Anisotropic Wave Guides - Propagation, Focusing and Dispersive Phenomena with Applications for Non-Destructive Testing

by

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Abstract

Acoustic Non-Destructive Testing (NDT) has a long history of applications in fatigue monitoring, fault testing, and more recently production control. A very large family of manufactured and raw materials consist of thin layers. Some examples include rolled aluminum, window glass, plywood, automobile bodies, plane wings, silicon wafers, bridge support beams, and paper. These layers can be viewed and modeled as acoustic waveguides.

This thesis will present the framework in which to analyze such layers. To this end, analytic solutions to the plane wave displacement and stress fields in a single layer monoclinic material will be presented.

The propagation, frequency, and dispersive characteristics of transmitted signals can be analyzed to determine various elastic properties of the layer or to identify faults. Wavelet (time-frequency), Fourier (frequency), and signal matching (time) techniques will be developed to analyze and extract features and properties of signals.

Several experimental examples will be presented.
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Chapter 1

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It has been a journey. Four topics later...
Chapter 2

Introduction

2.1 Introduction

Non-destructive testing and evaluation is the large body of work associated with gathering information from the signals of an interrogated system. The information that is to be determined from the signals gathered may be as simple as detecting a 'difference' between a known acceptable signal and a suspected 'bad' signal, i.e. flaw detection. The desired information may be something as complex as the elastic constants of an unknown crystalline material.

2.2 Background

There has been increased interest in the area of non-destructive testing on thin layers (Costley, Berthelot & Jacobs 1994), (Jansen & Hutchins 1992), (Rogers 1995), (Hutchins, Kundgren & Palmer 1989), (Bresse, Hutchins & Lundgren 1988), (Ditri, Pilarski, Pavlakovic & Rose n.d.). The increase in work in this area is due in part to availability of fast, cheap, and powerful computers. The physics of thin layers, has been addressed by many individuals, though complex it is becoming well understood (Li & Thompson 1990) (Nayfeh & Chimenti 1989) (Solie & Auld 1973) (Kausel 1986) (Bratton, Datta & Shah 1989) (Markus, Kaplan & Veremeenko 1985). The complexity of the physics of propagation (i.e. many dispersive modes of propagation) in thin layers, leads to very complex signals from which to determine anything useful.

Techniques that are applied to diagnostics of non-dispersive medium or of single-mode dispersive mediums are not directly applicable to the complex multi-mode dispersive layers. Generally to simplify this general complexity, relative phase measurements, or wave counting is used to slowly map out the modes of propagations. The modes curves are then inverted to determine the elastic constants. This can be performed in one of two fashions. The first method is a temporal frequency sweep where a transmitter and receiver are held at a fixed distance. The transmitted signal frequency is swept and the receiver records the temporal crests and troughs. The second method is a spatial frequency sweep where the transmitter emits a constant frequency signal, and the detector is slowly moved along the direction of propagation, mapping out the spatial crests and troughs.

Wave counting techniques are typically done in a region where the modes of propagation are known to be sensitive to the changes in the parameters that are of interest (Pilarski & Rose 1992). The exercise of mapping out the modes of propagation (dispersion relationship) and then inverting the modes to determine elastic constants (Rogers 1995) or quality of bonds (Lowe & Cawley 1994)) has solidly shown that this is a feasible method of extracting information from a medium under test. It can also be quite tedious, requiring accurate stepping devices and a very good experimental setup.

2.2.1 Time of Flight

Non-dispersive propagation occurs in medium governed by the wave equation. The velocity of a harmonic wave (tone) in a linear isotropic medium is the velocity that an observer must travel to follow the wave and always see a constant phase. For a linear non-dispersive medium this phase velocity, $C_0$, is constant for all frequencies.
This intuitive description arises mathematically from the (harmonic) solution of the wave equation, equation 2.1.

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_s^2} \frac{\partial^2 u}{\partial t^2}
\]  

(2.1)

Consequently, a propagating wave can be represented \( U(x, t) = U_0 \cos(kx - \omega t) \) or in harmonic exponentials as

\[
U(x, t) = X(x)e^{i\omega t}
\]

(2.2)

or in harmonic exponentials as

\[
U(x, t) = X(x)e^{ikx} + e^{-ikx}e^{i\omega t}.
\]

(2.3)

Another approach when solving the wave equation is to examine an initial value problem. Imagine that a medium is given some instantaneous localized disturbance. We want to know how that disturbance propagates. We want to detect that pulse at some time and distance removed from the source. The optimal detection correlator for the signal that we are trying to detect is the signal to be detected. So what is it?

Consider the wave equation, equation 2.1 and introduce the change of variables:

\[
\zeta = x - c_o t
\]

(2.4)

\[
\eta = x + c_o t
\]

(2.5)

By chain-rule differentiation:

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x}
\]

\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t}
\]

(2.6)

The second derivatives give:

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \zeta^2} + 2 \frac{\partial^2 u}{\partial \zeta \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}
\]

(2.7)

Substituting into the wave equation give:

\[
\frac{\partial^2 u(\zeta, \eta)}{\partial \zeta \partial \eta} = 0
\]

(2.10)

This may be integrated directly to give:

\[
\frac{\partial u}{\partial \zeta} = F(\zeta)
\]

(2.11)

\[
u(\zeta, \eta) = f(\zeta) + g(\eta)
\]

(2.12)

Finally, changing back to \( x, t \) variables produces the classical D’Alembert solution to the wave equation,

\[
u(x, t) = f(x - c_s t) + g(x + c_s t)
\]

(2.13)

Note that \( f \) and \( g \) are arbitrary functions of integration that are determined from initial conditions. These function represent right and left propagating disturbances. Increasing time requires
increasing $x$ to maintain the arguments of these functions. The shape of $f$ and $g$ as determined from initial conditions is maintained during propagation. They propagate without distortion.

D'Alembert's solution shows that given an initial disturbance in a linear medium (with no initial velocity), the disturbance will propagate undistorted with velocity $c_o$ for all time. So the optimal detector is a matched correlator that has the same shape as the initial disturbance.

**Problems with Time of Flight**

Time of flight techniques are much more difficult to use on dispersive medium. Any additions to the wave equation will result in solutions where the phase velocity is a non-linear function of frequency or wavelength. Consequently, different Fourier wave components composing an initial disturbance will propagate at different velocities. The name dispersion arises from the fact that a pulse will not maintain its shape as it propagates, it disperses. We can no longer shape the initial disturbance and observe the same shape when we detect it some distance and time later.

Dispersion can be caused by:

- Geometry, or the presence of boundaries
- Frequency dependence of material properties, density, elastic constants
- Scattering of waves by inhomogeneities or wavelengths on the order of the crystal/grain
- Absorption / dissipation

Time of flight techniques are rarely used when testing thin multi-layered targets. “The theory of ultrasound propagation in solids is more difficult under transient condition (i.e. short pulses), and when the waves undergo significant amounts of reflection, refraction, and diffraction” ¹. An initial pulse of energy will spread into several modes and disperse significantly as it propagates. Looking at this signal and extracting information from it can be very difficult due to the complexity of the physics of propagation. It is even more complex when the information that you are trying to extract is an absolute measurement as opposed to a relative measurement. A relative measurement would be looking at the change in a signal from a known reference or looking at the change in signal between two receivers. An absolute measurement consists of extracting an absolute value for elastic constants or thickness of a layer(s) from signals through some signal processing. A significant contribution to the complexity of absolute measurement is the requirement that the entire system be characterized, understood and taken account for, this includes system implementation details like filters, electronics, and transducers.

### 2.3 Overview

The goal of this body of work is to start to lay the framework for extracting information from time of flight signals associated with thin layers of anisotropic materials.

Transient, time of flight, signals must be used given the nature of the types of things that are to be monitored and the nature of the instrument that is to be used. The transduction device is a pulsed laser striking the top surface of one or several layers. The detection device is a continuous laser monitoring the top surface of the layers, some distance from the point(s) of excitation. The layers that are to be tested are rapidly growing thin films, spinning disks of semiconductor material, fast moving production lines of thin aluminum, and many others. The measurements will either be relative or absolute.

**2.3.1 Signal Processing Alternatives**

There are several signal processing techniques that could be used to extract information from short-time transient dispersed signals.

¹ page 398 of (Scruby & Drain 1990)
One method, as shown in figure 2-1, would be to extract the modes of propagation from the signals, through a frequency-time decomposition, and then fit the extracted curve to theoretical curves. Another technique, as shown in figure 2-3 would be to run a real signal through an array of theoretical signals and extract the one with the best fit. (Another possibility, if can be made fast enough, would be a feed-back convergence method of modifying model parameters until you find a best fit. This is just a more elegant implementation than the full signal filter bank method.)

**Group Velocity Matching / Inversion**

Figure 2-1 illustrates the group velocity matching technique. A real signal is decomposed into its frequency versus time components, via wavelets, short time windowed Fourier transform, or any other technique. The modes evident in the frequency-time decomposition are extracted. Then an iterative fitting procedure, (simplex, deepest decent, neural net, etc.) is performed modifying model parameters until the error between theory and reality is below an acceptable level. For now, just note that the group velocity is the first derivative of the dispersion relationship. Normally it would be displayed as group velocity versus frequency. The velocity is easily transformed into time versus frequency since we know the distance between excitation and detection.

One problem with this technique is that when more than one mode is present it is difficult to separate the individual curves, since they interfere with one another in the frequency time decomposition. Additionally we are fighting the uncertainty principal, it is difficult to detect the time of occurrence of low frequency signals, and it is difficult to determine the frequencies of short time events.

**Dispersion Curve Matching / Inversion**

Figure 2-2 illustrates the idea of dispersion curve matching or inversion. The concept is fairly simple. Data is collected at many different times and at many different locations. The (2D) Fourier transform of spatial and temporal data results in spatial and temporal frequencies. The transform yields the dispersion curves directly. The curves are extracted and an iterative routine is used to
modify the guesses for elastic constants and thickness(es) until the theoretical dispersion curves match the experimental.

**Dispersion Curve Extraction via 2D-Fourier Transform**

![Diagram of Dispersion Curve Extraction via 2D-Fourier Transform](image)

Figure 2-2: Dispersion Curve Extraction via a 2D-Fourier Transform

One problem with this technique is the requirement for multiple transducers or moving a single transducer. In many practical applications this is not possible, and we are limited to a single transducer at a single location.

**Model Filter Bank**

Figure 2-3 illustrates the filter bank approach. The number of layers, their thicknesses, and type(s) of material dictate the modes of propagation, i.e. the dispersion relationships. The transduction event (piezoelectric transducer, laser heating source, or electro-magnetic pulse) is characterized by the transducers spatial frequencies and the distribution of energy among the modes. The initial pulse’s spatial frequency components are propagated (as dictated by the dispersion relationship) through the medium under test to some detection point. A time series is monitored at that detection point. A matrix of theoretical signals over many thicknesses and elastic constants is generated. The real signal is then correlated with this matrix of theoretical signals. The thickness and elastic constants of the best fit theoretical signal are the thickness and elastic constants of the real signal. This correlation fit can be performed between the real signal and individual modes or the full theoretical signal in an attempt to increase accuracy.

Phenomena that occur in short distances and over small times are always difficult to model with a high degree of accuracy. The excitation event or source of a pulse is a short time event localized to a small area of a wave transmitting medium. One must spend a good deal of effort understanding and modeling the transducer and the 't-zero' transduction event. Understanding the initial development and evolution of a wave as energy gets partitioned into various modes and frequencies is another component of the t-zero to t-zero-plus time horizon that is difficult to model. The far field is much easier to understand.

One problem with this technique is the need to characterize the transduction event. Another problem is the memory requirements associated with storing such a wide range of theoretical signals. If memory is an issue or if we have a very fast computer we can modify the Time Based Filter Bank.
to use an iterative fitting routine to calculate signals 'on the fly'. This idea is illustrated in figure 2-4.

Some applications of absolute measurements have been successful (Ringermacher, Reed & Strife 1993), (Ringermacher & McKie 1995). When the elastic constants of a layer are well known, and what is of interest is the thickness of the layer, the 'a0' mode can be used with some success. The 'a0' mode or the lowest order anti-symmetric mode is relatively isolated from other modes, and is very sensitive to thickness. Additionally the beginning portion of the mode is fairly simple, and can be approximated quite nicely with an analytic expression (Hutchins et al. 1989). These 'niceties' make it possible to use the 'a0' mode in some absolute measurements.

2.3.2 Requirements

What are the theoretical requirements for these techniques? Mode extraction and inversion requires a reasonable model of the modes of propagation, the dispersion relationships. The signal propagation model requires the frequency composition of the initial disturbance (or its equivalent for each mode as seen in the far field), and the dispersion relationship.

One last time to iterate:

The dispersion relationship is the functional relationship between temporal and spatial frequency i.e. \( w = w(k) \). Alternatively it can be viewed as the functional relationship between phase speed (of propagation) and frequency i.e. \( c = c(w) \) or \( c = c(k) \). The fundamental relationship here is that phase speed is related to temporal and spatial frequencies in this manner, \( c(k) = w(k)/|k| \). The dispersion relationship gets its name from the fact that when \( c(w) \) is not constant, different frequencies travel at different speeds. An initial disturbance localized in time and space will distort or disperse as it propagates.

The initial disturbance will be represented as the Fourier transform of the initial spatially distributed transduction event.
Once we know what frequency components are present, and how each propagates, it is a simple manner to determine the field at some arbitrary point and time away from the initial disturbance by summing the field contributions over all frequencies as seen in equation 2.14.

\[ y(x_d, t) = \int_{-\infty}^{\infty} A_0(k)e^{-i(kx_d-w(k)t)}dk \]  

Equation 2.14

What will be presented here is an example of the groundwork that must be established in order to extract meaningful data from transient signals which traveled through a medium consisting of one or multiple layers, a wave guide, of potentially anisotropic material. The equations of motion must be developed. This leads to a highly non-linear transcendental equation, the dispersion equation, which must be solved for the modes of propagation. An adaptive root tracing algorithm is then presented for solving the dispersion equation.

The Filter Bank (of signals) Method will be used to extract the temperature and thickness of a single layer. The Filter Bank Method was chosen because of the accuracy of the measurement that we are trying to perform. The uncertainty associated with the group velocity extraction method appears to be too large for the measurements of interest. Experiment and instrument limitations prevents the application of a two-dimensional Fourier transform technique.

Several methods for eliminating the need to have an excellent model for the transduction event will be explained. A technique for approximating the initial spatial frequencies of several modes from a single observation point will be presented.

The applicability of this exercise to other thin targets will be emphasized.
Chapter 3

Thin Layer

We will first consider a monoclinic thin layer of material bounded on both sides by vacuum. We will develop the equations of motion in Cartesian coordinates for a plane wave propagating along a Cartesian axis. We will limit our analysis to systems that have monoclinic symmetry (a "naturally" cubic material rotated about one of the natural axis results in a monoclinic-symmetric matrix of elasticity.)

The results presented here can be extended to multiple layers by "stacking" the matrix of equations of motion for subsequent layers and satisfying the displacement and stress boundary conditions. Briefly the solution follows the following plan.

- We choose a coordinate system $<x_1, x_2, x_3>$ or more succinctly $<1, 2, 3>$ that is oriented arbitrarily with the "natural" crystal axes $<x', y', z'>$. We will be concerned with a plane wave propagating along the coordinate $<1>$ axis. The compact representation of the coefficients of elasticity along $<x'y'z'>$ are rotated into the coordinate $<123>$ frame, generally increasing the number of non-zero elements in the results elasticity matrix.

- We propose an exponential (harmonic) solution. We substitute the harmonic solution into the equation of motion.

- We solve for the determinate of the resulting matrix for the slowness surface. That is, we solve for the physically allowable relationship between the wave number parallel to the direction of propagation, the wave number perpendicular to the direction of propagation, and the frequency of oscillation. In general this will give us six roots (three complex pairs) relating the wave-vector, $K$, to the temporal frequency, $w$. Realizing that $K^2 = k_1^2 + k_2^2$, we can solve for $k_3$ as a function of $k_1$ and $w$.

- Then substituting back into the equation of motion we can solve for the, in general, six partial waves of displacement. For each partial wave, we only know the amplitude of displacement along each axis up to a multiplicative constant.

- We use displacement field as described by the summation of the partial waves to substitute back into the stress equation. Now, we have equations for six partial stress waves. Again, for any one wave we only know the amplitude of stress along each axis up to a multiplicative constant.

- For the free-free layer we use the stress wave equations to satisfy the stress-free boundary conditions.

3.1 Mathematical Formulation

As shown in figure 3-1 we choose a coordinate system $<x_1, x_2, x_3>$ that is oriented arbitrarily with the "natural" crystal axes $<x', y', z'>$. 

At a point in the medium the strain-displacement equation with respect to the $<123>$ axes is (in indicial notation):
The stress-strain (constitutive) relationship is:

\[ T_{ij} = C_{ijkl} \epsilon_{kl}; i, j, k, l = x_1, x_2, x_3 \]  

(3.2)

The dynamic equilibrium equation is (with no body force):

\[ \frac{\partial T_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}; i, j = 1, 2, 3 \]  

(3.3)

Substituting the stress-strain and strain-displacement equations into the dynamic equilibrium equation results in a system of three coupled differential equations for displacements \(u_1, u_2,\) and \(u_3,\) the displacement equation of motion.

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = \frac{1}{2} c_{ijkl} \left( \frac{\partial^2 u_k}{\partial x_l^2} + \frac{\partial u_l}{\partial x_k \partial x_l} \right); i, j = 1, 2, 3 \]  

(3.4)

At any boundary all wave vectors are in the same plane. This means that the response of the layer will be independent of the in-plane coordinate normal to the direction of propagation. The analysis can then be simplified to consider two components of the wave vector, one parallel to the direction of propagation, and the other normal to the direction of propagation and the layer surface. More succinctly, plane wave propagation along the \(x_1\) direction is independent of the \(x_2\) direction. A solution of the following form is proposed.

\[ u_j = U_j e^{i(k_1 x_1 + k_3 x_3 - \omega t)} \]  

(3.5)

Substituting this into the displacement equation of motion results in a matrix equation of the form. A more thorough development and expansion of this procedure can be followed in the appendix.

\[ \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]  

(3.6)

In compact notation.

\[ \mathbf{KU} = \mathbf{0} \]  

(3.7)

Where (for a triclinic material)
\[ k_{11} = c_{11} k_1^2 + c_{45} k_3^2 + 2 c_{15} k_3 k_1 - p w^2 \]  \hspace{1cm} (3.8)
\[ k_{22} = c_{66} k_1^2 + c_{44} k_3^2 + 2 c_{46} k_3 k_1 - p w^2 \]  \hspace{1cm} (3.9)
\[ k_{33} = c_{55} k_1^2 + c_{33} k_3^2 + 2 c_{35} k_3 k_1 - p w^2 \]  \hspace{1cm} (3.10)
\[ k_{12} = c_{16} k_1^2 + c_{45} k_3^2 + ((c_{14} + c_{56})) k_3 k_1 \]  \hspace{1cm} (3.11)
\[ k_{13} = c_{15} k_1^2 + c_{35} k_3^2 + ((c_{13} + c_{55})) k_3 k_1 \]  \hspace{1cm} (3.12)
\[ k_{23} = c_{56} k_1^2 + c_{34} k_3^2 + ((c_{36} + c_{45})) k_3 k_1 \]  \hspace{1cm} (3.13)

This is often referred to as the Christofell Equation.

Solvability dictates that the determinate of this system of equation must be equal to zero. The determinate is a sixth order polynomial in \( k_3 \), for which there is no analytical solution. Having an analytic solution is not a requirement nor a problem, it can be solved numerically.

Currently we are only interested in materials (coupled with directions of propagation) that are monoclinic in complexity or less. So this means that the following simplifications can be made.

\[
c_{14} = 0 \quad \text{(3.14)}
\]
\[
c_{24} = 0 \quad \text{(3.15)}
\]
\[
c_{34} = 0 \quad \text{(3.16)}
\]
\[
c_{15} = 0 \quad \text{(3.17)}
\]
\[
c_{25} = 0 \quad \text{(3.18)}
\]
\[
c_{35} = 0 \quad \text{(3.19)}
\]
\[
c_{46} = 0 \quad \text{(3.20)}
\]
\[
c_{56} = 0 \quad \text{(3.21)}
\]

Specifically we are interested in a cubic material. These additional elastic constants are zero.

\[
c_{36} = 0 \quad \text{(3.22)}
\]
\[
c_{45} = 0 \quad \text{(3.23)}
\]

So now the elements of the Christofell equation are

\[
k_{11} = c_{11} k_1^2 + c_{55} k_3^2 - p w^2 \]  \hspace{1cm} (3.24)
\[
k_{22} = c_{66} k_1^2 + c_{44} k_3^2 - p w^2 \]  \hspace{1cm} (3.25)
\[
k_{33} = c_{55} k_1^2 + c_{33} k_3^2 - p w^2 \]  \hspace{1cm} (3.26)
\[
k_{12} = c_{16} k_1^2 \]  \hspace{1cm} (3.27)
\[
k_{13} = (c_{13} + c_{55}) k_1 k_3 \]  \hspace{1cm} (3.28)
\[
k_{23} = 0 \]  \hspace{1cm} (3.29)

The determinate of equation 3.6 with these simplification to monoclinic complexity is a third order polynomial in \( k_3^2 \). Third order polynomials do have an analytic solution.

\[
A_6 k_3^6 + A_4 k_3^4 + A_2 k_3^2 + A_0
\]  \hspace{1cm} (3.30)

Where:

\[
A_6 = (-c_{33} c_{44}^2 + c_{33} c_{44} c_{65}) \]  \hspace{1cm} (3.31)
\[
A_4 = [-c_{13}^2 c_{44} + c_{11} c_{33} c_{44} - 2 c_{16} c_{33} c_{45} + 2 c_{13} c_{36} c_{45} + 2 c_{13} c_{45}^2 - c_{36} c_{55} -
\]
This equation has three roots in \( k_3^2 \), or six +/- roots in \( k_3 \) \( (k_{3+1}, k_{3-1}, k_{3+2}, k_{3-2}, k_{3+3}, k_{3-3}) \). The six roots relate the wave vector normal to the plate surface to the wave vector parallel to the direction of propagation and the frequency of oscillation. What do these relationships represent? A wave propagating along 1 with frequency \( w \) and wavenumber \( k_1 \) can only have these six possible wavenumber in the 3 direction, normal, to the layer surface. These are the physically allowable combinations of wave-vector and frequency often referred to as the slowness surface.

We initially proposed a solution that had displacement polarizations along each axis but which had wave vectors along only the direction of propagation and the direction normal to the surface of the plate. Substituting into the equation of motion and using the solvability condition on the resulting matrix, we find that six possible partial waves can exist in the medium, as illustrated in figure 3-2. Each partial wave still only has wave-vectors parallel to the direction of propagation and normal to the plate surface, but has displacement polarization components along each direction. For notational purposes note that \( U_{2+} \) is the amplitude of the displacement along the 2 direction from partial wave +1.

\[
A_2 = \frac{2c_{13}c_{44}c_{55} + c_{33}c_{55}c_{66}}{(-c_{33}c_{44} + c_{44}^2 - c_{33}c_{55} - c_{44}c_{55})w^2} k_1^2 + \frac{c_{16}c_{33} + 2c_{13}c_{16}c_{36} - c_{11}c_{36}^2}{2c_{13}c_{16}c_{45} - 2c_{11}c_{36}c_{45} - c_{11}c_{45}^2 + 2c_{16}c_{36} + c_{11}c_{44} + c_{16}c_{36}c_{55} + c_{11}c_{44}c_{55} + c_{13}c_{66} + c_{11}c_{33}c_{66} - 2c_{13}c_{55}c_{66}} k_1^4 \]
\[
A_0 = \frac{c_{16}^2 + c_{11}c_{55}c_{66}}{c_{33} + c_{44} + c_{55}} * w^4 (3.33)
\]

Using the first two equations from the matrix equation 3.6. We can solve for the amplitude ratios of the different polarization components.

\[
\frac{U_{2j}}{U_{1j}} = \frac{(k_{11}, k_{23}, -k_{13}, k_{12})}{(k_{13}, k_{22}, -k_{12}, k_{23})}; where j = \pm(1, 2, 3)
\]
\[
\frac{U_{3j}}{U_{1j}} = \frac{(k_{12}, k_{12}, -k_{11}, k_{22})}{(k_{13}, k_{22}, -k_{12}, k_{23})}; where j = \pm(1, 2, 3)
\]
Remember $U_1$ is the amplitude of oscillation along the direction of propagation. We could normalize all $U_1$’s to 1. However for better numerical stability in implementation ratios where the denominator could go to zero are avoided. So we will use the following expressions for the partial wave amplitudes.

$$U_1 = (k_{13}, k_{22}, k_{12})$$  \hspace{1cm} (3.38)  
$$U_2 = (k_{11}, k_{23}, k_{12})$$  \hspace{1cm} (3.39)  
$$U_3 = (k_{12}, k_{12}, k_{22})$$  \hspace{1cm} (3.40)

The ratios could be simplified using the values for $k_{ij}$ from the cubic assumption.

$$\frac{U_2}{U_1} = \frac{-c_{16}k^2}{c_{66}k_{11}^2 + c_{44}k_{33}^2 - pw^2}$$ \hspace{1cm} (3.41)

$$\frac{U_3}{U_1} = \frac{c_{16}k_{11}^4 - (c_{66}k_{11}^2 + c_{44}k_{33}^2 - pw^2)(c_{11}k_{11}^2 + c_{55}k_{33}^2 - pw^2)}{(c_{13} + c_{55})k_{11}k_{33}(c_{66}k_{11}^2 + c_{44}k_{33}^2 - pw^2)}$$ \hspace{1cm} (3.42)

So, finally up to a multiplicative constant the displacements in the bulk of a medium can be represented as follows.

$$u_1(k_1, w, x_1, x_3, t) = \sum_{j=\pm 1}^{\pm 3} U_{1j} e^{i(k_1x_1 + k_3x_3 - wt)}$$ \hspace{1cm} (3.43)

$$u_2(k_1, w, x_1, x_3, t) = \sum_{j=\pm 1}^{\pm 3} U_{2j} e^{i(k_1x_1 + k_3x_3 - wt)}$$ \hspace{1cm} (3.44)

$$u_3(k_1, w, x_1, x_3, t) = \sum_{j=\pm 1}^{\pm 3} U_{3j} e^{i(k_1x_1 + k_3x_3 - wt)}$$ \hspace{1cm} (3.45)

The displacement field in the bulk of the layer is obtained:

$$u(k_1, w, R, t) = u_1 + u_2 + u_3$$ \hspace{1cm} (3.46)

We now use these displacement relationships to find the relationships for stresses and to satisfy the stress-free boundary conditions of the layer surface.

Using equation 3.1, relating strain to displacement, and equation 3.2, relating strain to stress, we can develop equations for stress.

### 3.2 Satisfying Boundary Conditions

From equation 3.1:

- $\varepsilon_{x_1x_1} = \frac{\partial u_1}{\partial x_1} = (i k_1) u_1,$
- $\varepsilon_{x_2x_2} = \frac{\partial u_2}{\partial x_2} = 0,$
- $\varepsilon_{x_3x_3} = \frac{\partial u_3}{\partial x_3} = (i k_3) u_3,$
- $\varepsilon_{x_2x_3} = \frac{1}{2} (\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) = \frac{1}{2} ((i k_3) u_2, + 0)$
- $\varepsilon_{x_1x_3} = \frac{1}{2} (\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) = \frac{1}{2} ((i k_3) u_1, + (i k_1) u_3)$
- $\varepsilon_{x_1x_2} = \frac{1}{2} (\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) = \frac{1}{2} (0 + (i k_1) u_3)$
Where \( j = \pm(1, 2, 3) \)

From equation 3.2: (We are only concerned with stresses oriented on the \( x_3 \) direction.)

\[
T_{x_3 \epsilon_{3j}} = c_{31\epsilon_{x_1x_1j}} + \frac{1}{2}c_{36\epsilon_{x_1x_2x_2j}} + \frac{1}{2}c_{35\epsilon_{x_1x_3x_3j}} + \frac{1}{2}c_{36\epsilon_{x_2x_1x_1j}} + \frac{1}{2}c_{32\epsilon_{x_2x_2x_2j}} + \frac{1}{2}c_{34\epsilon_{x_2x_3x_3j}} + \frac{1}{2}c_{35\epsilon_{x_3x_1x_1j}} + \frac{1}{2}c_{34\epsilon_{x_3x_2x_2j}} + c_{33\epsilon_{x_3x_3x_3j}}
\]

\[\text{(3.47)}\]

\[
T_{x_3 \epsilon_{1j}} = c_{51\epsilon_{x_1x_1j}} + \frac{1}{2}c_{56\epsilon_{x_1x_2x_2j}} + \frac{1}{2}c_{55\epsilon_{x_1x_3x_3j}} + \frac{1}{2}c_{56\epsilon_{x_2x_1x_1j}} + \frac{1}{2}c_{52\epsilon_{x_2x_2x_2j}} + \frac{1}{2}c_{54\epsilon_{x_2x_3x_3j}} + \frac{1}{2}c_{55\epsilon_{x_3x_1x_1j}} + \frac{1}{2}c_{54\epsilon_{x_3x_2x_2j}} + c_{53\epsilon_{x_3x_3x_3j}}
\]

\[\text{(3.48)}\]

\[
T_{x_3 \epsilon_{2j}} = c_{41\epsilon_{x_1x_1j}} + \frac{1}{2}c_{46\epsilon_{x_1x_2x_2j}} + \frac{1}{2}c_{45\epsilon_{x_1x_3x_3j}} + \frac{1}{2}c_{46\epsilon_{x_2x_1x_1j}} + \frac{1}{2}c_{42\epsilon_{x_2x_2x_2j}} + \frac{1}{2}c_{44\epsilon_{x_2x_3x_3j}} + \frac{1}{2}c_{45\epsilon_{x_3x_1x_1j}} + \frac{1}{2}c_{44\epsilon_{x_3x_2x_2j}} + c_{43\epsilon_{x_3x_3x_3j}}
\]

\[\text{(3.49)}\]

Since we will be concerned only with stresses on the \( x_3 \) face, only those stresses are presented here.

\[
\begin{align*}
T_{x_3 x_3}(k_1, w, x_1, x_3, t) &= \sum_{j=\pm 1}^{\pm 3} T_{x_3 \epsilon_{3j}} e^{i(k_1x_1+k_3x_3-wt)} \\
T_{x_3 x_1}(k_1, w, x_1, x_3, t) &= \sum_{j=\pm 1}^{\pm 3} T_{x_3 \epsilon_{1j}} e^{i(k_1x_1+k_3x_3-wt)} \\
T_{x_3 x_2}(k_1, w, x_1, x_3, t) &= \sum_{j=\pm 1}^{\pm 3} T_{x_3 \epsilon_{2j}} e^{i(k_1x_1+k_3x_3-wt)}
\end{align*}
\]

\[\text{(3.50)}\]

\[\text{(3.51)}\]

\[\text{(3.52)}\]

Where, for a monoclinic material:

\[
\begin{align*}
T_{x_3 x_3} &= c_{13}k_1U_1 + c_{36}k_1U_2 + c_{33}k_3U_3 \\
T_{x_3 x_1} &= c_{55}k_1U_1 + c_{45}k_3U_2 + c_{55}k_1U_3 \\
T_{x_3 x_2} &= c_{45}k_3U_1 + c_{44}k_3U_2 + c_{45}k_1U_3
\end{align*}
\]

\[\text{(3.53)}\]

\[\text{(3.54)}\]

\[\text{(3.55)}\]

From these equations the \( x_3 \) axis oriented stresses at any point in the layer can be calculated. We want to satisfy the stress free boundary conditions at the top, \( +d \), and bottom, \( -d \), of the layer.

Before we proceed it is worthwhile to present the observation relating to the the complex pairs of partial waves. Specifically note that for the monoclinic case:

\[
\begin{align*}
U_{1 \pm 1} &= U_{1 \mp 1} \\
U_{2 \pm 1} &= U_{2 \mp 1} \\
U_{3 \pm 1} &= -U_{2 \mp 1}
\end{align*}
\]

\[\text{(3.56)}\]

\[\text{(3.57)}\]

\[\text{(3.58)}\]
These observations make it easier manipulate the matrix that results from satisfying the boundary conditions.

Now, using equation 3.50 to satisfy the stress-free boundary conditions at the top and bottom of the layer results in a system of equations in \( U_1 \), whose determinant follows.

\[
\begin{vmatrix}
T_{x x 3 1} E_{+1} & T_{x x 3 2} E_{-1} & T_{x x 3 3} E_{+2} & T_{x x 3 4} E_{-2} & T_{x x 3 5} E_{+3} & T_{x x 3 6} E_{-3} \\
-T_{x x 3 1} E_{+1} & -T_{x x 3 2} E_{+2} & T_{x x 3 3} E_{-2} & T_{x x 3 4} E_{+3} & T_{x x 3 5} E_{-3} & T_{x x 3 6} E_{+3} \\
T_{x x 3 1} E_{+1} & T_{x x 3 2} E_{-1} & T_{x x 3 3} E_{+2} & T_{x x 3 4} E_{+3} & T_{x x 3 5} E_{-3} & T_{x x 3 6} E_{+3} \\
T_{x x 3 1} E_{-1} & T_{x x 3 2} E_{+1} & T_{x x 3 3} E_{+2} & T_{x x 3 4} E_{+3} & T_{x x 3 5} E_{-3} & T_{x x 3 6} E_{+3} \\
-T_{x x 3 1} E_{+1} & -T_{x x 3 2} E_{-2} & -T_{x x 3 3} E_{+2} & -T_{x x 3 4} E_{+3} & -T_{x x 3 5} E_{-3} & T_{x x 3 6} E_{+3} \\
T_{x x 3 1} E_{+1} & T_{x x 3 2} E_{+2} & T_{x x 3 3} E_{+3} & T_{x x 3 4} E_{-3} & T_{x x 3 5} E_{-3} & T_{x x 3 6} E_{+3} \\
\end{vmatrix} = 0 \quad (3.62)
\]

where

\[
E_{+j} = e^{+ik_3 d} \\
E_{-j} = e^{-ik_3 d}
\]

Notice that there are numerous complex exponential pairs. It appears as if we might be able get this result into sines and cosines. The reason that the following manipulation can be performed is because of the three complex exponential pairs that arise during the monoclinic case.

\[
\text{Column}_1 = \text{Column}_1 + \text{Column}_2 \\
\text{Column}_2 = \text{Column}_1 - \text{Column}_2 \\
\text{Column}_3 = \text{Column}_3 + \text{Column}_4 \\
\text{Column}_4 = \text{Column}_3 - \text{Column}_4 \\
\text{Column}_5 = \text{Column}_5 + \text{Column}_6 \\
\text{Column}_6 = \text{Column}_5 - \text{Column}_6 \\
\]

Then

\[
\text{Row}_1 = \text{Row}_1 + \text{Row}_4 \\
\text{Row}_2 = \text{Row}_2 + \text{Row}_5 \\
\text{Row}_3 = \text{Row}_3 + \text{Row}_6 \\
\text{Row}_4 = \text{Row}_1 - \text{Row}_4 \\
\text{Row}_5 = \text{Row}_2 - \text{Row}_5 \\
\text{Row}_6 = \text{Row}_3 - \text{Row}_6 \\
\]

Then rearrange into two three-by-three sub-matrices. The result of these manipulations follows.

\[
\begin{vmatrix}
T_{x x 3 1}(e^{+ik_3 d} + e^{-ik_3 d}) & T_{x x 3 2}(e^{+ik_3 d} + e^{-ik_3 d}) & T_{x x 3 3}(e^{+ik_3 d} + e^{-ik_3 d}) \\
T_{x x 3 1}(e^{+ik_3 d} - e^{-ik_3 d}) & T_{x x 3 2}(e^{+ik_3 d} - e^{-ik_3 d}) & T_{x x 3 3}(e^{+ik_3 d} - e^{-ik_3 d}) \\
T_{x x 3 1}(e^{+ik_3 d} - e^{-ik_3 d}) & T_{x x 3 2}(e^{+ik_3 d} - e^{-ik_3 d}) & T_{x x 3 3}(e^{+ik_3 d} - e^{-ik_3 d}) \\
\end{vmatrix} = 0 \quad (3.65)
\]
We finally obtain two decoupled characteristic equations that correspond to symmetric and anti-symmetric modes of displacement.

The first sub-matrix corresponds to the symmetric modes.

\[
\begin{vmatrix}
T_{x_3x_1}(e^{+ik_3d} - e^{-ik_3d}) & T_{x_3x_2}(e^{+ik_3d} - e^{-ik_3d}) & T_{x_3x_3}(e^{+ik_3d} - e^{-ik_3d}) \\
T_{x_3x_1}(e^{+ik_3d} + e^{-ik_3d}) & T_{x_3x_2}(e^{+ik_3d} + e^{-ik_3d}) & T_{x_3x_3}(e^{+ik_3d} + e^{-ik_3d}) \\
T_{x_3x_1}(e^{+ik_3d} + e^{-ik_3d}) & T_{x_3x_2}(e^{+ik_3d} + e^{-ik_3d}) & T_{x_3x_3}(e^{+ik_3d} + e^{-ik_3d}) \\
\end{vmatrix} = 0 \quad (3.66)
\]

The second sub-matrix corresponds to the anti-symmetric modes.

\[
\begin{align*}
F_1 \sin(k_3,d) & \sin(k_3,d) \cos(k_3,d) - \\
F_2 \sin(k_3,d) & \sin(k_3,d) \cos(k_3,d) + \\
F_3 \sin(k_3,d) & \sin(k_3,d) \cos(k_3,d) = 0 \\
\end{align*} \quad (3.67)
\]

where

\[
\begin{align*}
F_1 &= T_{x_3x_1}(T_{x_3x_1}T_{x_3x_2}T_{x_3x_3} - T_{x_3x_2}T_{x_3x_3}) \\
F_2 &= T_{x_3x_2}(T_{x_3x_2}T_{x_3x_3} - T_{x_3x_1}T_{x_3x_3}) \\
F_3 &= T_{x_3x_3}(T_{x_3x_3}T_{x_3x_2} - T_{x_3x_1}T_{x_3x_2}) \\
\end{align*} \quad (3.69-3.71)
\]

So finally, we have two equations which relate spatial frequency of propagation \( k_1 \) to temporal frequency \( w \).
Chapter 4
Dispersion Curves

The dispersion relationships, found in the previous chapter, are a set of non-linear transcendental equations, with an infinite number of modal solutions. They have several singularities that real solutions may cross. They relate spatial frequency, wave number, along the direction of propagation to temporal frequency.

This chapter discusses the solutions to the dispersion relationships developed in the previous chapter.

A complex, adaptive root tracing algorithm necessary to fight the complexities of the dispersion equations is also presented.

4.1 Solution-Space Matrix

A simple minimum-scan of the dispersion relationship can be used to map out the entire solution space (all of the minimums) over a range of spatial and temporal frequency. A minimum-scan entails picking a spatial frequency, $k$, and finding all of the temporal frequencies, $w$’s for which the dispersion relationship is minimized (actually goes to zero) over the range of interest. Increase $k$ and repeat. The result is a matrix of the solution-space, where the row indices number corresponds to the $k$ increment, and the matrix values are the values of $w$ that minimize the dispersion relationship. Several examples of such a solution-space matrices are plotted in figures 4-1, 4-2, 4-3, and 4-4.

Figure 4-1: Solution space matrix of silicon at 100 degrees Celsius along an axis oriented with the $<010>$ direction of the cubic crystal. This is for the anti-symmetric modes. The horizontal axis is dimensionless spatial frequency. The vertical axis is dimensionless temporal frequency.
Figure 4-2: Solution space matrix of silicon at 550 degrees Celsius along an axis oriented with the <010> direction of the cubic crystal. This is for the symmetric modes. The horizontal axis is dimensionless spatial frequency. The vertical axis is dimensionless temporal frequency.

The entire dispersion relationship solution-space could then be used in a propagation model to generate theoretical signals. (Relative amplitude distributions would also be needed, that is a generation model.) This method is not very useful. If you use the entire solution space you might as well just perform a finite-element analysis of the layer (or multiple layers) which would automatically incorporate all of the modes in the frequency ranges of interest and not provide any insight.

4.2 Root Tracing

For several reasons, we are interested in isolating and studying individual modes. As will be discussed, we would like to be able to determine the absence and presence of different modes by comparison to theoretical signals consisting of single modes. We are trying to develop a more intuitive understanding and computationally fast model of the underlying physics. We are very interested in the group velocity of the modal solutions (so that we can use a frequency-time decomposition method). The group velocity is the first derivative along a modal solution curve. For all of these reasons we need to trace along the modal solutions of the dispersion relationship.

There are several problems associated with tracing the roots to the dispersion equations with which standard root finders can not cope.

**Crossing Singularity** Some of the modes of interest cross through regions where the dispersion relationship is singular. (figure, with matrix , with the region circled)

**Mode Converging** When dealing with a monoclinic material (i.e. a cubic material at an off axis) modes can get very close to one another and then rapidly diverge, resulting in an extremely sharp 1st derivative. (figure, maybe several, with regions that are close)

**Large Frequency-Thickness (fd) product** The dispersion equation is solved in a non-dimensional form, non-dimensionalized with the half thickness of the plate, d. For high frequencies, large fd-product, the roots of the dispersion equations asymptote along a singularity resulting in numerical instabilities. This problem has been dealt with historically in several fashions (Hosten & Castaings 1993), (Lowe 1995), and (Castaings & Hosten 1994) but is not really of concern here since this occurs at a much higher frequency than is of interest.
Figure 4-3: Solution space matrix of silicon at 100 degrees Celsius along an axis oriented with 30° away from the < 010 > direction of the cubic crystal. This is for the anti-symmetric modes. The horizontal axis is dimensionless spatial frequency. The vertical axis is dimensionless temporal frequency.

### 4.2.1 Algorithm Pseudo-code

An adaptive root-tracing algorithm has been written that deals with all of these problems. Currently it is customized for tracing the dispersion-relationship function, but it can be generalized for any function of two variables. It has been thoroughly tested and modified as problems arise.

**Initialization, Setup, or Input**
- Elastic Constants
- \( \Delta K \)
- \( K_0 \)
- Initial \( W \) range at \( K_0 \)
- minimum \( \Delta W \).

While (\( K \ LT \ maxK \))
  
  a) while (MinimumCount == 0)
      
      Do a (coarse) Scan. Returns MinimumCount (the number of minimums found)
      
      If (MinimumCount == 0)
      
      Increase search range.
      
      If (Increased Search Range to many times)
      
      - Issue a warning, and Stop.
      
      end
      
  end

b) for (1 to MinimumCount)

  While (Not Reached Converged Condition)
  
  Do a (fine) Scan. Returns MinimumCountFine

  If (Found more than One Minimum)
  
  - Increment MinimumCount
  
  - Modify List Containing information about minimums

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Figure 4-4: Solution space matrix of silicon at 550 degrees Celsius along an axis oriented with 30° away from the <010> direction of the cubic crystal. This is for the symmetric modes. The horizontal axis is dimensionless spatial frequency. The vertical axis is dimensionless temporal frequency.

```
end
   Modify (fine) Scan Parameters. (Left, Right, Middle)
end
end

c) If (MinimumCount GT 1)
   Pick Best Solution, based on smoothness of 1st derivative
end

d) Add Solution to Array of W's

e) Modify the Scan Parameters for the next (coarse) Scan, using more information for Prediction at is becomes available.
   -Whigh, Wlow, delW

f) If (2nd Derivative GT 2ndDerivative Threshold AND
    we aren't already using a fine delK)
   -Save state and set the flag indicating that we are now using a fine delK
   15 -set for fine delK
   -Seed fine delK search parameters with interpolated points from array already found
end

g) If (In fine delK AND 2nd Derivative LT 2nd Derivative Threshold - Hysteresis)
   -restore state
   -restore Prediction
end
end
```
4.2.2 Algorithm Pseudo-code explanation

Each section is now explained in further depth.

Coarse Scan

a) A coarse scan is performed over a range that spans the predicted root plus and minus a tolerance. If a minimum is not found, then the range is increased, until at least one minimum is found. If the range increasing occurs too many times, the routine is stopped and the user is warned.

It is possible that more than one minimum will be found. That would occur if searching in a region that is near another mode or a singularity. A list of all of the minimums is created. (The 'correct' minimum will be selected later.)

Iterative Sectioning

b) After the coarse scan we have the current location of the minimum and points on the left and right of the suspected minimum at some delta. The minimum(s) found during the coarse scan are now determined more accurately during a series of location estimate improvements (fine scans).

Convergence is guaranteed if we follow a bisection routine that iteratively examines points at half-deltas on the left and right of the suspected minimum until a low threshold delta-W is reached. We calculate points at half-delta on either side of the suspected minimum. If either of the newly examined points are less than the suspected minimum, the suspected minimum is updated; either the point on the left or the right become the former suspected minimum; and the other point is unchanged. If both of the newly examined points are larger than the suspected minimum the suspected minimum is unchanged; and the left and right points are updated.

We are also interested in the speed of the algorithm, since we have to generate dispersion curves over many angles and temperatures. Bisection is not necessarily the fastest method to increase the accuracy of the minimum estimate. Trisection or more might work faster. It is a function of the speed of the code that checks points on either side of the suspected minimum, and the number of times the search region is more finely meshed/searched (to reach the low threshold delta-W). This is illustrated in figure 4-5.

As the code stands at the moment trisection is fastest. Trisection works the same as the bisection described above, except now we examine points at third-delta on either side of the suspected minimum.

![Figure 4-5: Plot showing that bisection is not the fastest sectioning scheme for this algorithm](image_url)
It is again possible that fine-scanning will find an additional minimum not found during the coarse-scan. That would occur if searching in a region that is very near another mode or a singularity. The list of all of the minimums is updated if an additional potential minimum is found. (The 'correct' minimum will be selected later.)

It is also possible that a flat-bottomed minimum-well will be found during the iterative-search mesh-sectioning. If this occurs the search region is remeshed only for that given iteration until a pure minimum is found. If the remeshing occurs to frequently, the user is warned and the search is aborted.

The criteria that I am using to halt the iterative search is a lower bound on the “delta-W”. When the search region has been meshed to the minimum delta-W the zero is considered to be found. Currently the minimum delta-W is set to be about 0.001 Hz (which corresponds to the eight digit ASCII format of the dimensionless W). A threshold on the absolute value of the function (the dispersion relationship) is a bad idea because the trend of the dispersion function gets larger and larger for higher frequencies. If an absolute threshold were used the higher frequency roots would not have the same accuracy as the lower frequency roots.

Root Selection

c) At the completion of stepping through all potential minimums and calculating them to the low-threshold delta-W, we have to pick which minimum is the correct root. The correct root is chosen based on keeping the first derivative the smoothest. Obviously this requires that a portion of the curve has already been generated. This is why it is wise to start the routine in a region that is certain to contain a single root.

Add selected root to array
d) The list of roots along the curve is updated with the newly selected root.

Update search parameters for next coarse scan
e) The search region for the next coarse-scan must be calculated. The predicted value for the next root is calculated from the previous roots, and then a tolerance above and below the prediction sets the region. When first starting to trace the curve the prediction is linear, the prediction progresses to third order, as more roots along the curve are calculated.

Switching to fine step

f) In a region where two (or more) modes are in very close proximity, the curves might have a very sharp derivative. It may be so sharp that the predictions are very poor, and the root-tracing may oscillate between the two modes or trail off on a singularity. One fix, would be to run the search routine with a much finer delta-K, so that the sharp turns may be more closely followed.

Another alternative, is to adjust the delta-K when a region of sharp derivative is detected. That is what occurs in this section of the search algorithm. First the state of the root-tracing routine is saved so that is can be restored. Then the initial new state of the routine for a finer delta-K is interpolated from the coarse delta-K. Then the root-tracing advances as before. With the one difference of updating the saved state when necessary.

Exiting fine step

g) When we have left a region of high derivave we no longer need to be tracing with a fine delta-K. After the 2nd derive drops back below the derivative threshold level (minus some hysteresis) the state for a coarse delta-K is restored, the prediction is updated, and the root tracing proceeds as before.
4.3 Automation

We would like to be able to automate this root-tracing routine, to search out a similar mode over a variety of angles and temperatures (elastic constants).

It is inefficient to use the solution-space matrix as a crutch in the root tracing. We would like to find a region that for all angles and temperatures of interest the desired mode is isolated from all other modes and singularities. This region will then be the starting region for the root-tracing routine over the many conditions of interest. Otherwise the start of each new curve would require human intervention.

The change in elastic constants associated with a change in angle are very large. The change in elastic constants associated with a change in temperature (of one degree Kelvin) is very small. For the following exercise we can ignore the changes in elastic constants associated with change in temperature.

Over a wide range of spatial and temporal frequency, generate the solution-space matrix for all of the angles of interest. Then plot all of these matrices together in one plot. Typically, there will be a region that can be selected, where the one mode of interest is separate from all other modes and singularities. Select that region and use it as the starting point for the root-trace algorithm for all of the angle and temperatures of interest. Figures 4-6, 4-7,4-8, and 4-9 illustrate this idea.

![Figure 4-6:](image)

Solution space matrix of silicon at 100 degrees Celsius along axes oriented 0, 1, 5, 30, 35, 44, 45° from the <010> direction of the cubic crystal. This is for the anti-symmetric modes. The horizontal axis is dimensionless spatial frequency. The vertical axis is dimensionless temporal frequency.

The starting point for the “a0” mode was at \(k_{nd} = 1\) over a range of \(w_{nd}\) from 0.2 to 0.7, this region is expanded in figure 4-7. The starting point for the “s0b” mode was at \(k_{nd} = 1.4\) over a range of \(w_{nd}\) from 1.55 to 1.85, this region is expanded in figure 4-9. We could easily calculate the temporal frequencies for the case when the spatial frequency is zero. But it can be difficult starting from \((k = 0, w = 0)\) since the calculations can be very noisy. Attempting to start from \((k = 0, w = 0)\) is very troublesome, it is difficult to select the real roots from the singularities.

4.4 Dispersion Curves

Here are results of tracing similar modes over many angles and temperature for silicon.

Figures 4-10, 4-11,4-12,4-13, and 4-14 are plots of the lowest order anti-symmetric mode for a silicon layer at a temperature of 541 degrees Celsius and several different angles.
Figure 4-7:
Solution space matrix of silicon at 100 degrees Celsius along axes oriented 0, 1, 5, 30, 35, 44, 45° from the <010> direction of the cubic crystal. This is for the anti-symmetric modes. The horizontal axis is dimensionless spatial frequency. The vertical axis is dimensionless temporal frequency.

Figures 4-15, 4-16, 4-17, 4-18, and 4-19 are plots of the lowest order symmetric mode for a silicon layer at a temperature of 541 degrees Celsius and several different angles.

Figures 4-20 and 4-21 show the similar lowest order anti-symmetric mode at every angle from 0 to 45 degrees from the <010> axis at 1 degree intervals.

Figures 4-22 and 4-23 show the similar lowest order symmetric mode at every angle from 0 to 45 degrees from the <010> axis at 1 degree intervals.

One additional item should be noted. If we examine figure 4-24 we notice that as a function of temperature the dispersion curves for the lowest order symmetric and antisymmetric dispersion curves change more at the higher frequencies than the lower frequencies. That is \( \frac{d\omega}{dT} |_{k=small} < \frac{d\omega}{dT} |_{k=large} \). This suggests that we should use the higher frequencies when trying to fit theory to reality, as the higher frequencies will give a sharper correlation peak when attempting to perform a fit. The lower frequencies in the signal which are relatively insensitive to temperature are effectively noise.
Figure 4-8:
Solution space matrix of silicon at 550 degrees Celsius along axes oriented 0, 1, 5, 30, 35, 44, 45° from the <010> direction of the cubic crystal. This is for the anti-symmetric modes. The horizontal axis is dimensionless spatial frequency. The vertical axis is dimensionless temporal frequency.

Figure 4-9:
Solution space matrix of silicon at 550 degrees Celsius along axes oriented 0, 1, 5, 30, 35, 44, 45° from the <010> direction of the cubic crystal. This is for the symmetric modes. The horizontal axis is dimensionless spatial frequency. The vertical axis is dimensionless temporal frequency.
Figure 4-10:
Several different representation of a dispersion curve for a silicon layer. Mode a0. Along the <010> axis.

Figure 4-11:
Several different representation of a dispersion curve for a silicon layer. Mode a0. Along an axis 1° from <010>.
Figure 4-12:
Several different representation of a dispersion curve for a silicon layer. Mode a0. Along an axis $22^\circ$ from $< 010 >$.

Figure 4-13:
Several different representation of a dispersion curve for a silicon layer. Mode a0. Along an axis $44^\circ$ from $< 010 >$. 
Several different representation of a dispersion curve for a silicon layer. Mode $a_0$. Along an axis $45^\circ$ from $<010>$.

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Figure 4-15:
Several different representation of a dispersion curve for a silicon layer. Mode $s_0b$. Along the $<010>$ axis.
Figure 4-16:
Several different representation of a dispersion curve for a silicon layer. Mode $s_{0b}$. Along an axis $1^\circ$ from $<010>$.

Figure 4-17:
Several different representation of a dispersion curve for a silicon layer. Mode $s_{0b}$. Along an axis $22^\circ$ from $<010>$. 
Figure 4-18:
Several different representation of a dispersion curve for a silicon layer. Mode s0b. Along an axis 44° from <010>.

Figure 4-19:
Several different representation of a dispersion curve for a silicon layer. Mode s0b. Along an axis 45° from <010>.
Figure 4-20:
Dispersion curves for a silicon layer. Mode a0. Along axes 0:1:45° from <010>.

Figure 4-21:
Group Velocity curves for a silicon layer. Mode a0. Along axes 0:1:45° from <010>.
Figure 4-22:
Dispersion curves for a silicon layer. Mode s0b. Along axes 0: 1: 45° from <010>.

Figure 4-23:
Group Velocity curves for a silicon layer. Mode s0b. Along axes 0: 1: 45° from <010>.
Silicon Layer. Mode a0 and s0b sensitivity to temperature. Along axis 0° from <010>.
Chapter 5

Frequency Time Decomposition

It was decided very early to use model signals, instead of attempting to extract the group velocity curves from a frequency-time decomposition. A frequency-time decomposition is still very useful in observing the general frequency-time nature and characteristics of a signal. It will also be used later in an algorithm to determine the equivalent modal transducers.

The idea behind decomposing a signal into its frequency-time components is fairly simple; determine what frequencies are present in the signal at any time. The results of a frequency-time decomposition are best presented on a three-dimensional plot/image, where one axis is time, one axis is frequency, and the third is the amplitude of decomposition.

To detect the presence of any one frequency in signal we need at least one, preferably more, cycle of the frequency of interest. Several cycles of a high frequency signal can occur in a short time, narrow bandwidth, window. A low frequency signal requires a long time, wider temporal bandwidth, window for at least a single half cycle of that low frequency to be present and detectable. A constant frequency-bandwidth decomposition is a logical methodology for dealing with this requirement.

A continuous wavelet decomposition is a frame type of frequency-time sub-band decomposition. A signal is decomposed into its (single stretched and shifted) basis function components. A mother wavelet is chosen. Daughter wavelets are stretched, frequency-bandwidth constant, versions of the mother wavelet. This decomposition is represented in equation 5.2.

\[
B_{jk} = \int_{-\infty}^{\infty} S(t)W_{jk}(t)dt
\]  
\[W_{jk}(t) = W(jt - k)
\]

5.0.1 Determining Dispersion Relationship

We monitor a signal \(S(t)\) traveling through a dispersive medium at some point \(X_o\), and the dispersion relationship is what we would like to determine. If we perform a continuous wavelet decomposition on a signal that has already had the chance to disperse, we will have coefficients \(B_{jk}\).

What do these \(B_{jk}\)'s represent?

They represent the magnitude of a wavelet with a particular temporal center frequency at each point of time along the received dispersed signal. Each wave-packet (wavelet) that composes this signal traveled with the group velocity of the dispersive medium evaluated at the center frequency of the wavelet.

At times \(t = \frac{d_{sep}}{C_g(F_{cntr})}\) we should see a relative local maximum of the amplitude of the wavelet with center frequency \(F_{cntr}\). If we plot the magnitude of the coefficients as a function of their shift, (or arrival time,) and center frequency, we start to see some interesting results. What do we expect? We expect to see the dispersion curves emerging from this image.

5.0.2 The Wavelet - Morlet

The modulated Gaussian pulse is a natural basis for signal analysis since many physical phenomena obey Gaussian models. The Morlet wavelet is defined in equation 5.3.
The choice of $\sigma$ affects the temporal width of the wavelet. The shape of the wavelet is maintained as it is dilated by holding the frequency bandwidth, $\sigma f_s = \text{Constant}$. Indeed, the Heisenberg uncertainty principal tells us that this quantity is always equal to or greater than $1/2$. The constant is a design parameter for this wavelet. Adjusting it sets the number of cycles that will appear inside the Gaussian pulse.

In a continuous wavelet implementation, the band center frequency is changed continuously (up to sampling limits).

![Figure 5-1: Morlet wavelet at several different dilations.](image)

Figure 5-1 shows the Morlet wavelet at several different dilations. Notice that the shorter time signals have a large center frequency, and that the longer time signals have a small center frequency. The temporal band-width/center-frequency product is held constant. The amplitude is scaled to keep energy constant.

### 5.0.3 Example

Figure 5-2 shows a Morlet wavelet decomposition and the pulses of a material with dispersion relationship $ak = w^2$. Note the linear frequency-time relationship.

In a dispersive medium the phase velocity of any harmonic wave remains $c(w) = c(k) = w/k$. The group velocity, however, is the first derivative of the radial frequency, $w$, with respect to the spatial frequency, $k$. In this example the medium has a dispersion relation $ak = w^2$. The phase velocity is given $C_p = w/k = 1/aw$. The group velocity is given $C_g = dw/dk = 1/(2aw)$.

A medium with this dispersion relationship is impulsively loaded in time and space. (i.e. The initial disturbance can be represented $D(x,t) = \delta(x)\delta(t)$.) Figure 5-2 also shows this pulse monitored at several distances from the source. Notice how the pulse spreads as it propagates.

### 5.1 Waves in a Layer

The general equations of elasticity that lead to the dispersion relationships for a thin layer were developed in the previous chapter.

Theoretical plane-wave transient signals of the first two symmetric and first two anti-symmetric modes of a layer of aluminum with transmitter and receiver located 5 inches apart were calculated.
Morlet wavelet frequency-time decomposition of sample dispersive signals, with dispersion relationship \( 1 \cdot 10^{15}k = w^2 \). Signal monitored at distances of 1 foot, 8 feet, and 16 feet from transmitter. Spatial frequencies from 101/m to 12.51/m.

These signals were then decomposed into their frequency-time components. The frequency - group velocity (converted to time) curves are overlayed on the decomposition. Figures 5-3, 5-4, 5-5, and 5-6 show the individual theoretical modal signals with their decomposition. Figures 5-8 and 5-7 show composite signals with their decompositions.

Notice that it is relatively easy to see the frequency-time mode shape when only one mode is present. As more modes are excited and show their presence in a detected signal, they interfere with one another. It is more difficult to separate the characteristics and shape of one mode versus another.

**5.2 Curve Extraction**

We would like to extract the group velocity curves from the frequency-time decompositions.

Observe the decompositions where only one mode is present. It is a fairly straightforward procedure to extract a single mode. Now observe the decompositions where more than one mode is present. It is significantly more difficult to extract distinct modes. Where modes overlap and interfere with one another it is near impossible to extract individual undistorted curves.
Signal, group velocity, and Morlet wavelet decomposition an a0 pulse in a thin layer of aluminum. Layer thickness is .02 inches. Distance from transmitter to receiver is .5 inches. $C_t = 3130 \text{ m/sec}$. Poisson ratio is 0.334.

Signal, group velocity, and Morlet wavelet decomposition an a1 pulse in a thin layer of aluminum. Layer thickness is .02 inches. Distance from transmitter to receiver is .5 inches. $C_t = 3130 \text{ m/sec}$. Poisson ratio is 0.334.
Figure 5-5:
Signal, group velocity, and Morlet wavelet decomposition an s0 pulse in a thin layer of aluminum. Layer thickness is .02 inches. Distance from transmitter to receiver is .5 inches. \( C_t = 3130 \text{m/sec} \). Poisson ratio is 0.334.

Figure 5-6:
Signal, group velocity, and Morlet wavelet decomposition an s1 pulse in a thin layer of aluminum. Layer thickness is .02 inches. Distance from transmitter to receiver is .5 inches. \( C_t = 3130 \text{m/sec} \). Poisson ratio is 0.334.
Signal, group velocity, and Morlet wavelet decomposition an a0 and s0 pulse in a thin layer of aluminum. Layer thickness is \(0.02\) inches. Distance from transmitter to receiver is \(0.5\) inches. \(C_t = 3130 m/sec\). Poisson ratio is 0.334.

Figure 5-7:

Signal, group velocity, and Morlet wavelet decomposition an a0, a1, s0, s1 pulse in a thin layer of aluminum. Layer thickness is \(0.02\) inches. Distance from transmitter to receiver is \(0.5\) inches. \(C_t = 3130 m/sec\). Poisson ratio is 0.334.

Figure 5-8:
Chapter 6

Temporal Signals

Remember, the motivation for all of this is to use a model to extract information from real dispersive signals. The dispersion curves are only one portion of generating a model signal. The timing of the excitation pulse and its initial spatial frequency distribution is required to correctly calculate a detected signal some distance away from the excitation point.

Several question arise when trying to deal with several modes at the same time. Unless the initial disturbance is orthogonal to all but a single mode, how much energy from the initial disturbance goes into each mode? Is the same initial spatial distribution used for each mode? Will there be sympathetic excitation of other modes, even if one mode acquired all of the initial energy? All of these question are complicated by the fact that we have an imperfect spatial and temporal model of the transduction device.

6.1 Plane Waves

Beyond dispersion curves, the initial frequency distribution of the pulse (or the spatial impulse response of the transducer), is needed to model what would be seen as the received signal. The dispersion curves are relatively straight forward to calculate, and to understand. The timing and distribution of the initial disturbance is a difficult phenomena to model. Equally difficult, is the short-distance and small-time mode evolution as different modes are excited to varying degrees. Simply stated, the transduction event is a complex function of space and time which very difficult to model accurately with the myriad of details that must be addressed.

Figure 6-1:
This figure illustrates an ideal transduction function for a one dimensional transducer, a line.
This figure illustrates the idea of transforming a real transduction function as illustrated in figure 6-1 into equivalent modal excitation functions. The modal excitation function are an impulse in time with some spatial distribution.

To simplify the transduction event model, we can determine an equivalent initial spatial frequency distribution, \( A_0(k, \text{mode}) \), for every mode of propagation along with their corresponding t-zeros, this idea is illustrated in equation 6.1 and figures 6-1 and 6-2.

\[
D(R, t) = A_0(k, \text{mode1})\delta(t - t_{\text{mode1}}) + A_0(k, \text{mode2})\delta(t - t_{\text{mode2}}) + \cdots
\]  

(6.1)

The equivalent initial spatial frequency distributions are called Equivalent Modal Excitation Functions (EMEF's). This amounts to treating each mode as if it is impulsively loaded in time with some initial spatial distribution. The time of occurrence of the impulse is not necessarily when we would consider 't-zero' experimentally. There is some offset, that captures the mode generation and non-impulsive nature of the disturbance. The spatial distribution may or may not be similar to that of the actual initial pulse.

This conveniently separates the the initial transduction event into separate functions for each mode modeled as an impulse in time, a delta function, with some spatial distribution handled by its spatial frequency Fourier transform, \( A_0(k) \). This initial disturbance for each mode is then propagated through time and space, via an inverse Fourier integral, according to the physics of the dispersion relationship, equation 6.2.

\[
y(x_d, t) = \int_{-\infty}^{\infty} A_0(k)e^{-i(kx_d - w(k)t)}dk
\]

(6.2)

And for multiple modes,

\[
y(x_d, t) = \sum_{\text{modes}} \left( \int_{-\infty}^{\infty} A_{0, \text{mode}}(k)\ e^{-i(kx_d - w(k, \text{mode})t)}dk \right)
\]

(6.3)

It should be possible to determine equivalent spatial frequency and timing information empirically. Use a set of real signals and some processing to extract the information that characterizes the an equivalent transduction event for each mode.
6.2 Empirical Characterization of Transducer, Plane Wave

The techniques that were described as methods to extract thickness and elastic constants from a signal are some of the same methods that can be used to empirically model a transducer. When trying to characterize a transducer we use a layer under test that is a “known”. The elastic constants are known and the thickness is known and we want to determine the $A_k$'s transduction mechanism for each mode.

6.2.1 Single Mode - Single Detection Point (SM-SDP)

If the signal contained only a single mode the inverse transducer characterization would be relatively easy. This assumes that we have a reasonable dispersion relationship for that single mode. Detect the signal at some distance from the transduction event over time. Perform a Fourier transform on the time signal, which gives the temporal frequency distribution. Then map the temporal frequency into spatial frequency using the dispersion relationship. This spatial distribution is the equivalent, temporally impulsive, transduction event. This empirically determined transduction characterization can then be used to generate theoretical signals. This is illustrated in figure 6-3.

Figure 6-3:
This figure illustrates the process of characterizing the equivalent transduction function of a medium that supports one mode of propagation.
6.2.2 Multiple Modes - Multiple Detection Points (MM-MDP)

If the propagation medium supports more than one mode the equivalent transduction determination is more difficult. After a Fourier transform is performed on the time signal, we have no idea how the various frequency components are distributed across several modes. Therefore we can’t map the temporal frequencies into spatial frequencies using the dispersion relationship.

We don’t have a single unique transformation between spatial and temporal frequencies, we must gather information in time and space. Instead of detecting a signal at a single location over time, we must detect the signal in the medium at numerous locations along the propagation path over time. (Obviously the data is sampled in time and space, the Nyquist criteria of the temporal and spatial frequencies of interest must be met.) This two-dimensional data set in time and space is then transformed into spatial and temporal frequencies with a two-dimensional Fourier transform. An image of the results of the transform will show several distinct modes. The amplitudes along the mode curves are then extracted to be used as the equivalent transduction function for each mode.

6.2.3 Multiple Modes - Single Detection Points (MM-SDP)

Again we consider a medium that supports more than one mode. We might not be able to gather information over both time and space, due to experimental complexity or post collection realization that the signal is multi-modal.

Monitor the time history of a signal at a single location, \( x_d \). Perform a temporal-frequency/time decomposition on the signal. Remember that the first derivative of the dispersion relationship is the group velocity. The group velocities of the modes of interest are converted into times-of-flight through \( t_f = \frac{x_d}{c_g(w)} \). Extract the amplitudes of the frequency time decomposition along the frequency-time curves that belong to each mode.

This technique is a battle against the uncertainty principal, the more overlap that there is in the multiple modes, in time at low frequencies and in frequency at short times, the more difficult it is to extract detailed un-correlated information about each mode. Therefore, in practice this can be a very difficult technique to use. It is mentioned here for completeness and familiarity. It will be used later.

6.3 Axicon Wave

The specific system with which we are concerned with analyzing, non-destructively testing, is a thin layer of anisotropic cubic crystal. Both the excitation source and the detector must be non-contact so the only choices are electro-magnetic pulse or laser heating. We can’t get close to the test sample, so laser is the only option.

The main problem with the laser-generation of ultrasound is the low signal level. This typically necessitates strong laser pulses which may damage or ablate the surface. A technique first introduced by Maldague, Cielo & Jen (1986) uses a combination of conical and spherical lenses to shape a laser pulse into a ring. The surface of the medium under test is irradiated with this ring, distributing energy over a larger area, reducing the possibility of damage. Two thermoelastic waves then propagate from the ring, one converging to the center, the other expanding. The converging wave is interferometrically detected at the center of the ring, where the amplitude of the wave is amplified. This is illustrated in figure 6-4.

The transduction device is an axicon ring laser pulse. If an isotropic surface is impulsively loaded with a hoop, one circular wave will uniformly converge to the center of the hoop and one will diverge. If the detection device is located at the center of the hoop, all of the power that was distributed around the hoop becomes focused at the center. For an isotropic layer, it could be argued that a sufficient model for the wave as detected at the center of the hoop would be the same as a simple plane wave that traveled a distance equal to the radius of the hoop. This argument is based on the fact that the hoop could be modeled as the summation of plane waves all excited simultaneously with normals crossing at the center, as illustrated in figure 6-5.
This figure illustrates axicon ring excitation and interferometric detection at the center.

Ring Modelled as a Summation of Plane Waves around the Circumference

This figure illustrates the proposed model for an axicon ring.

The problem is that mathematically the center detection point is a singularity where the simple summation model of converging plane waves might break down. Converging phenomena are difficult to model accurately. The focused energy could force the propagation medium into a non-linear regime. Diffraction limiting is also a concern. One method around these problems would be to create a large spot size (or maybe a tiny axicon) that is detecting the majority of its energy some small distance from the center of the ring. The benefit of the geometric increase in signal amplitude can be had without the adverse diffraction and non-linear effects if the correct combination of initial ring pulse energy and detection at a small distance from the center is used.

For an isotropic material it is easy to experimentally verify the equivalence, in shape, between a ring source with radius, \( r \), generating a converging wave detected with a transducer at the center of the hoop and a line source generating a plane wave detected with a transducer located at a normal distance \( r \) from the source.

There are however numerous applications where the material or layer under test is not isotropic, but anisotropic. A plane wave propagating along any one direction is different from all other directions, see figure 6-6. That is because the dispersion relationships which dictate the physics of propagation, for an anisotropic material are very different depending on the direction of the plane wave propagation. A plane wave along any one of these direction and a circularly converging wave are no longer the same.

6.3.1 Axicon Model

The model proposed for the anisotropic converging wave is still a summation of plane waves for each mode around the circumference of the ring as in equation 6.4 and illustrated in figure 6-5.
Figure 6-6:

This figure illustrates the difference between a plane wave along any one axis and an axicon wave. There are no other models known in the literature for modeling a converging wave on an anisotropic layer(s). Actually, there does not appear to have been any attempt to use an axicon on an anisotropic material.

We can calculate dispersion curves $w(k, \theta, \text{mode})$, at any angle. The next question is what are the $A_0(k, \theta, \text{mode})$'s. Presumably we need to determine equivalent modal excitation functions for every mode at every angle.

It was attempted to use plane waves along several different directions and several distances and to use a frequency-time decomposition to empirically determine the $A_0(k)$'s for every mode and angle as discussed in the section on empirical characterization of transducers. The problem is that for the layers of interest the group velocities curves for several modes are very close in frequency and time and so it is difficult to determine into which mode to put energy. Additionally, the optics used to make a line are different enough from the optics that make an axicon ring that the results of the $A_0(k)$'s as determined empirically for a line source probably wouldn't be as accurate as desired for use in the axicon model.

The difference in the optics between line and axicon, the complexity of the experiment, and time limitations prevented any attempt at sampling at numerous points along many angles to use the two-dimensional Fourier transform approach.

Determining Equivalent Modal Excitation Functions

Theoretical single-mode axicons with a Cauchy and Gaussian initial spatial frequency distribution for every angle and mode, were calculated. A single-mode axicon signal is a theoretical axicon signal that is summation of one similar mode around the axicon for all directions of propagation.

The single-mode axicon signals generated with the Cauchy and Gaussian distribution were compared to real signals. The general shapes and structure of the real signal were represented. From this familiarization process it became evident that in the real signals of interest only two “single-mode axicons waves” appeared to be present. They were the lowest anti-symmetric mode and the (2nd) lowest symmetric mode. These correspond to the lowest order symmetric and lowest order anti-symmetric modes that are present when propagating along a cubic axis.
It is easy to imagine using any one of the Imperial Characterization of Transducer techniques described above in an iterative fashion. Use one of the techniques to determine a best guess for an initial spatial distribution, create a model signal, compare to the original signal, and tweak the extracted \( A_0(k) \) from the comparison of the model signal to the real signal (the details of how-to-tweak can be difficult). This iterative idea should be particularly valuable to the frequency-time decomposition method for a multiple-mode signal detected at a single point, since it is the one for which there is not a unique inversion.

The axicon model is discretized into one degree intervals, which actually means 46 unique angles given the eight-fold symmetry of the cubic material. That means 92 different modes. Given time and experimental constraints a single equivalent loading function for any one single-mode axicon was assumed. That means that \( A_0(k, \theta, \text{mode}) \) becomes \( A_0(k, \text{mode}) \).

With the assumption that each similar mode around the axicon has the same equivalent transduction function and the realization that only two single-mode axicons are present in the real signal, only two spatial frequency distributions had to determined.

The MM-SDP idea was implemented in an iterative fashion. The amplitude of all 92 modes is determined from the frequency-time decomposition on both the real and theoretical signal. Then the results for the similar modes around the axicon are collapsed into an “average” frequency-time decomposition along an “average dispersion curve”. This is not strictly valid, because of the inequality equation 6.5.

\[
\sum_{i=1}^{n} \left[ \int_{\text{inf}}^{\text{inf}} A_0(k)e^{-i(kx - w_{xt})} \right] \neq \int_{\text{inf}}^{\text{inf}} A_0(k)e^{-i(kx - \sum w_{xt})} (6.5)
\]

The difference in the average dispersion curves between the real signal and the theoretical is used to modify the amplitude distributions used to generate theoretical signals.

The process is illustrated in figure 6-7, and described here.

- An initial estimate is made for the for the symmetric, \( A_0(k, \text{sym}) \), and anti-symmetric, \( A_0(k, \text{anti-symm}) \)’s, similar modes.
- Two theoretical similar-mode axicons are generated.
- Due to electronics and mode evolution phenomena we don’t have good fix on t-zero for either mode. So we find the location of each theoretical single mode axicon in the real signal.
- The similar-mode axicons are shifted, scaled and added together to form a theoretical signal.
- The amplitude of the temporal frequency-time decomposition along all 92 modes is determined for both the theoretical signal and the real signal.
- The average amplitude of the two average dispersion curves for the real and theoretical signal is determined.
- The difference between the average dispersion curves of the real and the dispersion curves of the theoretical signal is calculated. This is the error between the guess for the initial transduction functions and what the equivalent transduction function actually is.
- Multiplied by some gain the errors are used to modify the guesses for \( A_0(k, \text{sym}) \) and \( A_0(k, \text{anti-symm}) \).
- The process is repeated until an acceptable error between the real and theoretical signal is realized.

Obviously, this process converges much faster the better the initial guesses for the \( A_0(k) \)’s. The laser pulse that is serving as the excitation transducer, is in the shape of a Gaussian beam. It is pinging the layer from one side. These two facts suggest that the dominant mode(s) should be anti-symmetric with a Gaussian shaped initial spatial frequency distribution. This provides an intuitive basis for a first estimate of the shape of the \( A_0(k, \text{anti-symm}) \).
As the initial loading is almost completely orthogonal to the symmetric modes, I postulated that the symmetric modes would appear only via sympathetic and noise-type excitation from the asymmetric modes, and that no one frequency range in the symmetric modes would be more prevalent than any other. As an initial guess for the symmetric modes a flat (white noise) distribution for the initial spatial frequencies, $A_0(k, symm)$ was used.

Figures 6-8 and 6-9 show the starting and ending predictions of the initial amplitude distributions, the equivalent modal excitation functions, for the a0 mode and the s0b mode.

Figures 6-10 and 6-11 show the starting and ending theoretical signals displayed along with the real signal that was being used to converge toward.

### 6.4 Conclusions

These excitation functions characterize the complex geometry, energy distribution and mode excitation associated with the initial transient pulse. They capture the transducer and medium excitation mechanism as observed in the far field. It also embodies the delays and other characteristics associated with system electronics.

Now that we have this empirical model, the knowledge of what modes are present, and the dispersion relationships we are in a position to calculate theoretical signals to act as the optimal signal correlators to detect and extract information from real signals.
Determining Equivalent Modal Excitation Functions and Timing

Figure 6-7:
Determining equivalent modal excitation functions and timing.
Figure 6-8:
Starting and ending theoretical Equivalent Modal Excitations Functions for the a0 axicon modal signal.

Figure 6-9:
Starting and ending theoretical Equivalent Modal Excitations Functions for the s0b axicon modal signal.
Figure 6-10:
Starting theoretical signal compared to real signal.

Figure 6-11:
Final theoretical signal compared to real signal.
Chapter 7

Comparing Theory to Reality

The dispersion curves and the EMEF's are used to generate a filter bank array of signals over the temperature and thickness ranges of interest.

Real signals taken on a thin layer of a known thickness and at a known temperature are then fed through this filter bank. The temperature and thickness are then extracted from the maximum amplitude of a massive correlation. An example of the output of such a correlation is presented in figures 7-1, 7-2, and 7-3.

![Correlation Filter Bank Output](image)

Figure 7-1: Filter bank output. The maximum amplitude is the location of the thickness and temperature of the input signal.

The results of temperature determination on a single silicon layer of thickness .61425mm is presented in figure 7-4.

These preliminary results indicate that the temperature measurements using this technique given the accuracy of the experimental set-up are accurate to better that 1.5 degrees at one standard deviation.

Figure 7-5 demonstrates an increase in sensitivity using higher frequencies which are more sensitive to changes in temperature.
Figure 7-2: Filter bank output. The maximum amplitude is the location of the thickness and temperature of the input signal.

Figure 7-3: Filter bank output thickness cross section. The maximum amplitude is the location of the thickness of the input signal.
Figure 7-4: Results of temperature determination on a layer of thickness 0.61425mm. The results are mean corrected.

Figure 7-5: The figure shows the increased sharpness of the correlation peak when using higher frequencies.
Chapter 8

Conclusions

The steps taken in this body of work are those that are necessary for any non-destructive testing of thin layered media. So this work is very useful and extendable. Numerous advances in understanding, modeling, and implementation were made.

Specifically,

- This body of work presented an implementation of one technique to analyze dispersive signals, specifically in a thin layer.
- An example of the development of the equations of motion for a thin anisotropic layer were presented. The dispersion curves were developed.
- A robust root-tracing algorithm was developed to trace the dispersion curves.
- A model was proposed for a ring excitation source.
- A iterative technique to empirically determine equivalent modal excitation was developed and implemented.
- Theoretical/model signals were used to extract the temperature and thickness information in real signals.

These are techniques and methodologies that are applicable to all types of non-destructive diagnostics of dispersive materials, specifically of thin-layered wave guides.
Bibliography


