45 ply group shown in Figure 6.34, predictions of crack initiation emulate the general trend of the data for temperatures above -100 °F, approximately. Predictions of crack accumulation over-predicts the observed edge data. However, crack density data within the volume for this ply group are closer to the predicted densities. For the 90 ply group shown in Figure 6.35, crack density predictions under-predict the experimental data. Initiation is predicted at a significantly lower temperature than exhibited by the data. However, accumulation predictions for temperatures below -100 °F agrees with the experimental data. For the -45₂ ply group shown in Figure 6.36, initiation is predicted at a higher temperature than the experimental data. Crack accumulation for the entire temperature range are over-predicted.

Crack densities as a function of depth into the width at -200 °F are shown in Figures 6.37 and 6.38. Figure 6.37 shows the experimental data collected by Park [87]. Figure 6.38 shows predictions from the current method. Comparing the two shows that predictions qualitatively emulate the behaviour of the data. In the thinner ply groups, the
Figure 6.34  Experimental data and predictions of crack density vs. temperature for the 45 ply group. Flaws are assumed to be seeded within the volume.
Figure 6.35  Experimental data and predictions of crack density vs. temperature for the 90 ply group. Flaws are assumed to be seeded within the volume.
Figure 6.36 Experimental data and predictions of crack density vs. temperature for the -45\(_2\) ply group. Flaws are assumed to be seeded within the volume.
Figure 6.37 Predictions of crack density vs. depth in the width direction at -200 °F.
Figure 6.38 Experimental crack density data vs. depth in the width direction at -200 °F.
densities vary erratically with depth. In contrast, the density in the thicker ply group is essentially constant.

Prediction of crack distributions assuming flaw seeding within the volume at select temperatures are illustrated in Figure 6.39. In the -45\textsubscript{2} ply group, mostly through cracks can be seen. In the thinner ply groups, especially early in the loading, the cracking pattern is random. Regions of weaker material allow cracks to congregate resulting in concentrated areas of cracks. However, as temperatures decrease, the opposite becomes true. Conditions for the initiation and extension of cracks is more favourable with decreasing temperature. As a result, regions of tougher material create islands where there are few cracks. Eventually, with the temperatures continuing to decrease, these islands will disappear and the cracking pattern will be more uniform as the thinner ply groups become saturated with cracks.

### 6.6 Qualitative Comparison with Experimental Fatigue Data

Results obtained by Lafarie-Frenot and Henaff-Gardin [45,46] in their fatigue study are qualitatively compared with mechanical loading results from this study. Two sets of results are compared. The first involves the different observed crack types and their populations. The second compares crack densities as functions of depth into the width.

Four types of cracks are classified by Lafarie-Frenot and Henaff-Gardin. These are new cracks, propagating cracks, stopped cracks, and through cracks. Each of these crack types has a counterpart classification in this study. New cracks correspond to starter cracks. Propagating cracks correspond to type E partial cracks. Stopped cracks correspond to type P partial cracks. Through cracks correspond to through cracks.

Figures 6.40 and 6.41 compare crack densities for the different crack types. Figure 6.40 shows the experimental results for a cross-ply laminate subject to mechanical fatigue.
Crack distributions for the [0/45/90/-45]s laminate at select temperatures. Flaw are assumed to be seeded within the volume.
Figure 6.40 Experimental crack densities for different crack types for the fatigue specimens in the study by LaFarie-Frenot and Henaff-Gardin [45,46].
Figure 6.41  Predictions of crack density for different crack types in the 90\textsubscript{2} ply group of the [0\textsubscript{2}/45\textsubscript{2}/90\textsubscript{2}/-45\textsubscript{2}]\textsubscript{1} laminate vs. applied load. Flaws are assumed to be seeded along the edges.
Experimental crack density as a function of depth into the width from the fatigue specimen in the study by LaFarie-Frenot and Henaff-Gardin [45,46].
Crack Density vs. Depth

Figure 6.43 Predictions of crack density as a function of depth into the width at specific loads for the 902 ply group.
loads. Figure 6.41 shows crack density predictions for the [0/45/90/-45]s laminate subject to monotonic mechanical loads. For the laminate shown in Figure 6.40, only three crack types were observed. Although through cracks were not seen to occur for this laminate, they have been observed and classified by the authors [46]. Even though the load types are different, the general trends for each of the different crack types agree. The population of new or starter cracks decreases until there are very few. The population of propagating or type E partial cracks increases until a maximum is reached, at which point, the population decreases to a low value. The population of stopped or type P partial cracks steadily increase. For the laminate in this study, a population of through cracks exists whose numbers increase as load increases.

Crack densities as functions of depth into the width are shown in Figures 6.42 and 6.43. Figure 6.42 shows experimental crack density at different cycles. Figure 6.43 shows predicted crack density assuming flaws seeding along the edges at different loads. Although the load types, materials, and laminates are different, the same qualitative behaviour seen in the experimental specimens of Lafaire-Frenot and Henaff-Gardin are reproduced. At the two lower loads, crack density decreases as distance from the edges increases. At the higher load, crack density is constant except in the middle where a bump exists. The behaviour exhibited by the predictions replicates that seen in the experimental specimens.
CHAPTER 7

DISCUSSION

In this chapter, a summary outline of Chapter 6 will be presented first. Then, results presented in Chapter 6 will be discussed. Topics include the effective flaw distribution, the material variations model, flaw seeding assumptions, and comparison of experimental crack density data and experimental observations with predictions of crack density and crack distributions from the current method.

7.1 SUMMARY OUTLINE OF CHAPTER 6

Chapter 6 is organized into four general parts. Section 6.1 described the selection and determination of the inputs and parameters necessary to execute the current method. Section 6.2 demonstrated the effects of different components of the method on predictions of crack densities and distributions. Four combinations were presented: random crack locations with uniform material, random crack locations and material variations, effective flaw distribution with uniform material, and effective flaw distribution and material variations. Section 6.3 presented parametric studies of the material variations model and the effective flaw distribution. Material variations model parameters that were investigated include the amplitude, $A$, and the length parameters, $\lambda_x'$ and $\lambda_y'$, respectively. Effective flaw distribution parameters that were investigated include the shape parameter, $m$, the characteristic effective flaw size, $d$, and the effective flaw density, $EFD$. Sections 6.4 - 6.6 compared experimental crack density data and experimental observations with predictions of crack density and crack distributions from the current method. This included crack densities and crack distributions in AS4/3501-6
laminates subject to mechanical and thermal loads, crack densities and crack distributions in P75/934 laminates subject to thermal loads, and qualitative comparisons of experimental fatigue results with monotonic load predictions.

7.2 DISTRIBUTION OF EFFECTIVE FLAWS

The primary impact of the effective flaw distribution model is that it controls crack initiation. In Section 6.2, crack density predictions using the effective flaw distributions with uniform material show earlier initiation and more gradual accumulation as compared to the sudden onset of cracking predicted by the deterministic method. However, density predictions using only random crack locations and material variations also show early initiation. Both effective flaws and material variations cause early initiation, independently or in combination. The effective flaw distribution component enables the current method to successfully replicate crack initiation and crack accumulation in both the thinner and thicker ply groups as demonstrated by results from Section 6.2 and Sections 6.4 - 6.6. In thin ply groups, weaker material (from the material variations model) or effective flaws along the edges (from the effective flaw model) will produce visible cracks at loads below those necessary to propagate through cracks. These early cracks are observed in practice. In the thicker, middle ply groups, crack initiation is retarded compared with predictions from the deterministic method. Controlling the energetic favorability of crack initiation through the effective flaw distribution component enables the current method to replicate this feature of the data as shown in Sections 6.4 - 6.6.

Examination of the crack distribution predictions show that the effective flaw distribution predicts different crack types for the thinner and thicker ply groups. In the thinner ply groups, the effective flaw distribution predicts starter cracks. Starter cracks occur because conditions are energetically favorable for the formation of cracks from effective flaws but are unfavorable for the extension of the starter cracks. This crack type
has been experimentally observed and categorized in a fatigue load study by LaFarie-Frenot and Henaff-Gardin [45,46]. In the thicker, middle ply groups, suppression of crack initiation results in only through cracks. When conditions are favorable for cracks to initiate from effective flaws, conditions are also favorable for their extension from one side to the other.

Using different combinations of the three effective flaw distribution parameters (the characteristic effective flaw size, \( d \), the shape parameter, \( m \), and the effective flaw density, \( EFD \)), an infinite number of distributions can be generated. In practice, \( d \) and \( m \) are fit to experimental data using a large number of flaws. However, only the largest effective flaws control crack initiation, as illustrated Figure 7.1. Provided that this minority population controls cracking, different combinations of the three parameters can produce essentially the same population of the largest effective flaws. Thus, the selection of a specific combination for a given laminate can be essentially arbitrary.

Parametric studies in Section 6.3 demonstrated the effects of individually varying each of three parameters on crack density predictions. Underlying the effects on the crack density predictions is the effect on the population of the largest effective flaws. Adjusting \( d \) moves the distribution along the abscissa by scaling the effective flaw sizes up or down while the shape and size of the distribution remains the same, as illustrated in Figure 7.2. More importantly, as a result, the largest effective flaws are scaled up or down. This is reflected in the crack density predictions. While the trends of crack accumulation are generally the same, the predictions move along the abscissa.

Adjusting \( m \) affects the shape of the distribution and the range of effective flaw sizes while the mean of the distribution remains stationary as illustrated in Figure 7.3. As a result, the largest effective flaws are smaller or larger and the shape of the distribution of the largest effective flaw population is affected. Adjusting \( EFD \) affects the total number of effective flaws taken into account by the distribution. The population of all the
Figure 7.1  Population of the largest effective flaws that control crack initiation.
Figure 7.2 Effect of changing $d$ on effective flaw population.
Figure 7.3 Effect of changing $m$ on effective flaw population.
effective flaws, including the largest ones, increases or decreases as illustrated in Figure 7.4.

Although using the effective flaw distribution is successful in replicating certain aspects of the data and observed behavior, the discretized Weibull distribution used in this method possesses two general drawbacks. The first includes the number of parameters needed to use the distribution and the fitting these parameters using experimental data. The second includes the need for a different characteristic effective flaw size for thickest laminate. $d$ for the $[0_4/45_4/90_4/-45_4]_5$ laminate in Table 6.3 is different compared to the other two thinner laminates. This problem may be resolvable by the correct combination of $d$, $m$, and $EFD$. However, a single combination of the three parameters was not achieved in this study.

On the positive side, the values of the effective flaw distribution parameters are physically reasonable. The characteristic effective flaw size, $d$, is on the scale of fiber diameters. The shape parameter, $m$, compares reasonably with those used in other stochastic studies. The effective flaw density, $EFD$, is also reasonable.

### 7.3 MATERIAL VARIATIONS MODEL

Like the effective flaw distribution, the material variations model also reproduces certain aspects seen in experimental data and observations. These aspects include the scatter seen in the experimental data, the different partial crack types observed experimentally, and the behavior of the crack distribution patterns in the laminates.

From results in Sections 6.2 and 6.3, crack density predictions show scatter similar to that of the experimental data. In Section 6.2, comparing density predictions assuming uniform material and using the material variations model shows that the material variations model produces a wider swath of predicted densities. The parametric studies in Section 6.3 demonstrate that the width of the swath is controlled by the amplitude of the material variations, $A$. The larger the amplitude, the wider the predicted
Figure 7.4  Effect of changing $EFD$ on effective flaw population.
density swath. The combination of random crack locations and the material variations model leads to variation in predictions between each simulation run. The scatter in the predictions corresponds reasonably well with those of the experimental data as seen in Sections 6.4 - 6.6. This suggests that the initial estimated value of the amplitude is reasonable.

Another aspect emulated using the material variations model is the earlier predicted initiation and more gradual crack accumulation. Regions of weaker material allow cracks to initiate earlier than predicted by the deterministic method as shown in Section 6.2. The effect of using the material variations model is also demonstrated in predictions of crack distributions. In Section 6.2, comparison of distributions assuming uniform material with those using the material variations model reveal that the material variations are responsible for the different crack types observed and the observed disorder in the cracking pattern. The material variations model replicates two different partial crack types, which are seen in distribution predictions in Sections 6.2 - 6.6. Type E partial cracks extend until the apparent toughness of the material renders conditions unfavorable. The gradual extension and development of partial cracks creates the opportunity for them to encounter one another, which results in type P partial cracks. The gradual development of cracks is responsible for the increased disorder in cracking patterns. The development of cracks and the degree of disorder in the cracking patterns is influenced by the amplitude of the material variations as demonstrated in Section 6.3. Amplitude parametric study crack distribution predictions show that the degree of disorder in the cracking patterns increases with amplitude of the material variations.

The amplitude controls most of the behavior caused by the material variations model. It affects the width of the swath of predicted densities, rate of growth of different crack types, and the degree of disorder in the crack distributions. In contrast, the length parameters, \( \lambda_x \) and \( \lambda_y \), respectively, have limited impact on the predictions. Parametric studies conjointly varying the values of the length parameters in Section 6.3 reveal that
the crack density predictions remain unaffected by different values. The effect of the length parameters can be seen in the distributions which reveal only slight, visual dissimilarities for different length values.

7.4 EFFECTIVE FLAW SEEDING ASSUMPTIONS

Experimental data and observations from Maddocks [88] and Park [87] suggest that effective flaws are seeded within the volume of the ply groups. The $[0\gamma/45\gamma/90\gamma/-45\gamma]_s$ laminate subject to thermal loads demonstrates constant crack densities into the width. Predictions emulate this behavior when effective flaws are seeded within the volumes of the ply groups. Conversely, predictions assuming effective flaws are seeded along the edges show higher densities at the edges than toward the middle of the ply groups. Interior crack density data collected by Park agree reasonably well with predictions assuming flaw seeding within the volume. Additionally, the behavior of crack density as a function of depth into the width is emulated when flaws are assumed to be seeded within the volume. However, others have observed cracking that initiates from flaws along the edges of the ply groups [45,46,66]. Predictions assuming seeding along the edges qualitatively correspond with those experimental crack density as a function of depth into the width for fatigue specimens [45,46].

The assumption used to seed effective flaws in the ply groups of a laminate can affect predictions of crack density and crack distributions. Observed edge density predictions for flaws seeded within the volume of the ply groups are lower than those assuming flaw seeding along the edges of the ply groups. As mentioned in Section 7.3, parametric studies in Section 6.3 revealed that material variations prevent some cracks from being seen on the edges, which results in the lower density predictions.

The seeding assumptions also affects the crack distribution predictions. Effective flaws seeded within the volume of the ply groups result in more disorderly cracking patterns as seen in predictions presented in Sections 6.2 and 6.4 - 6.6. Partial cracks
dominate in the thinner ply groups because cracks can initiate at random locations in the $x'-y'$ plane. As cracks extend, they encounter one another resulting in type P partial cracks. The disjointed type P partial cracks result larger average crack spacing which is reflected in the crack densities. Cracks that initiate from effective flaws seeded along the edges of the ply groups are restricted to extending from the edge. As a result, through cracks in addition to partial cracks create a more orderly cracking pattern.

7.5 COMPARISON OF PREDICTIONS AND DATA

The current method is successful in emulating features of the experimental data and observed behaviors that the deterministic method is incapable of capturing. These features include the general crack density trends, crack density scatter, suppression of cracking in the middle ply groups, the difference in crack distributions in the thinner and thicker ply groups, the different types of cracks, and cracking as a function of positions through the width.

Trends of the data emulated by the current method are early initiation and the gradual accumulation of cracks. As discussed in Section 7.3, early initiation is controlled by the effective flaw distribution. The large effective flaws control crack initiation, which results in a more gradual accumulation of cracks. This emulates the behavior exhibited by available data.

Features seen in the data scatter that are replicated by the predictions include the width of the plotted density swath generated by several independent tests or analyses. As discussed in Section 7.4, the width of the density swath is controlled by the amplitude parameter of the material variations model. The degree of predicted scatter produced by the initial estimated amplitude value agrees reasonably well with the amount exhibited by the data.

The laminates investigated in this study exhibit suppressed cracking in the thicker, middle ply groups. This characteristic is captured through the use of the effective flaw
distribution. Crack initiation is suppressed because the available effective flaws are too small or too few in number to allow rapid crack initiation.

Crack distributions predicted by the current method emulate the observed behavior of experimental specimens. Cracking patterns in the thinner ply groups and the thicker ply groups for the laminates in this study are basically different. In the thicker, middle ply groups, suppressed cracking results in through cracks. At the point cracks initiate in these ply groups, conditions are energetically favorable for the formation of through cracks. In the thinner ply groups, effective flaw distributions and material variations cause a variety of crack types, which results in disordered cracking patterns.

Predictions of crack distributions assuming flaw seeding along the edges subject to mechanical and thermal loads qualitatively agree with observations and experimental results of fatigue specimens in a study by LaFarie-Frenot and Henaff-Gardin [45,46]. Two aspects of their results reproduced by the current method include the different varieties of observed cracks and populations of the various crack types as a function of load.

In their studies and in the present work, four types of cracks are classified. These are new cracks or starter cracks, propagating cracks or type E partial cracks, stopped cracks or type P partial cracks, and through cracks. In Section 6.6, the general trends of the predictions of the populations of these four different cracks qualitatively correspond with the experimental data. The trends of the predictions can be explained through the current method. Starter cracks can initiate in regions where crack extension is unfavorable. As load increases, the population of starter cracks decreases because regions where conditions are energetically favorable for crack extension enlarge. Consequently, the population of type E partial cracks increases while the population of starter cracks decreases. As the load increases, the influence of the material variations on the extension of cracks steadily weakens, which results in the cresting in the type E partial crack population. Because of the weakening influence of the material variations on crack extension, cracks continue to extend until they stop one another or until they become
through cracks. This results in the steady increase in the population of type P partial cracks and through cracks while the population of type E partial cracks peaks and decreases.

Predictions of crack density as a function of position through the width from the current method qualitatively agree with experimental data collected by LaFarie-Frenot and Henaff-Gardin. The trends of the predictions can be explained in terms of the current method. At the lower loads, the density along the edges is greater than toward the middle of the ply group. Cracks initiate from flaws at the edges and material variations allows cracks to extend only a limited distance. As load increases, the density near the middle and along the edges becomes more even because cracks continue to extend toward the middle from the edges as the influence of the material variations weakens. At the higher loads, where cracking nears saturation and the influence of the material variations is small, the density along the edges and near the middle is the same because cracks have extended to become type P partial cracks or through cracks. Also, at these loads, because the cracks are extending from the edges, they encounter one another around the center of the ply group which results in the bump in the crack density around the center.

Predictions of crack density as a function of position through the width, which assume flaw seeding within the volume of ply groups, emulate the trends of the data collected by Park. For the thinner ply groups, crack densities vary considerably as a function of depth. This behavior can be explained by examining the crack distribution predictions. Regions of tougher material, where there are very few cracks, are surrounded by cracks. At -200 °F, depending on the depth, these tougher regions will cause the crack density to fluctuate. For the thicker ply groups, crack density is essentially constant with depth. Examination of the crack distribution predictions shows that through cracks dominate. As a result, the crack density will be constant with depth.
Although the current method is more successful than the earlier methods in emulating many feature of the data, there are some instances where it is not fully successful. These instances include under-prediction of data at lower loads and over-prediction of data at higher loads. Examples where the data is under-predicted include the 90\textsubscript{n} ply groups subject to mechanical loads and in the thin ply groups subject to thermal loads. This is especially prominent in predictions for P75/934. Larger effective flaws than those taken into account are a possible cause for the under-prediction. The under-prediction may also be caused by material variations amplitude greater than the one used in obtaining the predictions or temperature dependence of material properties.

Over-prediction of the data generally occurs at higher loads near saturation. Possible causes for the over-prediction include the fracture toughness value, variations in the effective flaw distribution, the effect of other damage modes, and edge effects. The fracture toughness value used in the predictions is optimized for the deterministic model. Optimizing the fracture toughness value for the current method could possibly result in better correlation between predictions and data near saturation. Variations in the effective flaw distribution similar to material variations [49] could possibly precipitate the over-predictions. As the loads increase, other damage modes such as delamination could affect the cracking behavior of the ply groups by making cracking less favorable. Failure to account for edge effects could also be responsible for the over-predictions. Crack densities within the volume could be higher than near the edges, as demonstrated by P75/934, resulting in lower observed densities along the edges.
CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

The impetus of this study was to attain a better understanding of transverse microcracking in composite laminates subject to mechanical and thermal loads. This is achieved through the development of a stochastic-analytical, predictive methodology. Conclusions drawn from the results of this study are presented in Section 8.1. Recommendations for further research into microcracking are presented in Section 8.2.

8.1 CONCLUSIONS

Discussion of the results presented in Chapter 7 leads to the following conclusions:

(1) The primary objective of this study is achieved. Expanded capabilities that include the effective flaw distribution and the material variations model enable the current method to replicate more features of observed behaviours than the deterministic method. Predictions from the current method capture the general trends of the available experimental data and correspond reasonably well to most of it. These include emulating the general trends of the collected data for crack density as a function of load and crack density as a function of position through the width. Also, the current method can predict crack distributions. Crack distribution predictions emulate observed cracking behaviour and patterns.

(2) The distribution of effective flaw sizes described by a Weibull probability function successfully replicates some aspects of cracking by controlling crack initiation. These include prediction of gradual crack initiation, emulating the
general trends of the experimental data, replicating starter cracks in distribution predictions, and providing a mechanism for crack suppression in the thicker, middle ply groups. Despite these successes, the discretized effective flaw distribution possesses some drawbacks. These drawbacks include the number of necessary parameters, the interaction of these parameters, and the need to fit these parameters using experimental data.

(3) The material variations model is a simple and effective tool that reproduces various aspects of experimentally observed cracking behaviour. This includes the scatter seen in experimental data, the different crack types, the populations of the various crack types, and experimentally observed cracking patterns.

(4) The two seeding assumptions reproduce experimental results and observed behaviour. Flaw seeding within the volume of the ply groups correlates with the results and observations of Park and Maddocks. Flaw seeding along the edges of the ply groups correlates with the results and observations of LaFarie-Frenot and Henaff-Gardin. These results, however, leave the question of where cracks actually initiate unanswered.

(5) The success of relatively simple models in replicating complex behaviors lends confidence to the idea that the models incorporated within the method are capturing the true mechanisms of cracking.

(6) Disagreements between predictions and data are possibly the result of factors not taken into account that affect cracking. For example, these could include interaction between ply groups, edge effects, other damage modes, and variability in the effective flaw distributions. Another factor is temperature-dependent material properties which were not used in obtaining the predictions. The disagreements between predictions and data for thermal loads is possibly due to the fact that the effective flaw distribution parameters were obtained using only mechanical load data.
8.2 RECOMMENDATIONS

Although the current method is an improvement compared to the deterministic method, improvements are needed for it to become more effective and efficient. In the process, we will continue to gain a better understanding of microcracking. Areas needing further attention are:

(1) Investigating the two different seeding assumptions to resolve the disparities in observed cracking behaviours. Experiments might reveal insights into the differences in the cracking behaviours. This might include determining the conditions under which cracks are more likely to initiate from flaws along the edges or within the volume. Areas that might be considered include different material systems, different laminate preparation techniques, different load types, and edge finishing effects.

(2) Resolving the consistent disagreement between predictions and data for thermal loads. This could include obtaining effective flaw distribution parameters using data from both load types, obtaining more thermal data that are more closely spaced and which cover a larger temperature range, collecting and using temperature-dependent data, and investigating possible differences in the cracking mechanisms between the two load types. These will allow for more and better comparisons between predictions and data.

(3) Modeling of other factors that affect cracking to improve the precision in predicting crack densities and distributions. Some of these could include using a free-edge analysis similar to Park's, modeling the interaction of cracks in different ply groups, and including shearing (Mode II) effects.

(4) Expanding the method to both thermal and mechanical cyclic loads.

(5) Understanding and improving the effective flaw distribution scheme. Among the improvements would be to eliminate some of the (possibly redundant) parameters and procuring a single set of parameters for all laminates and load types. Also, the
development of a spatially varying effective flaw distribution variation model (similar to the material variations model) might help.

(6) Developing a better, more accurate method for obtaining values for the effective flaw distribution and the material variations model. For the effective flaw distribution, this could entail a more formal fitting scheme using results from both mechanical and thermal loads. For the material variations model, X-radiography for a variety of experimental specimens at various load increments will allow for direct comparison between predicted cracking patterns and experimental cracking patterns. Experimental cracking patterns could serve as a way to obtain more accurate values of the amplitude and length parameters.
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Equilibrium, stress-strain, and shear stress are the groups of equations that represent the system of equations that must be solved for the stress and displacement fields. All three sets sum to a total of seven equations, Equations (A.1) - (A.7), that comprise the system of equations necessary for a solution. Figure 4.5 is the shear lag model used to obtain the system of equations.

Laminate equilibrium results in Equation (A.1).

\[ \sigma_o a_o = \sigma_r a_r + \sigma_c a_c \]  \hspace{1cm} (A.1)

Equilibrium of the cracked ply group gives

\[ q = \frac{a_c}{2} \frac{\partial \sigma_c}{\partial x'} \]  \hspace{1cm} (A.2)

Equilibrium of the rest of the laminate gives

\[ q = -\frac{a_r}{2} \frac{\partial \sigma_r}{\partial x'} \]  \hspace{1cm} (A.3)

Combining the stress-strain and strain-displacement equations for the cracked ply group and the rest of the laminate results in Equation (A.4) and Equation (A.5), respectively. For the cracked ply group,

\[ \frac{\sigma_c}{E_c} = \frac{\partial u_c}{\partial x'} - \alpha_c \Delta T \]  \hspace{1cm} (A.4a)

where
\[
\varepsilon_c = \frac{\partial u_c}{\partial x'} \quad (A.4b)
\]

For the rest of the laminate,

\[
\frac{\sigma_r}{E_r} = \frac{\partial u_r}{\partial x'} - \alpha_r \Delta T \quad (A.5a)
\]

where

\[
\varepsilon_r = \frac{\partial u_r}{\partial x'} \quad (A.5b)
\]

The shear stress between the cracked ply group and the rest of the laminate will be represented through a stress-strain equation of the form

\[
q = K(u_c - u_r) \quad (A.6)
\]

where \( K \) is a stiffness constant. \( K \) relates the shear stress between the cracked ply group and the rest of the laminate and the displacements of the two components. \( K \) is defined as

\[
K = \frac{G_{eff}}{a_q} \quad (A.7)
\]

By combining the previous seven equations, a nonhomogeneous, second order, linear, differential equation as a function of the stress in the cracked ply group, \( \sigma_c \), can be obtained. First, Equation (A.7) is substituted into Equations (A.2) and (A.3), respectively. Then taking the derivative of Equations (A.2) and (A.3), respectively, gives

\[
K\left(\frac{\partial u_c}{\partial x'} - \frac{\partial u_r}{\partial x'}\right) = \frac{a_c}{2} \frac{\partial^2}{(\partial x')^2} (\sigma_c) \quad (A.8)
\]

and

\[
K\left(\frac{\partial u_c}{\partial x'} - \frac{\partial u_r}{\partial x'}\right) = -\frac{a_r}{2} \frac{\partial^2}{(\partial x')^2} (\sigma_r) \quad (A.9)
\]

Next, subtracting Equation (A.5) from (A.4), multiplying the result by \( K \), and substituting this result into Equation (A.8) gives
\[ \frac{d^2}{(dx')^2}(\sigma_c) = \frac{2K}{a_c} \left[ \frac{\sigma_c - \sigma_r}{E_c - E_r} + (\alpha_c - \alpha_r)\Delta T \right] \]  \hspace{1cm} (A.10)

Solving the laminate equilibrium equation, Equation (A.1), for \( \sigma_r \), substituting that result for \( \sigma_r \) in the above equation, and manipulating gives

\[ \frac{d}{(dx')^2}(\sigma_c) - 2K \left( \frac{a_{r}E_{r} + a_{c}E_{c}}{a_{r}a_{c}E_{r}E_{c}} \right) \sigma_c = \frac{2K}{a_c} \left[ - \frac{\sigma_a a_o}{a_r E_r} + (\alpha_c - \alpha_r)\Delta T \right] \]  \hspace{1cm} (A.11)

Two parameters are defined in the next two equations. First is the shear lag parameter, \( \xi \), which is dimensionless. \( \xi \) is given by

\[ \xi = \sqrt{\frac{Ka_{c}a_{r}E_{c}}{2a_{r}E_{r}E_{c}}} \]  \hspace{1cm} (A.12)

The other parameter, \( \lambda \), represents side to the right of the equality in Equation (A.11). It is given by

\[ \lambda = \frac{2K}{a_c} \left[ \frac{a_{c}\sigma_a}{a_r E_r} - (\alpha_c - \alpha_r)\Delta T \right] \]  \hspace{1cm} (A.13)

Rule of mixtures, given in the the next two equations, are used to re-express Equation (A.13) in the form of Equation (A.17).

\[ a_{o}E_{o} = a_{r}E_{r} + a_{c}E_{c} \]  \hspace{1cm} (A.14)

\[ a_{o}E_{o}\alpha_{o} = a_{r}E_{r}\alpha_{r} + a_{c}E_{c}\alpha_{c} \]  \hspace{1cm} (A.15)

By expressing the far-field stress in the cracking ply group, \( \sigma_{oo} \), as

\[ \sigma_{oo} = \left[ \frac{E_{c}}{E_{o}}\sigma_a + E_{c}(\alpha_{o} - \alpha_c)\Delta T \right] \]  \hspace{1cm} (A.16)

\( \lambda \) can be re-expressed with some manipulation and rearrangement as

\[ \lambda = \frac{4\xi}{a_c} \sigma_{oo} \]  \hspace{1cm} (A.17)

Finally, using Equations (A.12) - (A.17), Equation (A.10) can be expressed as
\[
\frac{\partial^2}{(\partial x')^2} \left( \sigma_c \right) - \frac{4 \xi^2}{a_c^2} \sigma_c = -\lambda
\]  
(A.18)

which is the nonhomogeneous, second-order, linear, ordinary differential equation that can be used to solve for \( \sigma_c \). Using standard methods \[90\] with the boundary conditions given by \( \sigma_c(x' = \pm h) = 0 \), the solution is

\[
\sigma_c(x') = \begin{bmatrix}
\cosh\left(\frac{2 \xi x'}{a_c}\right) \\
1 - \frac{\cosh\left(\frac{2 \xi h}{a_c}\right)}{\cosh\left(\frac{2 \xi x'}{a_c}\right)}
\end{bmatrix} \sigma_{\infty}
\]  
(A.19)

Physically, upon inspection of Equation (A.19), the shear lag parameter scales the distance over which the stress in the cracking ply group rises to the far-field stress as measured from the crack.

Stress in the rest of the laminate, \( \sigma_r \), can be obtained by making use of Equation (A.19). This is accomplished by solving Equation (A.1) for \( \sigma_r \), substituting Equation (A.19) for \( \sigma_c \), and re-organizing. The result is

\[
\sigma_r(x') = \frac{\sigma_s a_s}{a_r} - \frac{a_c}{a_r} \left[ 1 - \frac{\cosh\left(\frac{2 \xi x'}{a_c}\right)}{\cosh\left(\frac{2 \xi h}{a_c}\right)} \right] \sigma_{\infty}
\]  
(A.20)

The displacements for the cracking ply group and the rest of the laminate are given by Equations (A.21) and (A.22), respectively. The cracking ply group displacement, \( u_c \), is found in the following manner: Equation (A.19) is substituted into Equation (A.4), Equation (A.16) is substituted into the result of the combination of Equations (A.19) and (A.4), this last resulting equation is solved for \( \partial u_c \), and integrated from 0 to \( x' \). The result is
\[ u_c(x') = \frac{a_r E_r}{a_o E_o} \left( a_o \sigma_a + \alpha_c \Delta T \right) \left[ 1 - a_c \frac{\sinh \left( \frac{2 \xi x'}{a_c} \right)}{\cosh \left( \frac{2 \xi h}{a_c} \right)} \right] \\
+ \alpha_c \Delta T x' \left[ 1 - a_c \frac{\sinh \left( \frac{2 \xi x'}{a_c} \right)}{\cosh \left( \frac{2 \xi h}{a_c} \right)} \right] \]

The displacement of the rest of the laminate, \( u_n \), is found in the following manner: Equation (A.20) is substituted into Equation (A.5), Equation (A.16) is substituted into the result of the combination of Equations (A.20) and (A.5), this last resulting equation is solved for \( du_r \) and integrated from 0 to \( x' \). The result is

\[ u_r(x') = \frac{\sigma_a x'}{a_c E_r} \left[ a_o E_o - a_c E_c \left( 1 - a_c \frac{\sinh \left( \frac{2 \xi x'}{a_c} \right)}{\cosh \left( \frac{2 \xi h}{a_c} \right)} \right) \right] \\
+ \frac{\Delta T x'}{a_o E_o} \left[ a_o E_o \alpha_r + a_c E_c \left( \alpha_c - \alpha_r \right) \left( 1 - a_c \frac{\sinh \left( \frac{2 \xi x'}{a_c} \right)}{\cosh \left( \frac{2 \xi h}{a_c} \right)} \right) \right] \]

(A.22)
In Section 4.4, the physical complexity of an extending crack is modeled more simply to facilitate the change in external work calculations. Solving the actual situation would require an encumbering numerical solution. This actual situation is modeled using two extreme models that are easier to calculate. One extreme model assumes that the average stresses in the uncracked and cracked portions are the same, as illustrated in Figure B.1. The other extreme model assumes that the average displacements in the uncracked and cracked portions are the same, as illustrated in Figure B.2. The change in external work for the actual situation is believed to exist between these two extremes. By demonstrating that the difference between the two extreme models is small, the solution from one of the extreme models will be sufficiently close to actual solution for the purposes of this study. In Section 4.4, the first extreme model, shown in Figure B.1, is used to calculate the change in external work.

In the first extreme model, the external work prior to the formation of the incremental partial crack extension is given by

\[ W_1 = \sigma \Delta y' u^* \]  \hspace{1cm} \text{(B.1)}

where \( \sigma \) is the average stress, \( \Delta y' \) is the incremental extension, and \( u^* \) is the displacement without the crack. The external work after the formation of the incremental crack extension is given by

\[ W_2 = \bar{\sigma} \Delta y' u \]  \hspace{1cm} \text{(B.2)}

\( u \) is the displacement with the crack.
Figure B.1 First extreme model used to calculate the change in external work. The average stress is the same in the uncracked and cracked portions. (a) Before crack extension. (b) After crack extension.
Figure B.2  Second extreme model used to calculate the change in external work. The average displacement is the same in the uncracked and cracked portions. (a) Before crack extension. (b) After crack extension.
The average stress in Equation (B.1) is given by

$$\bar{\sigma} = E^* \varepsilon^*$$  \hspace{1cm} (B.3)

where $E^*$ is the stiffness without the crack and

$$\varepsilon^* = \frac{u^*}{l}$$  \hspace{1cm} (B.4)

is the strain without the crack.

The average stress in Equation (B.2) is given by

$$\bar{\sigma} = E \varepsilon$$  \hspace{1cm} (B.5)

where $E$ is the stiffness with the crack and

$$\varepsilon = \frac{u}{l}$$  \hspace{1cm} (B.6)

is the strain with the crack.

The change in external work is obtained by subtracting Equation (B.1) from Equation (B.2).

$$\Delta W_{\sigma} = \bar{\sigma} \Delta y'(u - u^*)$$  \hspace{1cm} (B.7)

Using Equations (B.3) - (B.6), Equation (B.7) is manipulated into the following form.

$$\Delta W_{\sigma} = \bar{\sigma}^2 l \Delta y' \left( \frac{1}{E} - \frac{1}{E^*} \right)$$  \hspace{1cm} (B.8)

In the second extreme model, the external work prior to cracking is given by

$$W_i = \sigma^{*'}(w - a)\bar{u}^* + \sigma' a \bar{u}^*$$  \hspace{1cm} (B.9)

where $\sigma^{*'}$ is the stress in the uncracked portion before the incremental crack extension, $\sigma'$ is the stress in the cracked portion before the incremental crack extension, $a$ is the crack length, $w$ is the width, and $\bar{u}^*$ is the average displacement prior to the incremental crack extension.

$\bar{u}^*$ is given by
\[
\bar{u}^* = \frac{\sigma^* l}{E^*} = \frac{\sigma l}{E} = \frac{\bar{\sigma} l}{E}
\]  

(B.10)

\(\sigma^*\) is given by

\[
\sigma^* = \frac{E^*}{\bar{E}^*} \bar{\sigma}
\]  

(B.11)

\(\sigma\) is given by

\[
\sigma' = \frac{E}{\bar{E}^*} \bar{\sigma}
\]  

(B.12)

In Equations (B.11) and (B.12), \(\bar{E}^*\) is the average stiffness before the incremental crack extension. \(\bar{E}^*\) is obtained using the Rule of Mixtures:

\[
\bar{E}^* = \frac{E^*(w-a)+E_a}{w}
\]  

(B.13)

The external work after the formation of the incremental crack extension is given by

\[
W_2 = \sigma^*(w-a-\Delta y')\bar{u} + \sigma(a+\Delta y')\bar{u}
\]  

(B.14)

where \(\sigma^*\) is the stress in the uncracked portion after the incremental crack extension, \(\sigma\) is the stress in the cracked portion after the incremental crack extension, \(\bar{u}\) is the average displacement after the extension of the crack.

\(\bar{u}\) is given by

\[
\bar{u} = \frac{\sigma^* l}{E^*} = \frac{\sigma l}{E} = \frac{\bar{\sigma} l}{E}
\]  

(B.15)

\(\sigma^*\) is given by

\[
\sigma^* = \frac{E^*}{E} \bar{\sigma}
\]  

(B.16)

\(\sigma\) is given by

\[
\sigma = \frac{E}{E} \bar{\sigma}
\]  

(B.17)
In Equations (B.16) and (B.17), \( \bar{E} \) is the average stiffness after the incremental crack extension. \( \bar{E} \) is obtained using the Rule of Mixtures:

\[
\bar{E} = \frac{E^*(w-a-\Delta y') + E(a+\Delta y')}{w} \quad (B.18)
\]

The change in external work is obtained by subtracting Equation (B.9) from Equation (B.14).

\[
\Delta W_u = \bar{u} \left[ \sigma^* (w-a-\Delta y') + \sigma (a+\Delta y') \right] - \bar{u}^* \left[ \sigma^{*'} (w-a) + \sigma' a \right] \quad (B.19)
\]

Substituting Equations (B.10) - (B.12) and (B.15) - (B.17) into Equation (B.19) gives

\[
\Delta W_u = \bar{\sigma}^2 l \left\{ \frac{1}{E^2} \left[ E^* (w-a-\Delta y') + E(a+\Delta y') \right] - \frac{1}{(E^*)^2} \left[ E^* (w-a) + Ea \right] \right\} \quad (B.20)
\]

after some manipulation. Performing some more algebraic manipulations on Equation (B.20) gives

\[
\Delta W_u = \bar{\sigma}^2 l \left\{ \left( \frac{1}{E^2} - \frac{1}{(E^*)^2} \right) \left[ E^* (w-a) + Ea \right] - \frac{\Delta y'}{E^2} (E - E^*) \right\} \quad (B.21)
\]

\( E^* \) and \( E \) are related through

\[
E^* = E + \Delta e \quad (B.22)
\]

where \( \Delta e \) is the change in stiffness of the cracked portion from the uncracked portion. Similarly, \( E^* \) and \( \bar{E} \) are related through

\[
\bar{E}^* = \bar{E} + \Delta \bar{e} \quad (B.23)
\]

where \( \Delta \bar{e} \) is the average change in stiffness of the cracked portion from the uncracked portion. \( \Delta \bar{e} \) can be expressed in terms of \( \Delta e \) using Equations (B.13), (B.18), and (B.22)
\[ \Delta \bar{e} = \frac{\Delta y'}{w} \Delta e \]  

(B.24)

Generally, \( \Delta e \), is relatively small, on the order of 2% of the \( E^* \). In Equation (B.24), since \( \Delta y' \) is small compared to \( w \), \( \Delta \bar{e} \) is very small. Substituting Equation (B.24) into Equation (B.19) gives

\[ \Delta W_u = \frac{\Delta y' \Delta e}{Ew + \Delta e(w - a)} - \frac{\Delta y' \Delta e}{E^2} \]  

(B.25)

Expanding \( \frac{1}{(E^*)^2} \) in the above Equation gives

\[ \frac{1}{(E^*)^2} = \frac{1}{E^2 + 2E\Delta \bar{e} + \Delta \bar{e}^2} \]  

(B.26)

Expanding the denominator and eliminating the small, higher-order term \( \Delta \bar{e}^2 \), Equation (B.26) can be approximated as

\[ \frac{1}{(E^*)^2} = \frac{1}{E^2} \left( 1 - 2 \frac{\Delta \bar{e}}{E} \right) \]  

(B.27)

Substituting Equation (B.27) into Equation (B.25) gives

\[ \Delta W_u = \frac{\Delta y' \Delta e}{E^2} \left( 2 \frac{\Delta \bar{e}}{E} \left[ Ew + \Delta e(w - a) \right] - \Delta y' \Delta e \right) \]  

(B.28)

Substituting Equation (B.24) into the above Equation

\[ \Delta W_u = \frac{\Delta y' \Delta e}{E^2} \left( 2 \frac{E}{E} + 2 \frac{\Delta e(w - a)}{Ew} - 1 \right) \]  

(B.29)

\( \frac{E}{E} \) in the above Equation can be approximated as

\[ \frac{E}{E} \approx \frac{1}{1 + \frac{\Delta e(w - a)}{Ew}} \approx 1 - \frac{\Delta e(w - a)}{Ew} \]  

(B.30)
The above Equation is a grosser approximation because $\Delta e$ is small but not very small like $\Delta \overline{e}$. Substituting Equation (B.30) into Equation (B.29) gives

$$\Delta W_u = \frac{\overline{\sigma}^2 l \Delta y' \Delta e}{E^2} \left\{ 1 - 2 \frac{\Delta e (w - a)}{w} \left( \frac{1}{E} - \frac{1}{\overline{E}} \right) \right\} \quad (B.31)$$

Using Equation (B.30) and eliminating higher order terms, Equation (B.31) is approximately

$$\Delta W_u = \frac{\overline{\sigma}^2 l \Delta y' \Delta e}{E^2} \quad (B.32)$$

Inverting Equation (B.30) gives approximately

$$\frac{\overline{E}}{E} \approx 1 + \frac{\Delta e (w - a)}{Ew} \quad (B.33)$$

Squaring Equation (B.33), eliminating higher order terms, and substituting into Equation (B.32) is approximately

$$\Delta W_u = \frac{\overline{\sigma}^2 l \Delta y' \Delta e}{E^2} \left[ 1 + 2 \frac{\Delta e (w - a)}{Ew} \right] \quad (B.34)$$

By comparing Equations (B.32) and (B.34), the maximum difference between the two is

$$\text{max error} \approx 2 \frac{\Delta e}{E} \quad (B.35)$$

Since the ratio of $\Delta e/E$ is small, the change in external work described by Equation (B.34) is sufficiently close to the actual change in external work and can be used in Section 4.4 calculations safely.
APPENDIX C

INCREMENTAL PARTIAL CRACK EXTENSION ENERGY EXPRESSION

The change in strain energy release rate of an incremental extension of a partial crack is a slightly modified version of the change in strain energy release rate derived by Maddocks [88]. Change in strain energy release rate for an incremental extension of a partial crack is the change in total energy from an uncracked state to a cracked state for a self-similar extension of length $\Delta y'$. This is given by

$$\Delta G_I = \frac{\Delta W - \Delta U}{a_c \Delta y'} \quad (C.1)$$

In the following sections, the expressions for $\Delta W$ and $\Delta U$ will be given. The expression $\Delta G_I$ will be expressed using $\Delta W$ and $\Delta U$.

C.1 $\Delta W$

The change in external work due to an incremental extension of a partial crack is

$$\Delta W = (W|_{h_1} + W|_{h_2}) - W|_{2h} \quad (C.2)$$

where $W|_{2h}$ is the work done by the applied load before the extension of the crack and $(W|_{h_1} + W|_{h_2})$ is the work done by the applied load after the extension of the crack. $W|_{2h}$ is given by

$$W|_{2h} = 2a_o \sigma_o \Delta y'u_o(x' = h) \quad (C.3)$$

Using Equation (A.22), $u_o(x' = h)$ is
Substituting Equation (C.4) into Equation (C.3) gives

\[
W_{2h} = \left( \frac{2h_a \sigma_a^2}{a, E_r} - \left[ \frac{a_c^2 a_o^2 E_o \sigma_a^2 - a_c^2 a_c a_o E_o E_r (\alpha_c - \alpha_r) \Delta T \sigma_a}{\xi a_o E_o} \right] \Delta y' \right) 
\]

\[
W_i, \text{ where } i = 1, 2, \text{ is given by}
\]

\[
W_h = 2a_o \sigma_a \Delta y' u_r \left( x' = \frac{h_i}{2} \right)
\]

The displacement, \( u_r(x' = h_i/2) \), can be obtained by re-solving Equation (A.18) using the boundary conditions: \( \sigma_r(x' = \pm h_i/2) = 0 \). Placing the solution of \( u_r(x' = h_i/2) \) into Equation (C.6) gives

\[
W_{h} = \left( \frac{h_i a_o \sigma_a^2}{a, E_r} - \left[ \frac{a_c^2 a_o^2 E_o \sigma_a^2 - a_c^2 a_c a_o E_o E_r (\alpha_c - \alpha_r) \Delta T \sigma_a}{\xi a_o E_o} \right] \right) \Delta y'
\]

where \( i = 1, 2 \).

The summation \( (W_{h_i} + W_{h_2}) \) using Equation (C.7) gives

\[
(W_{h_1} + W_{h_2}) = \left( \frac{a_o \sigma_a^2}{a, E_r} \left( h_1 + h_2 \right) - \left[ \frac{a_c^2 a_o^2 E_o \sigma_a^2 - a_c^2 a_c a_o E_o E_r (\alpha_c - \alpha_r) \Delta T \sigma_a}{\xi a_o E_o} \right] \right) \Delta y'
\]

Substituting \( h_1 + h_2 = 2h \) into Equation (C.8) gives
\[ \left( W_l + W_b \right) = \left( \begin{array}{c} 2h_{a_0}\frac{\sigma_a^2}{a_0E_r} - \left[ \frac{a_c^2a_c^2E_c\sigma_a^2}{\xi a_0a_0E_o} - \frac{a_c^2a_rE_cE_o(\alpha_c - \alpha_r)\Delta T\sigma_a}{\xi a_0E_oE_o} \right] \end{array} \right) \Delta y' \] (C.9)

Substituting Equations (C.5) and (C.9) into Equation (C.2) gives

\[ \Delta W = \left[ \begin{array}{c} \frac{a_c^2a_c^2E_c\sigma_a^2}{\xi a_0a_0E_o} - \frac{a_c^2a_rE_cE_o(\alpha_c - \alpha_r)\Delta T\sigma_a}{\xi a_0E_oE_o} \end{array} \right] \Delta y' \] (C.10)

The first term in Equation (C.10) can be reduced and re-arranged into a more convenient form. For the sake of efficiency, the first term will be represented through a temporary variable, \( \Gamma \).

\[ \Gamma = \left[ \begin{array}{c} a_c^2a_c^2E_c\sigma_a^2 \end{array} \right] \frac{\xi a_0a_0E_oE_o}{a_c^2a_rE_cE_o(\alpha_c - \alpha_r)\Delta T\sigma_a} \] (C.11)

In re-arranging Equation (C.11), the Rule of Mixtures given by Equations (A.14) and (A.15) are used at various stages.

\[ \Gamma = \frac{a_c^2E_c}{\xi a_0E_o} - \frac{a_c^2E_c}{\xi E_o}(\alpha_c - \alpha_r)\Delta T\sigma_a \] (C.12)

\[ \Gamma = \frac{a_c^2E_c}{\xi a_0E_o} \left( \frac{E_cE_o\sigma_a}{E_oa_r} - \frac{E_cE_o}{E_o}(\alpha_c - \alpha_r)\Delta T \right) \] (C.13)

\[ \Gamma = \frac{a_c^2a_oE_c}{\xi a_0E_r} \left( \frac{E_cE_o\sigma_a}{E_oa_r} - \frac{E_cE_o}{E_o}(\alpha_c - \alpha_r)\Delta T \right) \] (C.14)

Using the far-field stress given by Equation (A.16) and manipulating it with the use of Equations (A.14) and (A.15) gives

\[ \Gamma = \frac{a_c^2a_oE_c}{\xi a_0E_r} \sigma_a \sigma_{cw} \] (C.15)
where

\[ \sigma_{\infty} = \frac{E_c}{E_o} \sigma_a + E_c (\alpha_o - \alpha_c) \Delta T = \frac{E_c}{E_o} \sigma_a + \frac{E_c E_r a_r}{a_o E_o} (\alpha_r - \alpha_c) \Delta T \] (C.16)

Replacing \( \Gamma \), in Equation (C.15), into Equation (C.10), \( \Delta W \) is expressed as

\[ \Delta W = \left( \frac{a^2 a_o}{\xi a_o E_r} \sigma_{\infty} \sigma_o \left[ \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) - \tanh \left( \frac{2h_2}{a_c} \right) \right] \right) \Delta y' \] (C.17)

The change in external work employs an approximation to simplify the calculations. Details are given in Appendix B.

**C.2 \( \Delta U \)**

The change in internal energy for an incremental extension of a partial crack has two contributing components: the change in internal strain energy due to the normal stresses and the change in internal strain energy due to the shear stresses residing in the shear transfer region. The change in total internal energy is given by

\[ \Delta U = \Delta U_\sigma + \Delta U_q \] (C.18)

where \( \Delta U \) is the total change in internal strain energy due to the incremental extension of a partial crack, \( \Delta U_\sigma \) is the change in internal strain energy due to the normal stresses, and \( \Delta U_q \) is the change in internal strain energy due to the shear stresses located in the shear transfer region. Details of \( \Delta U_\sigma \) and \( \Delta U_q \) will be separated into two Sub-Sections.

**C.2.1 \( \Delta U_\sigma \)**

The change in internal strain energy due to the normal stresses, \( \Delta U_\sigma \), is given by

\[ \Delta U_\sigma = \left( U_\sigma \bigg|_{h_1} + U_\sigma \bigg|_{h_2} \right) - U_\sigma \bigg|_{2h} \] (C.19)

The internal strain energy from the normal stresses, \( U_\sigma \), is given by
The components of Equation (C.19) can be expressed in terms of Equation (C.20). \( U_{\sigma |_{2h} } \), the internal strain energy of the volume of the partial crack due to normal stresses before extending, can be expressed as

\[
U_{\sigma |_{2h} } = 2 \left( \frac{1}{2} a_i \Delta y' \int_0^{h} \frac{\sigma_i^2}{E} \, dx' + \frac{1}{2} a_c \Delta y' \int_0^{h} \frac{\sigma_c^2}{E_c} \, dx' \right)
\]  

(C.21)

The volume in Equation (C.21) is integrated from the location of the partial crack extension to neighbouring cracks in the \( x' \)-direction, over the length of the incremental extension, \( \Delta y' \), in the \( y' \)-direction, and over the combined thicknesses of the cracking ply group and the rest of the laminate in the \( z \)-direction. Due to symmetry, Equation (C.21) can be integrated from 0 to \( h \) and the integrals can be multiplied by 2. This gives the same results as integrating from \(-h\) to \( h \). The result of the integrations in Equation (C.21) is

\[
U_{\sigma |_{2h} } = \left\{ \frac{\sigma_a^2 a_i^2 h}{a_i E_i} + \frac{a_c a_a E_i}{a_c E_c} \sigma_{cm}^2 \left[ h - \frac{a_c}{8} \tanh \left( \frac{2h \xi}{a_c} \right) + \frac{a_c \sinh \left( \frac{4h \xi}{a_c} \right) + h}{8} \right] \right\} \Delta y'
\]

(C.22)

\( U_{\sigma |_{h} } \) where \( i = 1, 2 \), the internal strain energy of the volumes of the partial crack due to normal stresses after an incremental extension, can be expressed as

\[
U_{\sigma |_{h} } = 2 \left( \frac{1}{2} a_i \Delta y' \int_0^{h/2} \frac{\sigma_i^2}{E} \, dx' + \frac{1}{2} a_c \Delta y' \int_0^{h/2} \frac{\sigma_c^2}{E_c} \, dx' \right)
\]

(C.23)

The same symmetry considerations applied to Equation (C.21) can be applied to Equation (C.23). The normal stresses, \( \sigma_i \) and \( \sigma_c \), used in Equation (C.23) are obtained by solving Equation (A.18) using the boundary conditions \( \sigma_c(x' = \pm h/2) = 0 \). The result of the integrations in Equation (C.23) is
\[
U_{\sigma} \mid_{h_{i}} = \left\{ \begin{array}{l}
\frac{\sigma_{a}^{2} a_{o}^{2}}{2 a, E, r} - h_{i} + \frac{a_{c} a_{o} E_{o}}{a_{r}, E, E_{c}} \left[ \frac{h_{i}}{2} - \frac{1}{2} \tanh \left( \frac{\xi h_{i}}{a_{c}} \right) + \frac{a_{c}}{8 \xi} \sinh \left( \frac{2 \xi h_{i}}{a_{c}} \right) + \frac{h_{i}}{4} \frac{1}{\cosh^{2} \left( \frac{\xi h_{i}}{a_{c}} \right)} \right] \sigma_{c_{o}}^{2}
\end{array} \right.
\]
\[\Delta y' \ (C.24)\]

where \( i = 1, 2. \)

The summation \( U_{\sigma} \mid_{h_{i}} + U_{\sigma} \mid_{h_{j}} \) using Equation (C.24) gives

\[
\left( U_{\sigma} \mid_{h_{i}} + U_{\sigma} \mid_{h_{j}} \right) = \left\{ \begin{array}{l}
\frac{\sigma_{a}^{2} a_{o}^{2}}{2 a, E, r} (h_{1} + h_{2}) + \frac{a_{c} a_{o} E_{o}}{a_{r}, E, E_{c}} \frac{h_{1} + h_{2}}{2} - \frac{a_{c}}{2} \tanh \left( \frac{\xi h_{1}}{a_{c}} \right) + \tanh \left( \frac{\xi h_{2}}{a_{c}} \right) + \frac{a_{c}}{8 \xi} \sinh \left( \frac{2 \xi h_{1}}{a_{c}} \right) + \frac{h_{1}}{4} \frac{1}{\cosh^{2} \left( \frac{\xi h_{1}}{a_{c}} \right)} + \frac{a_{c}}{8 \xi} \sinh \left( \frac{2 \xi h_{2}}{a_{c}} \right) + \frac{h_{2}}{4} \frac{1}{\cosh^{2} \left( \frac{\xi h_{2}}{a_{c}} \right)} \sigma_{c_{o}}^{2}
\end{array} \right.
\[\Delta y' \ (C.25)\]

Substituting Equations (C.22) and (C.25) into Equation (C.19) gives
\[ \Delta U_\sigma = \left[ \begin{array}{c} - \frac{a_c}{\xi} \left( \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) \right) + \\ \frac{a_c}{8\xi} \sinh \left( \frac{2\xi h_1}{a_c} \right) + \frac{h_1}{4} + \frac{a_c}{8\xi} \sinh \left( \frac{2\xi h_2}{a_c} \right) + \frac{h_2}{4} \\ \cosh^2 \left( \frac{\xi h_1}{a_c} \right) - \cosh^2 \left( \frac{\xi h_2}{a_c} \right) + \\ \frac{a_c}{\xi} \tanh \left( \frac{2h_2}{a_c} \right) - \frac{a_c}{8\xi} \sinh \left( \frac{4h_2}{a_c} \right) + \frac{h}{2} + \frac{2a_c}{8\xi} \cosh^2 \left( \frac{2h_2}{a_c} \right) + \\ \frac{a_c}{2\xi} \tanh \left( \frac{2h_2}{a_c} \right) \end{array} \right] \Delta y' + \left[ \begin{array}{c} \frac{a_c}{2\xi} \left( \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) \right) \end{array} \right] \]  \\
(C.26)

Equation (C.26) can be simplified greatly by using the trigonometric function \( \sinh(2x) = 2\sinh(x)\cosh(x) \). Applying this relationship reduces Equation (C.26) to

\[ \Delta U_\sigma = \left[ \begin{array}{c} \frac{3}{4} \tanh \left( \frac{2h_2}{a_c} \right) - \frac{3}{4} \tanh \left( \frac{\xi h_1}{a_c} \right) + \\ \frac{a_c^2}{\xi a_c E_o} \left( \frac{\xi h_1}{a_c} \right) + \frac{h_1}{4a_c} \sech^2 \left( \frac{\xi h_1}{a_c} \right) + \\ \frac{h_2}{4a_c} \sech^2 \left( \frac{2h_2}{a_c} \right) \end{array} \right] \Delta y' + \left[ \begin{array}{c} \frac{a_c^2}{\xi a_c E_o} \left( \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) - \tanh \left( \frac{2h_2}{a_c} \right) \right) \sigma_\infty \end{array} \right] \]

\[ (C.27) \]

C.2.2 \( \Delta U_q \)

The change in internal strain energy due to the shear stresses residing in the shear transfer region, \( \Delta U_q \), is given by
\[ \Delta U_q = \left( U_q \bigg|_{h_1} + U_q \bigg|_{h_2} \right) - U_q \bigg|_{2h} \] (C.28)

The strain energy from the shear stresses, \( U_q \), are given by

\[ U_q = \frac{1}{2} \int \gamma^2 \frac{q}{G_{eff}} \, dV \] (C.29)

Equation (C.29) is integrated over the shear transfer regions between the cracking ply group and the rest of the laminate. The thickness of the shear transfer regions sum to \( 2a_q \) since shear stresses are transferred at the top and bottom of the cracking ply group.

The components of Equation (C.28) can be expressed in terms of Equation (C.29). \( U_q \bigg|_{2h} \), the internal strain energy of the volume of the partial crack due to shear stresses before extending, can be expressed as

\[ U_q \bigg|_{2h} = \frac{2}{K} \Delta y' \int_0^h q^2 \, dq' \] (C.30)

Equation (C.30) is integrated from the location of the partial crack extension to neighbouring cracks in the \( x' \)-direction, over the length of the incremental extension, \( \Delta y' \), in the \( y' \)-direction, and over the shear transfer regions between the cracking ply group and the rest of the laminate in the \( z \)-direction. Due to symmetry, Equation (C.30) can be integrated from 0 to \( h \) and the integrals can be multiplied by 2. This gives the same results as integrating from \(-h\) to \( h\). An additional multiplicative factor of 2 is included in Equation (C.30) as well as Equation (C.31) because, as noted previously, the integrated volumes include both the top and the bottom of the cracking ply group which have a combined thickness of \( 2a_q \). The result of the integration in Equation (C.30) is

\[ U_q \bigg|_{2h} = \left( \frac{K a_c a_o^2 E_o^2}{16 a_c^2 E_r E_c^2 \xi^3} \left[ 2 \tanh \left( \frac{2h \xi}{a_c} \right) - \frac{4h \xi}{a_c} \right] \right) \frac{\sigma_{c0}^2}{2} \Delta y' \] (C.31)

Substituting for \( K \) in the above Equation with

\[ K = \frac{2a_c E_r E_c \xi^2}{a_o a_c E_o} \] (C.32)
results in

\[ U_q \bigg|_{2h} = \left( \frac{a_c^2 a_o E_o}{4a_c E_c E_c} \right) \left[ \tanh \left( \frac{2h_2}{a_c} \right) - \tanh \left( \frac{2h_2}{a_c} \right) \right] \Delta y' \quad (C.33) \]

The internal strain energy of the volumes of the partial crack due to shear stresses after an incremental extension, \( U_q \big|_{h_i} \) where \( i = 1, 2 \), can be expressed as

\[ U_q \big|_{h_i} = \frac{2}{K} \Delta y' \int_0^{h_i/2} q^2 dx' \quad (C.34) \]

The same symmetry considerations applied to Equation (C.30) can be applied to Equation (C.34). The shear stress, \( q \), used in Equation (C.34) is obtained by solving Equation (A.18) using the boundary conditions \( \sigma_c (x' = \pm h/2) = 0 \). The result of the integrations in Equation (C.34) is

\[ U_q \bigg|_{h_i} = \left( \frac{Ka_c^3 a_o^2 E_o^2}{8a_c E_c E_c^2 E_c^2} \right) \left[ \tanh \left( \frac{\xi h_i}{a_c} \right) - \tanh \left( \frac{\xi h_i}{a_c} \right) \right] \Delta y' \quad (C.35) \]

Substituting for \( K \) in Equation (C.35) using Equation (C.32) gives

\[ U_q \big|_{h_i} = \left( \frac{a_c^2 a_o E_o}{4a_c E_c^2 E_c^2} \right) \left[ \tanh \left( \frac{\xi h_i}{a_c} \right) - \tanh \left( \frac{\xi h_i}{a_c} \right) \right] \Delta y' \quad (C.36) \]

where \( i = 1, 2 \).

Placing Equations (C.33) and (C.36) into Equation (C.28) gives

\[ \Delta U_q = \left( \frac{a_c^2 a_o E_o}{4a_c E_c E_c} \right) \begin{bmatrix} \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) - \\ \tanh \left( \frac{2h_2}{a_c} \right) - \tanh \left( \frac{2h_2}{a_c} \right) \end{bmatrix} \Delta y' \quad (C.37) \]
C.2.3 $\Delta U$

Using the results from Sub-Sections C.2.1 and C.2.2, Equations (C.27) and (C.37), respectively, can be substituted into Equation (C.18) which gives

$$\Delta U = \left[ \frac{a_c^2 E_o}{\xi a_c E_r} \sigma_{cm} - \frac{a_c^2 a_o E_o}{2 \xi a_c E_c E_r} \sigma_{cm} \right] \left[ \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) - \tanh \left( \frac{2h \xi}{a_c} \right) \right] \Delta y'$$ (C.38)

after some manipulation and rearrangement.

C.3 CHANGE IN STRAIN ENERGY RELEASE RATE IN THE $y'$-DIRECTION

Using the results of Sections C.2 and C.3 by substituting Equations (C.17) and (C.38) into Equation (C.1) gives

$$\Delta G_1 = \frac{a_c a_o E_o}{2 \xi a_c E_c} \sigma_{cm}^2 \left[ \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) - \tanh \left( \frac{2h \xi}{a_c} \right) \right]$$ (C.39)

Placing Equation (C.16) into Equation (C.39) gives

$$\Delta G_1 = \frac{a_c E_c}{2 \xi a_c E_r a_o E_o} \left[ a_o \sigma_a - a_c E_r (\alpha_c - \alpha_r) \Delta T \right]^2 \left[ \tanh \left( \frac{\xi h_1}{a_c} \right) + \tanh \left( \frac{\xi h_2}{a_c} \right) - \tanh \left( \frac{2h \xi}{a_c} \right) \right]$$ (C.40)
APPENDIX D

COMPARISON OF PREDICTIONS BETWEEN THE STOCHASTIC-ANALYTICAL METHOD AND DETERMINISTIC METHOD
Crack density predictions that depend only on geometry with uniform material for the $45_2$ ply group. Flaws are assumed to be seeded along the edges.

Figure D.1
Crack density predictions that depend only on geometry with uniform material for the 90\textsubscript{2} ply group. Flaws are assumed to be seeded along the edges.
Crack density predictions that depend only on geometry with uniform material for the -45_4 ply group. Flaws are assumed to be seeded along the edges.

Figure D.3
Crack density predictions that depend only on geometry with uniform material for the 45_2 ply group. Flaws are assumed to be seeded within the volume.

Figure D.4
Crack density predictions that depend only on geometry with uniform material for the 90\textsubscript{2} ply group. Flaws are assumed to be seeded within the volume.

Figure D.5
Crack density predictions that depend only on geometry with uniform material for the -45_{4} ply group. Flaws are assumed to be seeded within the volume.

Figure D.6
Crack distribution history for the $[0_{2}/45_{2}/90_{2}/-45_{2}]_{s}$ at select applied loads. Predictions are assumed to depend only on geometry. Uniform material assumed. Flaws seeded along the edges of the ply groups.
Load = 17.7 kN

Load = 21.8 kN

Load = 25.2 kN

Load = 29.3 kN

Figure D.8 Crack distribution history for the $[0_2/45_2/90_2/-45_2]_s$ at select applied loads. Predictions are assumed to depend only on geometry. Uniform material assumed. Flaws seeded within the volume of the ply groups.
Crack density predictions that random crack locations and material variations for the $45_2$ ply group. Flaws are assumed to be seeded along the edges.
Figure D.10  Crack density predictions that random crack locations and material variations for the 90₂ ply group. Flaws are assumed to be seeded along the edges.
Crack density predictions that random crack locations and material variations for the $-45_4$ ply group. Flaws are assumed to be seeded along the edges.
Figure D.12 Crack density predictions that random crack locations and material variations for the 45_2 ply group. Flaws are assumed to be seeded within the volume.
Figure D.13 Crack density predictions that random crack locations and material variations for the 90₂ ply group. Flaws are assumed to be seeded within the volume.
Crack density predictions that random crack locations and material variations for the -45<sub>4</sub> ply group. Flaws are assumed to be seeded along the volume.

Figure D.14
Figure D.15 Crack distribution history for the [0°/45°/90°/-45°]s at select applied loads. Predictions are assumed to depend only on geometry. Material variations assumed. Flaws seeded along the edges of the ply groups.
Figure D.16 Crack distribution history for the $[0_{2}/45_{2}/90_{2}/-45_{2}]_{s}$ at select applied loads. Predictions are assumed to depend only on geometry. Material variations assumed. Flaws seeded within the volume of the ply groups.
Figure D.17  Crack density predictions using a Weibull distribution of effective flaws and assuming uniform material for the $45_2$ ply group. Flaws are assumed to be seeded along the edges.
Figure D.18 Crack density predictions using a Weibull distribution of effective flaws and assuming uniform material for the 90_2 ply group. Flaws are assumed to be seeded along the edges.
Figure D.19 Crack density predictions using a Weibull distribution of effective flaws and assuming uniform material for the -45\textsubscript{4} ply group. Flaws are assumed to be seeded along the edges.
Figure D.20 Crack density predictions using a Weibull distribution of effective flaws and assuming uniform material for the 452 ply group. Flaws are assumed to be seeded within the volume.
[0₂/₄₅₂/₉₀₂/-₄₅₂]ₙ

90₂ Ply Group

Crack Density (1/cm)

Applied Load (kN)

Maddocks

Sim. Runs

Figure D.21 Crack density predictions using a Weibull distribution of effective flaws and assuming uniform material for the 90₂ ply group. Flaws are assumed to be seeded within the volume.
Figure D.22 Crack density predictions using a Weibull distribution of effective flaws and assuming uniform material for the -45\(_4\) ply group. Flaws are assumed to be seeded within the volume.
Figure D.23 Crack distribution history for the $[0_2/45_2/90_2/-45_2]_s$ at select applied loads. A Weibull distribution of effective flaw are used. Uniform material assumed. Flaws seeded along the edges of the ply groups.
Figure D.24 Crack distribution history for the $\{0/45_2/90_2/-45_2\}_s$ at select applied loads. A Weibull distribution of effective flaw is used. Uniform material assumed. Flaws seeded within the volume of the ply groups.
Figure D.25 Crack density predictions using a Weibull distribution of effective flaws and assuming material variations for the $45_2$ ply group. Flaws are assumed to be seeded along the edges.
Figure D.26 Crack density predictions using a Weibull distribution of effective flaws and assuming material variations for the 90₂ ply group. Flaws are assumed to be seeded along the edges.
Crack density predictions using a Weibull distribution of effective flaws and assuming material variations for the -45₂ ply group. Flaws are assumed to be seeded along the edges.

Figure D.27

Crack Density (1/cm) vs. Applied Load (kN)

[0₂/45₂/90₂/-45₂]ₚ
-45₄ Ply Group

- Maddocks
× Sim. Runs
Crack density predictions using a Weibull distribution of effective flaws and assuming material variations for the 45₂ ply group. Flaws are assumed to be seeded within the volume.
Figure D.29 Crack density predictions using a Weibull distribution of effective flaws and assuming material variations for the 902 ply group. Flaws are assumed to be seeded within the volume.
Figure D.30  Crack density predictions using a Weibull distribution of effective flaws and assuming material variations for the -45\textsubscript{4} ply group. Flaws are assumed to be seeded within the volume.
Figure D.31 Crack distribution history for the $[0_2/45_2/90_2/-45_2]_s$ at select applied loads. Flaws are assumed to be seeded along the edges of the ply groups.
Figure D.32 Crack distribution history for the $[0_2/45_2/90_2/-45_2]_s$ at select applied loads. Flaws are assumed to be seeded within the volume of the ply groups.
APPENDIX E

PREDICTIONS FROM THE PARAMETRIC STUDIES
Figure E.1  Material variations model amplitude parametric study. Predictions of crack density vs. applied load for the $45_2$ ply group. Flaws are assumed to be seeded along the edges.
[0₂/45₂/90₂/-45₂]ₜ₄

45₂ Ply Group

Maddocks

- A = 0%
- A = 10%
- A = 20%

Figure E.2  Material variations model amplitude parametric study. Predictions of crack density vs. applied load for the 45₂ ply group. Flaws are assumed to be seeded within the volume.
Crack distribution history for the 45\_2 ply group at select applied loads for the material variations amplitude parametric study. Flaws seeded along the edges of the ply groups. Case 1 - 0\%. Case 2 - 10\%. Case 3 - 20\%. 

Figure E.3
Figure E.4 Crack distribution history for the \([0_2/45_2/90_2/-45_2]_s\) at select applied loads. Flaws seeded along the edges of the ply groups. Uniform material assumed.
Crack distribution history for the -45\textsubscript{4} ply group at select applied loads for the material variations amplitude parametric study. Flaws seeded along the edges of the ply groups. Case 1 - 0\%. Case 2 - 10\%. Case 3 - 20\%.
Crack distribution history for the $45_2$ at select applied loads for the material variations amplitude parametric study. Flaws seeded within the volume of the ply groups. Case 1 - 0%. Case 2 - 10%. Case 3 - 20%.
Crack distribution history for the 902 at select applied loads for the material variations amplitude parametric study. Flaws seeded within the volume of the ply groups. Case 1 - 0%. Case 2 - 10%. Case 3 - 20%.

Figure E.7
Crack distribution history for the 90\textsubscript{2} at select applied loads for the material variations amplitude parametric study. Flaws seeded within the volume of the ply groups. Case 1 - 0\%. Case 2 - 10\%. Case 3 - 20\%.
Figure E.9 Material variations model length parameter study. Predictions of crack density vs. applied load for the 45₂ ply group. Flaws are assumed to be seeded along the edges.
Figure E.10  Case 1 ($\lambda_x = \lambda_y = 0.4$ cm.) predictions of crack distribution for the $[0_2/45_2/90_2/-45_2]_6$ at select applied loads for the material variations length parameter study. Edge flaw seeding assumed.
Figure E.11 Case 2 ($\lambda_v = \lambda_c = 0.667$ cm.) predictions of crack distribution for the $[0_2/45_2/90_2/-45_2]_3$ at select applied loads for the material variations length parameter study. Edge flaw seeding assumed.
Figure E.12   Case 3 ($\lambda_\alpha = \lambda_\gamma = 1$ cm.) predictions of crack distribution for the [0$_2$/45$_2$/90$_2$/-45$_2$]$_s$ at select applied loads for the material variations length parameter study. Edge flaw seeding assumed.
Figure E.13 Case 4 ($\lambda_x = \lambda_y = 2$ cm.) predictions of crack distribution for the $[0_{2}/45_{2}/90_{2}/-45_{2}]_s$ at select applied loads for the material variations length parameter study. Edge flaw seeding assumed.
Figure E.14 Length parameter study crack distribution history for the 45_2 ply group of the [0_2/45_2/90_2/-45_2]_s laminate at select applied loads. Flaws seeded along the edges of the ply groups.
Figure E.15  Experimental data and predictions from the multi-dimensional method of crack density vs. applied load for the 452 ply group. Flaws are assumed to be seeded within the volume of the ply group.
Figure E.16 Crack distribution history for the $[0_2/45_2/90_2/-45_2]_s$ at select applied loads. Edge flaw seeding. Material variations model depends only on the $x'$-direction length parameter.
Figure E.17 Crack distribution history for the \([0_2/45_2/90_2/-45_2]_s\) at select applied loads. Volume flaw seeding. Material variations model depends only on the \(x'\)-direction length parameter.
Figure E.18 Crack distribution history for the $[0_2/45_2/90_2/-45_2]_s$ at select applied loads. Edge flaw seeding. Material variations model depends only on the $y'$-direction length parameter.
Figure E.19  Crack distribution history for the [0/45₂/90₂/-45₂]ₙ at select applied loads. Volume flaw seeding. Material variations model depends only on the y'-direction length parameter.
Figure E.20  Crack distribution history for the 45\textsubscript{2} ply group of the [0\textsubscript{2}/45\textsubscript{2}/90\textsubscript{2}/-45\textsubscript{2}]\textsubscript{s} laminate at 29.3 kN assuming the material variations model is a function of only one direction. (a) Edge seeding, $x'$-dependent. (b) Volume seeding, $x'$-dependent. (c) Edge seeding, $y'$-dependent. (d) Volume seeding, $y'$-dependent.
Figure E.21 Predictions of crack density vs. applied load for the 45_2 ply group for the Weibull parameter study. Flaws are assumed to be seeded along the edges. Case 1: \( d = 0.00225 \). Case 2: \( d = 0.00425 \).
Figure E.22 Predictions of crack density vs. applied load for the 45_2 ply group for the Weibull parameter study. Flaws are assumed to be seeded along the edges. Case 3: m = 5. Case 4: m = 15.
Predictions of crack density vs. applied load for the 45₂ ply group for the effective flaw density parameter study. Flaws are assumed to be seeded along the edges. Case 1: \( EFD = 20 \text{ cm.} \). Case 2: \( EFD = 1000 \text{ cm.} \).
APPENDIX F

COMPARISON OF PREDICTIONS TO EXPERIMENTAL DATA
Experimental data and predictions of crack density vs. applied load for the 45_2 ply group. Flaws are assumed to be seeded along the edges.
Experimental data and predictions of crack density vs. applied load for the 90_2 ply group. Flaws are assumed to be seeded along the edges.
Experimental data and predictions of crack density vs. applied load for the -45₄ ply group. Flaws are assumed to be seeded along the edges.

Figure F.3
Experimental data and predictions of crack density vs. applied load for the 452 ply group. Flaws are assumed to be seeded within the volume.
Experimental data and predictions of crack density vs. applied load for the 90\textsubscript{2} ply group. Flaws are assumed to be seeded within the volume.
Experimental data and predictions of crack density vs. applied load for the -45₄ ply group. Flaws are assumed to be seeded within the volume.
Figure F.7  Experimental data and predictions of crack density vs. applied load for the -45\textdegree ply group. Flaws are assumed to be seeded along the edges.
Experimental data and predictions of crack density vs. applied load for the $-45_8$ ply group. Flaws are assumed to be seeded along the edges.
Experimental data and predictions of crack density vs. applied load for the -458 ply group. Flaws are assumed to be seeded along the edges.
Figure F.10 Experimental data and predictions of crack density vs. applied load for the $45_4$ ply group. Flaws are assumed to be seeded within the volume.
Figure F.11 Experimental data and predictions of crack density vs. applied load for the 90\(_4\) ply group. Flaws are assumed to be seeded within the volume.
Figure F.12  Experimental data and predictions of crack density vs. applied load for the -45\textsubscript{8} ply group. Flaws are assumed to be seeded within the volume.
Figure F.13 Experimental data and predictions of crack density vs. applied load for the 60₂ ply group. Flaws are assumed to be seeded along the edges.
Figure F.14  Experimental data and predictions of crack density vs. applied load for the -604 ply group. Flaws are assumed to be seeded along the edges.
Figure F.15 Experimental data and predictions of crack density vs. applied load for the $60_2$ ply group. Flaws are assumed to be seeded within the volume.
Figure F.16  Experimental data and predictions of crack density vs. applied load for the -60₄ ply group. Flaws are assumed to be seeded within the volume.
Figure F.17 Crack distribution history for the $[0_{2}/45_{2}/90_{2}/-45_{2}]_s$ at select applied loads. Flaws are assumed to be seeded along the edges of the ply groups.
Crack distribution history for the $[0_2/45_2/90_2/-45_2]$, at select applied loads. Flaws are assumed to be seeded within the volume of the ply groups.
Figure F.19 Crack distribution history for the $[0_4/45_4/90_4/-45_4]_s$ at select applied loads. Flaws are assumed to be seeded along the edges of the ply groups.
Load = 16.8 kN

Load = 23.6 kN

Load = 30.3 kN

Load = 37.0 kN

Figure F.20 Crack distribution history for the [0/45/90/45]s at select applied loads. Flaws are assumed to be seeded within the volume of the ply groups.
Figure F.21 Crack distribution history for the $[0_2/\pm 60_2]_s$ at select applied loads. Flaws are assumed to be seeded along the edges of the ply groups.
Figure F.22  Crack distribution history for the \([0_2/\pm 60_2]_s\) at select applied loads. Flaws are assumed to be seeded within the volume of the ply groups.
Figure F.23 Experimental data and predictions of crack density vs. temperature for the 45\(_2\) ply group. Flaws are assumed to be seeded along the edges.
Figure F.24  Experimental data and predictions of crack density vs. temperature for the 902 ply group. Flaws are assumed to be seeded along the edges.
Figure F.25 Experimental data and predictions of crack density vs. temperature for the -45\(_4\) ply group. Flaws are assumed to be seeded along the edges.
Experimental data and predictions of crack density vs. temperature for the 45₂ ply group. Flaws are assumed to be seeded within the volume.
Figure F.27  Experimental data and predictions of crack density vs. temperature for the 902 ply group. Flaws are assumed to be seeded within the volume.
Figure F.28 Experimental data and predictions of crack density vs. temperature for the -45₄ ply group. Flaws are assumed to be seeded within the volume.
Figure F.29  Experimental data and predictions of crack density vs. temperature for the $45_4$ ply group. Flaws are assumed to be seeded along the edges.
Figure F.30  Experimental data and predictions of crack density vs. temperature for the 90₄ ply group. Flaws are assumed to be seeded along the edges.
Figure F.31  Experimental data and predictions of crack density vs. temperature for the -45_8 ply group. Flaws are assumed to be seeded along the edges.
Figure F.32  Experimental data and predictions of crack density vs. temperature for the $45_4$ ply group. Flaws are assumed to be seeded within the volume.
Figure F.33 Experimental data and predictions of crack density vs. temperature for the 90₄ ply group. Flaws are assumed to be seeded within the volume.
Figure F.34 Experimental data and predictions of crack density vs. temperature for the -45\textsubscript{8} ply group. Flaws are assumed to be seeded within the volume.
Figure F.35 Crack distribution history for the \([0/45_2/90_2/-45_2]_s\) at select temperatures. Flaws are assumed to be seeded along the edges of the ply groups.
Figure F.36 Crack distribution history for the [0₂/45₂/90₂/-45₂]ₙ at select temperatures. Flaws are assumed to be seeded within the volume of the ply groups.
Figure F.37 Crack distribution history for the [0/45/90/-45]s at select temperatures. Flaws are assumed to be seeded along the edges of the ply groups.
Figure F.38  Crack distribution history for the $[0_{d}/45_{d}/90_{d}/-45_{d}]_{s}$ at select temperatures. Flaws are assumed to be seeded within the volume of the ply groups.
APPENDIX G

COMPARISON OF PREDICTIONS TO P75/934 EXPERIMENTAL DATA AND QUALITATIVE COMPARISON OF PREDICTIONS TO FATIGUE LOAD RESULTS
Experimental data and predictions of crack density vs. temperature for the 45 ply group. Flaws are assumed to be seeded within the volume.
Experimental data and predictions of crack density vs. temperature for the 90 ply group. Flaws are assumed to be seeded within the volume.
Figure G.3  Experimental data and predictions of crack density vs. temperature for the -45₂ ply group. Flaws are assumed to be seeded within the volume.
Figure G.4  Predictions of crack density vs. depth in the width direction at -200 °F.
Figure G.5  Experimental crack density data vs. depth in the width direction at -200 °F.
Temp = -40 °F

Temp = -110 °F

Temp = -180 °F

Temp = -250 °F

Figure G.6 Crack distribution history for the [0/45/90/-45]_s composed of P75/934 at select temperatures. Volume flaw seeding is assumed.
Experimental crack densities for different crack types for the fatigue specimens in the study by LaFarié-Frenot and Henaff-Gardin [45,46].
Different Crack Types

Predictions of crack density for different crack types in the 90\textsubscript{2} ply group of the [0\textsubscript{2}/45\textsubscript{2}/90\textsubscript{2}/-45\textsubscript{2}]\textsubscript{s} laminate vs. applied load. Flaws are assumed to be seeded along the edges.
Experimental crack density as a function of depth into the width from the fatigue specimen in the study by LaFrie-Frenot and Henaff-Gardin [45,46].
Crack Density vs. Depth

Figure G.10  Predictions of crack density as a function of depth into the width at specific loads for the 902 ply group.