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by

Manuel Cerrillo

Abstract

This work deals with the study of the transient phenomena of electromagnetic waves in hollow metallic cylinders of a general geometric cross section when excited under different initial and terminal conditions. A semi-infinite waveguide, the cross-sectional dimensions of which are small in comparison with its length, is excited at one end, taken as the origin, by an electromagnetic field of rather arbitrary waveform. The solution given here shows (a) the distortion of the waves in amplitude and frequency as they propagate along the waveguide; (b) the speeds, signal and group velocities, and the time of formation (time in which the internal fields build up) at the given point of observation, or the spatial distribution of the fields at a given instant of time; (c) general methods of solution for complicated waveforms of the incoming signals; (d) reduction of all transients to a typical one by means of generating functions. Exact, asymptotic, and graphical solutions are given as solutions of the transient behavior, and applications are made to some typical cases. To accomplish the above results, it was necessary to give a detailed and complete discussion of the motion of electromagnetic waves in systems of cylindrical configurations and the new results were obtained in this connection. Laplace transformations are used as the basic mathematical tool in this investigation.

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* This report is a slight modification of a thesis of the same title submitted by the author in partial fulfillment of the requirements for the Degree of Doctor of Philosophy at the Massachusetts Institute of Technology, June, 1947.

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INTRODUCTION

0.0 This study will deal with the transient phenomena of electromagnetic waves inside hollow cylinders of a general geometric cross section under different initial and terminal conditions. A general solution will be worked out which is sufficiently complete to meet all possible cases of initial conditions, in space and time, appropriate to the analytic character of this Dirichlet's type problem.

Since in wave guides the propagation constant is a function of frequency, such a guide behaves like a dispersive medium and this situation complicates the solution to the transient problem. For dispersive media, phase, signal and group velocities have been defined, and it is intended to investigate their meaning in connection with the present problem.

The solution of the transient problem is of particular interest in connection with the linear accelerator. Electric charges are injected, with a certain initial velocity, into a circular wave guide, or series of cavities, in which electromagnetic wave pulses of TH type propagate along the axis. The interrelation of the pulses and charges must be such that the particles, acted upon by the longitudinal electric field, suffer a unilateral acceleration in the direction of the waveguide axis. Since this particle must be accelerated by the internal electromagnetic field, it is important to know how this field propagates as well as the velocity of signal formations and main energy build-up along the accelerator. In order that the charges can acquire very large velocities, the main bulk of the pulse energy must be propagated at such speed that the particle is always acted upon by an electric field of sufficient intensity. Although the velocity of propagation of electromagnetic disturbances is that of light, it does not follow that the main body of the pulse will be formed at the same speed. In general, the precursor of the first wave is of such small intensity that its effect on the charge may be negligible.

As an application of this general transient theory numerical examples will be given for specific cases, showing the distortion of the pulses as they travel along, the form of the signal at a given cross section, and the surface of equal phase, when a sinusoidal pulse of definite frequency and duration is applied at a terminal cross section of the wave guide.

An attempt will be made to unify, as much as possible, the mathematical procedure. Laplace transform theory will be the basic tool.

0-0.1 The main problem solved here can be briefly defined as a lossless and semi-infinite wave guide, with cross-sectional demensions small in comparison with its length, which is excited in one of its modes at a given cross section taken as the origin. By hypothesis, it will be assumed that the time of formation of the transverse field is much shorter than the one required for the electromagnetic perturbation to reach an internal point P far away from the origin. The requirements to be obtained are: a solution for the elementary waves, the speeds with which the field builds up at the point of observation, the distortion in amplitude and phase of the original waves when they propagate along the cylinder due to the dispersive action of the guide.

Once the solution for elementary waves is obtained, solution for other fields of excitation can be found by a linear superposition of those waves.

The input wave forms of excitation are unlimited except for some analytical restriction. From the practical point of view there are some waves which have a predominant importance. They are, for example, oscillation-modulated by pulses of different form and duration, amplitude-and frequency-modulated waves, etc. The solution of the problem must be such that it can cover all the cases of practical application.

0-0.2 The study of transient phenomena in wave guides is far more involved than was expected. The principal difficulties are of mathematical character. The analytic process is complicated and delicate to handle. Besides, it is necessary to deal with a vector field and with a large number of possible initial conditions of excitation.

At the start, one of the simplest cases of wave-guide excitation was considered. Serious integration difficulties were encountered. After considerable trouble, one component of the field was obtained as an asymptotic series development in the Poincaré sense. The solution was practically useless because the other components of the field could not be derived from it, since the operation of differentiation is forbidden with such series. Besides, this series solution could not be valid when the applied frequency was too close to the cut-off frequency of the guide, which is a case of practical importance.

Now, if the type of initial condition changes, a new problem arises and it would be necessary to repeat a litany of mathematical troubles. Then, it was concluded that to solve any particular cases was not an appropriate method of attack; it was, therefore, abandoned.

0-0.3 A more general method of tackling this investigation was needed and it was necessary to go beyond the limitations of the scope of this work, starting the search from the fundamental aspect of propagation of waves in cylinders, up to a stage in which satisfactory solutions of the propagation of waves in hollow cylinders can be obtained.

This analytic study is, therefore, not limited to wave guides. A solution will be obtained for the instantaneous fields in cylindrical systems or configurations whose cross-section geometry is not limited to a special form or to a single set of the walls of the cylinder. The only restriction is that the field propagates without dissipation. The mathematical method used here is such that it can be easily extended to the case of dissipation.

0-0.4 We will start from the set of Maxwell equations in generalized cylindrical coordinates. The mathematical tools consistently and systematically used are Laplace's transformations.

This investigation was conducted as follows:

1. The set of Maxwell equations are subjected to a Laplace transformation with respect to the time, by introducing the definition of a vector space S. These transformed equations are expressed explicitly in terms of the initial spatial distribution at $t=0$.

2. The corresponding hypothesis for cylindrical configuration is introduced into Maxwell's equation in this S domain.

3. A new vector space R is defined and Maxwell's equations will be subjected to a new Laplace transformation with respect to x_3 (x_3 being the longitudinal coordinate along the generator of the cylinder). This second transformation introduced the initial condition, as a function of time, at $x_3=0$.

4. Here Maxwell's equations are solved and a vector field is obtained which represents elementary waves in this space. This field satisfies boundary conditions at the walls of the cylinder.

5. The vectors of this field are subjected now to an inverse Laplace transformation. A vector field is so obtained, in the S space, in terms of the initial conditions of all vectors at $x_3=0$ and $t=0$. This field represents two independent sets of waves traveling in the positive and negative directions of x_3.

6. This vector field is not necessarily electromagnetic for arbitrary initial values, since under this condition this mathematical field does not satisfy Maxwell's equations in S space.

   If this mathematical field is to be an electromagnetic one, the initial conditions are not all independent. They must satisfy simple relations. When these relations are introduced in the mathematical field, it will break up in three independent electromagnetic fields corresponding to TE, TH, and TEH waves.

7. The next natural step is to transform these fields from the S domain into the instantaneous t domain.

   a. If there is no dissipation, the TEH system does not offer any special problem of inversion. The propagation is merely the movement of the incoming signal without distortion and takes place with the speed of the light in the medium.

   b. The inversion of the TE and TH fields is very difficult to perform. Most of this work is devoted to this operation.

8. a. A systematic study of the transforms of the TE and TH waves was made to find the analytical connection between them. In this way a considerable reduction was made to the number of transforms which have to be inverted.

   b. A general survey was made to find a group of possible practical initial conditions. This study revealed that one has to deal with transforms of the type

$$F(s,\sqrt{s^2+c^2})\,e^{-k\sqrt{s^2+c^2}}$$

   where F is the ratio of two polynomials. This type covered almost all practical cases of amplitude-modulated signals. In case of frequency modulation, the transforms are more complicated meromorphic functions. By means of well-known theorems the last case can be reduced to expansions of the first case.

9. Conditions and analytical requirements on these transforms were investigated to secure a field which is electromagnetic upon the inversion into the instantaneous time domain. This condition proved to be significant.

10. Several methods of inversion were first tried out. Most of this work was done using the inverse integral. This inversion in the S plane proved to be very difficult to obtain for all these transforms.

11. Complex transformations were introduced. One simple complex transformation proved suitable for obtaining the inverse function corresponding to transforms of the type indicated in 9.

    At first, the results of the solutions were uniform convergent series expansions of the Neumann type. Later, these series were recognized as Lommel functions of the order of zero and one.

12. Since Lommel's functions are not tabulated, except for a few, it was necessary to obtain an appropriate expansion for the purpose of numerical computation. This expansion was made by introducing a new complex transformation.

13. Analytical expressions were obtained for the envelope and phase functions of the inverse functions of the transforms indicated in 9.

14. Study of the meaning of signal, group velocity and time of signal formation for all transforms of type 9.

15. Numerical computation of the associated functions.

16. Application to transients in wave guides.

This work is divided into four chapters. The first covers from articles 1 to 6 in this summary; the second, articles 7 to 10; the third, article 11; and the fourth, articles 12 to 16.

0-1.0 We can summarize the results of this investigation as follows:

1. The TE, TH and TEM fields in cylindrical configurations can be obtained without the introduction of three different potentials.

2. The analytical requirements which are necessary to satisfy the initial conditions in order to excite electromagnetic waves in hollow cylinders are given.

3. The appropriate interpretation of the branch cuts is given as secondary waves, which vanish in the permanent state.

4. The inverse functions of transforms of the type indicated under Art. 9 p. 6 can be obtained in a compact form and in terms of Lommel's functions.

5. The existence of a generating function which produces these inverse transforms. In other words, the reduction of all these functions to a single one, if appropriate transformations are introduced.

6. A theorem on inversion was found which proves useful to compute a family of transforms.

7. Simple approximate formulas were obtained to compute inverse functions of the above-mentioned type. If the transient period is divided into three intervals known as Precursor, Main Signal Formation and Coda Regions, appropriate simple expansions are given for each subinterval.

8. The reduction of the main signal formation interval, of all transforms, to a generating function is given. This is closely related to Cornu's Spiral.

9. Universal curves to construct envelopes and phase functions are given.

10. By using the above-mentioned universal curves, a graphical method of construction of the envelope and phase functions for particular transforms were obtained. This method saves a large amount of labor in numerical computations.

11. The determination of the signal and group velocities, time of formation and slope of formation for all these transforms is considered. An association of these velocities to the poles of the original transforms are indicated. Definitions of these concepts in terms of a generalized variable are given. The independence of these concepts to the cross section of the wave guide is shown.

12. The application is made to complicated forms of wave excitation in hollow tubes.

0-2.0 At the time of writing the results of this report, it was noticed that the Newmann series expansion of the above transforms corresponded to the functions of Lommel. This discovery enables one to make a short cut in the mathematical development of Ch. III.* Unfortunately since this chapter was already written and due to lack of time it will not be possible to incorporate these changes.

0-3.0 The mathematical method used in this work can be readily and easily extended to investigate the case in which dissipation exists. In this case, two sets of Maxwell's equations and a set of boundary conditions will be transformed. The subsequent methods of inversion are almost the same as those indicated here.

In addition, the analytic requirements on the transforms necessary to excite electromagnetic fields (see Ch. II) in hollow pipes might be used to work out the problem of discontinuities inside the guides.

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My thanks and deepest appreciation: to Professors Guillemin and Chu for their valuable guidance of this investigation; to Professors Stratton, Hazen, Wiener and Gardner for reviewing this work; to my distinguished friend, Professor Brillouin, for his careful checks on some mathematical steps.

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* This short-cut procedure is inserted in Appendix I.

CHAPTER I

Section 0 - Introduction and method followed in this chapter.

I-0.1 The material presented in this chapter will be limited to an analytical study, in its basic aspect, of the propagation of electromagnetic waves in systems of cylindrical configuration. The fundamental assumptions used here are: a. the propagation takes place without dissipation; b. the medium is uniform and isotropic; c. the effect, if any, of external charges on the internal field is negligible. The procedure followed in these chapters is indicated in the following paragraphs.

I-0.2 Section 1 deals with: Laplace transformation of the set of Maxwell's equations with respect to the time. Transformation to the S domain. Introduction in explicit form of the initial spatial condition at $t=0$.

I-0.3 Section 2 deals with: A second Laplace transformation of Maxwell's equations with respect to the axial coordinate when the equations are expressed in cylindrical generalized coordinates. (Transformation to the R domain.) Introduction of initial time conditions at $x_3 = 0$. Derivation of a general expression of the vector field and the corresponding differential equations.

I-0.4 Section 3 will comprise: Introduction in the R domain of the transverse boundary conditions. Yield solutions for the fields in the form of elementary waves. These field solutions result from solving a well-known differential equation of the Poisson type together with a system of algebraic equations.

I-0.5 Section 4 will be comprised of: A study of the initial time conditions at $x_3 = 0$ only. The first inverse transformation of the vector fields from the R domain into the S domain. A study of the general character of these transformed fields. Separation of the field into wave components moving along the positive and negative directions of the longitudinal axis.

I-0.6 Section 5 will be comprised of: A discussion of the mathematical character of the field in the S domain and its decomposition into three independent electromagnetic fields, corresponding to TE, TH and TEH waves.

I-0.7 Section 6 will be comprised of: An inverse transformation of the instantaneous TEH fields from the S domain to the time domain. Configurations suitable for TEH waves. Undistorted propagation in the medium with the speed of light.

I-0.8 Section 7 will be comprised of: Dispersive character of the TE and TH fields. Presentation of the problem of inverse transformation of the instantaneous TE and TH fields from the S time domain. Analytical relations between these s transforms and a first reduction and classification into types.

I-0.9 Section 8 will contain: Analytical survey of the initial time conditions, their introduction to the S domain and the resultant structure of s transforms. Amplitude- and frequency-modulated excitations. Prototype transforms.

I-0.10 Section 9 will contain: Further analytical requirements on the s transforms to secure an electromagnetic field after their inverse transformation into the time domain. Condition of guide excitation.

I-0.11 Section 10 will contain: Review of the methods yielding inverse transforms appropriate to the present problem of wave propagation in cylindrical systems. (The actual process of carrying out these transforms is discussed in detail in the following three chapters.)

Section 1 - First Laplace transformation of Maxwell's equations and the introduction of initial space distribution at t = 0.

I-1.0 Under the assumptions indicated in I-0.1, Maxwell's equations have the form

$$\nabla\times\vec{\mathcal{E}}+\mu\frac{\partial\vec{\mathcal{H}}}{\partial t}=0;\quad \nabla\times\vec{\mathcal{H}}-\varepsilon\frac{\partial\vec{\mathcal{E}}}{\partial t}=0$$
$$\nabla\cdot\vec{\mathcal{E}}=0\quad;\quad \nabla\cdot\vec{\mathcal{H}}=0 \qquad (1)\text{I1}$$

in which

$\vec{\mathcal{E}}=\vec{\mathcal{E}}(x_1,x_2,x_3,t)$ - instantaneous electric intensity vector

$\vec{\mathcal{H}}=\vec{\mathcal{H}}(x_1,x_2,x_3,t)$ - instantaneous magnetic intensity vector

μ = magnetic permeability

ε = electric permittivity

x_n; $n=1,2,3$; - generalized coordinates of the point of observation

t = time

MKS system of units

(2)I1

Let $\vec{\mathcal{F}}(x_1,x_2,x_3,t)$ be a vector such that the following set of integrals exist.

$$F_n = \int_0^\infty \mathscr{F}_n e^{-st} dt; \; n = 1,2,3 \tag{3)I1}$$

in which $\mathscr{F}_n(x_1,x_2,x_3,t)$ are its components along the coordinate axis and $s = \sigma + i\omega$ is the complex frequency; $i = \sqrt{-1}$. This functional transformation will define a vector space S. In this sense it is said that the vector F is the transform of the vector $\vec{\mathscr{F}}$ when

$$F = i_1F_1 + i_2F_2 + i_3F_3 = \int_0^\infty \vec{\mathscr{F}} e^{-st} dt \; . \tag{4)I1}$$

Then

$$E = \mathscr{L}_{(t)} \vec{\mathscr{E}}$$
$$H = \mathscr{L}_{(t)} \vec{\mathscr{H}} \tag{5)I1}$$

I-1.1 Now transform Maxwell's equations. Since t is independent of the space coordinates, the symbol ∇ is independent of t. Then

$$\mathscr{L}_{(t)} \nabla \times \vec{\mathscr{E}} = \nabla \times \mathscr{L}_{(t)} \vec{\mathscr{E}} = \nabla \times \mathbf{E} \; ;$$
$$\mathscr{L}_{(t)} \nabla \times \vec{\mathscr{H}} = \nabla \times \mathscr{L}_{(t)} \vec{\mathscr{H}} = \nabla \times \mathbf{H} \; ;$$
$$\mathscr{L}_{(t)} \nabla \cdot \vec{\mathscr{E}} = \nabla \cdot \mathbf{E} \; ; \quad \mathscr{L}_{(t)} \nabla \cdot \vec{\mathscr{H}} = \nabla \cdot \mathrm{H} \; .$$

Also

$$\mathscr{L}_{(t)} \frac{\partial \vec{\mathscr{E}}}{\partial t} = -\vec{\mathscr{E}}(x_1,x_2,x_3,0) + sE$$
$$\mathscr{L}_{(t)} \frac{\partial \vec{\mathscr{H}}}{\partial t} = -\vec{\mathscr{H}}(x_1,x_2,x_3,0) + sH$$

and therefore, Maxwell's equations transform as

$$\nabla \times E + s\mu H = \mu \vec{\mathscr{H}}(x_1,x_2,x_3,t)_{t=0} \; ;$$
$$\nabla \times E - s\varepsilon E = -\varepsilon \vec{\mathscr{E}}(x_1,x_2,x_3,t)_{t=0} \; ; \tag{6)I1}$$
$$\nabla \cdot E = 0 \; ; \; \nabla \cdot H = 0 \; .$$

The vectors $\vec{\mathscr{H}}(x_1,x_2,x_3,t)_{t=0}$ and $\vec{\mathscr{E}}(x_1,x_2,x_3,t)_{t=0}$ represent the initial spatial distribution of the field at $t = 0$. They may or may not be independent, or exist simultaneously, or be zero.

Section 2 - Laplace transformation with respect to x_3 and introduction of initial time conditions at $x_3 = 0$.

I-2.0 We will now suppose that the geometrical configuration of the system is cylindrical. Let h_1, h_2, h_3, designate the metric coefficients. If x_3 is taken as the distance along the axis of the cylinder from a point P to a given cross section, which is taken as the origin, then the geometry of the system is characterized by

$$h_3 = 1 \tag{(1)I2}$$

$$h_1(x_1,x_2) \ ; \ h_2(x_1,x_2) \ ; \quad \text{(independent of } x_3\text{)}.$$

It will be assumed that the propagation takes place along the x_3 axis.

Now, let $F = F(x_1,x_2,x_3,s)$, a vector and $\emptyset = \emptyset(x_1,x_2,x_3,s)$, a scalar. Under (1)I2, the expressions $\nabla \times F$, $\nabla \cdot F$, $\nabla^2 F$ and $\nabla \emptyset$ have the form

$$\left.\begin{aligned}
\nabla \times F &= i_1\left[\frac{1}{h_2}\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right] + i_2\left[\frac{\partial F_1}{\partial x_3} - \frac{1}{h_1}\frac{\partial F_3}{\partial x_1}\right] + i_3\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2) - \frac{\partial}{\partial x_2}(h_1F_1)\right] \\
\nabla \cdot F &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2F_1) + \frac{\partial}{\partial x_2}(h_1F_2)\right] + \frac{\partial F_3}{\partial x_3} \\
\nabla^2 F &= i_1\nabla^2F_1 + i_2\nabla^2F_2 + i_3\nabla^2F_3 \\
\nabla^2 F_n &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial F_n}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial F_n}{\partial x_2}\right)\right] + \frac{\partial^2 F_n}{\partial x_3} \ ; n = 1,2,3 \\
\nabla \emptyset &= \frac{i_1}{h_1}\frac{\partial \emptyset}{\partial x_1} + \frac{i_1}{h_2}\frac{\partial \emptyset}{\partial x_2} + i_3\frac{\partial \emptyset}{\partial x_3}
\end{aligned}\right\} \tag{(2)I2}$$

I-2.1 A second Laplace transformation with respect to x_3 is now to be introduced, as follows:

$$\left.\begin{aligned} F^* &= \mathcal{L}_{x_3} F = \int_0^\infty F e^{-rx_3} dx_3 \quad \text{where} \\ F_n^* &= \mathcal{L}_{x_3} F_n = \int_0^\infty F_n e^{-rx_3} dx_3 ; \; r = \text{complex variable.} \end{aligned}\right\} \quad (3)\text{I2}$$

Thus a new vector space R is defined. An asterisk is used to denote these x_3 transforms.

The transformation of (2)I2 yields:

$$\left.\begin{aligned} \mathcal{L}_{x_3}(\nabla \times F) &= i_1 \left\{ \frac{1}{h_2} \frac{\partial F_3^*}{\partial x_2} - \left[rF_2^* - F_2(0) \right] \right\} + i_2 \left\{ \left[rF_1^* - F_1(0) \right] - \frac{1}{h_1} \frac{\partial F_3^*}{\partial x_1} \right\} + \frac{i_3}{h_1 h_2} \left\{ \frac{\partial}{\partial x_1}(h_2 F_2^*) - \frac{\partial}{\partial x_2}(h_1 F_1^*) \right\} \\ \mathcal{L}_{x_3}(\nabla \cdot F) &= \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial x_1}(h_2 F_1^*) + \frac{\partial}{\partial x_2}(h_1 F_2^*) \right\} + rF_3^* - F_3(0) \\ \mathcal{L}_{x_3}(\nabla^2 F_n) &= \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1} \frac{\partial F_n^*}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2} \frac{\partial F_n^*}{\partial x_2}\right) \right\} + r^2 F_n^* - rF_n(0) - \left.\frac{\partial}{\partial x_3} F_n\right)_{x_3=0} ; \; n=1,2,3 \\ \mathcal{L}_{x_3}(\nabla \emptyset) &= \frac{i_1}{h_1} \frac{\partial \emptyset^*}{\partial x_1} + \frac{i_2}{h_2} \frac{\partial \emptyset^*}{\partial x_2} + i_3 \left[r\emptyset^* - \emptyset(0) \right] \end{aligned}\right\} \quad (4)\text{I2}$$

since x_1, x_2, h_1 and h_2 are independent of x_3. The notation is further explained by

$$\left.\begin{aligned} F_n(0) &= F_n(x_1, x_2, x_3, s)\big)_{x_3=0} = F_n(x_1, x_2, +0, s) \\ \left.\frac{\partial F_n}{\partial x_3}\right)_{x_3=0} &= \frac{\partial}{\partial x_3} F_n(x_1, x_2, x_3, s)\big)_{x_3=0} \\ \emptyset(0) &= \emptyset(x_1, x_2, x_3, s)_{x_3=0} = \emptyset(x_1, x_2, +0, s) \; . \end{aligned}\right\} \quad (5)\text{I2}$$

I-2.2 Using the above results, Maxwell's equations transform as

$\mathcal{L}_{x_3}\left[\nabla\times E+s\mu H=\mu\vec{\mathcal{H}}(x_1,x_2,x_3,0)\right]$ goes into

$$\frac{1}{h_2}\frac{\partial E_3^*}{\partial x_2}-\left[rE_2^*-E_2(x_1,x_2,0,s)\right]=-s\mu H_1^*+\mu\mathcal{H}_1^*(x_1,x_2,r,0)$$

$$\left[rE_1^*-E_1(x_1,x_2,0,s)\right]-\frac{1}{h_1}\frac{\partial E_3^*}{\partial x_1}=-s\mu H_2^*+\mu\mathcal{H}_2^*(x_1,x_2,r,0)$$

$$\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2E_2^*)-\frac{\partial}{\partial x_2}(h_1E_1^*)\right]=-s\mu H_3^*+\mu\mathcal{H}_3^*(x_1,x_2,r,0).$$

$\mathcal{L}_{x_3}\left[\nabla\times H-s\varepsilon E=-\varepsilon\vec{\mathcal{E}}(x_1,x_2,x_3,0)\right]$ goes into

$$\frac{1}{h_2}\frac{\partial H_3^*}{\partial x_2}-\left[rH_2^*-H_2(x_1,x_2,0,s)\right]=s\varepsilon E_1^*-\varepsilon\mathcal{E}_1^*(x_1,x_2,r,0)$$

$$\left[rH_1^*-H_1(x_1,x_2,0,s)\right]-\frac{1}{h_1}\frac{\partial H_3^*}{\partial x_1}=s\varepsilon E_2^*-\varepsilon\mathcal{E}_2^*(x_1,x_2,r,0)$$

$$\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2H_2^*)-\frac{\partial}{\partial x_2}(h_1H_1^*)\right]=s\varepsilon E_3^*-\varepsilon\mathcal{E}_3^*(x_1,x_2,r,0)\ .$$

$\mathcal{L}_{x_3}(\nabla\cdot E=0)$ goes into

$$\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2E_1^*)+\frac{\partial}{\partial x_2}(h_1E_2^*)\right]+\left[rE_3^*-E_3(x_1,x_2,0,s)\right]=0\ .$$

$\mathcal{L}_{x_3}(\nabla\cdot H=0)$ goes into

$$\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2H_1^*)+\frac{\partial}{\partial x_2}(h_1H_2^*)\right]+\left[rH_3^*-H_3(x_1,x_2,0,s)\right]=0$$

in which

$$\mathcal{E}_n^*(x_1,x_2,r,0)=\int_0^\infty\mathcal{E}_n(x_1,x_2,x_3,0)\,e^{-rx_3}dx_3\ ,$$

$$\mathcal{H}_n^*(x_1,x_2,r,0)=\int_0^\infty\mathcal{H}_n(x_1,x_2,x_3,0)\,e^{-rx_3}dx_3\ . \qquad (6)\text{I2}$$

I-2.3 From (6)I2, E_1^*, E_2^*, H_1^* and H_2^* can be expressed in terms of the partial space derivation of E_3^* and H_3^* and the group of initial conditions, as follows:

$$\left.\begin{aligned}
E_1^* &= \frac{1}{r^2 - s^2\mu\varepsilon}\left\{-s\mu\left[\frac{1}{h_2}\frac{\partial H_3^*}{\partial x_2} + H_2(x_1,x_2,0,s)\right] + r\left[\frac{1}{h_1}\frac{\partial E_3^*}{\partial x_1} + E_1(x_1,x_2,0,s)\right] + \mu\left[r\mathcal{H}_2^*(x_1,x_2,r,0) - s\varepsilon\mathcal{E}_1^*(x_1,x_2,r,0)\right]\right\} \\
E_2^* &= \frac{1}{r^2 - s^2\mu\varepsilon}\left\{s\mu\left[\frac{1}{h_1}\frac{\partial H_3^*}{\partial x_1} + H_1(x_1,x_2,0,s)\right] + r\left[\frac{1}{h_2}\frac{\partial E_3^*}{\partial x_2} + E_2(x_1,x_2,0,s)\right] - \mu\left[r\mathcal{H}_1^*(x_1,x_2,r,0) + s\varepsilon\mathcal{E}_2^*(x_1,x_2,r,0)\right]\right\} \\
H_1^* &= \frac{1}{r^2 - s^2\mu\varepsilon}\left\{s\varepsilon\left[\frac{1}{h_2}\frac{\partial E_3^*}{\partial x_2} + E_2(x_1,x_2,0,s)\right] + r\left[\frac{1}{h_1}\frac{\partial H_3^*}{\partial x_1} + H_1(x_1,x_2,0,s)\right] - \varepsilon\left[r\mathcal{E}_2^*(x_1,x_2,r,0) + s\mu\mathcal{H}_1^*(x_1,x_2,r,0)\right]\right\} \\
H_2^* &= \frac{1}{r^2 - s^2\mu\varepsilon}\left\{-s\varepsilon\left[\frac{1}{h_1}\frac{\partial E_3^*}{\partial x_1} + E_1(x_1,x_2,0,s)\right] + r\left[\frac{1}{h_2}\frac{\partial H_3^*}{\partial x_2} + H_2(x_1,x_2,0,s)\right] + \varepsilon\left[r\mathcal{E}_1^*(x_1,x_2,r,0) - s\mu\mathcal{H}_2^*(x_1,x_2,r,0)\right]\right\}
\end{aligned}\right\} \text{(7)I2}$$

Now, the next step is to determine the values of E_3^*, H_3^* and their partial space derivatives.

I-2.4 In this paragraph one will derive the differential equations for E_3^* and H_3^*. In the next section, I-3, we will find the solutions for these components and their partial derivatives.

To make a short cut the well-known theorem $\nabla\times\nabla\times F = \nabla(\nabla\cdot F) - \nabla^2 F$ will be used, in connection with equations (6)I1, (1)I1 and (2)I2. After some algebraic manipulation one gets

$$\left.\begin{aligned}
&\frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial E_3^*}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial E_3^*}{\partial x_2}\right)\right\} + (r^2 - s^2\mu\varepsilon)E_3^* = \\
&= \left\{rE_3(x_1,x_2,0,s)\left.\frac{\partial E_3(x_1,x_2,x_3,s)}{\partial x_3}\right)_{x_3=0}\right\} - \mu\varepsilon\left\{s\mathcal{E}_3^*(x_1,x_2,r,0) + \left.\frac{\partial \mathcal{E}_3^*(x_1,x_2,r,t)}{\partial t}\right)_{t=0}\right\} \\
&\frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial H_3^*}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial H_3^*}{\partial x_2}\right)\right\} + (r^2 - s^2\mu\varepsilon)H_3^* = \\
&= \left\{rH_3(x_1,x_2,0,s)\left.\frac{\partial H_3(x_1,x_2,x_3,s)}{\partial x_3}\right)_{x_3=0}\right\} - \mu\varepsilon\left\{s\mathcal{H}_3^*(x_1,x_2,r,0) + \left.\frac{\partial \mathcal{H}_3^*(x_1,x_2,r,t)}{\partial t}\right)_{t=0}\right\}.
\end{aligned}\right\} \text{(8)I2}$$

The brackets in the second members contain the initial conditions in space and time. These initial conditions are not all necessarily independent and may, or may not, exist, simultaneously. In Section 5, Ch. I, one will determine their interrelations necessary to insure that the final fields are electromagnetic (satisfy Maxwell's equations). In Section 8, Ch. I, a general study will be made with regard to these initial conditions and it will be shown how to obtain their respective transforms.

Section 3 - Elementary wave transforms and transverse boundary conditions.

I-3.0 In order to obtain elementary solutions of equations (8)I2, which satisfy the boundary conditions at the limiting walls we will write the electric and magnetic intensities as:

$$\left.\begin{aligned} \mathcal{E}_n &= \mathcal{V}_n(x_1,x_2)\mathcal{A}_n(x_3,t) \\ \mathcal{H}_n &= \Theta_n(x_1,x_2)\mathcal{B}_n(x_3,t) \end{aligned}\quad n=1,\ 2,\ 3 \right\} \qquad (1)\text{I3}$$

in which the quantities $\mathcal{V}_n$, $\mathcal{A}_n$, Θ_n and $\mathcal{B}_n$ are functions of the indicated variables only. These functions will be determined.

Let us take, in succession, their Laplace transforms

$$\left.\begin{aligned} E_n &= \mathcal{V}_n(x_1,x_2)A_n(x_3,s) \\ H_n &= \Theta_n(x_1,x_2)B_n(x_3,s) \end{aligned}\quad n=1,\ 2,\ 3 \right\} \qquad (2)\text{I3}$$

and

$$\left.\begin{aligned} E_n^* &= \mathcal{V}_n(x_1,x_2)A_n^*(r,s) \\ H_n^* &= \Theta_n(x_1,x_2)B_n^*(r,s)\ . \end{aligned}\right\} \qquad (3)\text{I3}$$

From (2)I3 we get

$$\left.\frac{\partial E_n}{\partial x_3}\right)_{x_3=0} = \psi_n \frac{\partial}{\partial x_3} A_n(x_3,s)\Big)_{x_3=0} = \psi_n A'_{(x_3)}(0,s)$$
$$\left.\frac{\partial H_n}{\partial x_3}\right)_{x_3=0} = \theta_n \frac{\partial}{\partial x_3} B_n(x_3,s)\Big)_{x_3=0} = \theta_n B'_{(x_3)}(0,s) \qquad (4)I3$$

Notice that $A'_{(x_3)}(0,s)$ and $B'_{(x_3)}(0,s)$ are functions of s alone.

Now, let us take the Laplace transform of (1)I3 directly with respect to x_3.

$$\mathcal{E}^*_n = \psi_n(x_1,x_2)\mathcal{A}^*_n(r,t)$$
$$\mathcal{H}^*_n = \theta_n(x_1,x_2)\mathcal{B}^*_n(r,t) \qquad (5)I3$$

from which

$$\left.\frac{\partial \mathcal{E}^*_n}{\partial t}\right)_{t=0} = \psi_n \frac{\partial}{\partial t}\mathcal{A}^*_n(r,t)\Big)_{t=0} = \psi_n \mathcal{A}'^*_{(t)}(r,0)$$
$$\left.\frac{\partial \mathcal{H}^*_n}{\partial t}\right)_{t=0} = \theta_n \frac{\partial}{\partial t}\mathcal{B}^*_n(r,t)\Big)_{t=0} = \theta_n \mathcal{B}'^*_{(t)}(r,0) \,. \qquad (6)I3$$

Notice that $\mathcal{A}'^*_{(t)}(r,0)$ and $\mathcal{B}'^*_{(t)}(r,0)$ are functions of r alone.

I-3.1 By using these relations and substituting them in (8)I2, we attain, after some arrangement of terms,

$$-p^2 = \frac{1}{\psi_3(x_1,x_2)}\nabla^2_{(x_1x_2)}\psi_3 = \frac{1}{A^*_3(r,s)}\Big\{-(r^2-s^2\mu\varepsilon)\,A^*_3(r,s) +$$
$$+\left[rA_3(0,s)+A'_{3(x_3)}(0,s)\right]-\mu\varepsilon\left[s\mathcal{A}^*_3(r,0)+\mathcal{A}'_{3(t)}(r,0)\right]\Big\} \qquad (7)I3$$

and a similar equation for the magnetic vector. p^2 is a separation constant and $\nabla^2_{(x_1,x_2)}\psi_3$ denotes the Laplacian with respect to the transverse coordinates.

After the proper separation of these equations we get, finally:

(8)I3

Electric vector intensity

$$\nabla^2_{(x_1,x_2)}\psi_3 + p^2\psi_3(x_1,x_2) = 0$$

$$A_3^*(r,s) = \frac{-\mu\varepsilon\left[s\mathcal{A}_3^*(r,s) + \mathcal{A}_{3(t)}^{\prime *}(r,0)\right] + \left[rA_3(0,s) + A'_{3(x_3)}(0,s)\right]}{r^2 - k^2}$$

$$k^2 = s^2\mu\varepsilon + p^2$$

Magnetic vector intensity

$$\nabla^2_{(x_1,x_2)}\theta_3 + p^2\theta_3(x_1,x_2) = 0$$

$$B_3^*(r,s) = \frac{-\mu\varepsilon\left[s\mathcal{B}_3^*(r,0) + \mathcal{B}_{3(t)}^{\prime *}(r,0)\right] + \left[rB_3^*(0,s) + B'_{3(x_3)}(0,s)\right]}{r^2 - k^2}$$

$$k^2 = s^2\mu\varepsilon + p^2$$

I-3.2 In order to obtain an elementary solution for the x_3 components of the field, in R space, we have to solve first (8)I3.

$A_3^*(r,s)$ and $B_3^*(r,s)$ are already solved in terms of initial conditions of the field for $x_3 = 0$ and $t = 0$. Notice that these components are functions only of the initial conditions pertaining to the longitudinal components. The initial conditions which appear in the equations for $A_3^*(r,s)$ and $B_3^*(r,s)$ are not those specified in the ordinary t space. Rather they are their s or r transforms. It is easy to derive the transformation of these initial conditions from the original data. In Section 8, Ch. I, these transformations will be reviewed again and will be illustrated for the way of computing them from the original data. For the moment it will be assumed that the transforms of the initial conditions appearing in the second members of $A_3^*(r,s)$ and $B_3^*(r,s)$ are known and therefore that the factors $A_3^*(r,s)$ and $B_3^*(r,s)$ are already determined in the R space.

I-3.3 The next step is to solve for ψ_3 and Θ_3 from the differential equations of the Poisson type as well as to fix the proper values for the separation constant p.

The explicit form of this differential equation depends on the cross-sectional geometry of the guide. Notice that $A_3^*(r,s)$ and $B_3^*(r,s)$ are independent of this cross-sectional form. The separation constant p must be such that the field satisfies the transverse boundary conditions at the walls.

In wave guides, primary interest centers in the axial propagation. If the dimension of the cross section is small compared with its length, it can be assumed that the transverse field is already formed. Under this assumption it is easy to find elementary solutions for ψ_3 or Θ_3. The equation of the Poisson type referred to above is the same as the one obtained in the study of the propagation in the permanent state in wave guides.

This equation has been solved already for some typical cross sections of wave guides. These computations are not going to be repeated since they have already been done. These solutions can be found in several books on wave guides. A special mention is made of the important work on this subject of Drs. Chu and Barrow. It will be assumed, therefore, that ψ_3, Θ_3 and p^2 are already obtained and consider them as known quantities. In Ch. IV a table of those functions is given for some typical cross sections.

I-3.4 The axial component of the electric and magnetic intensity vector, in the R space, is therefore given by

$$\left.\begin{aligned} E_3^*(x_1,x_2,r,s) &= \left\{\frac{\left[rA_3(0,s)+A'_{3(x_3)}(0,s)\right]}{r^2-k^2} - \frac{\mu\varepsilon\left[s\mathcal{A}_3^*(r,0)+\mathcal{A}'^*_{3(t)}(r,0)\right]}{r^2-k^2}\right\}\psi_3(x_1,x_2) \\ H_3^*(x_1,x_2,r,s) &= \left\{\frac{\left[rB_3(0,s)+B'_{3(x_3)}(0,s)\right]}{r^2-k^2} - \frac{\mu\varepsilon\left[s\mathcal{B}_3^*(r,0)+\mathcal{B}'^*_{3(t)}(r,0)\right]}{r^2-k^2}\right\}\Theta_3(x_1,x_2). \end{aligned}\right\} \quad (9)I3$$

The partial derivatives with respect to x_1 and x_2 are

$$\left.\begin{aligned} \left.\begin{matrix} \dfrac{\partial E_3^*}{\partial x_1} = \\ \dfrac{\partial E_3^*}{\partial x_2} = \end{matrix}\right\} &\left[\frac{rA_3(0,s)+A'_{3(x_3)}(0,s)}{r^2-k^2} - \frac{\mu\varepsilon\left[s\mathcal{A}_3^*(r,0)+\mathcal{A}'^*_{3(t)}(r,0)\right]}{r^2-k^2}\right]\left\{\begin{matrix} \dfrac{\partial \psi_3}{\partial x_1} \\ \dfrac{\partial \psi_3}{\partial x_2} \end{matrix}\right. \\ \left.\begin{matrix} \dfrac{\partial H_3^*}{\partial x_1} = \\ \dfrac{\partial H_3^*}{\partial x_2} = \end{matrix}\right\} &\left[\frac{rB_3(0,s)+B'_{3(x_3)}(0,s)}{r^2-k^2} - \frac{\mu\varepsilon\left[s\mathcal{B}_3^*(r,0)+\mathcal{B}'^*_{3(t)}(r,0)\right]}{r^2-k^2}\right]\left\{\begin{matrix} \dfrac{\partial \Theta_3}{\partial x_1} \\ \dfrac{\partial \Theta_3}{\partial x_2}. \end{matrix}\right. \end{aligned}\right\} \quad (10)I3$$

Introducing these partial derivatives in (7)I2 and using (9)I3 we obtain the field solution, in the R space, of the intensity vectors in terms of the initial spatial distribution at $t = 0$ and the initial time condition at $x_3 = 0$. This field propagates along x_3 axis. Note the Dirichlet's character of the solutions.

$$E_1^* = \left(\frac{1}{h_1}\frac{\partial\psi_3}{\partial x_1}\right) r M_3^*(r,s) - \left(\frac{1}{h_2}\frac{\partial\theta_3}{\partial x_2}\right) s\mu N_3^*(r,s) + E_1(0,s)\frac{r}{r^2-s^2\mu\varepsilon} - H_2(0,s)s\mu\frac{1}{r^2-s^2\mu\varepsilon} -$$

$$-\left(\frac{1}{h_1}\frac{\partial\psi_3}{\partial x_1}\right)\frac{1}{c^2} r\mathcal{M}_3^*(r,s) + \left(\frac{1}{h_2}\frac{\partial\theta_3}{\partial x_2}\right)\frac{s\mu}{c^2}\mathcal{N}_3^*(r,s) - \frac{s}{c^2}\frac{\mathcal{E}_1^*(x_1,x_2,r,0)}{r^2-s^2\mu\varepsilon} + \mu\frac{r\mathcal{H}_2^*(x_1,x_2,r,0)}{r^2-s^2\mu\varepsilon}$$

$$E_2^* = \left(\frac{1}{h_1}\frac{\partial\psi_3}{\partial x_1}\right) r M_3^*(r,s) + \left(\frac{1}{h_1}\frac{\partial\theta_3}{\partial x_1}\right) s\mu N_3^*(r,s) + E_2(0,s)\frac{r}{r^2-s^2\mu\varepsilon} + H_1(0,s)s\mu\frac{1}{r^2-s^2\mu\varepsilon} -$$

$$-\left(\frac{1}{h_2}\frac{\partial\psi_3}{\partial x_2}\right)\frac{1}{c^2} r\mathcal{M}_3^*(r,s) - \left(\frac{1}{h_1}\frac{\partial\theta_3}{\partial x_1}\right)\frac{1}{c^2}\mu s\mathcal{N}^*(r,s) - \frac{s}{c^2}\frac{\mathcal{E}_2^*(x_1,x_2,r,0)}{r^2-s^2\mu\varepsilon} - \mu\frac{r\mathcal{H}_1^*(x_1,x_2,r,0)}{r^2-s^2\mu\varepsilon}$$

$$E_3^* = \psi_3(r^2-s^2\mu\varepsilon)M_3^*(r,s) -$$

$$-\psi_3\frac{1}{c^2}(r^2-s^2\mu\varepsilon)\mathcal{M}_3^*(r,s)$$

$$H_1^* = \left(\frac{1}{h_1}\frac{\partial\theta_3}{\partial x_1}\right) r N_3^*(r,s) + \left(\frac{1}{h_2}\frac{\partial\psi_3}{\partial x_2}\right) s\varepsilon M_3^*(r,s) + H_1(0,s)\frac{r}{r^2-s^2\mu\varepsilon} + E_2(0,s)s\varepsilon\frac{r}{r^2-s^2\mu\varepsilon} -$$

$$-\left(\frac{1}{h_1}\frac{\partial\theta_3}{\partial x_1}\right)\frac{1}{c^2}\mathcal{N}_3^*(r,s) - \left(\frac{1}{h_2}\frac{\partial\psi_3}{\partial x_3}\right)\frac{1}{c^2}s\varepsilon\mathcal{M}_3^*(r,s) - \frac{s}{c^2}\frac{\mathcal{H}_1^*(x_1,x_2,r,0)}{r^2-s^2\mu\varepsilon} - \varepsilon\frac{r\mathcal{E}_2^*(x_1,x_2,r,0)}{r^2-s^2\mu\varepsilon}$$

$$H_2^* = \left(\frac{1}{h_2}\frac{\partial\theta_3}{\partial x_2}\right) r N_3^*(r,s) - \left(\frac{1}{h_1}\frac{\partial\psi_3}{\partial x_1}\right) s\varepsilon M_3^*(r,s) + H_2(0,s)\frac{r}{r^2-s^2\mu\varepsilon} - E_1(0,s)s\varepsilon\frac{r}{r^2-s^2\mu\varepsilon} - \qquad (11)\text{I } 3$$

$$-\left(\frac{1}{h_2}\frac{\partial\theta_3}{\partial x_2}\right)\frac{1}{c^2} r\mathcal{N}_3^*(r,s) + \left(\frac{1}{h_1}\frac{\partial\psi_3}{\partial x_1}\right)\frac{1}{c^2}s\varepsilon\mathcal{M}_3^*(r,s) - \frac{s}{c^2}\frac{\mathcal{H}_2^*(x_1,x_2,r,0)}{r^2-s^2\mu\varepsilon} + \varepsilon\frac{r\mathcal{E}_1^*(x_1,x_2,r,0)}{r^2-s^2\mu\varepsilon}$$

$$H_3^* = \theta_3(r^2-s^2\mu\varepsilon)N_3^*(r,s) -$$

$$-\theta_3\frac{1}{c^2}(r^2-s^2\mu\varepsilon)\mathcal{N}_3^*(r,s)$$

in which $c^2 = \frac{1}{\mu\varepsilon}$ speed of light in the medium.

$$M_3^*(r,s) = \frac{rA_3(0,s) + A'_{3(x_3)}(0,s)}{(r^2-k^2)(r^2-s^2\mu\varepsilon)}; \quad N_3^*(r,s) = \frac{rB_3(0,s) + B'_{3(x_3)}(0,s)}{(r^2-k^2)(r^2-s^2\mu\varepsilon)}$$

$$\mathcal{M}_3^*(r,s) = \frac{r\mathcal{A}_3^*(r,0) + \mathcal{A}'_{3(t)}(r,0)}{(r^2-k^2)(r^2-s^2\mu\varepsilon)}; \quad \mathcal{N}_3^*(r,s) = \frac{r\mathcal{B}_3^*(r,0) + \mathcal{B}'_{3(t)}(r,0)}{(r^2-k^2)(r^2-s^2\mu\varepsilon)}$$

(11)I3 represents a mathematical field. It is composed of the superposition of two general and independent electromagnetic fields corresponding respectively to initial conditions at $x_3 = 0$ and $t = 0$. The first line in the equation for each vector represents the field generated for excitation at $x_3 = 0$, while the second line represents the one generated for the initial spatial distribution at $t = 0$. To separate them it is only necessary to equate to zero the initial conditions indicated with Roman letters or the ones indicated with the script type. A further separation of fields will be made in Section 5 of this chapter.

Section 4 - First inversion of the field (11)I3 with respect to r. Initial conditions at $x_3 = 0$.

I-4.0 This section will deal with the inversion of the field from R space into S space. This can be accomplished by taking an inverse Laplace transformation, with respect to r, of each of the above components.

I-4.1 It can be observed in (11)I3 that the inversion of the terms indicated with script type letters, can not be performed unless the initial conditions at $t = 0$, $\mathcal{A}_3^*(x_1,x_2,r,0)$, $\mathcal{A}_{3(t)}'^*(x_1,x_2,r,0)$, $\mathcal{E}_1^*(x_1,x_2,r,0)$, $\mathcal{E}_2^*(x_1,x_2,r,0)$ $\mathcal{H}_1^*(x_1,x_2,r,0)$ and $\mathcal{H}_2^*(x_1,x_2,r,0)$ are expressed explicitly with respect to r. This means that one has to specify a definite initial spatial distribution. Once a spatial distribution is specified, it is possible to proceed with the inversion.

The situation is quite different with the initial conditions at $x_3 = 0$. The inversion with respect to r can be performed even if these conditions are not specified, since they are independent of r. Initial time conditions at x_3 constitute an important branch of problems of wave propagation in such systems. Therefore, this last problem will be dealt with in much of this work.

I-4.2 By simple inspection of (11)I3, and initial conditions at $x_3 = 0$, one can obtain the types of r transforms which must be inverted. They are:

$$R_1(r) = \frac{r^2}{(r^2-s^2\mu\varepsilon)(r^2-k^2)} \; ; \; R_4(r) = \frac{r}{r^2-a^2}$$

$$R_2(r) = \frac{r}{(r^2-s^2\mu\varepsilon)(r^2-k^2)} \; ; \; R_5(r) = \frac{1}{r^2-a^2}$$

$$R_3(r) = \frac{1}{(r^2-s^2\mu\varepsilon)(r^2-k^2)} \; ; \; a^2 = s^2\mu\varepsilon \text{ or } k^2 \; .$$

All these transforms behave as $1/r^{\gamma}$ when $r \to \infty$ with $\gamma \geqq 2$ except $R_4(r)$. This means that the inverse functions are zero at $x_3 = 0$ except R_4 which approaches a finite constant.

By the simple and well-known process of inversion, it is found that

$$\mathcal{L}_r^{-1} R_1(r) = \frac{1}{p^2}\left[k \sinh kx_3 - \frac{s}{c} \sinh s\frac{x_3}{c}\right]; \; \mathcal{L}_r^{-1} R_4(r) = \cosh ax_3$$

$$\mathcal{L}_r^{-1} R_2(r) = \frac{1}{p^2}\left[\cosh kx_3 - \cosh s\frac{x_3}{c}\right] \; ; \mathcal{L}_r^{-1} R_5(r) = \frac{1}{a} \sinh ax_3$$

$$\mathcal{L}_r^{-1} R_3(r) = \frac{1}{p^2}\left[\frac{1}{k} \sinh kx_3 - \frac{c}{s} \sinh s\frac{x_3}{c}\right];$$

and the final mathematical field, in the S domain, is given by

$$E_1=\frac{1}{p^2}\left(\frac{1}{h_1}\frac{\partial\psi_3}{\partial x_1}\right)\left\{A_3(0,s)\left[k\sinh kx_3-\frac{s}{c}\sinh\frac{sx_3}{c}\right]+A'_{3(x_3)}(0,s)\left[\cosh kx_3-\cosh\frac{sx_3}{c}\right]\right\}-$$

$$-\frac{\mu}{p^2}\left(\frac{1}{h_2}\frac{\partial\theta_3}{\partial x_2}\right)\left\{sB_3(0,s)\left[\cosh kx_3-\cosh\frac{sx_3}{c}\right]+B'_{3(x_3)}(0,s)\left[\frac{s}{k}\sinh kx_3-c\sinh\frac{sx_3}{c}\right]\right\}+$$

$$+E_1(0,s)\cosh\frac{sx_3}{c}-\sqrt{\frac{\mu}{\varepsilon}}\,H_2(0,s)\sinh\frac{sx_3}{c}$$

$$E_2=\frac{1}{p^2}\left(\frac{1}{h_2}\frac{\partial\psi_3}{\partial x_2}\right)\left\{A_3(0,s)\left[k\sinh kx_3-\frac{s}{c}\sinh\frac{sx_3}{c}\right]+A'_{3(x_3)}(0,s)\left[\cosh kx_3-\cosh\frac{sx_3}{c}\right]\right\}+$$

$$+\frac{\mu}{p^2}\left(\frac{1}{h_1}\frac{\partial\theta_3}{\partial x_1}\right)\left\{sB_3(0,s)\left[\cosh kx_3-\cosh\frac{sx_3}{c}\right]+B'_{3(x_3)}(0,s)\left[\frac{s}{k}\sinh kx_3-c\sinh\frac{sx_3}{c}\right]\right\}+$$

$$+E_2(0,s)\cosh\frac{sx_3}{c}+\sqrt{\frac{\mu}{\varepsilon}}\,H_1(0,s)\sinh\frac{sx_3}{c}$$

$$E_3=\psi_3\left\{A_3(0,s)\cosh kx_3+A'_{3(x_3)}(0,s)\frac{1}{k}\sinh kx_3\right\}\qquad (1)\,I4$$

$$H_1=\frac{1}{p^2}\left(\frac{1}{h_1}\frac{\partial\theta_3}{\partial x_1}\right)\left\{B_3(0,s)\left[k\sinh kx_3-\frac{s}{c}\sinh\frac{sx_3}{c}\right]+B'_{3(x_3)}(0,s)\left[\cosh kx_3-\cosh\frac{sx_3}{c}\right]\right\}+$$

$$+\frac{\varepsilon}{p^2}\left(\frac{1}{h_2}\frac{\partial\psi_3}{\partial x_2}\right)\left\{sA_3(0,s)\left[\cosh kx_3-\cosh\frac{sx_3}{c}\right]+A'_{3(x_3)}(0,s)\left[\frac{s}{k}\sinh kx_3-c\sinh\frac{sx_3}{c}\right]\right\}+$$

$$+H_1(0,s)\cosh\frac{sx_3}{c}+\sqrt{\frac{\varepsilon}{\mu}}\,E_2(0,s)\sinh\frac{sx_3}{c}$$

$$H_2=\frac{1}{p^2}\left(\frac{1}{h_2}\frac{\partial\theta_3}{\partial x_2}\right)\left\{B_3(0,s)\left[k\sinh kx_3-\frac{s}{c}\sinh\frac{sx_3}{c}\right]+B'_{3(x_3)}(0,s)\left[\cosh kx_3-\cosh\frac{sx_3}{c}\right]\right\}-$$

$$-\frac{\varepsilon}{p^2}\left(\frac{1}{h_1}\frac{\partial\psi_1}{\partial x_1}\right)\left\{sA_3(0,s)\left[\cosh kx_3-\cosh\frac{sx_3}{c}\right]+A'_{3(x_3)}(0,s)\left[\frac{s}{k}\sinh kx_3-c\sinh\frac{sx_3}{c}\right]\right\}+$$

$$+H_2(0,s)\cosh\frac{sx_3}{c}-\sqrt{\frac{\varepsilon}{\mu}}\,E_1(0,s)\sinh\frac{sx_3}{c}$$

$$H_3=\theta_3\left\{B_3(0,s)\cosh kx_3+B'_{3(x_3)}\frac{1}{k}\sinh kx_3\right\}.$$

Section 5 - Direct and reflected waves. Independent initial conditions at $x_3 = 0$. The electromagnetic fields TH, TE and TEH.

I-5.0 The mathematical field (1)I4 can be decomposed into two fields: the direct and reflected components. Each one of these fields can be obtained by using the well-known trigonometric relations.

$$\cosh kx_3 = \frac{e^{kx_3} + e^{-kx_3}}{2}$$

$$\sinh kx_3 = \frac{e^{kx_3} - e^{-kx_3}}{2}$$

$$\cosh \frac{sx_3}{c} = \frac{e^{\frac{x_3}{c}s} + e^{-\frac{x_3}{c}s}}{2}$$

$$\sinh \frac{sx_3}{c} = \frac{e^{\frac{x_3}{c}s} - e^{-\frac{x_3}{c}s}}{2}$$

in (1)I4. The direct field is composed of the terms which contain the exponential with negative sign, and the reflected field is composed of the terms with positive exponent. After some simple collection of terms, (1)I4 can be written as

$$\left.\begin{aligned} E_n &= E_{nd} + E_{nr} \\ H_n &= H_{nd} + H_{nr} \end{aligned}\right\} n = 1,\ 2,\ 3 \qquad (1)I5$$

in which the indices d and r mean direct and reflected components. The field (1)I4 is then formed by a linear superposition of the direct and the reflected components.

I-5.1 The direct and reflected fields may have independent existence. In fact

$$\left.\begin{aligned} \mathcal{L}(x_3)\, E_{nd} &= E^*_{nd} \\ \mathcal{L}(x_3)\, H_{nd} &= H^*_{nd} \end{aligned}\right\} n = 1,\ 2,\ 3$$

or

$$\left.\begin{aligned}\mathcal{L}_{(x_3)}E_{nr} &= E^*_{nr}\\ \mathcal{L}_{(x_3)}H_{nr} &= H^*_{nr}\end{aligned}\right\} n = 1, 2, 3$$

satisfy independently Maxwell's equations in R. In paragraphs I-5.2 and I-5.4 the conditions will be investigated under which

$$\left.\begin{aligned}&E_{nd}\\ &H_{nd}\end{aligned}\right\} n = 1, 2, 3$$

and

$$\left.\begin{aligned}&E_{nr}\\ &H_{nr}\end{aligned}\right\} n = 1, 2, 3$$

satisfy Maxwell's equations in S as independent fields.

I-5.2 The terms "direct" and "reflected" are inappropriate in the S domain. They are, nevertheless, used because when the fields, by inversion, are carried over into the time domain, they represent waves moving in the positive and negative directions respectively of the x_3 axis. This character is distinguished in the S domain by the presence of the exponential function in the transforms.

If, by nature of the problem, the actual propagation takes place in the positive or negative direction of the longitudinal axis, then only the direct or the reflected field respectively is taken. In some cases the presence of these two fields is required; for example: first, an infinite wave guide is excited in its middle cross section and the energy flows in two directions along the x_3 axis; second, two different sources may exist in the guide or there may be discontinuities in the cross section which produce internal reflections, etc.

I-5.3 A simple inspection of (1)I4 reveals two types of exponents of e: $\frac{x_3}{c}$ s and $x_3k = \frac{x_3}{c}\sqrt{s^2+p^2c^2}$.

The terms associated with the first exponent propagate wholly with the speed c. For those terms associated with the second exponent the propagation occurs in a dispersive media; although the perturbation moves with the speed c this does not mean that the main bulk of energy moves with the same speed. For these terms the concepts of group and signal velocities and time of signal formation will be introduced. If Eqs. (1)I4 are looked upon with physical eyes, they will give the strange appearance that two types of propagation exist simultaneously for the same initial condition. This circumstance must not be interpreted to the effect that there is something wrong with these solutions. They are, in general, correct solutions from the mathematical point of view. The fact is that they are not necessarily electromagnetic solutions.

The next paragraph is devoted to investigating the conditions under which these solutions are electromagnetic ones.

I-5.4 The vectors E and H, in the S space, are said to be electromagnetic ones if they satisfy the set of s-transformed Maxwell's equations.

If we keep all the initial functions in the S domain arbitrary and independent from each other, then by a simple substitution of (1)I4 into Maxwell's equations it is revealed that they are not electromagnetic vectors.

The conditions under which (1)I4 are electromagnetic fields are attained when Maxwell's equations are identically satisfied.

The equations

$\nabla \times \mathbf{E} + s\mu \mathrm{H} = 0$ and

$\nabla \times \mathrm{H} - s\epsilon \mathbf{E} = 0$

are satisfied identically by the vectors **E** and H(1)I4, for all arbitrary and independent values of the initial conditions; but the relations

$\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathrm{H} = 0$

are not satisfied unless the initial conditions are properly related.

The process of substitution is long and requires tiresome algebraic manipulations, but otherwise is simple. It is omitted here in order to keep the presentation compact.

In order to make a systematic discussion of this situation, three typical cases will be considered.

a. Only A_3 and A_3' are given as independent initial conditions and $B_3 = B_3' = 0$.

From (1)I4 it can be seen immediately that the corresponding field has a TH character. Under the above assumptions this field is an electromagnetic one if

$$\left.\begin{aligned} H_1(0,s) &= +\left(\frac{1}{h_2}\frac{\partial\psi_3}{\partial x_2}\right) \\ H_2(0,s) &= -\left(\frac{1}{h_1}\frac{\partial\psi_3}{\partial x_1}\right) \end{aligned}\right\} \frac{1}{p^2 c}\, sA_3(0,s)$$
$$\left.\begin{aligned} E_1(0,s) &= +\left(\frac{1}{h_1}\frac{\partial\psi_3}{\partial x_1}\right) \\ E_2(0,s) &= +\left(\frac{1}{h_2}\frac{\partial\psi_3}{\partial x_2}\right) \end{aligned}\right\} \frac{1}{p^2}\, A'_{3(x_3)}(0,s) \qquad (2)I5$$

It is clear that case a can be divided into two sub-cases, a_1 and a_2, defined as

a_1. $A_3(0,s) \neq 0$ and $A'_{3(x_3)}(0,s) = 0$

a_2. $A_3(0,s) = 0$ and $A'_{3(x_3)}(0,s) \neq 0$.

(Notice that it is equivalent to specify $A'_{3(x_3)}$ or a transverse component of E.)

b. Only $B_3(0,s)$ and $B'_{3(x_3)}(0,s)$ are given independent initial conditions and $A_3(0,s) = A'_{3(x_3)}(0,s) = 0$.

From (1)I4 it can be seen that the corresponding field has a TE character. Under the above assumptions this field is an electromagnetic one if

$$\left.\begin{aligned} E_1(0,s) &= -\left(\frac{1}{h_2}\frac{\partial\theta_3}{\partial x_2}\right) \\ E_2(0,s) &= +\left(\frac{1}{h_1}\frac{\partial\theta_3}{\partial x_1}\right) \end{aligned}\right\} \frac{1}{p^2 c}\, sB_3(0,s)$$
$$\left.\begin{aligned} H_1(0,s) &= +\left(\frac{1}{h_1}\frac{\partial\theta_3}{\partial x_1}\right) \\ H_2(0,s) &= +\left(\frac{1}{h_2}\frac{\partial\theta_3}{\partial x_2}\right) \end{aligned}\right\} \frac{1}{p^2}\, B'_3(0,s) \qquad (3)I5$$

It is clear that case b can be divided into two subcases, b_1 and b_2, defined as

b_1. $B_3(0,s) \neq 0$ and $B'_{3(x_3)}(0,s) = 0$

b_2. $B_3(0,s) = 0$ and $B'_{3(x_3)}(0,s) \neq 0$.

(Notice that it is equivalent to specify $B'_{3(x_3)}$ or a transverse component of H.)

c. This case corresponds to the assumption: $A_3(0,s) = A'_{3(x_3)}(0,s) = B_3(0,s) = B'_{3(x_3)}(0,s) = 0$ but the transverse initial conditions are not necessarily zero. The resultant field has a TEH character. To be electromagnetic the requirements are

$$\left.\begin{aligned} &\frac{\partial}{\partial x_1}(h_2 E_1(0,s) + \frac{\partial}{\partial x_2}(h_1 E_2(0,s) = 0 \text{ or} \\ &\frac{\partial}{\partial x_1}(h_2 H_1(0,s) + \frac{\partial}{\partial x_2}(h_2 H_2(0,s) = 0 \ . \end{aligned}\right\} \quad (4)\text{I5}$$

This case c can be divided in two subcases, c_1 and c_2, defined as

c_1. $E_1(0,s)$ or $E_2(0,s)$ and $H_1(0,s) = H_2(0,s) = 0$

c_2. $H_1(0,s)$ or $H_2(0,s)$ and $E_1(0,s) = E_2(0,s) = 0$.

Case c_1 will be called "electric initial excitation".
Case c_2 will be called "magnetic initial excitation".

I-5.5 By inserting (2)I5, (3)I5 and (4)I5 in (1)I4, this mathematical field breaks up into three independent electromagnetic fields which correspond to TH, TE and TEH waves. It can be noticed that these fields were obtained directly without the introduction of three different potentials.

A summarization of this result is given in Table I for the direct waves. The proper solutions for the reflected waves can easily be obtained.

ELEMENTARY ELECTROMAGNETIC WAVES IN THE GENERALIZED CYLINDER.-S SPACE

CONSERVATIVE SYSTEM — **PROPAGATION ALONG(+) DIRECTION OF x_3 AXIS**

DISPERSIVE PROPAGATION

a.– T.H. WAVES

INITIAL COND.	FIELD VECTORS	BOUND COND.
$E_3)_{x_3=0} = \psi_3 A_3(0,s)$ OR $A_3(0,s)$	$\left.\begin{array}{l} E_1 = -\left(\frac{1}{h_1}\frac{\partial \psi_3}{\partial x_1}\right) \\ E_2 = -\left(\frac{1}{h_2}\frac{\partial \psi_3}{\partial x_2}\right)\end{array}\right\} \frac{1}{p^2 c} A_3(0,s)\sqrt{s^2+\omega_c^2}\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}$ $E_3 = +\psi_3 A_3(0,s) e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}$ $\left.\begin{array}{l} H_1 = +\left(\frac{1}{h_2}\frac{\partial \psi_3}{\partial x_2}\right) \\ H_2 = -\left(\frac{1}{h_1}\frac{\partial \psi_3}{\partial x_1}\right)\end{array}\right\} \frac{\epsilon s}{p^2} A_3(0,s) e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}$ $H_3 = 0$	$\nabla^2_{(x_1,x_2)} \psi_3(x_1,x_2) + p^2 \psi_3(x_1,x_2) = 0$
$E_3')_{x_3=0} = \psi_3 A_3'(0,s)$ OR $A_3'(0,s)$	$\left.\begin{array}{l} E_1 = +\left(\frac{1}{h_1}\frac{\partial \psi_3}{\partial x_1}\right) \\ E_2 = +\frac{1}{h_2}\frac{\partial \psi_3}{\partial x_2}\end{array}\right\} \frac{1}{p^2} A_3'(0,s) e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}$ $E_3 = -\psi_3 A_3'(0,s) \frac{c\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{\sqrt{s^2+\omega_c^2}}$ $\left.\begin{array}{l} H_1 = -\left(\frac{1}{h_2}\frac{\partial \psi_3}{\partial x_2}\right) \\ H_2 = +\left(\frac{1}{h_1}\frac{\partial \psi_1}{\partial x_1}\right)\end{array}\right\} \frac{\epsilon s c\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{p^2\sqrt{s^2+\omega_c^2}} A_3'(0,s)$ $H_3 = 0$	

b.–T.E. WAVES

INITIAL COND.	FIELD VECTORS	BOUND COND.
$H_3)_{x_3=0} = \theta_3 B_3(0,s)$ OR $B_3(0,s)$	$\left.\begin{array}{l} E_1 = -\left(\frac{1}{h_2}\frac{\partial \theta_3}{\partial x_2}\right) \\ E_2 = +\left(\frac{1}{h_1}\frac{\partial \theta_3}{\partial x_1}\right)\end{array}\right\} \frac{\mu s B_3(0,s)}{p^2} e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}$ $E_3 = 0$ $\left.\begin{array}{l} H_1 = -\left(\frac{1}{h_1}\frac{\partial \theta_3}{\partial x_1}\right) \\ H_2 = -\left(\frac{1}{h_2}\frac{\partial \theta_3}{\partial x_2}\right)\end{array}\right\} \frac{\sqrt{s^2+\omega_c^2}\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{c p^2} B_3(0,s)$ $H_3 = +\theta_3 B_3(0,s) e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}$	$\nabla^2_{(x_1,x_2)} \theta_3(x_1,x_2) + p^2 \theta_3(x_1,x_2) = 0$
$H_3')_{x_3=0} = \theta_3 B_3'(0,s)$ OR $B_3'(0,s)$	$\left.\begin{array}{l} E_1 = +\left(\frac{1}{h_2}\frac{\partial \theta_3}{\partial x_2}\right) \\ E_2 = -\left(\frac{1}{h_1}\frac{\partial \theta_3}{\partial x_1}\right)\end{array}\right\} \frac{\mu s c\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{p^2\sqrt{s^2+\omega_c^2}} B_3'(0,s)$ $E_3 = 0$ $\left.\begin{array}{l} H_1 = +\left(\frac{1}{h_1}\frac{\partial \theta_1}{\partial x_1}\right) \\ H_2 = +\left(\frac{1}{h_2}\frac{\partial \theta_2}{\partial x_2}\right)\end{array}\right\} \frac{1}{p^2} B_3'(0,s) e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}$ $H_3 = -\theta_3 B_3'(0,s) \frac{c\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{\sqrt{s^2+\omega_c^2}}$	

c.– T.E.H. WAVES.

UNDISTORTED PROPAGATION

(DO NOT EXIST IN WAVE GUIDE)

c_1 ELECTRIC EXCITATION

INITIAL COND.	FIELD VECTORS	BOUND COND.
$E_1(0,s)$ OR $E_2(0,s)$	$E_1 = E_1(0,s) e^{-\frac{x_3}{c}s}$ $E_2 = E_2(0,s) e^{-\frac{x_3}{c}s}$ $H_1 = -\sqrt{\frac{\epsilon}{\mu}} E_2(0,s) e^{-\frac{x_3}{c}s}$ $H_2 = +\sqrt{\frac{\epsilon}{\mu}} E_1(0,s) e^{-\frac{x_3}{c}s}$	$\frac{\partial}{\partial x_1}[h_2 E_1(0,s)] + \frac{\partial}{\partial x_2}[h_1 E_2(0,s)] = 0$

c_2 MAGNETIC EXCITATION

INITIAL COND.	FIELD VECTORS	BOUND COND.
$H_1(0,S)$ OR $H_2(0,S)$	$E_1 = +\sqrt{\frac{\mu}{\epsilon}} H_2(0,s) e^{-\frac{x_3}{c}s}$ $E_2 = -\sqrt{\frac{\mu}{\epsilon}} H_1(0,s) e^{-\frac{x_3}{c}s}$ $H_1 = H_1(0,s) e^{-\frac{x_3}{c}s}$ $H_2 = H_2(0,s) e^{-\frac{x_3}{c}s}$	$\frac{\partial}{\partial x_1}[h_2 H_1(0,s)] + \frac{\partial}{\partial x_2}[h_1 H_2(0,s)] = 0$

TABLE N° 11.

TE and TH fields exist in configurations equivalent to wave guides. TEH fields exist in configurations equivalent to transmission lines and coaxial cables. It is presumed, but not proved, that TE and TH modes exist in systems whose cross section is topologically equivalent to wave guides, and TEH modes in those topologically equivalent to transmission lines and coaxial cables.

Section 6 - The TEH field and its inversion into the instantaneous time domain.

I-6.0 The TEH fields can be transformed back into the time domain in a simple manner by using the well-known theorem of inversion.

If f(t) exists as the corresponding inverse transform of F(s) then

$$\mathcal{L}^{-1}_{(t)} F(s) e^{-\alpha s} = \begin{cases} 0 & \text{for } t < \alpha \\ f(t-\alpha) & \text{for } t > \alpha. \end{cases}$$

In this case $\alpha = \frac{x_3}{c} = t_0$. This means that if f(t) is the initial time function applied to the system at $x_3 = 0$, then this perturbation reaches a cross section at a distance $x_3 = a$ after an interval of time equal to a/c. The functions f(t) and $f(t-\alpha)$ have the same form but they are only shifted in time.

Applying the above principle to the TEH field, Table I, the undistorted character of its propagation can be seen immediately. This problem is simple and the results are already known. Therefore, no more attention will be paid to these TEH fields.

Section 7 - Dispersive character of the TE and TH fields. The problem of inversion. Basic analytical links of the corresponding transforms.

I-7.0 The inversion of the TE and TH fields constitutes a difficult mathematical problem. It is difficult because of the integrals that must be handled and because of its mathematical instrumentation which is delicate and involved.

The irrational form of the exponent of e indicates that these cylinders act as a dispersive medium. The waves, during their propagation, suffer deformation of their amplitudes and changes in their frequencies. Natural modes of propagation and cut-off frequencies exist. Although the wave precursor moves with the speed of light in the medium, this velocity tells nothing for itself. It is necessary to introduce new concepts in velocity, mainly group and signal velocities, to have a correct quantitative idea of how the propagation occurs. In this investigation these velocity concepts are carefully studied to see if they make proper sense in wave guides and mainly to discover how these concepts are influenced by the form of the incoming waves. This last aspect is quite delicate and not very well known.

I-7.1 Even in the most simple case of excitation the problem of inversion is very hard to perform. The difficulties are of a mathematical character. To solve a particular problem or to integrate the corresponding expressions for one field component is not a satisfactory and practical solution. A mathematical method, rather a simple one, has to be found such that a fair number of practical problems can be solved with it.

Most of this work is devoted to obtaining this method. In Chs. II, III and IV the problem of inversion is attacked in its different aspects.

Chapter II : General analytic study and the integration in the s plane.

Chapter III: Introduction of complex transformations and integration of fundamental transforms.

Chapter IV : Asymptotic and graphical solutions.

I-7.2 A systematic method of attack will always be followed. The first natural step is to find the general analytic relations between the transforms which appear in Table I. This procedure enables one to find typical transforms which generate the others. In this way the possible number of inversions will be reduced. In this section one will study these relations, which are independent of the type of initial conditions. Only the parts of these transforms which are functions of s will be referred to and the geometric factor or other constants will be omitted. For strategic reasons only will a prototype such as the following transform be used

$$F_o(s) = F(s)\ \frac{e^{-\alpha k}}{k} \qquad (1)I7$$

in which $k = \sqrt{s^2+\omega_c^2}$; $\omega_c^2 = p^2c^2$; F(s) is a function of s which can replace any one of the initial conditions in

Table I. A rapid inspection of the table reveals that the following type of transforms are available.

$$\left.\begin{aligned}
F_0(s) &= F(s)\,\frac{e^{-\alpha k}}{k} \\
F_1(s) &= sF(s)\,\frac{e^{-\alpha k}}{k} = sF_0(s) \\
F_2(s) &= F(s)e^{-\alpha k} = -\frac{\partial}{\partial\alpha}\,F_0(s) \\
F_3(s) &= sF(s)e^{-\alpha k} = -s\frac{\partial}{\partial\alpha}\,F_0(s) \\
F_4(s) &= F(s)ke^{-\alpha k} = +\frac{\partial^2}{\partial\alpha^2}\,F_0(s)\;.
\end{aligned}\right\} \qquad (2)I7$$

The meaning of (2)I7 is as follows. If one can find the inverse transform of $F_0(s)$ then, the corresponding inverse function of $F_1(s)$, $F_2(s)$, $F_3(s)$ and $F_4(s)$ can be obtained by a simple process of differentiation in the time domain. The above statement, although nice and simple, can not be utilized in the case when $\mathcal{L}^{-1} F_0(s)$ comes out as an asymptotic series expansion in the Poincaré sense, since the term by term differentiation is not then permitted.

The equations (2)I7 represent the analytic relations between the s transforms of the different components of the field vectors. The last statement indicates that we cannot consider the solution of only one component of the field, since sometimes the other cannot be obtained by differentiation. Most of the time the inverse transform comes out in the form of an asymptotic expansion.

Section 8 - The initial conditions expressed in the s domain. General type of s transforms to be handled.

I-8.0 In the preceding section, Section 7, the prototype transforms (1)I7 were obtained. Nothing has been said so far about F(s), which plays the role of an initial condition in the s domain. In this section, what may be the analytic structure of F(s), for a fairly large number of practical initial conditions will be investigated.

I-8.1 Some simple examples of initial conditions will clarify this situation.

1. Suppose that one specifies only the axial component of the electric vector as

$$\mathcal{E}_3\Big)_{x_3=0} = \begin{cases} 0 & t<0 \\ \mathcal{V}_3(1-\cos\omega_o t) & t>0 \end{cases} \qquad \underline{\text{TH field}}$$

in which $\mathcal{V}_3$ is a function only of the transverse coordinates x_1 and x_2 and is such that it satisfies the boundary conditions.

Then:

$$F(s) = A_3(0,s) = \int_0^\infty (1-\cos\omega_o t)e^{-st}dt = \frac{\omega_o^2}{s(s^2+\omega_o^2)} \ .$$

2. Suppose now that $\frac{\partial \mathcal{H}_3}{\partial x_3}\Big)_{x_3=0}$ is given as

$$\frac{\partial \mathcal{H}_3}{\partial x_3}\Big)_{x_3=0} = \begin{cases} 0 & t<0 \\ \Theta_3(x_1,x_2)\sin\omega_o t & t>0 \end{cases} \qquad \underline{\text{TE waves}}$$

Then:

$$F(s) = B_3'(0,s) = \int_0^\infty \sin\omega_o t\ e^{-st}dt = \frac{\omega_o}{s^2+\omega_o^2} \ .$$

3. Suppose here that

$$\frac{\partial \mathcal{E}_3}{\partial x_3}\Big)_{x_3=0} = \begin{cases} 0 & t<0 \\ \mathcal{V}_3 \sin\lambda t \sin\beta t & t>0 . \end{cases} \qquad \underline{\text{TH waves}}$$

Then:

$$F(s) = A_3'(0,s) = \frac{2\lambda\beta s}{[s^2+(\beta+\lambda)^2][s^2+(\beta-\lambda)^2]} \ .$$

This elementary example illustrates the procedure necessary to obtain the corresponding transforms of the initial time conditions.

I-8.2 If the initial time condition is given by an amplitude-modulated wave, whose time variation elements are given by the general form

$$f(t) = m(t) \sin \omega_0 t ,$$

let

$$\mathcal{L} f(t) = F(s) \text{ and}$$

$$\mathcal{L} m(t) = M(s) \text{ is known,}$$

then:

$$F(s) = \frac{1}{2i}\left[M(s - i\omega_0) - M(s + i\omega_0)\right] \text{ in which } i = \sqrt{-1}.$$

See "Transients in Linear System", Gardner and Barnes, Vol. I, P. 248.

The modulating function m(t) can have a large variety of forms. By a proper combination of simple exponential functions a large variety of modulating functions m(t) can be obtained. A simple discussion of this subject is found in the book "Traveling Waves on Transmission Systems", by Loyal Vivian Bewley, J. Wiley, 1933, Ch. I, P. 16. The use of exponentials in the construction of envelopes means that M(s) will be formed as the ratio of two polynomials in s.

I-8.3 For frequency-modulated forms of excitation, the corresponding time function has the form

$$f(t) = \mathcal{X}_0 \cos (\omega_0 t + m \sin at)$$

$$= \sum_{0}^{\infty} J_n(m) \left[\cos(\omega_0 + na)t + (-1)^n \cos (\omega_0 - na) t\right],$$

$$m = \text{constant.}$$

For practical purposes it is only necessary to use a finite number of terms. The Laplace transformation of f(t), for m constant, will be again the sum of rational fractions of s.

I-8.4 If it is considered that F(s) has the form, in general of the ratio of two polynomials in s, the number of possible wave forms of excitation becomes unlimited. By a proper combination of these forms still a larger number of new forms of initial time functions can be obtained. See, for example, Gardner and Barnes, P. 338 and following pages or any other table of Fourier transforms.

I-8.5 Sometimes the time parts of an initial condition are expressed as the product of two factors as in I-8.2. Let this initial time function be expressed as

$$f(t) = f_1(t) f_2(t)$$

and assume that the Laplace transform of each factor is already known. The theorem of complex convolution allows us to compute the transform of f(t) as

$$\mathcal{L}_{(t)}\left[f(t)\right] = \mathcal{L}_{(t)}\left[f_1(t) \times f_2(t)\right] = F(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_1(s-\omega) F_2(\omega)\, d\omega \,.$$

in which

$$\mathcal{L}_{(t)} f_1(t) = F_1(s) \text{ and}$$

$$\mathcal{L}_{(t)} f_2(t) = F_2(s) \,.$$

The use of some theorems allows the computing F(s) without performing the above integral in some simple, but important cases.

The next two theorems were taken from the book "Transients in Linear Systems", Gardner and Barnes, Vol. I, Pgs. 275 - 280.

Theorem I - If $f_1(t)$ and $f_2(t)$ are $\mathcal{L}$-transformable functions having the $\mathcal{L}$ transforms $F_1(s)$ and $F_2(s)$, respectively, and if $F_1(s) = \frac{A_1(s)}{B_2(s)}$ is a rational algebraic function having q first-order poles and no other, then

$$F(s) = \sum_{k=1}^{k=q} \frac{A_1(s_k)}{B_1'(s_k)} F_2(s - s_k) \; .$$

Theorem II - Let $f_1(t)$ and $f_2(t)$ be $\mathcal{L}$-transformable functions having the $\mathcal{L}$ transforms $F_1(s)$ and $F_2(s)$, respectively, and let $F_1(s)$ be a rational algebraic function having n distinct poles $s_1, s_2, \cdots s_n$ with

s_1 of multiplicity m_1

s_2 of multiplicity m_2

---

s_n of multiplicity m_n .

Subject to the restriction $m_1 + m_2 + \cdots + m_n = q$; then

$$F(s) = \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{(-1)^{m_k - j} K_{kj}}{(m_k - j)!} \left[\frac{d^{m_k - j}}{ds^{m_k - j}} F_2(s) \right]_{s = s - s_k}$$

in which

$$K_{kj} = \frac{1}{(j-1)!} \left[\frac{d^{j-1}}{ds^{j-1}} (s - s_k)^{m_k} F_1(s) \right]_{s = s_k} \; .$$

These theorems allow us to investigate further the analytical structure of F(s) for cases of practical interest. If the original time function $f_2(t)$ is formed by a linear combination of terms of the form $Kt^n e^{-\alpha t}$, then for different values of k, n and α then it is clear that $F_2(s)$ is a linear combination of expression $\frac{n!k}{(s+\alpha)^{n+1}}$ and therefore F(s) is again the ratio of two integer polynomials. If $F_2(s)$ is a meromorphic function, it can be expanded, if possible, by means of the theorem of Mittag-Leffler and the result is that F(s) will be expressed as a series of terms which are rational algebraic functions.

I-8.6 In this paragraph we will consider a more complicated case of the initial time function f(t) when expressed in terms of Bessel's functions of the first kind or in terms of series of the Neumann type. This type of excitation may occur, when the output of a wave guide excites a second one. The transforms of these functions are expressed in terms of $\sqrt{s^2+\alpha^2}$ as can be seen from a table of Laplace transforms. For example see Gardner and Barnes, P. 352.

I-8.7 An important case of wave guide excitation occurs when the initial time condition has the form of pulses. In this case in F(s) factors of the type $e^{-\vartheta s}$ will appear. The presence of such factors does not produce new types of F(s) but indicates a shift in the time domain.

Thus let

$$f(t)=\mathcal{L}_t^{-1}F(s,\sqrt{s^2+1})e^{-k\sqrt{s^2+1}} \text{ for } t>\alpha .$$

Then it is well known that

$$\mathcal{L}_t^{+1}F(s,\sqrt{s^2+1})e^{-\nu s}e^{-k\sqrt{s^2+1}}=f(t-\nu) \qquad t-\nu>\alpha .$$

I-8.7 As a summary of Section I-8 we can say that for a rather unlimited type of wave forms of guide excitation at $x_3=0$, one can consider that

F(s) has the form of the ratio of two polynomials in s and $\sqrt{s^2+1}$

and so $F(s,\sqrt{s^2+1})e^{-k\sqrt{s^2+1}}$ shall be written as our typical transform. This statement should not be interpreted to mean that one has proved that <u>all</u> possible cases of excitation must have the above structure. Rather it may be said that a practically unlimited variety of forms of excitation are contained in that transform structure. Of course, time functions whose transforms do not have this specific structure can be found, but these cases are unusual ones and perhaps of no practical interest.

Section 9 - Further analytical restrictions on $F(s,\sqrt{s^2+1})$ to assure electromagnetic solutions in the t domain.

I-9.0 In Section 5 of this chapter it was found that when the field corresponding to the vectors in the R domain were transformed back into the S domain, the transformed vectors do not necessarily satisfy Maxwell's equations in the S domain.. Only after some constraints were imposed on the initial conditions (Eqs. (2)I5, (3)I5 and (4)I5) was the resultant field electromagnetic in the S domain.

We now face a similar problem. The electromagnetic field in the S domain must be transformed into the t domain and we will investigate when this transformed field is electromagnetic in t in this section.

We will prove that the transformed field is, in general, <u>not electromagnetic</u>. It will be only if some restrictions are introduced in the vector components at $t = +x_3/c$.

I-9.1 This investigation will be based on two fundamental theorems on inverse transforms. Although they are well known, they will be repeated here.

<u>Theorem A</u>: Let f(t,k) be a function of the two independent variables t and k. It will be assumed that

a. - f(t,k) is at least of class C_1.

b. -
$$f(t,k) = \begin{cases} =0 & t < k \\ \neq 0 & t > k \end{cases}$$

c. - The function and its time derivative is $\mathcal{L}_{(t)}$ transformable having respectively

$$F(s) = \int_0^\infty f(t,k)\, e^{-st} dt$$
$$G(s) = \int_0^\infty \frac{\partial}{\partial t}\left[f(t,k)\right] e^{-st} dt$$

as transforms.

We will prove that

$$f'_{(t)}(t,k) = f(+k,k)\,u_0(t-k) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} sF(s,k)\, e^{st} ds .$$

To prove the theorem we have

<u>1st</u> From b and c we can write

$$F(s) = \int_k^\infty f(t,k)\, e^{-st} dt \quad \text{and}$$

$$G(s) = \int_k^\infty f'_{(t)}(t,k)\, e^{-st} dt \quad .$$

2nd Integrating by parts one obtains

$$G(s)=\mathcal{L}_t f'_{(t)}(t,k)=f(t,k)e^{-st}\Big]_k^{\infty}+s\int_k^{\infty} e^{-st}f(t,k)\,dt$$

$$=-e^{-ks}f(+k)+sF(s)\ .$$

3rd By hypothesis

$$\mathcal{L}^{-1}_{(t)}G(s) = f'_{(t)}(t,k)$$

and therefore, by the introduction of the singular unit impulse function μ

$$\mu_0(t)=\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{st}ds$$

we get

$$f'_{(t)}(t)=-\frac{1}{2\pi i}\ f(+k)\int_{c_0-i\infty}^{c_0+i\infty} e^{-ks}e^{st}ds+\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} sF(s)e^{st}ds$$

or

$$f'_{(t)}(t)=-f(+k)\mu_0(t-k)+\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} sF(s)e^{st}ds$$

and the theorem is proved.

Theorem B: Let $f(t,k)$ be a function of the two independent variables t and k.

By hypothesis we will assume that

a. - $f(t,k)$ is at least of class C_1

b. -
$$f(t,k)=\begin{cases}=0 & \text{for } t<k\\ \neq 0 & \text{for } t>k\ .\end{cases}$$

c.- This function and its derivative with respect to k is $\mathcal{L}_{(t)}$ transformable having respectively

$$F(s,k)=\int_0^{\infty} f(t,k)e^{-st}dt$$

$$h(s,k)=\int_0^{\infty} f'_{(k)}(t,k)e^{-st}dt$$

as transforms.

Then:

$$f'_{(k)}(t,k) = f(+k,k)\mu_0(t-k)+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F'_{(k)}(s,k)e^{st}ds\ .$$

To prove the theorem we have:

<u>1st</u> $F(s,k)=\int_k^{\infty} f(t,k)\, e^{-st}dt$

$$H(s,k)=\int_k^{\infty} f'_{(k)}(t,k)\, e^{-st}dt \; .$$

<u>2nd</u>

$$F'_{(k)}(s,k)=\frac{\partial}{\partial k}\int_k^{\infty} f(t,k)\, e^{-st}dt=-e^{-sk}f(+k)+\int_0^{\infty} e^{-st} f'_{(k)}(t,k)dt=$$

$$=-e^{sk}f(tk)+\mathcal{L}_{(t)} f'_k(t,k)\,dt \; .$$

<u>3rd</u> From the last equation we get

$$f'_k(t,k)=+\frac{f(+k)}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{-sk}e^{st}ds+\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} F'_{(k)}(s,k)\,e^{st}ds$$

so that finally

$$f'_k(t,k)=f(t,k)_{t=+k}\,\mathcal{U}_0(t-k)+\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} F'_k(s,k)\,e^{st}ds \qquad (2)I9$$

and the theorem is proved.

I-9.2 These theorems will be applied here to find the required restrictions on the initial transforms. The following discussion is concerned only with the TH and TE fields, <u>not with the TEH fields</u>. The TH and TE fields will be considered separately. The matter to be investigated can be expressed briefly by saying: The vector fields, Table I, for the TH and TE systems respectively will be transformed back into the t domain. From E_1, E_2, E_3, H_1, H_2, and H_3 we will pass into the t domain obtaining $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$. The question is: What conditions shall be imposed on $A_3(0,s)$; $A'_{3(x_3)}(0,s)$; $B_3(0,s)$; $B'_{3(x_3)}(0,s)$ in order that the last set of vectors verify the equations:

$$\nabla\times\vec{\mathcal{E}}+\mu\frac{\partial\vec{\mathcal{H}}}{\partial t}=0; \quad \nabla\times\vec{\mathcal{H}}-\varepsilon\frac{\partial\vec{\mathcal{E}}}{\partial t}=0$$

$$\nabla\cdot\vec{\mathcal{E}}=0 \quad ; \quad \nabla\cdot\vec{\mathcal{H}}=0 \; .$$

Since we have four possible initial conditions we will investigate them in order.

I-9.3 Take the TH field from Table I, Section I-5.5

Case a. - Initial condition $E_3(0,s)$ or its equivalent $A_3(0,s)$.

We consider the vector components, in the t domain, in accordance with the integral expressions

$$\left.\begin{aligned} \mathcal{E}_1 &= -\left(\frac{1}{h_1}\frac{\partial \psi_3}{\partial x_1}\right) \\ \mathcal{E}_2 &= -\left(\frac{1}{h_2}\frac{\partial \psi_3}{\partial x_2}\right) \end{aligned}\right\} \frac{1}{cp^2}\,\frac{1}{2\pi i}\int_{c_0 + i\infty}^{c_0+i\infty} A_3(0,s)\sqrt{s^2+\omega_c^2}\; e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}\, e^{st}ds$$

$$\mathcal{E}_3 = \psi_3 \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} A_3(0,s)\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}\, e^{st}ds$$

$$\left.\begin{aligned} \mathcal{H}_1 &= \left(\frac{1}{h_2}\frac{\partial \psi_3}{\partial x_2}\right) \\ \mathcal{H}_2 &= -\left(\frac{1}{h_1}\frac{\partial \psi_3}{\partial x_1}\right) \end{aligned}\right\} \frac{\varepsilon}{p^2}\int_{c_0-i\infty}^{c_0+i\infty} sA_3(0,s)\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}\, e^{st}ds$$

$$\mathcal{H}_3 = 0$$

a_1. - Consider first the equation

$$\nabla\cdot\mathcal{E} = 0$$

In cylindrical coordinates we have

$$\nabla\cdot\mathcal{E} = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial x_1}(h_2\mathcal{E}_1) + \frac{\partial}{\partial x_2}(h_1\mathcal{E}_2)\right\} + \frac{\partial \mathcal{E}_3}{\partial x_3} = 0 \ .$$

If we set $x_3 = k/c$ and we apply theorem B Section I-9.1 P. 45, we have

$$\frac{\partial \mathcal{E}_3}{\partial x_3} = \mathcal{E}_3(t,x_3)\Big|_{t=\frac{x_3}{c}}\mu_0\!\left(t-\frac{x_3}{c}\right) - \psi_3\frac{1}{2\pi i c}\int_{c_0-i\infty}^{c_0+i\infty} A_3(0,s)\sqrt{s^2+\omega_c^2}\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}\, e^{st}ds$$

$$\left.\begin{aligned}\frac{\partial}{\partial x_1}(h_2\mathcal{E}_1) &= -\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial \psi_3}{\partial x_1}\right)\\ \frac{\partial}{\partial x_2}(h_1\mathcal{E}_2) &= -\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial \psi_3}{\partial x_2}\right)\end{aligned}\right\}\frac{1}{cp^2}\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} A_3(0,s)\sqrt{s^2+\omega_c^2}\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}e^{st}ds$$

and therefore

$$\nabla\cdot\vec{\mathcal{E}} = \mathcal{E}_3(+\frac{x_3}{c},x_3)u_0(t-\frac{x_3}{c}) - \frac{1}{p^2c}\left\{\nabla^2_{(x_1x_2)}\psi_3 + p^2\psi_3\right\}\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}A_3(0,s)e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}e^{st}ds$$

and remembering that $\nabla^2\psi_3 + p^2\psi_3 = 0$

$$\nabla\cdot\vec{\mathcal{E}} = \mathcal{E}_3(+\frac{x_3}{c},x_3)u_0(t-\frac{x_3}{c})\,.$$

This means that in the wave front this equation is not satisfied unless

$$\mathcal{E}_3(+\frac{x_3}{c},x_3) = 0\,. \qquad (3)19$$

a_2. - It can be verified readily that $\nabla\cdot\vec{\mathcal{H}}=0$ will be satisfied without any further restriction on $A_3(0,s)$.

a_3. - Consider the two curl equations now. After a similar substitution and by the appropriate use of Theorems A and B we get the following results, which can be checked easily.

$$\left.\begin{aligned}&\nabla\times\vec{\mathcal{E}}+\mu\frac{\partial\vec{\mathcal{H}}}{\partial t}=0 \text{ requires that }\begin{cases}\mathcal{E}_2(+\frac{x_3}{c},x_3) - \sqrt{\frac{\mu}{\varepsilon}}\mathcal{H}_1(+\frac{x_3}{c},x_3) = 0\\ \mathcal{E}_1(+\frac{x_3}{c},x_3) + \sqrt{\frac{\mu}{\varepsilon}}\mathcal{H}_2(+\frac{x_3}{c},x_3) = 0\end{cases}\\ &\nabla\times\vec{\mathcal{H}}-\varepsilon\frac{\partial\vec{\mathcal{E}}}{\partial t}=0 \text{ requires that }\begin{cases}\mathcal{E}_3(+\frac{x_3}{c},x_3) = 0\\ \mathcal{E}_2(+\frac{x_3}{c},x_3) - \sqrt{\frac{\mu}{\varepsilon}}\mathcal{H}_1(+\frac{x_3}{c},x_3) = 0\\ \mathcal{E}_1(+\frac{x_3}{c},x_3) + \sqrt{\frac{\mu}{\varepsilon}}\mathcal{H}_2(+\frac{x_3}{c},x_3) = 0\end{cases}\end{aligned}\right\}$$

Summarizing it can be said that the field created by the initial condition $\gamma_3 A_3(0,s) = E_3(0,s)$ will be electromagnetic in the t domain if

$$\left.\begin{aligned} &\mathcal{E}_3(x_1,x_2,x_3,t)_{t=+\frac{x_3}{c}} = 0 \\ &\mathcal{E}_2(x_1,x_2,x_3,t)_{t=+\frac{x_3}{c}} - \sqrt{\tfrac{\mu}{\varepsilon}}\mathcal{H}_1(x_1,x_2,x_3,t)_{t=+\frac{x_3}{c}} = 0 \\ &\mathcal{E}_1(x_1,x_2,x_3,t)_{t=+\frac{x_3}{c}} + \sqrt{\tfrac{\mu}{\varepsilon}}\mathcal{H}_2(x_1,x_2,x_3,t)_{t=+\frac{x_3}{c}} = 0 \end{aligned}\right\} \qquad (5)\text{I9}$$

a_4. - Now our problem is to investigate what must be the analytical structure of $E_3(0,s)$, or its equivalent $A_3(0,s)$, to fulfill (5)I9. One has to observe that (5)I9 yields the condition at $t=+x_3/c$; that is among the initial values of the field vectors. Then the well-known "theorem of initial value", in the theory of Laplace transforms, is the natural tool to be used at this point. An approach to the "initial value theorem" can be reached as follows. Let $\varphi_n(t,k)$ be a function of the independent variables t and k and such that it satisfies the conditions in Theorem A. Its derivative with respect to t will be indicated by $\varphi'_{n(t)}(t,k)$ and its transform by $\phi(s,k)$.

First, from Theorem A the following equation can be written:

$$G_n(s) = \int_k^{\infty} \varphi'_{n(t)}(k,t)\, e^{-st} dt = -\, e^{-sk}\varphi_n(+k,k) + s\phi_n(s,k) \quad . \qquad (6)\text{I9}$$

Second, we will assume that $\phi_n(s,k)$ is our standard transform

$$\phi_n(s,k) = F_n(s, \sqrt{s^2+\omega_c^2})\, e^{-k\sqrt{s^2+\omega_c^2}}$$

in which $F_n(s,\sqrt{s^2+\omega_c^2})$ is the ratio of two polynomials in s and $\sqrt{s^2+\omega_c^2}$. Substituting in (6)I9 and multiplying both members by e^{+sk} the following expression is obtained:

$$\int_k^\infty \varphi'_{n(t)}(k,t)\,e^{-s(t-k)}dt = -\varphi_n(+k,k) + sF_n(s,\sqrt{s^2+\omega_c^2})\,e^{k(s-\sqrt{s^2+\omega_c^2})}$$

Now let $s\to\infty$, then $\sqrt{s^2+\omega_c^2}\to s$ and $s-\sqrt{s^2+\omega_c^2}\to 0$ and $F_n(s,\sqrt{s^2+\omega_c^2})\to F_n(s,s)$ which is rational function in S. Then

$$\lim_{s\to\infty}\int_k^\infty \varphi'_{n(t)}(k,t)\,e^{-s(t-k)}dt = \int_k^\infty \varphi'_{n(t)}(k,t)\lim_{s\to\infty}e^{-s(t-k)}dt = 0 =$$
$$= -\varphi(+k,k) + \lim_{s\to\infty} sF_n(s,s)$$

$$\therefore \lim_{s\to\infty}\left[sF_n(s,s)\right] = \varphi(+k,k) = \varphi(t,k)_{t=+k}$$

when we approach from the right side of t. This is the proper form of the "initial value theorem" for the type of transforms we have considered as standard type.

Since, by hypothesis, $F_n(s,s)$ is the ratio of two polynomials in s, its behavior for large values of s will be like

$$sF_n(s,s)\underset{s\to\infty}{\longrightarrow} s\,\frac{M_n}{s^\gamma} = \frac{M_n}{s^{\gamma-1}}$$

in which M_n is a constant and γ is an integer.

a_5. - Let us apply this theorem to the set of conditions (5)I9. To satisfy the first one must have

$$F_3(s,\sqrt{s^2+\omega_c^2}) = A_3(0,s) \;,$$

and therefore it is required that

$A_3(0,s)_{s\to\infty}$ behaves at least as $\frac{M_3}{\gamma^2}$ when $s\to\infty$.

a_6. - Let this situation for the last two conditions in (5)I9 be investigated.

We will prove first that: if $A_3(0,s)$, or its equivalent $E_3(0,s)$, is the only initial condition of excitation, that is, all others equal zero, then

$$\mathcal{E}_2(x_1,x_2,x_3,t)_{t=+\frac{x_3}{c}}=0$$

$$\mathcal{E}_1(x_1,x_2,x_3,t)_{t=+\frac{x_3}{c}}=0$$

Suppose that $\mathcal{E}_2$ and $\mathcal{E}_1$ at $t=+x_3/c$ were not zero. Then, since x_3 can be any cross section of the guide it can be $x_3=0$. But at $x_3=0$, $E_1(0,s)$ and $E_2(0,s)$ are identically zero, since in the TH system these components are proportional to $A_3'(0,s)$ which is zero. See conditions (2)I5.

Then

$$\lim_{s\to\infty} sE_1(0,s)=0$$

$$\lim_{s\to\infty} sE_2(0,s)=0$$

and therefore

$$\left.\begin{array}{l}\mathcal{E}_1(x_1,x_2,x_3,t)_{t=+\frac{x_3}{c}}=0\\ \mathcal{E}_2(x_1,x_2,x_3,t)_{t=+\frac{x_3}{c}}=0\end{array}\right\}\text{ if } A_3'(0,s)=0\ ;$$

which was to be proved.

As a consequence of the above property and from (5)I9 we can conclude immediately that if the field is to be electromagnetic, then

$$\mathcal{H}_1(x_1,x_2,x_3,t)_{t=+\frac{x_3}{c}}=\mathcal{H}_2(x_1,x_2,x_3,t)=0\ .$$

a_7. - Since the transverse components of the field at $t=+x_3/c$ (wave front) must vanish, their corresponding transforms must behave properly when $s\to\infty$. From Table I (TH waves excited only by $E_3(0,s)$ we can readily find that this will be the case if

$$E_3(0,s)\underset{s\to\infty}{\to}\frac{M}{s^\gamma} \qquad \gamma\geq 3$$

$$A_3(0,s)\underset{s\to\infty}{\to}\frac{N}{s^\gamma} \qquad \gamma\geq 3$$

which is the required condition of excitation when $E_3(0,s)$ is specified.

Case b. - Initial condition $E'_{3(x_3)}(0,s)$ or its equivalent $A'_{3(x_3)}(0,s)$

Following a method similar to the one indicated in Case a, it can be found readily that the field in the t domain will be electromagnetic for all values of x_3 and t if

$$E'_{3(x_3)}(0,s)\underset{s\to\infty}{\to}\frac{M}{s^\gamma} \qquad \gamma\geq 2$$

or its equivalent, M and N = constants

$$A'_{3(x_3)}(0,s)\to\frac{N}{s^\gamma} \qquad \gamma\geq 2$$

I-9.4 In this paragraph fields of the TE type will be considered.

As before, two cases will be regarded:

Case a. - $H_3(0,s)$ or its equivalent $B_3(0,s)$ is given.

Case b. - $H'_{3(x_3)}(0,s)$ or its equivalent $B'_{3(x_3)}(0,s)$ is given.

By a procedure similar to that followed in I-9.3 or by a comparison of the transforms of the TH and TE fields in Table I, the conclusion can readily be reached that:

Case a. - $H_3(0,s)\underset{s\to\infty}{\to}\frac{M}{s^\gamma}$ or its equivalent $B_3(0,s)\to\frac{N}{s^\gamma}$ if $\gamma\geq 3$, and

<u>Case b</u>. - $H'_{3(x_3)}(0,s) \rightarrow \frac{M}{s^\gamma}$ or its equivalent
$B'_{3(x_3)}(0,s) \rightarrow \frac{N}{s^\gamma}$ if $\gamma \geq 2$.

I-9.5 In this paragraph the results obtained in Section I-9 will be summarized.

<u>1st</u> - If the axial components of the field vectors, $\mathcal{E}_3$ at $x_3 = 0$, for TH fields or $\mathcal{H}_3$ at $x_3 = 0$ for TE fields, are respectively given as initial conditions, these initial conditions must be such that

$E_3(0,s)_{s\to\infty}$ or its equivalent $A_3(0,s)_{s\to\infty}$ for TH fields or
$H_3(0,s)_{s\to\infty}$ or its equivalent $B_3(0,s)_{s\to\infty}$ for TE fields
must behave as $\frac{M}{s^\gamma}$ where $\gamma \geq 3$. M = constant.

<u>2nd</u> - If the space derivations with respect to x_3 of the axial components of the field vectors, $\mathcal{E}'_{3(x_3)}$ at $x_3 = 0$ for TH fields or $\mathcal{H}'_{3(x_3)}$ at $x_3 = 0$ for TE fields, are respectively given as initial conditions, their corresponding transforms must be such that

$E'_{3(x_3)}(0,s)$ or its equivalent $A'_{3(x_3)}(0,s)$ for TH fields or
$H'_{3(x_3)}(0,s)$ or its equivalent $B'_{3(x_3)}(0,s)$ for TE fields
must behave as $\frac{N}{s^\gamma}$ where $\gamma \geq 2$. N = constant.

<u>3rd</u> - If $A'_{3(x_3)}(0,s)$ or $B'_{3(x_3)}(0,s)$ are respectively given initial conditions in the S domain, it is equivalent to give the transverse electric components or respectively the transverse magnetic components of the field at $x_3 = 0$.

The justification of the last statement is found in the last two equations of condition given in (2)I5 and (3)I5, Section 5 of this chapter.

Section 10 - Generalities on the problem of inversion of the TH and TE fields.

I-10.0 We are confronted now with the problem of the inversion of the TH and TE fields from the S domain into the time domain. This is the difficult problem of this investigation. Chapters II, III and IV are devoted to this task.

As a result of the basic discussion of Ch. I, it is known that one has to deal with the inversion into the t space of transforms of the type

$$F(s,\sqrt{s^2+\omega_c^2})\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}$$

in which $F(s,\sqrt{s^2+\omega_c^2})$ is the ratio of two polynomials in s and $\sqrt{s^2+\omega_c^2}$. $F(s,\sqrt{s^2+\omega_c^2})$ is also restricted to show a definite behavior when $s\to\infty$.

It is important to point out that the problem of this investigation is not merely to find the inverse Laplace transform of a specific function F(s). We have to find a practical method to obtain the corresponding functions in the time domain of a large family of Laplace transforms of complicated functions of s. That is why three complete chapters are devoted to this purpose.

I-10.1 The most natural starting point for the problem of inversion is the use of a table of Laplace transforms. Unfortunately, there is practically nothing which might be of some help. Only a very few are tabulated and can not be used directly even in the simplest problem.

The use of well-known simple theorems on inversion does not produce anything of any practical value. The Laplace Resultant or Convolution theorem, has some theoretical value only for a subtype of our basic transforms.

This subtype is

$$F_1(s) = \frac{e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{\sqrt{s^2+\omega_c^2}}$$

in which $F_1(s)$ is the ratio of two polynomials in s. If, under this assumption, $F_1(s)$ is expanded in partial fractions and, if the terms of this partial expansion are considered, then the above transform breaks up in two types.

$$\frac{A_k}{s-s_k} \frac{e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{\sqrt{s^2+\omega_c^2}} \quad \text{for simple poles}$$

and

$$\frac{A_{\beta\alpha}}{(s-s_\beta)^\alpha} \frac{e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{\sqrt{s^2+\omega_c^2}} \quad \text{for multiple poles .}$$

Now from the tables

$$\mathcal{L}^{-1} \frac{A_k}{(s-s_k)} = A_k e^{s_k t} \quad \text{for } t > 0$$

$$\mathcal{L}^{-1} \frac{A_{\beta\alpha}}{(s-s_\beta)^\alpha} = A_{\beta\alpha} \frac{t^{(\alpha-1)}}{(\alpha-1)!} e^{+s_\beta t} \quad \text{for } t > 0$$

$$\mathcal{L}^{-1} \frac{e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{\sqrt{s^2+\omega_c^2}} = \begin{cases} =0 & \text{for } t < \frac{x_3}{c} \\ =J_0\left[\omega_c \sqrt{t^2 - \left(\frac{x_3}{c}\right)^2}\right] & \text{for } t > \frac{x_3}{c} \end{cases}$$

and by the use of the convolution theorem, one obtains,

$$\mathcal{L}^{-1}\frac{A_k}{(s-s_k)}\,\frac{e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{\sqrt{s^2+\omega_c^2}}=\begin{cases}=0 \quad \text{for } t<\dfrac{x_3}{c}\\[2ex] =A_k e^{s_k t}\displaystyle\int_{\frac{x_3}{c}}^{t} e^{-s_k\tau} J_0\left[\omega_c\sqrt{\tau^2-\left(\frac{x_3}{c}\right)^2}\right]d\tau\end{cases}$$

$$\mathcal{L}^{-1}\frac{A_{\beta\alpha}}{(s-s_\beta)^\alpha}\,\frac{e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{\sqrt{s^2+\omega_c^2}}=\begin{cases}=0 \quad \text{for } t<\dfrac{x_3}{c}\\[2ex] =\dfrac{A_{\beta\alpha}e^{s_\beta t}}{(\alpha-1)!}\displaystyle\int_{\frac{x_3}{c}}^{t}(t-\tau)^{\alpha-1}e^{-s_\beta\tau}J_0\left[\omega_c\sqrt{\tau^2-\omega_c^2}\right]d\tau\,.\end{cases}$$

The above integral, as well as others which appear in a similar way, were carefully studied and some expansions were made in series. These series converge, sometimes, very slowly and give no information about the intrinsic character of the propagation. The convolution method was then abandoned. The results of this type of integration will not be mentioned here.

I-10.2 The inverse integral

$$f(t)=\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}F(s)e^{st}dt \ ; \quad s=\sigma+j\omega$$

is the main mathematical tool used to solve this problem. In the beginning the investigation was conducted in the s plane. Chapter II deals with a systematic study of the analytical properties. In this plane no valuable results were achieved. Only when some complex transformations were introduced, were the inversions of the prototype transforms obtained. The inverse functions came out in terms of compact expressions of Lommel's functions. The presentation of this part of the work is given in Ch. III.

Since Lommel's functions are not tabulated and not very well studied, appropriate expansions were made in order to perform numerical computations. A generating function was found for all these inverse transforms and a graphical method was developed to obtain the envelope and phase functions of the corresponding wave form. A general discussion was easily made for the signal and group velocities in terms of this generating function. This part is presented in Ch. IV.

CHAPTER II

Section 0 - Chapter contents and procedure

II-0.0 Chapter II develops the first analytical steps required to obtain the inverse transform functions of

$$F(s,\sqrt{s^2+\omega_c^2})\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}$$

under the fundamental assumptions:

1. $F(s,\sqrt{s^2+\omega_c^2})$ is the ratio of two polynomials in s and $\sqrt{s^2+\omega_c^2}$
2. $F(s,\sqrt{s^2+\omega_c^2})\quad \frac{M}{s^{\gamma}}$ for $\gamma \geq 2$ and $M =$ constant.

The mathematical investigation in this chapter will be less restricted since it will be valid for $\gamma \geq 1$.

II-0.1 The discussions of Ch. II shall be confined to the s plane. Complex transformations will be introduced afterwards in Chs. III and IV. Not many final results may be expected from this chapter. Its purpose is to give a basic systematic discussion about the analytical properties of this transform in the s plane, in such a way that all the requirements for the inversion are properly satisfied. A search will also be made for a simple starting point for this problem.

II-0.2 Specific transforms of the standard form can be worked out easily in the s plane. Since the main idea is to obtain a method of inversion which is applicable to most cases, these cases will not be given attention. The presence of the radical in F greatly complicates the problem in the s plane.

II-0.3 The readers who are acquainted with the basic theorems of existence of the inverse Laplace transforms and the techniques of evaluating the corresponding contour integral can omit this chapter. However, some confusion may arise when certain multivalued complex transformations are introduced. Misinterpretation of the Riemann surface or transformed contours will not lead to the desired results. This is the only reason this chapter is included.

II-0.4 Section 1 contains: Frequency normalization. Normalized transforms. Singularities. Fundamental theorem of existence of the inverse transform. Inversion integral and the Br_1 contour.

Section 2 contains: Riemann surfaces. Suitable branch cuts. Riemann surfaces of the exponential function.

Section 3 contains: Abscissa of uniform convergence. Br_1 and Br_2 contours. Integration for $t < \frac{x_3}{c}$ and $t > \frac{x_3}{c}$. Br_2 contours of integration for different cuts and specific transforms.

Section 4 contains: Aspect of the inverse transformation integral for some typical transforms and different types of s plane cutting.

Section 5 contains: The branch cuts in the s plane and their physical interpretation as secondary waves.

Section 1 - Frequency normalization. Normalized transforms. Inverse transform integral. Fundamental theorem and Br_1 contour. Singularities.

II-1.0 Let $\Phi(s)$ be a Laplace transform and $\varphi(t)$ its inverse Laplace transform; thus

$$\varphi(t) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} \phi(s)e^{st}ds; \quad s=\sigma+i\omega, \quad c_0>\sigma_0 \qquad (1)\text{II1}$$

in which σ_0 is the abscissa of uniform convergence and the contour of integration can be a line between the points $c_0 - i\infty$ and $c_0 + i\infty$ in such a way that all the singularities of $\phi(s)$ remain at the left-hand side of this line. Usually this contour is a straight line parallel to the ω axis and is sometimes the Br_1 ($Brownich_1$) contour of integration.

It will be assumed that the reader is acquainted with the theorems of existence, uniqueness, convergence, etc. of the inverse integral. Just one basic theorem will be repeated here without proof: "Let $\phi(s)$ be an analytic function of the complex variable s of order $O(\frac{1}{s^\gamma})$ in some half plane $R(s) \geq \sigma_0$, where σ_0 and γ are finite constants and $\gamma > 1$. Then the inversion integral along any line $\sigma = c_0$ where $c_0 > \sigma_0$ converges to a function $\varphi(t)$, which is independent of c_0 and such that $\mathcal{L}\varphi(t) = \phi(s)$."

By hypothesis our transforms satisfy the condition $\gamma > 1$; (see Section 9, Ch. I, P. 43). The above theorem shows that the immediate problem of this investigation is to search for the singularities and the position of σ_0.

II-1.1 Before proceeding further with this search, a normalization of the complex frequency will be introduced for convenience. Let:

$$s = \frac{S}{\omega_c} = \frac{\sigma}{\omega_c} + i\frac{\omega}{\omega_c} = \rho + i\nu \qquad (2)\text{II1}$$

in which $\omega_c = cp$; $p =$ separation constant.

The standard transform will become

$$F(s,\sqrt{s^2+\omega_c^2})\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}} = F(s,\sqrt{s^2+1})\, e^{-\kappa\sqrt{s^2+1}} \qquad (3)\text{II1}$$

Now, suppose that we denote by Γ the Br_1 contour of integration of the inverse integral with respect to s and by γ the corresponding transformed contour with respect to s, then

$$\frac{1}{2\pi i}\int_{\Gamma} F(s,\sqrt{s^2+\omega_c^2})\, e^{st-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}\, ds = \frac{\omega_c}{2\pi i}\int_{\gamma} F(s,\sqrt{s^2+1})\, e^{+s\tau-\kappa\sqrt{s^2+1}}\, ds = \omega_0 I; \qquad (4)\text{II1}$$

in which

$$\tau = \omega_c t = 2\pi t f_c = 2\pi\frac{t}{T_c}$$

$$\kappa = \omega_c\frac{x_3}{c} = 2\pi\frac{x_3}{\frac{c}{f_3}} = 2\pi\frac{x_3}{\lambda_c}$$

$T_c =$ cut-off period for the corresponding mode

$\lambda_c =$ cut-off wave length for the corresponding mode.

In the future only the integral I in terms of the normalized angular velocity will be considered.

Section 2 - Branch points, branch cuts, the function $\sqrt{s^2+1}$.

II-2.1 The standard transform has two branch points in the plane given by

$$s = \pm i$$

The s plane is composed of two Riemann surfaces connected by a branch cut. The cutting of the s plane is one of two types: 1st, by joining +i and -i through the point at infinity or, 2nd, by joining +i and -i by a line whose points remain at finite distance from the origin. This latter type of cutting is justified by the fact that $\sqrt{s^2+1} \rightarrow s$ when $s \rightarrow \infty$.

To fulfill the conditions of inversion indicated in the theorem given in II-1.0, all the points of the cutting lines must remain at the left of the Br_1 contour.

The integration of (4)II1 is difficult to perform. The analytical structure of the integrand changes with the way in which the s plane is cut. Much of the success of the integration depends on the choice of a proper cut. For this reason, several cuts are here studied and their effect on the form of the integrand will be observed.

Figures 1II and 2II show the selected cuts of the first and second types.

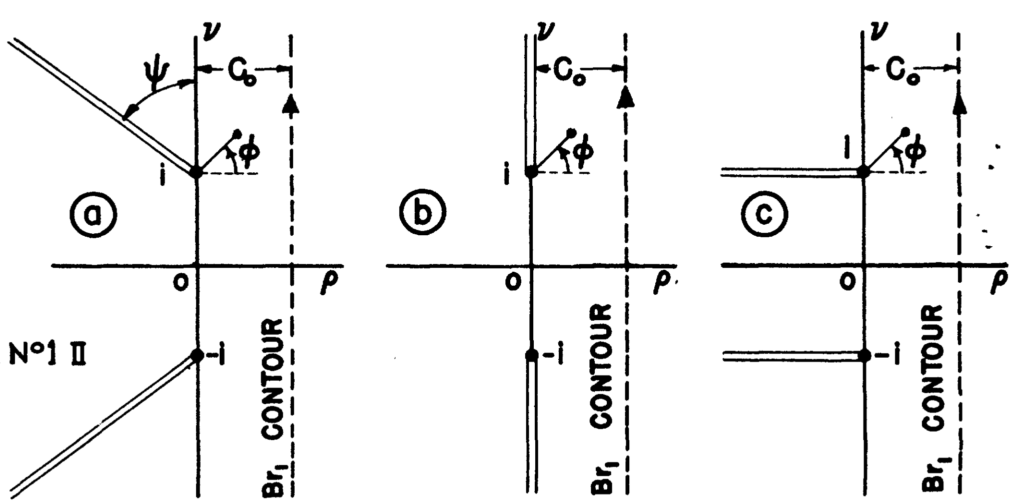


FIG. N°1 II

II-2.2 The presence of the radical $\sqrt{s^2+1}$ in the transform (3)II1 is responsible for its multivalued character. The branches of the transform are closely related to the branches of the radical. For this reason the branches of the radical will be studied with some detail. Consider the function:

$$W=u+jv= -\sqrt{s^2+1}.$$

When the variable point s moves in the s plane, the functions u and v change. One passes from one leaf to the other when the branch cut is crossed. The signs of u and v may change suddenly from one side of the cut to the other. This change in signs of u and v is important. When a contour is followed we have to be sure to take the correct sign for these functions. Besides, the knowledge of this sign distribution is a big help when other complex transformations are introduced, as in Chs. III and IV.

The two sheets of the Riemann surface, in the s plane will be denoted by $\mathcal{S}_I$ and $\mathcal{S}_{II}$. The Roman numbers indicate respectively the leaves one and two. This sign distribution is given in Figs. 3II and 4II corresponding to the branch cuts indicated in Figs. 1II and 2II. Although these signs are plotted in the s plane, they correspond to the resultant sign of u and v <u>and not</u> to ρ and ν. A zero indicates the points for which u or v vanish.

The Riemann surfaces $\mathcal{S}_I$ and $\mathcal{S}_{II}$ were defined in the way indicated below. This definition is made at the branch point +i. It is simple to find the corresponding

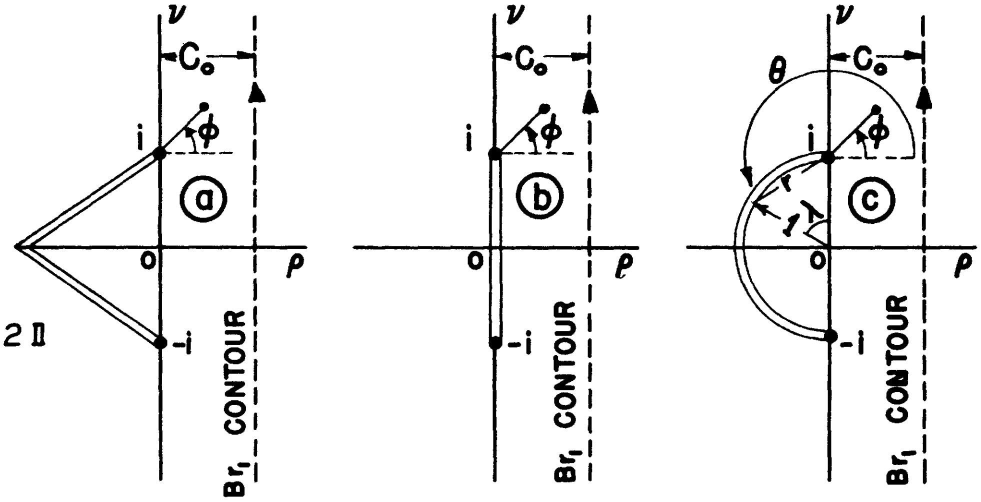
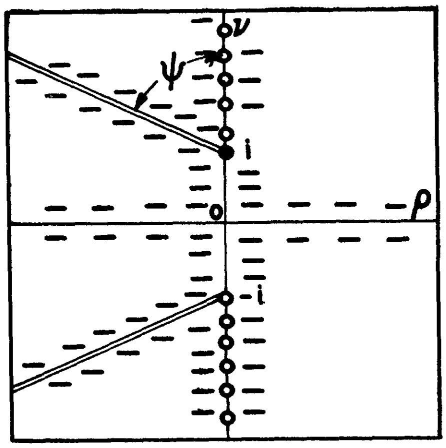
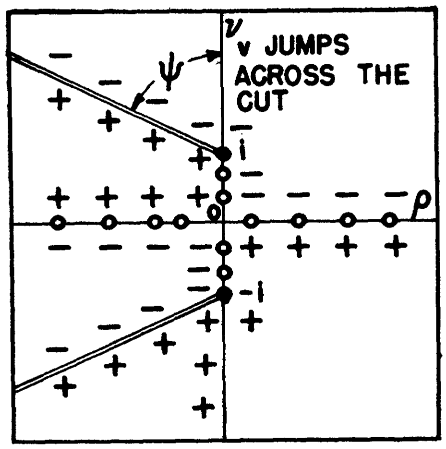
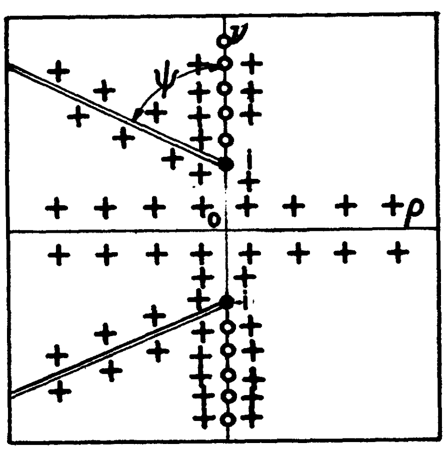
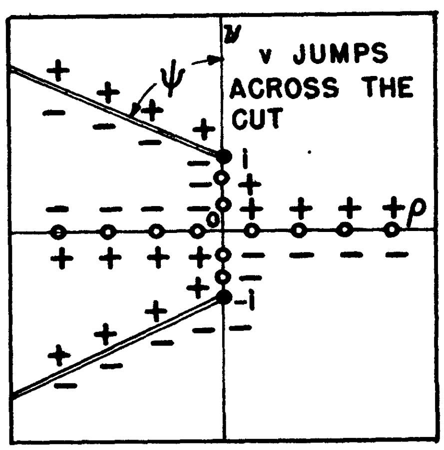


FIG. N°2 II

PROPOSED CUTS IN THE
λ PLANE

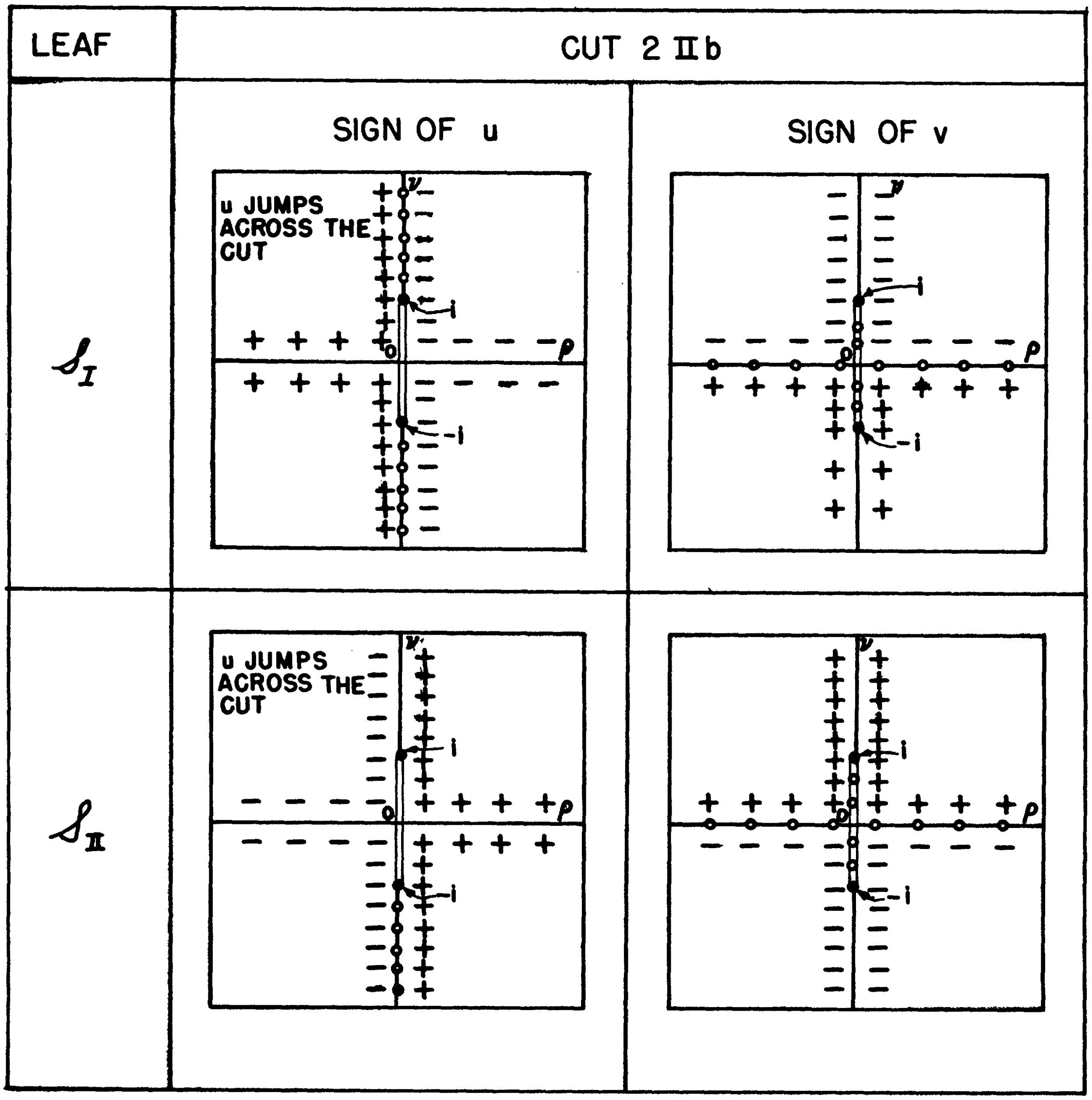
FIG N° 3 II

LEAF #	CUTS 1 II a,b,c	
	SIGN OF u	SIGN OF v
S_I		
S_{II}		

expressions for the branch point -i. The signs of u and v are in accordance with this definition.

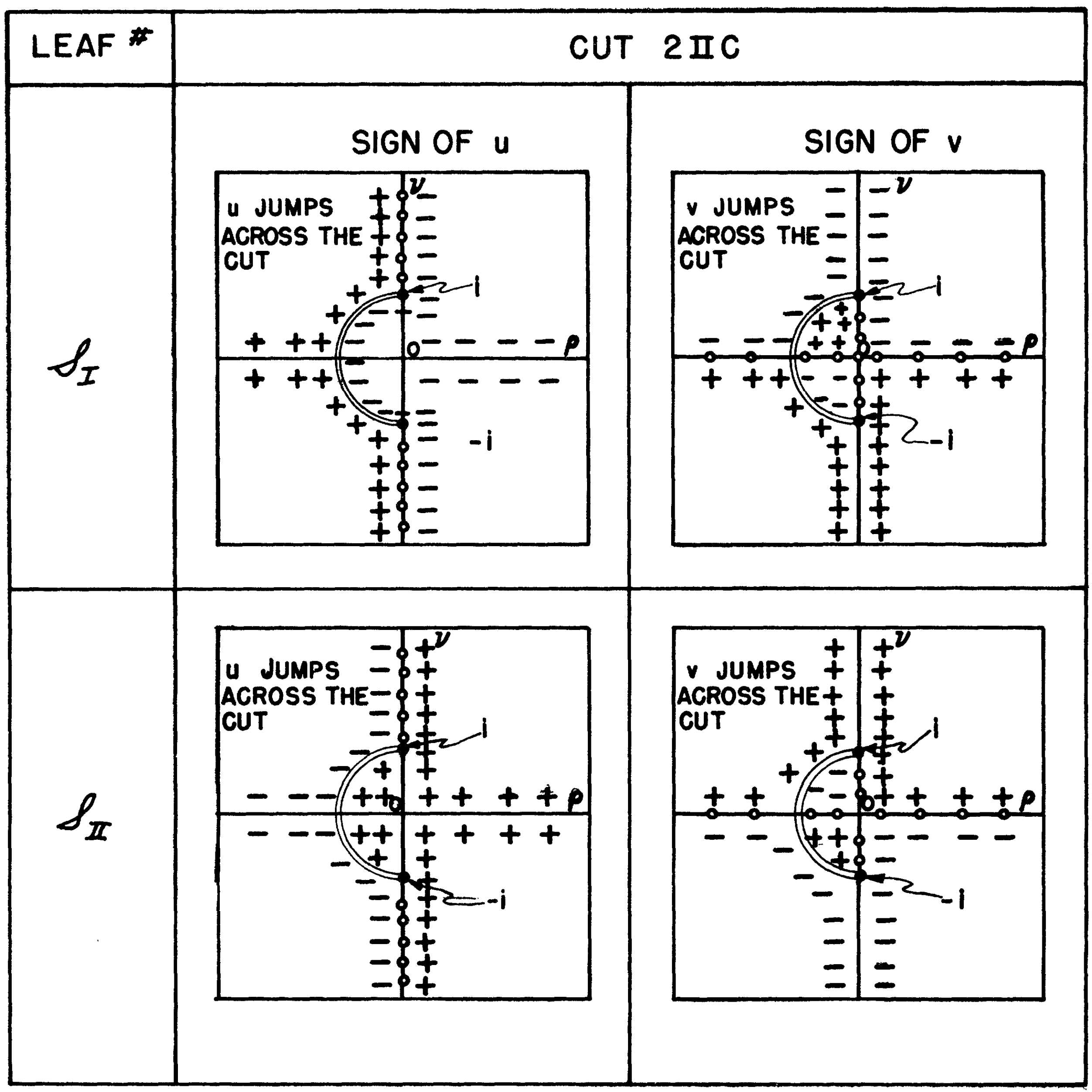
The corresponding sign distribution for cut 2IIa is similar to the one indicated in Fig. 4IIc. These results will be applied in Section II-4 and also in Chs. III and IV.

FIG. N° 4 II b



II-2.3 Poles. The poles of the function $F(s,\sqrt{s^2+1})$ are poles of the transform. Let n be its number and let them be at s_k; k=1, 2,•••,n. It is required that all poles be contained in some half s plane for which Real$s \leq \rho_0$. In the half plane Real$s \geq \rho_0$ the function $F(s,\sqrt{s^2+1})$ must be analytic. See fundamental theorem in II-1.0.

FIG. N° 4 IIc



The function $F(s,\sqrt{s^2+1})$ can be expanded by Laurent series around its poles. When this expansion is made the integral breaks up into integrals of the type

$$\frac{A_{k\alpha}}{2\pi i}\int_{\gamma}\frac{1}{(s-s_k)^{\alpha}}\, e^{s\tau-\chi\sqrt{s^2+1}}\,ds \qquad (2)\text{II}2$$

in which α is the multiplicity of the pole.

It can happen, and is often the case, that the points +i and -i are also poles of the transform. Here the Laurent expansion contains fractional powers of α and the integration of (2)II2 becomes more complicated. It is very difficult to integrate (2)II2 in the s plane. This is so even when F is a function of s alone and $\alpha=1$. When $F=F(s,\sqrt{s^2+1})$ then the situation becomes more complicated and it will not be carried out in this chapter. One of the objectives of Ch. III is to get rid of this radical by means of suitable complex transformations.

In Section II-4 some examples of the integral structure in simple cases of $F(s,\sqrt{s^2+1})$ and for different ways of cutting the s plane are given.

Section 3 - Abscissa of uniform convergence. Integration for $\tau<\chi$. Br_2 contour and integration for $\tau>\chi$. Br_2 contours for different cuts.

II-3.0 The abscissa of uniform convergence can be found after all the singularities of the transform have been located. Let s_M be the position of a singularity such that

$$\text{Real}\, s_M \geq \text{Real}\, s_k$$

It is evident that for all s, such that $\text{Real}\, s > \text{Real}\, s_M$, the function will be analytic. Therefore,

$c_0 \geq \text{Real}\, s_M$;

The equal sign is valid if the Br_1 contour, in the vicinity of s_M, is deformed by a semicircle leaving s_M at the left. Consider, as an example, the transforms

$$\frac{e^{-\kappa\sqrt{s^2+1}}}{s-j\nu_0} ; \quad \frac{e^{-\kappa\sqrt{s^2+1}}}{s^2+\nu_0^2} ; \quad \frac{e^{-\kappa\sqrt{s^2+1}}}{\sqrt{s^2+1}}$$

in which ν_0 and κ are real quantities.

The Fig. 5II shows the limiting position of the Br_1 contour with a dotted line for each of the above transforms. Any straight line which is parallel to the imaginary axis and to the right of this limiting contour, yields to the same inverse transformation. A similar process can be followed for other cases.

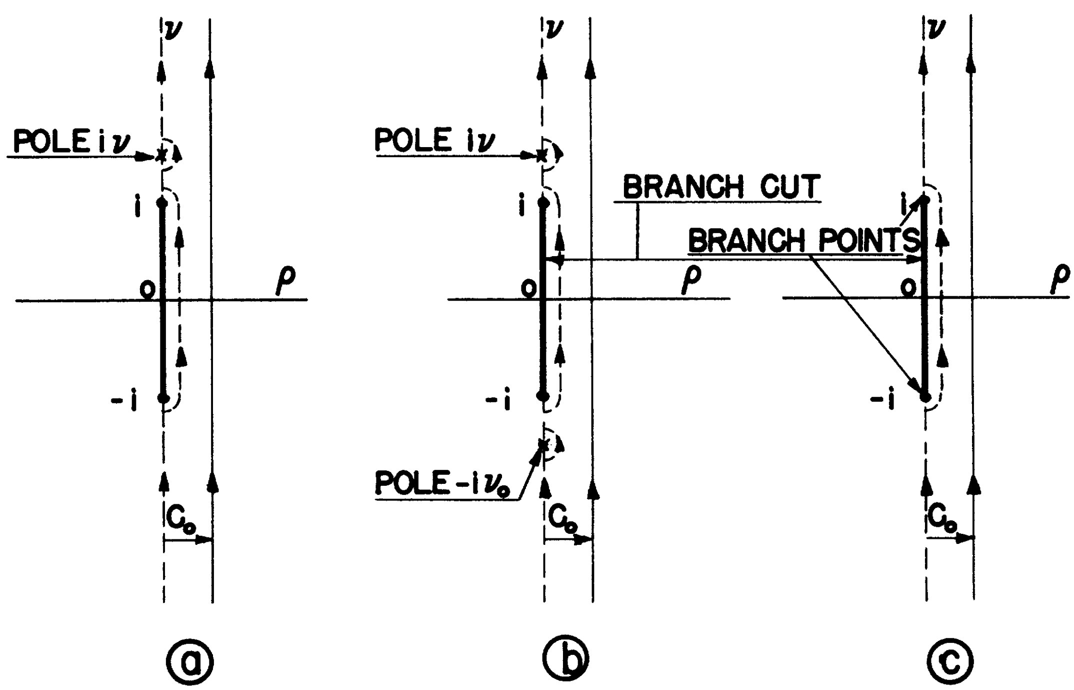


FIG. N° 5 II.

II-3.1 In this section we shall consider the inverse transform integral

$$I(\tau,\kappa)=\frac{1}{2\pi i}\int_{c_0+i\infty}^{c_0-i\infty} F(s,\sqrt{s^2+1})\, e^{s\tau-\kappa\sqrt{s^2+1}}\, ds$$

for $\tau<\kappa$, that is $\frac{t}{T_c}<\frac{x_3}{c}$ and show that

$$I(\tau,\kappa)\equiv 0 \text{ for } \tau<\kappa. \qquad (1)\text{II3}$$

To prove this the well-known process of computing the integral

$$\frac{1}{2\pi i}\oint_{G_\ell} F(s,\sqrt{s^2+1})\, e^{\tau s-\kappa\sqrt{s^2+1}}\, ds$$

is followed along the closed contour G_ℓ, indicated in Fig. 6II. Since $c_0>\rho_0$, the integrand is analytic inside and on the contour and, therefore, the above integral is zero. From this and from Fig. 6II we may write:

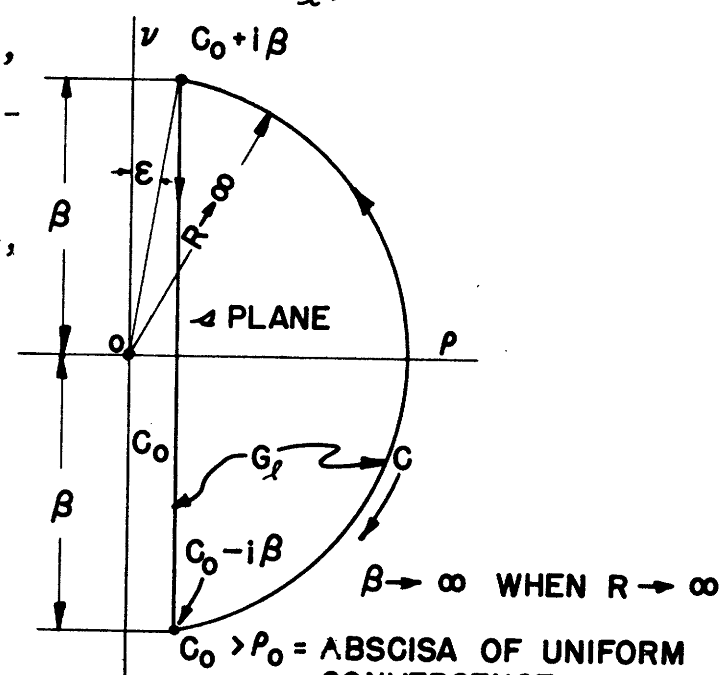


FIG. N° 6 II.

$$\int_{c_0-i\beta}^{c_0+i\beta} F(s,\sqrt{s^2+1})\, e^{s\tau-\kappa\sqrt{s^2+1}}\, ds=\int_C F(s,\sqrt{s^2+1})\, e^{s\tau-\kappa\sqrt{s^2+1}}\, ds.$$

Now, let us make $R\to\infty$. Since it has been assumed that $F(s,\sqrt{s^2+1})$ behaves as $\frac{M}{s^\gamma}$ for $\gamma>1$, and by making $s=Re^{j\varphi}$, the following is obtained by a well-known theorem of functions;

$$\left|\int_C F(s,\sqrt{s^2+1})\, e^{s\tau-\kappa\sqrt{s^2+1}}\, ds\right| \underset{R\to\infty}{\to} \left|\int_C \frac{M}{s^\gamma} e^{s(\tau-\kappa)}\, ds\right| \le |M| \int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} \frac{e^{-R(\kappa-\tau)\cos\varphi}}{R^\gamma}\, d\varphi \underset{R\to\infty}{\to} 0$$

since the exponent $-R(\kappa-\tau)\cos\varphi$ remains negative and tends to $-\infty$ when $R\to\infty$. Hence, (1)II3 is proved.

This property is interpreted by saying, that, at a cross section x_3 distance from the origin, no perturbation arrives for values of the time $t<\frac{x_3}{c}$, in which c is the speed of light in the medium. The interval $0 \le t < \frac{x_3}{c}$ will be called the silence zone, (Ch. IV).

II-3.2 The contour Br_1, that is a straight line from $c_0-\infty$ to $c_0+\infty$, has a rather theoretical value. Very seldom, and only in simple cases, can the integration be performed along this line. A deformation of this contour around the singularities of the transform simplifies the process of integration. To obtain the integration for $\tau>\kappa$, usually the Br_1 contour to the left is closed by adding a new contour, Γ_r formed by a very large semicircle connected with partial contours surrounding the singularities. The whole closed contour must be so located, that the integrand is analytic inside and on this contour. Let us call this closed contour G_r. Under the condition above the integral along G_r is zero. Since G_r is formed by the union of the Br_1 and Br_2 contours, the integral along Br_1 is equal and of opposite sign to the integral along Br_2.

$$\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} F(s,\sqrt{s^2+1})\, e^{s\tau-\kappa\sqrt{s^2+1}}\, ds = -\frac{1}{2\pi i}\int_{\substack{\Gamma_r\\ R\to\infty}} F(s,\sqrt{s^2+1})\, e^{s\tau-\kappa\sqrt{s^2+1}}\, ds$$

This procedure is illustrated with a particular example of a transform whose singularities are indicated in Fig. 7II.

It is well known that when $R \to \infty$, the semicircle at the left and the segments ab and gh contribute nothing to the integral, if $\tau > \kappa$ and $F(s,\sqrt{s^2+1})$ is of order $O(\frac{M}{s^\gamma})$ and $\gamma \geq 1$. This is independent of the type of branch cutting selected.

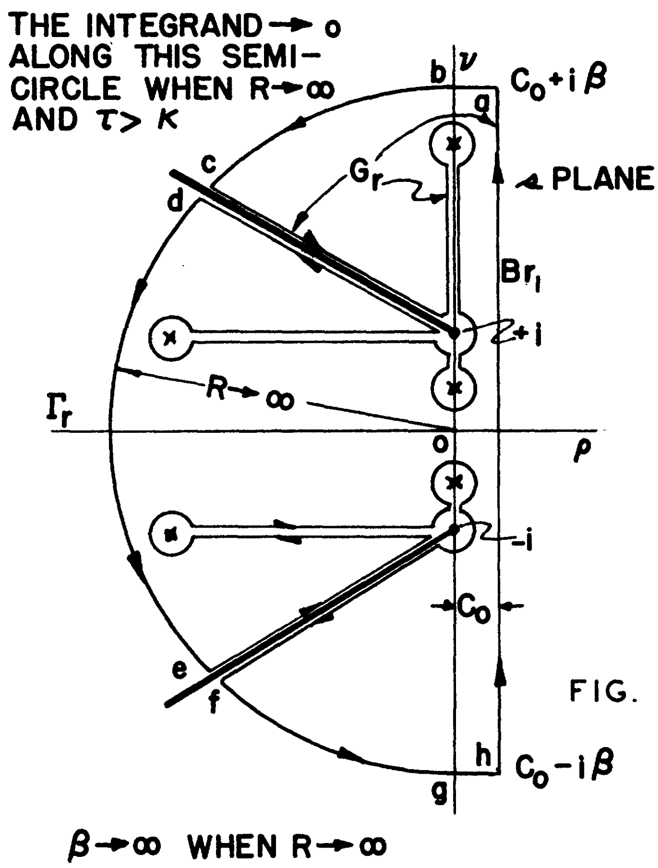


FIG. N° 7 II

Since this proof is simple and given in all text books on Laplace transform, it will not be repeated here.

The integration along the connecting channel, without branch cuts in the inside, is clearly equal to zero. Then the integral along the contour Γ_r reduces to the integral <u>around</u> the singularities and to the banks of the branch cuts. The contour <u>around</u> the singularities, in the proper direction, is sometimes called the Br_2 contour.

II-3.3 Integration around the poles and branch points is easy but along the banks of the branch cuts it is most difficult to perform. For a given transform the analytical structure of the integrand along the sides of the branch cuts changes when the type of branch cuts is changed. If the cuts are made in such a way that the conditions of the theorem given in II-1.0 are not violated, then all the integrals must yield the same results. Therefore, if the s plane is cut in several ways, different types of integrals will be obtained and perhaps one which can be integrated will be found. In this investigation a large variety of integrals must be dealt with and it is convenient to consider not only one but several types of branch cutting.

In Fig. 8II the Br_2 contours of integration are considered for some simple typical transforms

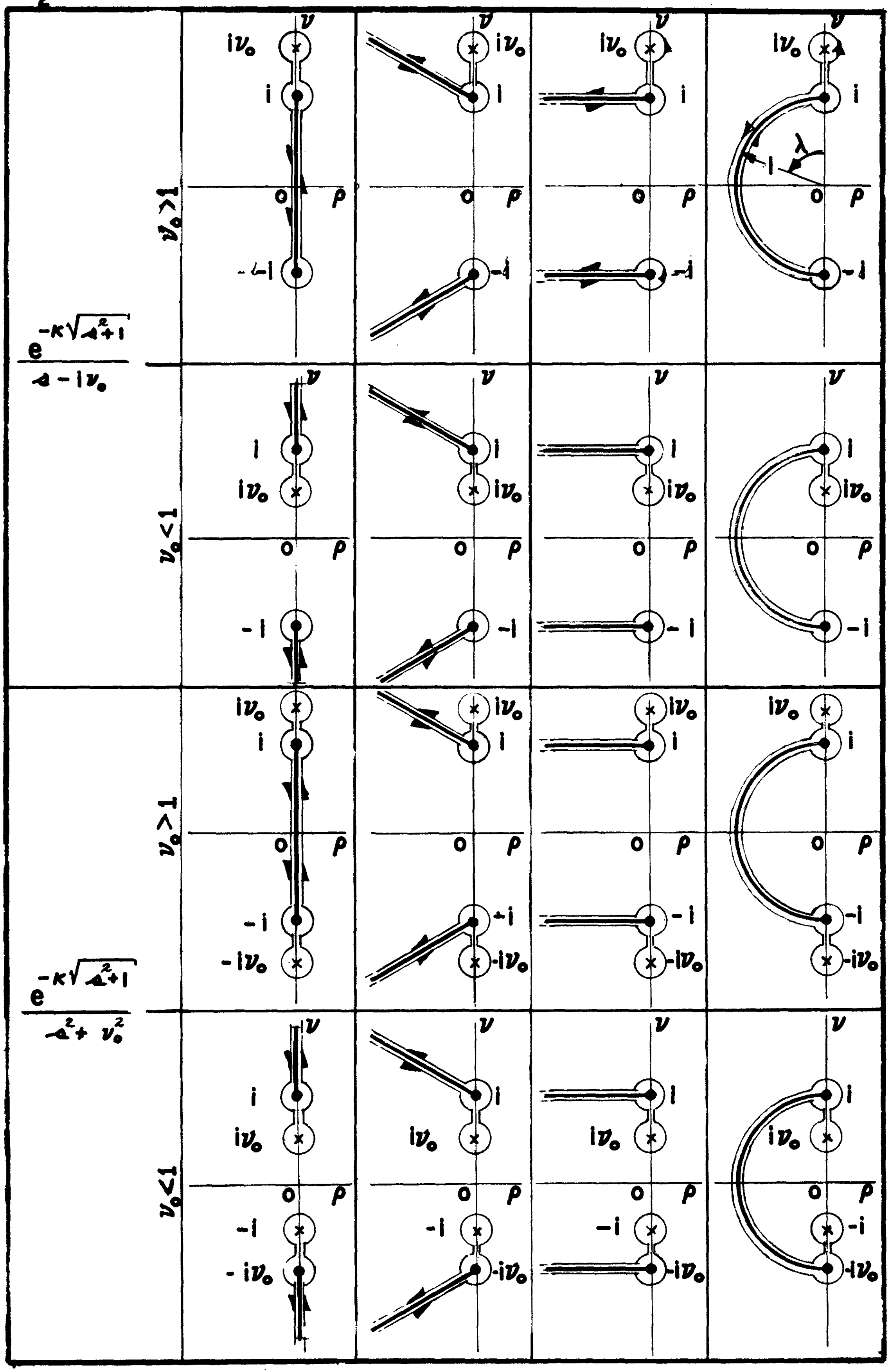
$$\frac{e^{-\kappa\sqrt{s^2+1}}}{s - j\nu_0}$$

$$\frac{e^{-\kappa\sqrt{s^2+1}}}{s^2 + \nu_0^2}$$

In Section II-3.3 the analytical expression for the integral corresponding to different types of branch cutting is given.

Br_2 CONTOUR

FIG. N° 8 II.



II-3.4 In this subsection some illustrative examples will be given showing the change of the analytical form of the integrand when the type of branch cutting of the s plane is altered.

One elementary transform was intentionally selected to show, at the same time, how involved the integrand becomes even in the simplest cases. Our purpose in this example is merely to illustrate how the selection of branch cutting affects the corresponding integrals, but they will not be integrated here.

All simple intermediate steps of algebra will be omitted and only the final results will be given.

Of course, the sign distribution plots in Figs. 3II, 4IIb and 4IIc must be used.

Take, for example, the simple transform

$$\phi(s) = \frac{e^{-\varkappa\sqrt{s^2+1}}}{s - j\nu_0}$$

and write

$$\varphi(\tau) = \mathcal{L}^{-1}_{(t)}\phi(s).$$

<u>Cut 2IIb. $\nu_0 > 1$</u>

$$\varphi(\tau) \begin{cases} = 0 & \text{for } \tau < \varkappa \\ = e^{i(\nu_0\tau - \varkappa\sqrt{\nu_0^2-1})} - \frac{2i}{\pi}\int_0^1 \frac{\sinh \varkappa\sqrt{1-\nu^2}}{\nu_0^2-\nu^2}\left[\nu_0 \cos\nu\tau + is \sin\nu\tau\right] d\nu = & \\ = e^{i(\nu_0\tau - \varkappa\sqrt{\nu_0^2-1})} - \frac{1}{\pi}\int_{-1}^{+1} \frac{e^{i\nu\tau}}{\nu_0-\nu} \sinh\varkappa\sqrt{1-\nu^2}\, d\nu & \text{for } \tau > \varkappa \end{cases} \qquad (2)\text{II}3$$

<u>Cut 2IIc. $\nu_0 \lessgtr 1$</u>

$$\varphi(\tau)\begin{cases} =0 & \text{for } \tau<\kappa \\ =e^{i(\nu_0\tau-\kappa\sqrt{\nu_0^2-1})}-\dfrac{i}{\pi}\displaystyle\int_0^{\pi}\frac{(1-\nu_0 e^{i\lambda})e^{i\tau e^{i\lambda}}}{1+\nu_0^2-2\nu_0\cos\lambda}\sin\left\{\kappa\sqrt{2\sin\lambda}\,e^{i(\frac{\pi}{4}+\lambda)}\right\}d\lambda & \text{for } \tau>\kappa \end{cases} \quad (3)\text{II}3$$

λ = variable of integration. See Fig. 8II.

<u>Cut 1IIc. $\nu_0<1$ and $\nu_0>1$</u>

$$\varphi(\tau)\begin{cases} =0 \quad \text{for } \tau<\kappa \\ =e^{i(\nu_0\tau-\kappa\sqrt{\nu_0^2-1})}\underline{\ }\dfrac{2}{\pi}\cos\nu_0\tau\displaystyle\int_0^{\infty}\frac{(\rho+j\nu_0)e^{-\rho\left(\tau+\kappa\sqrt{\frac{1+\sqrt{1+(\frac{2}{\rho})^2}}{2}}\right)}}{\rho^2+1-\nu_0^2+2\rho\nu_0 i}\sin\kappa\rho\sqrt{\frac{-1+\sqrt{1+(\frac{2}{\rho})^2}}{2}}\,d\rho\,+ \\ \quad +\dfrac{2}{\pi}\sin\nu_0\tau\displaystyle\int_0^{\infty}\frac{e^{-\rho\left(\tau+\kappa\sqrt{\frac{1+\sqrt{1+(\frac{2}{\rho})^2}}{2}}\right)}}{\rho^2+1-\nu_0^2+2\rho\nu_0 i}\sin\kappa\rho\sqrt{\frac{-1+\sqrt{1+(\frac{2}{\rho})^2}}{2}}\,d\rho \quad \text{for } \tau>\kappa \end{cases} \quad (4)\text{II}3$$

in which $s=\rho+i\nu$; $\nu=1$ and ρ variable of integration.

It can be seen that in cut 1IIc, the integrals represent envelopes of $\cos\nu\tau$ and $\sin\nu\tau$ respectively. Besides the integrand goes to zero very rapidly when τ, κ and ρ increase. It is clearly suitable for numerical integration, and asymptotic solutions may be obtained easily.

<u>Cut 1IIb. $\nu_0<1$</u>

$$\varphi(\tau)\begin{cases} =0 & \text{for } \tau<\kappa \\ =e^{-\kappa\sqrt{1-\nu_0^2}}e^{i\nu_0\tau}-\dfrac{2}{\pi}\displaystyle\int_1^{\infty}\frac{\sin\kappa\sqrt{\nu^2-1}}{\nu^2-\nu_0^2}\left[\nu\cos\nu\tau+i\nu_0\sin\nu\tau\right]d\tau & \text{for } \tau>\kappa \end{cases} \quad (5)\text{II}3$$

The corresponding discussion for cut 1IIa and some curved cuts will be given in the next subsection.

II-3.5 Branch cutting 1IIa offers the advantage of allowing a rotation of the cut lines around i and -i. We can look for a possible value of ψ for which the integrand becomes simpler and more suggestive.

It can be proved, after a rather long development that the real and imaginary parts of the radical can be written as

$$\left.\begin{aligned} u &= \pm\sqrt{rR}\,\sin\frac{\psi+\theta}{2} \\ v &= \pm\sqrt{rR}\,\cos\frac{\psi+\theta}{2} \end{aligned}\right\} \qquad (6)\text{II}3$$

the corresponding sign being taken in accordance with Fig. 3II. The meaning of the above notation is given in Fig. 9II. The corresponding integral which yields

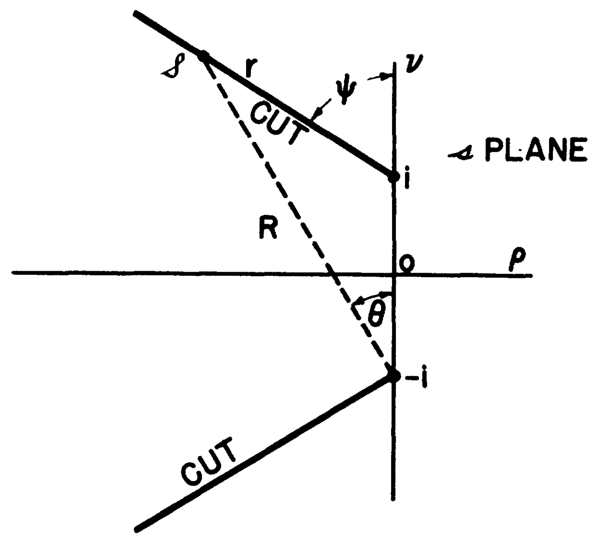


FIG N° 9 II.

$$\mathcal{L}^{-1}\,\frac{e^{-\kappa\sqrt{s^2+1}}}{s-j\nu_0}$$

is given by

Cut 1IIa. $\nu_0>1$ or $\nu_0<1$

$$\varphi(\tau)\begin{cases} =0 & \text{for } \tau<\kappa \\ =e^{i(\nu_0\tau-\kappa\sqrt{\nu_0^2-1})} & \\ =-\dfrac{e^{i(\nu\tau+\psi)}}{\pi}\displaystyle\int_0^\infty\frac{e^{ir\tau e^{i\psi}}}{(1-\nu_0+re^{i\psi})}\,e^{-\kappa\sqrt{rR}\sin(\frac{\psi+\theta}{2})}\times\sin\kappa\sqrt{rR}\cos\left(\frac{\psi+\theta}{2}\right)dr & \text{for } \tau>\kappa \\ =-\dfrac{e^{-i(\nu\tau+\psi)}}{\pi}\displaystyle\int_0^\infty\frac{e^{-ir\tau e^{i\psi}}}{(1+\nu_0+re^{-i\psi})}\,e^{-\kappa\sqrt{rR}\sin(\frac{\psi+\theta}{2})}\times\sin\kappa\sqrt{rR}\cos\left(\frac{\psi+\theta}{2}\right)dr & \end{cases} \qquad (7)\text{II}4$$

The integral in the above expression represents the envelope of the rapidly oscillating functions $e^{i(\nu\tau+\psi)}$

and $e^{-i(\nu\tau+\psi)}$. This is a convenient form for numerical computations. Unfortunately, no particular value of ψ makes the integrand simple and suitable.

Nevertheless, the above integral suggests the next step. Since r, R, Θ and ψ are available, a cutting line can be chosen so that the integrand becomes simpler. The relations (6)II3 are independent of the form of the cutting line, so that it can be used for curved branch cuts as well. In this way several curved branch cuts were carefully investigated. Some of them work well for only one particular transform but are no good for the general prototype $F(s,\sqrt{s^2+1})$.

A careful and almost exhaustive investigation was performed in the s plane in order to obtain a constructive solution of the expression

$$\varphi(\tau,x)=\mathcal{L}^{-1}F(s,\sqrt{s^2+1})\,e^{-x\sqrt{s^2+1}}$$

by considering branch cuts made of straight or curved lines. Particular solutions for some simple transforms can be obtained. The solutions are series expansions usually unsuitable for computation. Sometimes the series expansion is not valid when the applied frequency is very close to the cut-off frequency of the wave guide; this is a common and very important case in practice.

The solution of the problem of inversion will be given in Chs. III and IV with the aid of some complex transformations.

Section 4 - Transient formation. Possible interpretation of the integrals along the banks of the branch cuts. Secondary transient waves.

II-4.0 It has been shown that the inverse transformation of a given transform can be obtained by the integration of the inverse integral along contours which surround the singularities and branch cuts in the s plane. These lines of integration are sometimes referred to as the Brownich's Br_2 contours. The object of Section 4 is to make an overall investigation of the integrals along the branch cuts in order to find the characteristic behavior of the function when time changes.

II-4.1 For clarity the following simple transform will be used as a beginning:

$$\frac{e^{-\varkappa\sqrt{s^2+1}}}{s^2+\nu_0^2} .$$

This transform was selected because of its behavior as $\frac{M}{s^2}$ when $s \to \infty$. The branch cut 1IIb is used.

It can be found that

$$\mathcal{L}^{-1}\frac{e^{-\varkappa\sqrt{s^2+1}}}{s^2+\nu_0^2} = \varphi(\tau,\varkappa) = \left\{ \sin\left(\nu_0\tau - \varkappa\sqrt{\nu_0^2-1}\right) - \frac{\nu_0}{\pi}\int_{-1}^{+1} \frac{\sinh \varkappa\sqrt{1-\nu^2}}{\nu_0^2-\nu^2}\cos\nu\tau\, d\nu \right\} u_{-1}(\tau-\varkappa) \qquad (1)\text{II4}$$

in which $u_{-1}(\tau-\varkappa)$ is the unit step function shifted by $\varkappa$. The first term inside the bracket will become the steady state only when $\tau \to \infty$. The integral, which is the contribution of the banks of the cut, represents a transient term. We will make an overall discussion of its behavior as a function of time. In order to do so, let a function be defined as follows:

$$\theta(\nu)=\begin{cases}0 & \text{for } -\infty<\nu<-1\\ \dfrac{2\,\sinh\kappa\sqrt{1-\nu^2}}{\nu_0^2-\nu^2} & \text{for } -1\leq\nu\leq 1\\ 0 & \text{for } 1<\nu<\infty\end{cases}\ . \qquad (2)\text{II4}$$

It will be recalled also that ν represents the real normalized frequency $(\nu=\frac{\omega}{\omega_c})$. The above function will be considered as a continuous frequency spectrum which exists only inside the band width -1 to +1.

The corresponding time function of (2)II4 can be found. The Fourier transform of $\theta(\nu)$ is:

$$\alpha(T)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\theta(\nu)\,e^{\nu T}d\nu=\frac{1}{\pi}\int_{-1}^{+1}\theta(\nu)\,e^{\nu T}d\nu=+\frac{\nu_0}{\pi}\int_{-1}^{+1}\frac{\sinh\kappa\sqrt{1-\nu^2}}{\nu_0^2-\nu^2}\cos\nu T\,d\nu \qquad (3)\text{II4}$$

since $\theta(\nu)$ is an even function of ν.

Since $\theta(\nu)$ exists only within a finite band width, it is expected that $\alpha(T)$ will spread on both sides of the time origin and will show a rapid monotonic decay in amplitude as $|t|\to\infty$. Observe that the zero of T corresponds to the value κ of τ.

In (1)II4 the integral (3)II4 is multiplied by $u_{-1}(\tau-\kappa)$. Therefore $\alpha(t)u_{-1}(\tau-\kappa)$ has a value zero for $\tau<\kappa$. In Figs. 10IIa, b and c, the second member and the complete expression for $\varphi(\tau,\kappa)$ are shown, not the exact representations but only sketches of the expected functions. The exact curves are given in Chs. III and IV.

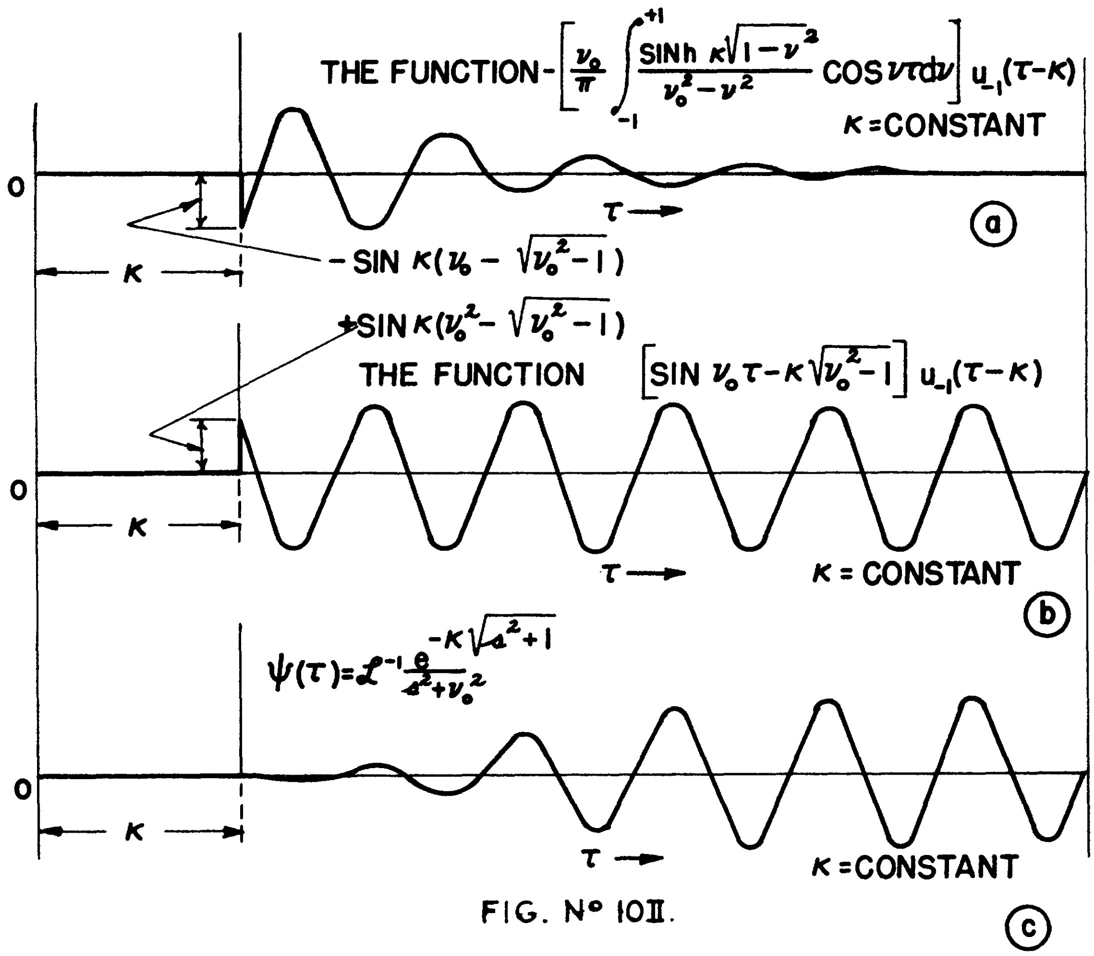


FIG. N° 10II.

II-4.11 The transform selected in this example is of order $O(\frac{M}{s^2})$ when $s \to \infty$. This means for $\tau = \kappa$ one must have

$$0 = \varphi(+\kappa, \kappa) \quad ,$$

whence

$$\frac{\pi \sin \kappa (\nu_0 - \sqrt{\nu_0^2 - 1})}{\nu_0} = \int_{-1}^{+1} \frac{\sinh \kappa \sqrt{1-\nu^2}}{\nu_0^2 - \nu^2} \cos(\nu\kappa)\, d\nu \quad . \qquad (4)\text{II4}$$

By this simple method values of a complicated definite integral are obtained. That (4)II4 is true can be shown by direct method (Ch. III) but it requires considerable labor.

By using the initial value theorem, some integrals can be directly evaluated which are in general quite involved. For example, take (2)II3, (3)II3, (4)II3 and

(5)II3 and use the property of the corresponding first member

$$\varphi(\tau,\kappa)_{\tau=+\kappa}=1+i0 \ ;$$

then other interesting integrals can be evaluated easily.

During this investigation a rather large family of involved integrals is evaluated by this method. The results are not given here because they are only side products and have no direct importance for the main body of the investigation.

II-4.12 Evaluate the corresponding integral around the branch cut and the branch points.

a. - The contributions to the integral around the branch points are both zero.

b. - The contribution of the left bank of the cut is given by

$$+\frac{\nu_o}{2\pi}\int_{-1}^{+1}\frac{e^{i\nu\tau}e^{-\kappa\sqrt{1-\nu^2}}}{\nu_o^2-\nu^2}\,d\nu \ . \qquad (5)\text{II4}$$

c. - The contribution of the right bank of the cut is given by

$$-\frac{\nu_o}{2\pi}\int_{-1}^{+1}\frac{e^{i\nu\tau}e^{+\kappa\sqrt{1-\nu^2}}}{\nu_o^2-\nu^2}\,d\nu \ . \qquad (6)\text{II4}$$

If attention is paid to the sign of the radical in the exponent, it is immediately recognizable that it represents waves moving along the positive and negative direction of the x_3 axis ($\kappa = 2\pi\frac{x_3}{\lambda_c}$). These waves move in a dispersive media and have a continuous frequency spectrum which exists only within the band -1 and +1 of the normalized frequency. The above integral can be looked upon as secondary waves which exist only in the transient state condition in the guide and disappear in the steady state.

A tentative explanation of the existence of these transient waves can be given as follows:

The term $\sin(\nu_0\tau - x\sqrt{\nu_0^2-1})u_{-1}(\tau - x)$ can be considered as an incident wave which moves with the speed of light. The wave front excites the cross sections of the guide which then reradiate. All these cross sections form a continuum of sources and the sum of their effects is given by the integrals (5)II4 and (6)II4.

The convergence of the integral (6)II4 when $x \to \infty$ will be discussed fully in the next chapter.

II-4.2 Consider the basic transform of this study

$$F(s, \sqrt{s^2+1})e^{-x\sqrt{s^2+1}}.$$

The inverse transform function is obtained by performing the corresponding integration around the singularities and along the banks of the branch cuts. Now take the integral along the bank; the sign of the radical in the exponent will change from one bank to the other. Therefore the secondary waves moving along each direction can be separated.

The above interpretation of the integrals along the branch cuts is only a tentative one. What was said before does not constitute a proof and still needs a full discussion of the convergence of the corresponding integrals. This problem will be treated in the next chapter.

CHAPTER III

The inverse Laplace transform of the prototype and the complex transformation $Z = s - \sqrt{s^2+1}$.

Section 0. - Object of the chapter.

III-0.0 This chapter develops the inverse Laplace transforms of a basic family of transforms which appear in the propagation of electromagnetic waves in systems of generalized cylindrical configurations. These results will be directly applied to transient phenomena in wave guides, excited at a cross section considered as the origin.

In Ch. I the basic transform was found to be

$$\phi(s) = F(s, \sqrt{s^2+1})\, e^{-k\sqrt{s^2+1}} \qquad (1)\text{III0}$$

in which F is the ratio of two polynomials in s and $\sqrt{s^2+1}$. This transform contains a large number of cases of practical application. For the study of electromagnetic waves in cylinders, $F(s,\sqrt{s^2+1})$ is restricted to $O(\frac{M}{s^\gamma})$ for $\gamma \geq 2$ as $s \to \infty$.

In Ch. II, the problem of the inversion in the s plane using different types of branch cutting was considered. It was discovered that, 1st, the integrals in the simplest cases are rather involved and difficult to compute, and 2nd, the presence of the radical in $F(s,\sqrt{s^2+1})$ is undesirable, mainly when the branch points +i and -i are also poles.

In Ch. III a complex transformation, namely $Z = s - \sqrt{s^2+1}$, will be introduced which eliminates the radical and yields a simple method of obtaining the inverse Laplace transform of the prototype. The solutions first obtained were in the form of uniformly convergent series of the Neumann type. It was found that these series solutions converge extremely slowly and are not suitable for numerical computations. In Ch. IV the problem of the summation of these series is considered.

During the preparation of this manuscript it became apparent that the above series were Lommel's functions of order 0 and 1. With this knowledge a short cut can be taken so that a much more compact presentation of this chapter is made possible.* Nevertheless, the original mathematical derivation will be kept because first, there is little time to make the required changes, and second, Lommel's functions are neither well known nor well studied and the method presented here is a simple way of approaching them. Finally, a new integral representation for this function is given.

Lommel's functions are not tabulated. The corresponding series expansions commonly given converge very slowly. The object of Ch. IV is to develop the formulas and processes to compute these functions.

The material presented in this chapter is outlined in the next paragraphs.

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* See Appendix I.

III-0.1 Section 1 will contain: Introduction of the complex transformation $Z = s - \sqrt{s^2+1}$ in the basic Laplace transform. Poles and other properties of the G(Z) function. Partial fraction expansion and decomposition of the general transform in subtypes. Derivation of the fundamental types of integrals.

III-0.2 Section 2 will contain: A general discussion of the contours of integration (Γ_2) in the Z plane. The reciprocal transformation $z = s + \sqrt{s^2+1}$ is also studied. An investigation is conducted in order to find some mapping properties of the s into the Z plane. New and typical paths of integration in the Z and z planes.

III-0.3 Section 3 will include: The integration of typical integrals found in Section 1. The solutions given in this section have the form of series expansion of the Neumann type.

III-0.4 Section 4 will include: The introduction of compact solutions in terms of functions of Lommel. Solutions when the poles of the transform are simple ones. A study of properties of Lommel's functions and different forms of the solutions. Behavior of the solutions at $\tau = \kappa$ and $\tau \to \infty$. Introduction of the generating functions of the inverse transforms. The compact solution for multiple poles. The concept of the group velocity and its expression in terms of the poles.

III-0.5 Section 5 will contain: An application of this theory in computing some useful transforms.

III-0.6 Section 6 will include: A direct application of this theory to wave guides with some examples.

Section 1 - The complex transformation $Z = s - \sqrt{s^2+1}$. Poles and properties of G(Z). Partial fraction expansion of G(Z). Derivation of typical integrals.

III-1.0 Let the complex transformation

$$Z = s - \sqrt{s^2+1} \quad \text{in which}$$

$$Z = x + iy \qquad (1)\text{III1}$$

be introduced. From it one can write

$$(s - Z)^2 = s^2 + 1 \text{ , or}$$

$$Z^2 - 2sZ - 1 = 0 \text{ .}$$

This is a second-degree equation in Z; call the roots Z' and Z''. Since the last term is -1,

$$Z'Z'' = -1 \text{ or}$$

$$Z'' = -\frac{1}{Z'} \text{ .}$$

It is known from (1)III1 on one of the roots, that

$$Z' = Z = s - \sqrt{s^2+1} \text{ ,}$$

and therefore

$$Z'' = -\frac{1}{Z} = s + \sqrt{s^2+1} \text{ ;}$$

adding and subtracting them,

$$s = \frac{1}{2}\left(Z - \frac{1}{Z}\right) \text{ and}$$

$$\sqrt{s^2+1} = -\frac{1}{2}\left(Z + \frac{1}{Z}\right) \text{ .} \qquad (2)\text{III1}$$

This last equation shows that the radical can be expressed as a rational function of Z.

III-1.1 The substitution of s and $\sqrt{s^2+1}$ in the basic transform leads to

$$F(Z)\,e^{-\frac{\kappa}{2}(Z+\frac{1}{Z})} \qquad (3)\text{III1}$$

and notice that F(Z) is now equal to the ratio of two integral polynomials in Z. Let these polynomials be called $F_N(Z)$ and $F_D(Z)$ so that

$$F(Z) = \frac{F_N(Z)}{F_D(Z)} \qquad (4)\text{III1}$$

III-1.2 The inverse Laplace transform of (1)III0 is

$$\mathcal{L}^{-1} F(s,\sqrt{s^2+1})\,e^{-\kappa\sqrt{s^2+1}} = \varphi(\tau,\kappa) = \frac{1}{2\pi i}\int_{\gamma_s} F(s,\sqrt{s^2+1})\,e^{\tau s-\kappa\sqrt{s^2+1}}\,ds, \qquad (5)\text{III1}$$

γ_s being a contour in the s plane, which gives the correct solution. It may be, for example, the Br_1 or the Br_2 contour indicated in Ch. II. If the transformation (1)III1 is introduced, the contour γ_s transforms into a contour γ_Z in the Z plane. Then

$$\varphi(\tau,\kappa) = \frac{1}{2\pi i}\int_{\gamma_Z} F(Z)\,e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\kappa}{2}(Z+\frac{1}{Z})}\,\frac{1}{2}\,\frac{1+Z^2}{Z^2}\,dZ =$$

$$= \varphi(\tau,\kappa) = \frac{1}{2\pi i}\int_{\gamma_Z} G(Z)\,e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\kappa}{2}(Z+\frac{1}{Z})}\,dZ\,, \qquad (6)\text{III1}$$

in which

$$G(Z) = \frac{1+Z^2}{2Z^2}\,F(Z) = \frac{G_N(Z)}{G_D(Z)} \qquad (7)\text{III1}$$

which is also the ratio of two integer polynomials in Z.

In the next section some important properties of the function G(Z) will be discussed and in Section 3 the transformed contour γ_Z will be studied.

III-1.3 Some important properties of the poles of G(Z).

Since G(Z) is an rational function of Z the only singularities are poles. These poles are the roots of $G_D(Z)$. From (7)IIIl and (4)IIIl, the roots of $G_D(Z)$ are the roots of $F_D(Z)$ and possibly $Z=0$.

Theorem 1. - If $F(s,\sqrt{s^2+1})$ can be written as

$$F(s,\sqrt{s^2+1}) = \frac{F_N(s,\sqrt{s^2+1})}{(s-s_k)\,\Theta_D(s,\sqrt{s^2+1})} \qquad (8)\text{IIIl}$$

and s_k is not a root of $F_N(s,\sqrt{s_k^2+1})$, then:

$$Z_k = s_k - \sqrt{s_k^2+1}$$
$$Z_k^* = s_k + \sqrt{s_k^2+1} \qquad (9)\text{IIIl}$$

are simple poles of G(Z); and are such that

$$Z_k^* \times Z_k = -1 \qquad (10)\text{IIIl}$$

Proof: The factor $s - s_k$:

$$s - s_k = \frac{1}{2}(Z - \frac{1}{Z}) - s_k =$$

$$= \frac{Z^2 - 2Zs_k - 1}{2Z}$$

$$= \frac{(Z-Z_k)(Z-Z_k^*)}{2Z}$$, after the transformation

in which

$$Z_k = s_k - \sqrt{s_k^2+1}, \text{ and}$$
$$Z_k^* = s_k + \sqrt{s_k^2+1};$$

by multiplication:

$$Z_k \times Z_k^* = -1 .$$

Under the hypothesis of the theorem, G(Z) can be written as

$$G(Z) = \frac{(1+Z^2)F_1(Z)}{(Z-Z_k)(Z-Z_k^*)\,\Theta_D(Z)}\,\frac{1}{Z}; \qquad (11)\text{IIIl}$$

thus the theorem is completely proved.

Some simple corollaries of this theorem will be indicated (s_k lies on δ_I):

a. If s_k is real Z_k and Z_k^* are also real.

b. If s_k is zero then $Z_k = -1$ and $Z_k^* = 1$.

c. If s_k is pure imaginary three cases will be distinguished.

c_1. $|s_k| > 1$

c_2. $|s_k| < 1$

c_3. $|s_k| = 1$.

If c_1, then: Z_k and Z_k^* are purely imaginary.

If c_2, then: Z_k and Z_k^* are complex and lie on the unit circle.

If c_3, then: $Z_k^* = -i$ and $Z_k = i$.

d. If $s_k \to \infty$ then $Z_k \to 0$ and $Z_k^* \to \infty$.

Theorem 2. - If $F(s, \sqrt{s^2+1})$ can be written as:

$$F(s,\sqrt{s^2+1}) = \frac{F_N(s,\sqrt{s^2+1})}{(\sqrt{s^2+1}-s_\lambda)\Theta_D(s,\sqrt{s^2+1})} \quad (12)\text{III1}$$

and if this factor does not appear in the numerator, then:

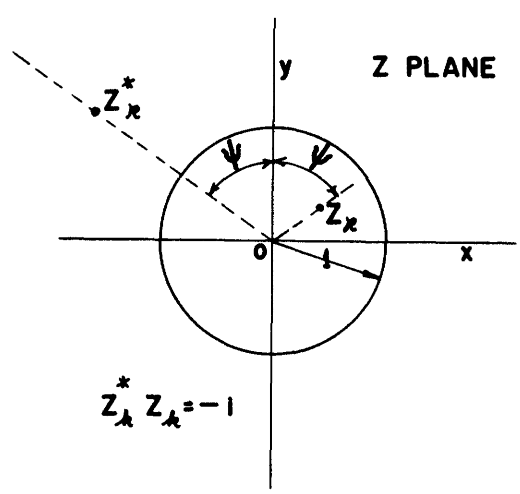
$$\left.\begin{aligned} Z_\lambda &= s_\lambda + \sqrt{s_\lambda^2 - 1}, \text{ and} \\ Z_\lambda^* &= s_\lambda - \sqrt{s_\lambda^2 - 1} \end{aligned}\right\} \quad (13)\text{III1}$$

are simple poles of G(Z) such that

$$Z_\lambda Z_\lambda^* = 1. \quad (14)\text{III1}$$

The proof of this theorem is similar to the one given for Theorem 1 and is, therefore, omitted.

FIG. I III



The most important use of this theorem is for $s_\lambda = 0$, and the roots become

$$Z_\lambda = i \text{ , and}$$
$$Z_\lambda^* = -i \text{ .}$$

<u>Theorem 3</u>. - If $F(s,\sqrt{s^2+1})$ can be written as

$$F(s,\sqrt{s^2+1}) = \frac{F_N(s,\sqrt{s^2+1})}{(s-s_k)^\alpha \, \theta_D(s,\sqrt{s^2+1})} \qquad (15)\text{IIIl}$$

and if no cancellation of this factor can be made, then:

$$\left.\begin{aligned} Z_k &= s_k - \sqrt{s_k+1} \\ Z_k^* &= s_k + \sqrt{s_k+1} \end{aligned}\right\} \qquad (16)\text{IIIl}$$

are both roots of $G(Z)$ and have a multiplicity α. Also

$$Z_k \times Z_k^* = -1 \text{ .} \qquad (17)\text{IIIl}$$

<u>Theorem 4</u>. - If $F(s,\sqrt{s^2+1})$ can be written as

$$F(s,\sqrt{s^2+1}) = \frac{F_N(s,\sqrt{s^2+1})}{(\sqrt{s^2+1}-s_\lambda)^\alpha \, \theta_D(s,\sqrt{s^2+1})} \qquad (18)\text{IIIl}$$

and a further simplification of the factor is not possible, then

$$\left.\begin{aligned} Z_\lambda^* &= s_\lambda - \sqrt{s_\lambda^2-1} \\ Z_\lambda &= s_\lambda + \sqrt{s_\lambda^2-1} \end{aligned}\right\} \qquad (19)\text{IIIl}$$

are both roots of $G_D(Z)$ and have a multiplicity α.

Here also

$$Z_\lambda^* \times Z_\lambda = 1. \qquad (20)\text{IIIl}$$

<u>Theorem 5</u>. - For each factor of the form $(s-s_k)$ or $(\sqrt{s^2+1}-s_\lambda)$ in the denominator of $F(s,\sqrt{s^2+1})$ a factor $2Z$ will appear in the numerator of $F(Z)$.

This theorem follows immediately.

An important result is: If $F(s,\sqrt{s^2+1})$ admits at least two simple factors such as those indicated above, then in

$$G(Z) = \frac{1}{2}\,\frac{1+Z^2}{Z^2}\,F(Z)$$

the term Z^2 in the denominator may be cancelled out.

Theorem 6. - The poles Z_k^* and Z_k (or Z_s and Z_s^*) can not coincide.

The last six theorems give the connection between singularities of $F(s,\sqrt{s^2+1})$ and the poles of G(Z). In Section 3 some other properties will be discussed.

III-1.4 Now a substantial simplification of the problem of the Laplace inversion of the basic transform (1)IIIO can be made.

Since G(Z) is the ratio of two integer polynomials in Z, it can be expanded in partial fractions, (finite number of terms), and a term by term integration can be performed.

Theorem 7. - "The degree of $G_D(Z)$ exceeds that of $G_N(Z)$ by γ." To prove this proceed as follows:

a. - Assume, by hypothesis, that $F(s,\sqrt{s^2+1})$ is of the order $O(\frac{M}{s^\gamma})\,\gamma \geq 1$ as $s \to \infty$. This means that if n is the larger degree of $F_N(s,\sqrt{s^2+1})$ then the degree of $F_D(s,\sqrt{s^2+1})$ must be $n+\gamma = m$.

b. - The largest power of Z in $F_N(s,\sqrt{s^2+1})$ is n. When s^n or $(\sqrt{s^2+1})^n$ is expressed in terms of Z a term of the form $\frac{(Z^2\pm1)^{2n}}{Z}$ and $F_N(Z)$ can be written as

$$F_N(Z) = \frac{1}{Z^n} F_N'(Z) \;;$$

in which $F_N'(Z)$ is an integral polynomial in Z of 2n degree.

The largest power of s, in $F_D(s, \sqrt{s^2+1})$, is $m = n+\gamma$. When $s^{n+\gamma}$ or $(\sqrt{s^2+1})^{n+\gamma}$ is expressed in terms of Z a term of the form $\frac{(Z^2\pm1)^{n+\gamma}}{Z^{n+\gamma}}$ is obtained and $F_D(Z)$ can be written as

$$F_D(Z) = \frac{1}{Z^{n+\gamma}} F_D'(Z)$$

in which F_D' is an integral polynomial in Z of $2(m+\gamma)$ degree.

Then:

$$F(Z) = \frac{Z^{\gamma} F_N'(Z)}{F_D'(Z)}$$

c. - Consider now G(Z). By (7)III1 the following expression is obtained.

$$G(Z) = \frac{Z^{\gamma-2}(1+Z^2)F_N'(Z)}{2F_D'(Z)} = \frac{G_N(Z)}{G_D(Z)} \;.$$

The difference between the degree of the denominator and numerator of G(Z) is given by

$$m - n = 2(n+\gamma) - 2n - \gamma = \gamma$$

and Theorem 7 is proved.

<u>Theorem 8.</u> - This is another simple but useful theorem concerning the number of poles of G(Z).

The number of poles of $G(Z)$ is equal to

$2m$ if $\gamma \geq 2$

$2m+1$ if $\gamma = 1$

in which m is the degree of $F_D(s, \sqrt{s^2+1})$.

The proof of this statement follows immediately from c of the last theorem. For if $\gamma \geq 2$ the polynomial $G_D(Z)$ is of the 2m degree. But if $\gamma = 1$, the difference $\gamma - 2 = -1$ introduces a new root, $Z = 0$, in the denominator.

Now the partial fraction expansion of $G(Z)$ can be written in terms of its poles.

$$\left.\begin{array}{l} \text{Let the poles at} \\ \quad Z_1 \text{ have a multiplicity } \alpha_1 \\ \quad Z_2 \text{ have a multiplicity } \alpha_2 \\ \quad \text{-------------------------} \\ \quad Z_q \text{ have a multiplicity } \alpha_q \\ \text{with the restriction that } \alpha_1 + \alpha_2 + \cdots + \alpha_q = m. \end{array}\right\} \qquad (21)\ \text{III}\ 1$$

By using the well-known form of the partial fraction expansion and considering these theorems:

$$G(Z) = \sum_{k=1}^{q} \sum_{j=1}^{\alpha_k} \frac{K_{kj}}{(Z-Z_k)^{\alpha_k - j+1}} + \sum_{k=1}^{q} \sum_{j=1}^{\alpha_k} \frac{K^*_{kj}}{(Z-Z^*_k)^{\alpha_k - j+1}} + \begin{cases} \dfrac{K_0}{Z} & \text{for } \gamma = 1 \\ \text{Zero for } \gamma \geq 2 \end{cases} \qquad (22)\ \text{III}\ 1$$

in which the constants

$$K_{kj} = \frac{1}{(j-1)!} \left\{ \frac{d^{(j-1)}}{dZ^{(j-1)}} \left[(Z-Z_k) G(Z) \right] \right\}_{Z=Z_k}$$

$$K^*_{kj} = \frac{1}{(j-1)!} \left\{ \frac{d^{(j-1)}}{dZ^{(j-1)}} \left[(Z-Z^*_k) G(Z) \right] \right\}_{Z=Z^*_k} \qquad (23)\ \text{III}\ 1$$

$$K_0 = \left[Z G(Z) \right]_{Z=0} \quad \text{for } \gamma = 1$$

It must be recalled that $Z_k Z_k^* = \pm 1$; the (+) or (-) signs being given in accordance with Theorems 1, 2, 3, and 4.

The expression of the inverse transform is found with these results.

$$\varphi(\tau,\kappa) = \begin{cases} 0 & \text{for } \tau < \kappa \\ \sum_{k=1}^{q} \sum_{j=1}^{\alpha_q} \frac{K_{kj}}{2\pi i} \int_{\gamma_Z} \frac{e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\kappa}{2}(Z+\frac{1}{Z})}}{(Z-Z_k)^{\alpha_k - j+1}}\, dZ + \begin{cases} 0 & ; \gamma > 1 \\ \frac{K_0}{2\pi i} \int_{\gamma_Z} e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\kappa}{2}(Z+\frac{1}{Z})} \frac{dZ}{Z} & ; \gamma = 1 \end{cases} & \tau > \kappa \end{cases} \qquad (24)\,\text{III}\,1$$

Notice that the terms with (*) have been omitted. Of course, the integrals

$$\sum_{k=1}^{q} \sum_{j=1}^{\alpha_q} \frac{K_{kj}^*}{2\pi i} \int_{\gamma_Z} \frac{e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\kappa}{2}(Z+\frac{1}{Z})}}{(Z-Z_k^*)^{\alpha_k - j+1}}\, dZ \neq 0 \,, \qquad (25)\,\text{III}\,1$$

but they represent the circuitation around the branch cut, in the s plane, in the Riemann surface $\mathcal{S}_{II}$. This integral does not occur in the value of $\varphi(\tau,\kappa)$ and, therefore, is omitted here. These last integrals represent waves moving in opposite directions since they are equivalent to one integration in $\mathcal{S}_{II}$ of the s plane. In Section 3 the truth of this statement will be checked.

Although the integrals (25) III 1 are not needed to obtain the value of the inverse transform $\varphi(\tau,\kappa)$, they will be computed. These integrals (25) III 1 generate Lommel's functions which are very useful in this investigation. Besides, the mathematical treatment is far more complete.

The foregoing can now be summarized.

The problem of the inversion of the basic transform $F(s,\sqrt{s^2+1})e^{-\tau\sqrt{s^2+1}}$, in which F is by hypothesis the ratio of two polynomials in s and $\sqrt{s^2+1}$ and such that F is of the order $O(\frac{M}{s^\gamma})$, $\gamma \geq 1$, has been reduced to the integration of the simpler integrals:

$$\frac{1}{2\pi i}\int_{\gamma_Z}\frac{e^{\frac{\xi}{2}(Z-\frac{1}{Z})+\frac{\tau}{2}(Z+\frac{1}{Z})}}{(Z-Z_k)^\alpha}dZ;\quad \frac{1}{2\pi i}\int_{\gamma_Z}\frac{e^{\frac{\xi}{2}(Z-\frac{1}{Z})+\frac{\tau}{2}(Z+\frac{1}{Z})}}{(Z-Z_k^*)^\alpha}dZ;\quad \frac{1}{2\pi i}\int_{\gamma_Z}\frac{e^{\frac{\xi}{2}(Z-\frac{1}{Z})+\frac{\tau}{2}(Z+\frac{1}{Z})}}{Z}dZ; \quad (26)\text{III }1$$

in which $\alpha \geq 1$ is a positive integer. The most important case is $\alpha = 1$. Apparently there is no analytical difference between the first and second integrals in (25)III1. Nevertheless, it is convenient to consider them separately.

Nothing definite has been said about the contour of integration γ_Z. In Section 2 this γ_Z contour will be studied. In Sections 3 and 4 of this chapter the writer will come back to the integration.

Section 2 - Branch cutting. Mapping properties. Contour of integration γ_Z for typical integrals. Introduction of the reciprocal transformation and resulting γ_Z contours.

III-2.0 In the course of this chapter, types 1IIb and 2IIb of branch cutting of the s plane will be systematically used. The sheets $\mathcal{S}_I$ and $\mathcal{S}_{II}$ of the s plane will be adopted as they were defined in Table I of Ch. II. The sign distribution diagrams given in Figs. 3II, 4IIb,c will be employed.

In this section the mapping properties of the introduced complex transformation will be studied as will the resulting γ_Z contours of integration for the typical integrals (24)III. The above process is repeated for $z = s + \sqrt{s^2+1}$.

III-2.1 Consider first the type 2IIb of branch cutting and define the Riemann Surfaces $\mathcal{S}_I$ and $\mathcal{S}_{II}$ as in Ch. II. Take first the sheet $\mathcal{S}_I$. The manner in which points of this surface map in the Z plane will be investigated.

a. - Large semicircle to the left-half plane, sheet $\mathcal{S}_I$.

Let

$$Z = s - \sqrt{s^2+1} \;.$$

If $|s| = 1$, then

$$(s^2+1)^{\frac{1}{2}} = \tau + \frac{1}{2} s^{-1} - \frac{1}{8} s^{-3} + \frac{1}{16} s^{-5} \;\ldots\ldots$$

when $|s|$ is very large;

$$Z = s - \sqrt{s^2+1} = -\frac{1}{2s} \;.$$

Take, for the left semicircle,

$$s = Re^{j\varphi} \;, \; \frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2} \;;$$

Then

$$Z = re^{j\theta} = -\frac{e^{-j\varphi}}{R}$$

$$\therefore \quad r \approx \frac{1}{R} \quad \text{and} \quad \theta \approx \pi - \varphi.$$

This result is indicated in Fig. 2III a.

b. - Real axis, negative side, sheet $\mathcal{S}_I$.

$s = -\rho$ and

$\sqrt{\rho^2+1}$ is pure real.

The sign of the radical must be +. See sign distribution diagram. Then

$$Z = -\rho + \sqrt{\rho^2+1} \;.$$

Then for

$\rho = 0 \quad Z = 1$, and for

$|\rho| = \infty \quad Z = 0.$

Real axis, positive side, $\mathcal{S}_I$ sheet

$s = +\rho$

$-\sqrt{\rho^2+1}$ is pure real.

The sign of the radical must be (-).

Then for

$\rho = 0 \quad Z = -1$ and for

$|\rho| = \infty \quad Z = 0.$

These results are indicated in Fig. 2III b.

c. - <u>Imaginary axis sheet $\mathcal{S}_I$</u>

c_1. - <u>Positive part above +i</u>

$s = i\nu \; ; \; |\nu| > 1$

$\sqrt{s^2+1} = i\sqrt{\nu^2-1}$ pure imaginary

$Z = i(\nu - \sqrt{\nu^2-1})$

When $\nu \to \infty \quad Z \to 0,$

$\nu = 1 \quad Z = i.$

c_2. - <u>Negative part below -i</u>

$s = -i \quad |\nu| > 1$

$\sqrt{s^2+1} = i\sqrt{\nu^2+1}$ pure imaginary. The sign is + from the corresponding sign diagram, so that

$Z = -i(\nu - \sqrt{\nu^2-1}).$

When: $|\nu| \to \infty \quad Z \to 0$
$|\nu| = 1 \quad Z = -i$.

These results are indicated in Fig. 2III c.

c_3. - <u>Imaginary axis between $+i$ and $-i$. Left approach sheet $\mathcal{S}_I$.</u>

$s = i\nu \quad |\nu| \leq 1$.

The radical $\sqrt{s^2+1} = \sqrt{1-\nu^2}$ is pure real. From the corresponding sign distribution diagram the following is obtained.

$Z = i\nu + \sqrt{1-\nu^2}$, since

$Z = x + iy$.

Then:

$x = \sqrt{1-\nu^2}$

$y = \nu$,

and therefore

$x^2 + y^2 = 1$. Circle of unit radius.

Also when:

$\nu = i$ then $Z = i$;

$\nu = 0$ then $Z = +1$;

$\nu = -i$ then $Z = -i$;

as indicated in Fig. 3III a.

c_4. - <u>Imaginary axis between $+i$ and $-i$. Right approach sheet $\mathcal{S}_I$.</u>

$s = j\nu$

$x = -\sqrt{1-y^2}$; $y = \nu$; $x^2 + y^2 = 1$

and, therefore, when:

$\nu = 1$ then $Z = +i$;

$\nu = 0$ then $Z = -1$;

$\nu = -1$ then $Z = -i$;

as indicated in Fig. 3III b.

d. - From the above results it can readily be seen that the sheet $\mathcal{S}_I$ maps in the inside of the unit circle in the Z plane. Therefore, it can be shown immediately that $\mathcal{S}_{II}$ will map outside of this unit circle.

e. - It is important to notice that the contour around and in the vicinity of the branch cut, Fig. 2III will map on the unit circle of the Z plane.

III-2.2 In this subsection a typical contour of integration will be considered. Figure 4IIIa represents a closed contour, Γ_s , which can be used in connection with the branch cutting 2IIb. Because of the partial fraction expansion of the basic transform, the integration with only one pole can now be considered. Let this be s_K. The closed contour Γ_s contains the Br_1 and Br_2 contours in the s plane. Supposing that this contour lies in the Riemann surface $\mathcal{S}_I$, one sees that then Γ_s transforms into Γ_Z in the Z plane, as indicated in Fig. 4IIIb. This statement can be justified with the discussion given in Subsection II-2.1. Notice that the whole contour Γ_s maps in the inside of the unit circle of the Z plane. Of course, in case of the supposition that Γ_s lies in $\mathcal{S}_{II}$, the whole contour will map in the outside of that unit circle.

FIG. 2 III

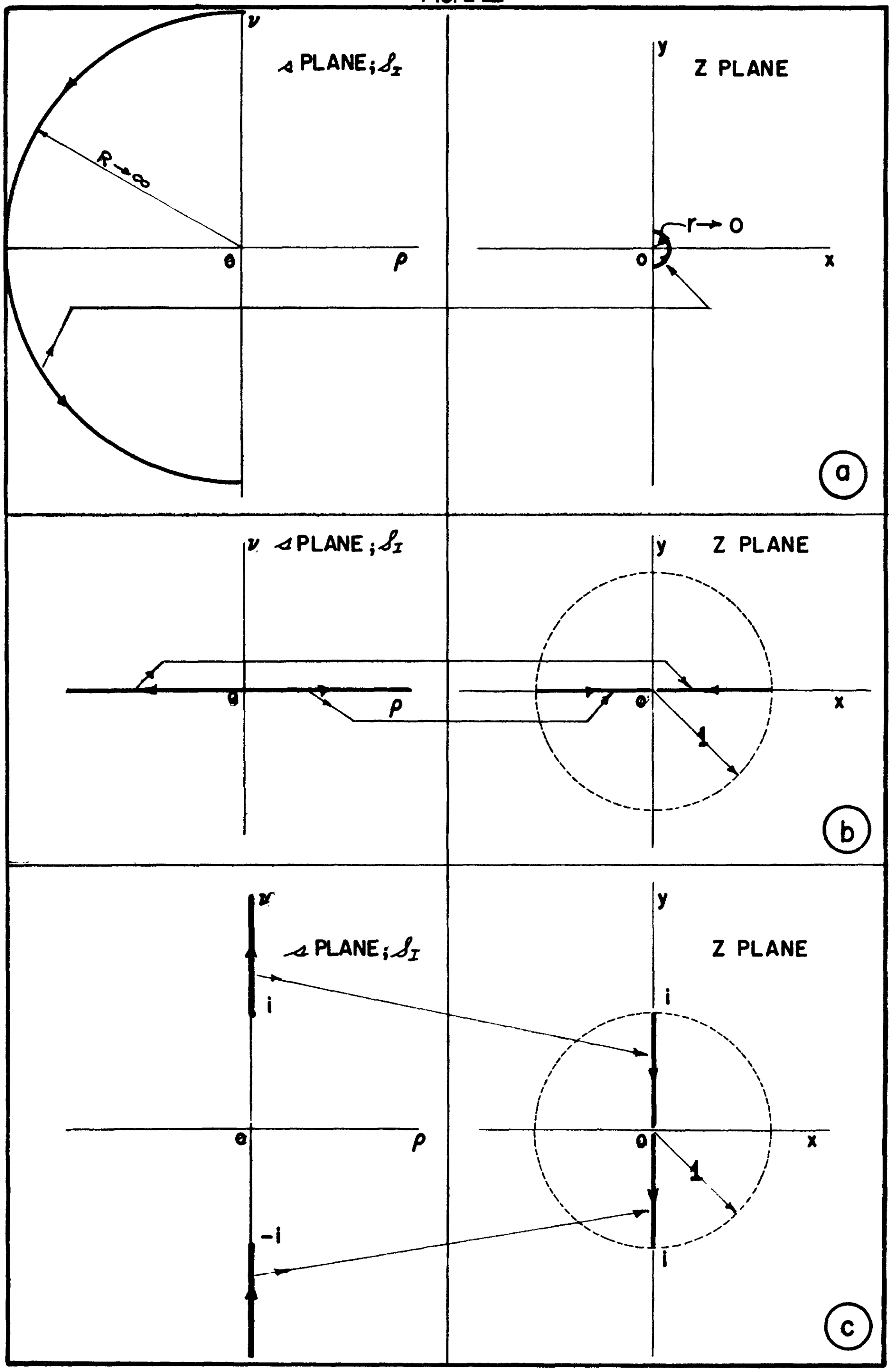
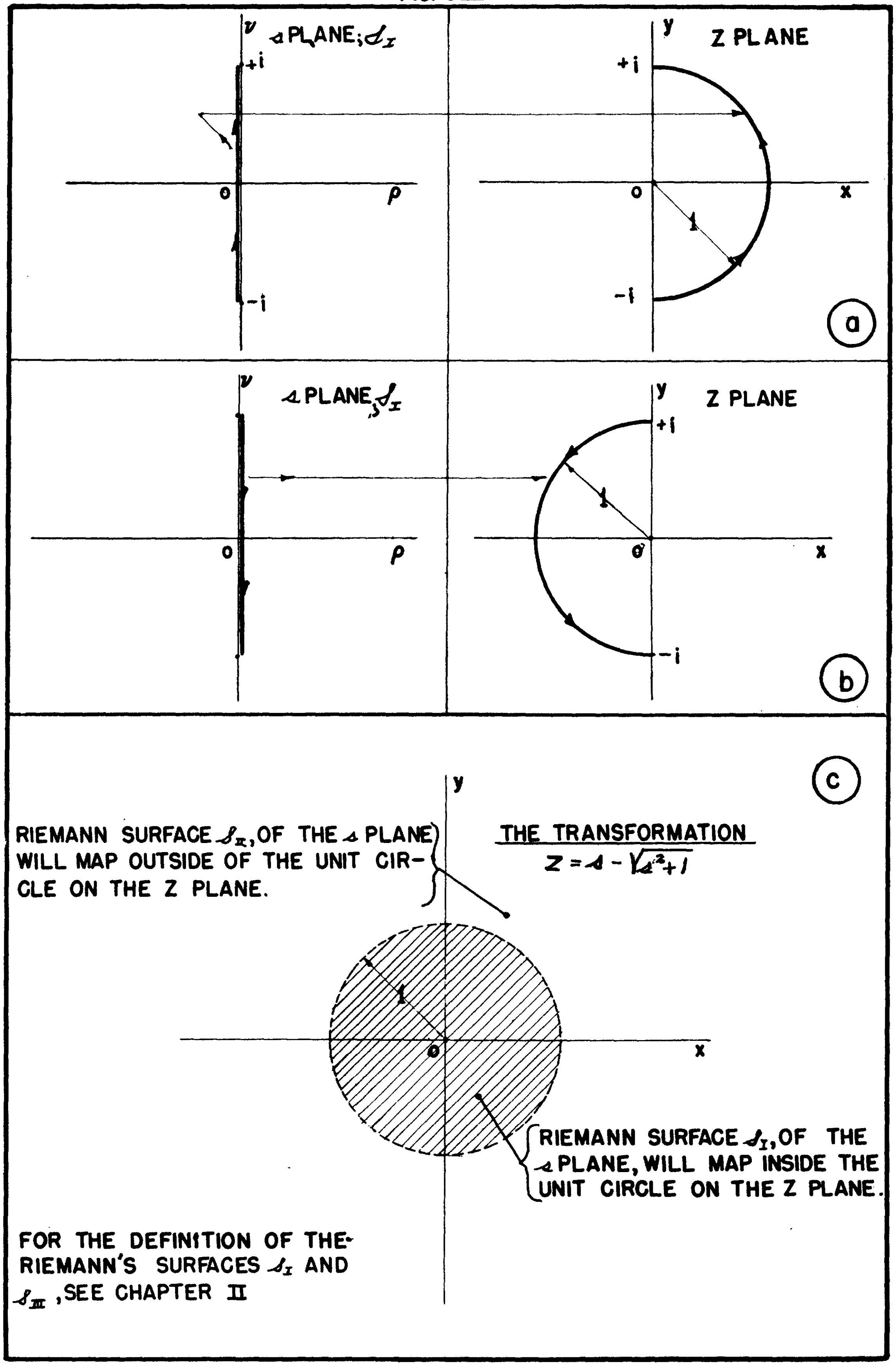
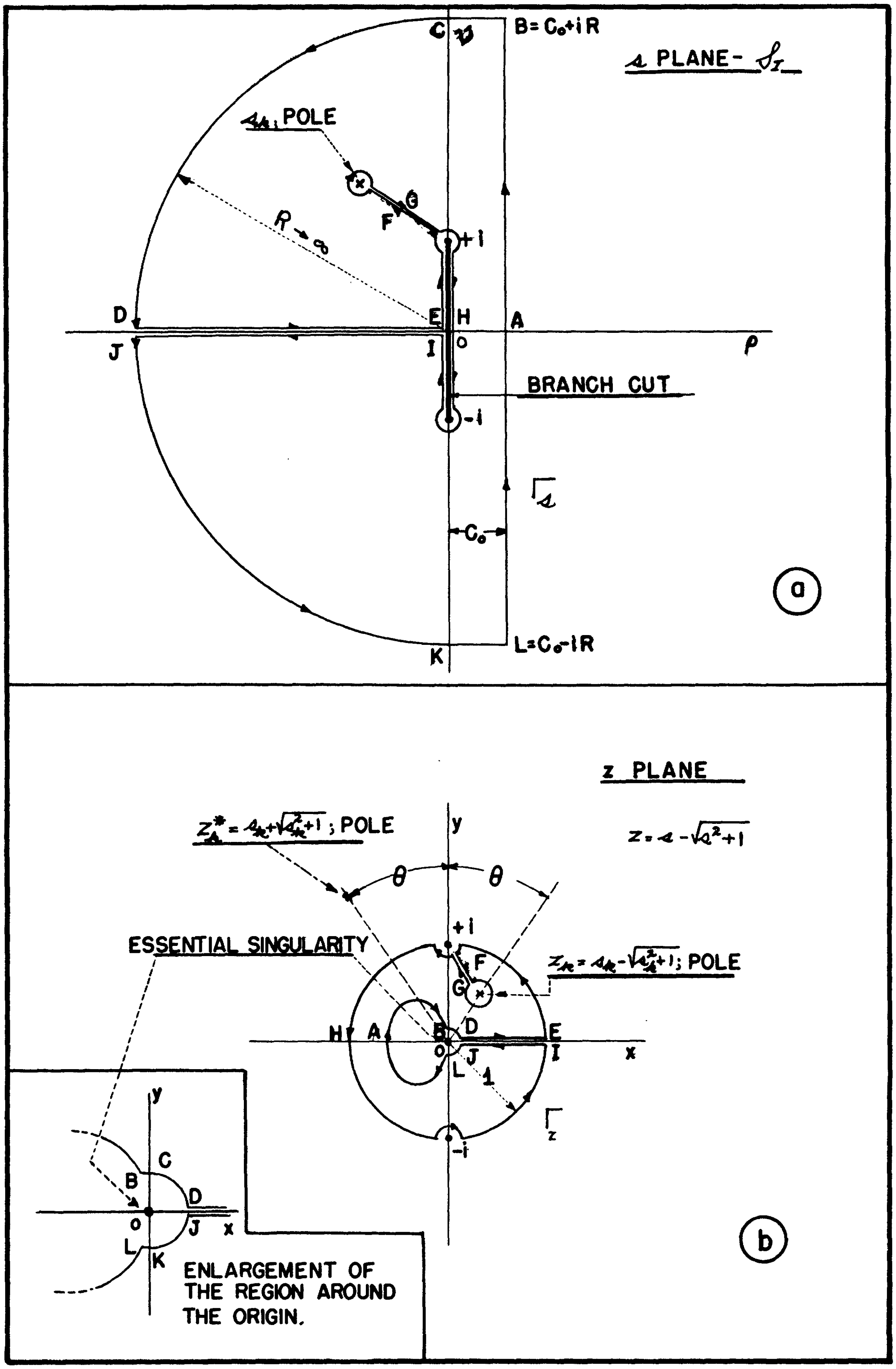

ν
s PLANE; S_I
R → ∞
0
ρ
y
Z PLANE
r → 0
0
x
a
ν
s PLANE; S_I
0
ρ
y
Z PLANE
0
x
1
b
ν
s PLANE; S_I
i
0
ρ
-i
y
Z PLANE
i
0
x
1
i
c

FIG. 3III



s PLANE; s_I
+i
o
ρ
-i
v
Z PLANE
y
+i
o
x
1
-i
a
s PLANE, s_I
v
o
ρ
Z PLANE
y
+i
1
o
x
-i
b
c
y
RIEMANN SURFACE s_{II}, OF THE s PLANE WILL MAP OUTSIDE OF THE UNIT CIRCLE ON THE Z PLANE.
THE TRANSFORMATION
$Z = s - \sqrt{s^2+1}$
1
o
x
RIEMANN SURFACE s_I, OF THE s PLANE, WILL MAP INSIDE THE UNIT CIRCLE ON THE Z PLANE.
FOR THE DEFINITION OF THE RIEMANN'S SURFACES s_I AND s_{II}, SEE CHAPTER II

FIG. 4 III



B = C_0+iR
s PLANE - S_I
s_k; POLE
$R \to \infty$
BRANCH CUT
L = C_0-iR
C_0
Γ_s
a
z PLANE
$z = s - \sqrt{s^2+1}$
$z_k^* = s_k + \sqrt{s_k^2+1}$; POLE
$z_k = s_k - \sqrt{s_k^2+1}$; POLE
ESSENTIAL SINGULARITY
Γ_z
ENLARGEMENT OF THE REGION AROUND THE ORIGIN.
b

It is important to notice that $Z=0$ and $Z=\infty$ are essential singularities of the integrand in all three typical integrals given in (25)III1. The essential singularity $Z=0$ disappears when $\tau=\varkappa$, (wave front).

If instead of taking the above type of branch cutting, 1IIb is adopted, the result given in Figs. 5IIIa and 5IIIb will be obtained. After a systematic study of the mapping properties and if the sign distribution diagrams given in Fig. 4 II b are considered, then it will be found that $\mathcal{S}_I$ maps on the left half of the Z plane and $\mathcal{S}_{II}$ maps on the right half of the Z plane. It is not surprising that in cut 1IIb the Riemann surfaces do not map inside and outside the unit circle. The reason for this is in the fact that in each case the surfaces $\mathcal{S}_I$ and $\mathcal{S}_{II}$ are defined in a different way.

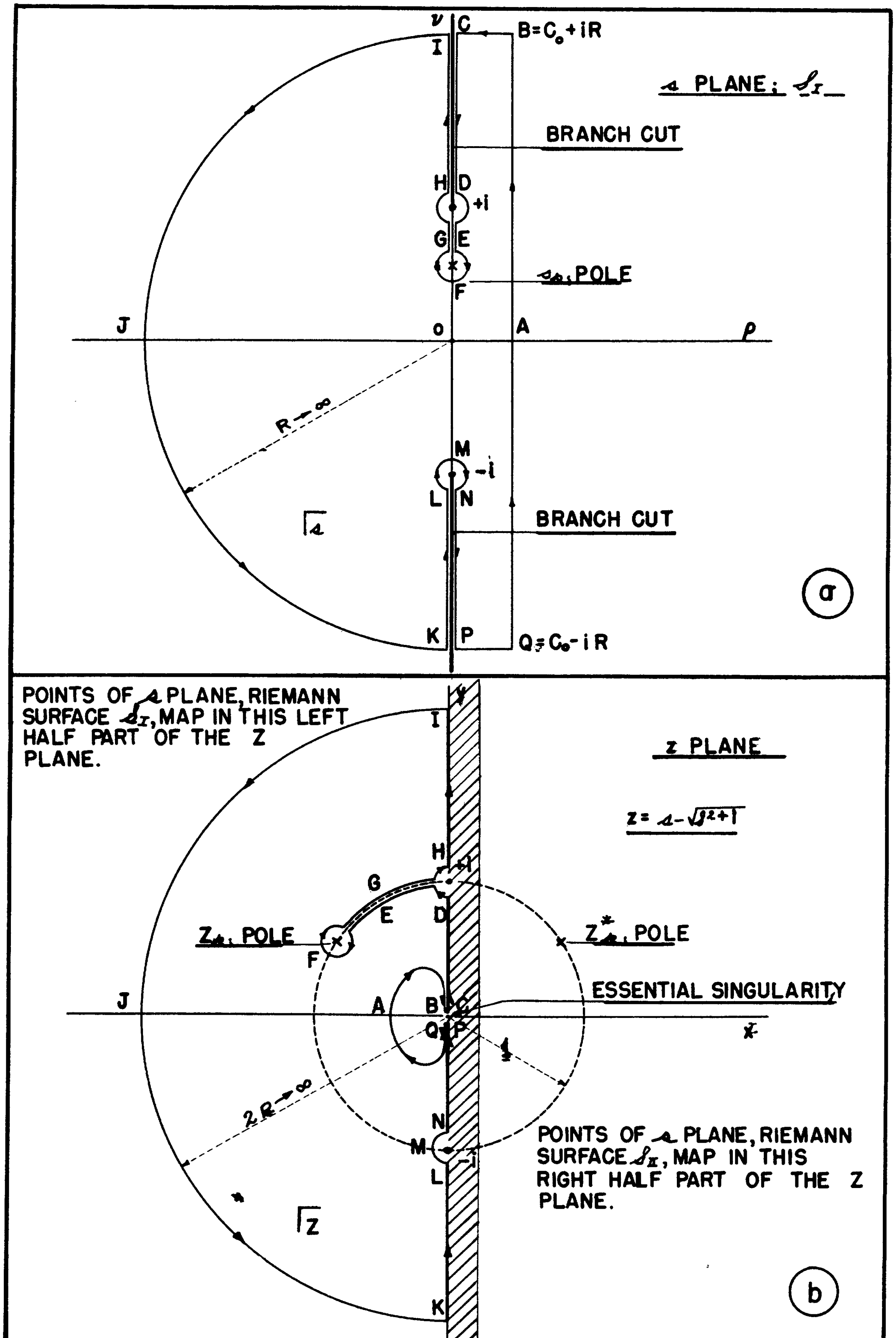
III-2.3 Suppose, instead of using the complex transformation (1)III1 the reciprocal transformation

$$z = s+\sqrt{s^2+1} \qquad (1)\text{III}2$$

is chosen. After a discussion similar to the one given in Section 3 of this chapter, it is found that the typical integrals in this case are:

$$\left.\begin{aligned}
&\frac{1}{2\pi i}\int_{\gamma_z} \frac{e^{\frac{\tau}{2}(z-\frac{1}{z})-\frac{\varkappa}{2}(z+\frac{1}{z})}}{(z-z_K)^{\alpha}}\,dz \\
&\frac{1}{2\pi i}\int_{\gamma_z} \frac{e^{\frac{\tau}{2}(z-\frac{1}{z})-\frac{\varkappa}{2}(z+\frac{1}{z})}}{(z-z_K^{*})^{\alpha}}\,dz \\
&\frac{1}{2\pi i}\int_{\gamma_z} \frac{e^{\frac{\tau}{2}(z-\frac{1}{z})-\frac{\varkappa}{2}(z+\frac{1}{z})}}{z}\,dz
\end{aligned}\right\} \qquad (2)\text{III}2$$

FIG. 5 III


B=C₀+iR
PLANE; S_I
BRANCH CUT
H D
+i
G E
POLE
F
J
o
A
ρ
R→∞
M
-i
L N
BRANCH CUT
K P
Q=C₀-iR
a
POINTS OF s PLANE, RIEMANN SURFACE S_I, MAP IN THIS LEFT HALF PART OF THE Z PLANE.
z PLANE
$z = s - \sqrt{s^2+1}$
H
+i
G
E
D
Zs, POLE
F
Z*s, POLE
ESSENTIAL SINGULARITY
J
A
B C
Q P
x
2R→∞
N
M
-i
L
POINTS OF s PLANE, RIEMANN SURFACE S_{II}, MAP IN THIS RIGHT HALF PART OF THE Z PLANE.
K
b

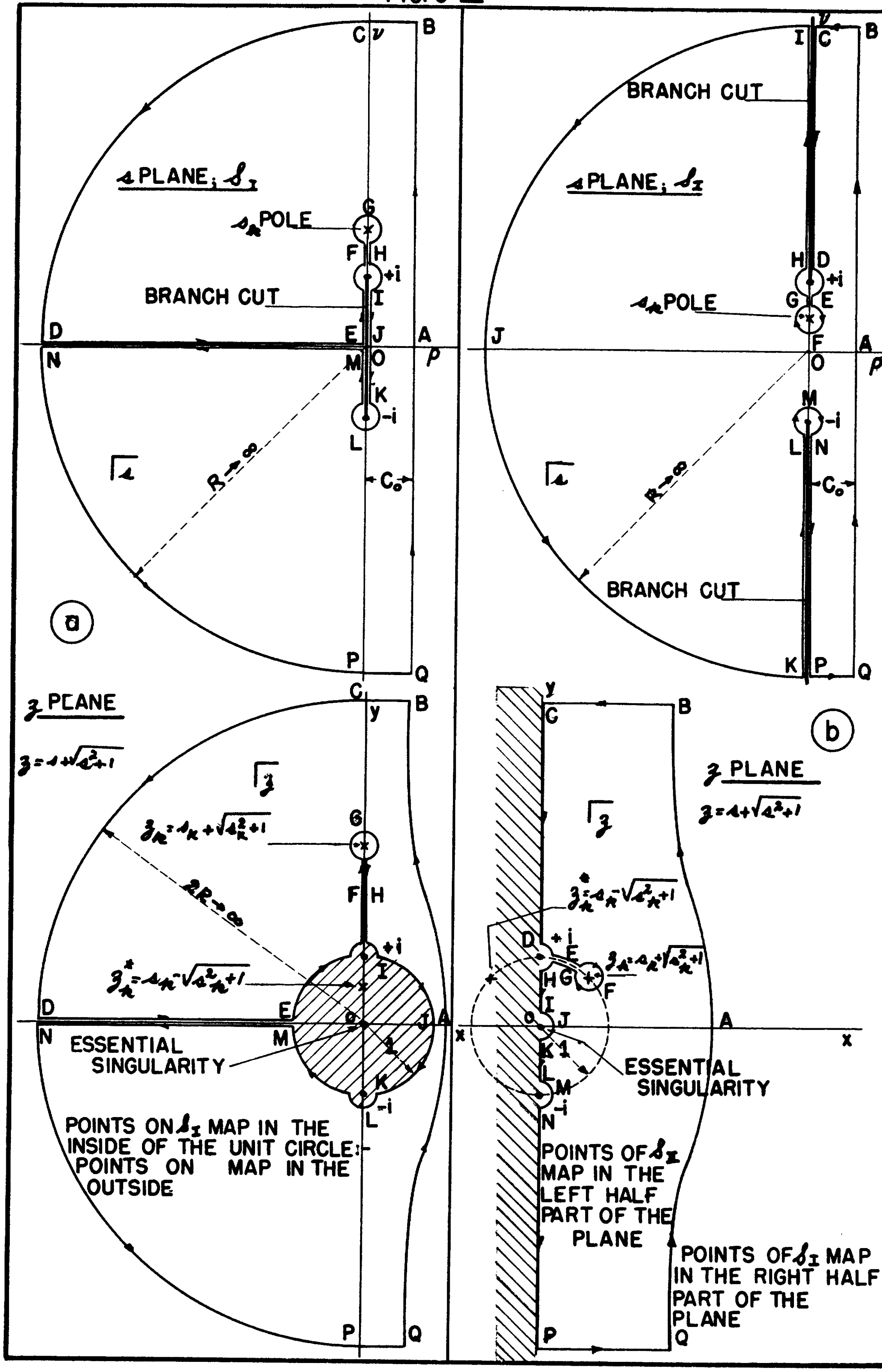
Since $Zz = -1$ it is expected that the new complex transformation z will have the effect of interchanging $\mathscr{S}_I$ and $\mathscr{S}_{II}$. This is the case, and the transformation of the contour Γ_s into the z plane is indicated in Fig. 6III. For branch cutting 2IIb, use Fig. 6IIIa; for branch cutting 1IIb use Fig. 6IIIb.

Notice that the points $z = 0$ and $z = \infty$ are essential singularities of the integrand in (2)II2. The essential singularity at $z = 0$ is removed when $\tau = \hbar$ (wave front).

The contours in Fig. 6III are drawn for the case in which s_k is purely imaginary. If this is not the case then it is simple to put the pole in the proper position by a simple displacement. (Compare Figs. 4III, 5III and 6III.)

III-2.4 The contour γ_Z or γ_z along which (25)III1 or (2)III2 must be taken in order to obtain the correct inverse Laplace transforms has not been given yet. This goal is very close since the transformation of Γ_s into the Z or the z plane has already been given. Contour Γ_s is formed by the union of Br_1 and Br_2 contours. Consequently, the part of Γ_Z or Γ_z which corresponds to one of them must now be chosen. By a simple inspection of Figs. 4III, 5III and 6III one immediately discovers that it is γ_Z or γ_z. For example, in Fig. 4III γ_Z is formed by the integration along the unit circle between E and J and around the pole at Z_k. In Figs. 5IIIb

FIG. 6 III



s PLANE; δ_I
s_k POLE
BRANCH CUT
$R \to \infty$
C_o
a
s PLANE; δ_I
BRANCH CUT
s_k POLE
$R \to \infty$
C_o
BRANCH CUT
z PLANE
$z = s + \sqrt{s^2+1}$
$z_k = s_k + \sqrt{s_k^2+1}$
$2R \to \infty$
$z_k^* = s_k - \sqrt{s_k^2+1}$
ESSENTIAL SINGULARITY
POINTS ON δ_I MAP IN THE INSIDE OF THE UNIT CIRCLE:- POINTS ON MAP IN THE OUTSIDE
b
z PLANE
$z = s + \sqrt{s^2+1}$
$z_k^* = s_k - \sqrt{s_k^2+1}$
$z_k = s_k + \sqrt{s_k^2+1}$
ESSENTIAL SINGULARITY
POINTS OF δ_I MAP IN THE LEFT HALF PART OF THE PLANE
POINTS OF δ_I MAP IN THE RIGHT HALF PART OF THE PLANE

and 6IIIb no integration is made along the whole unit circle; γ_Z or γ_z is along the imaginary axis. Even in those cases in which the contour γ_Z or γ_z lies on the imaginary axis, one can reduce, by elementary contour deformation, to the integration around the unit circle and the corresponding poles.

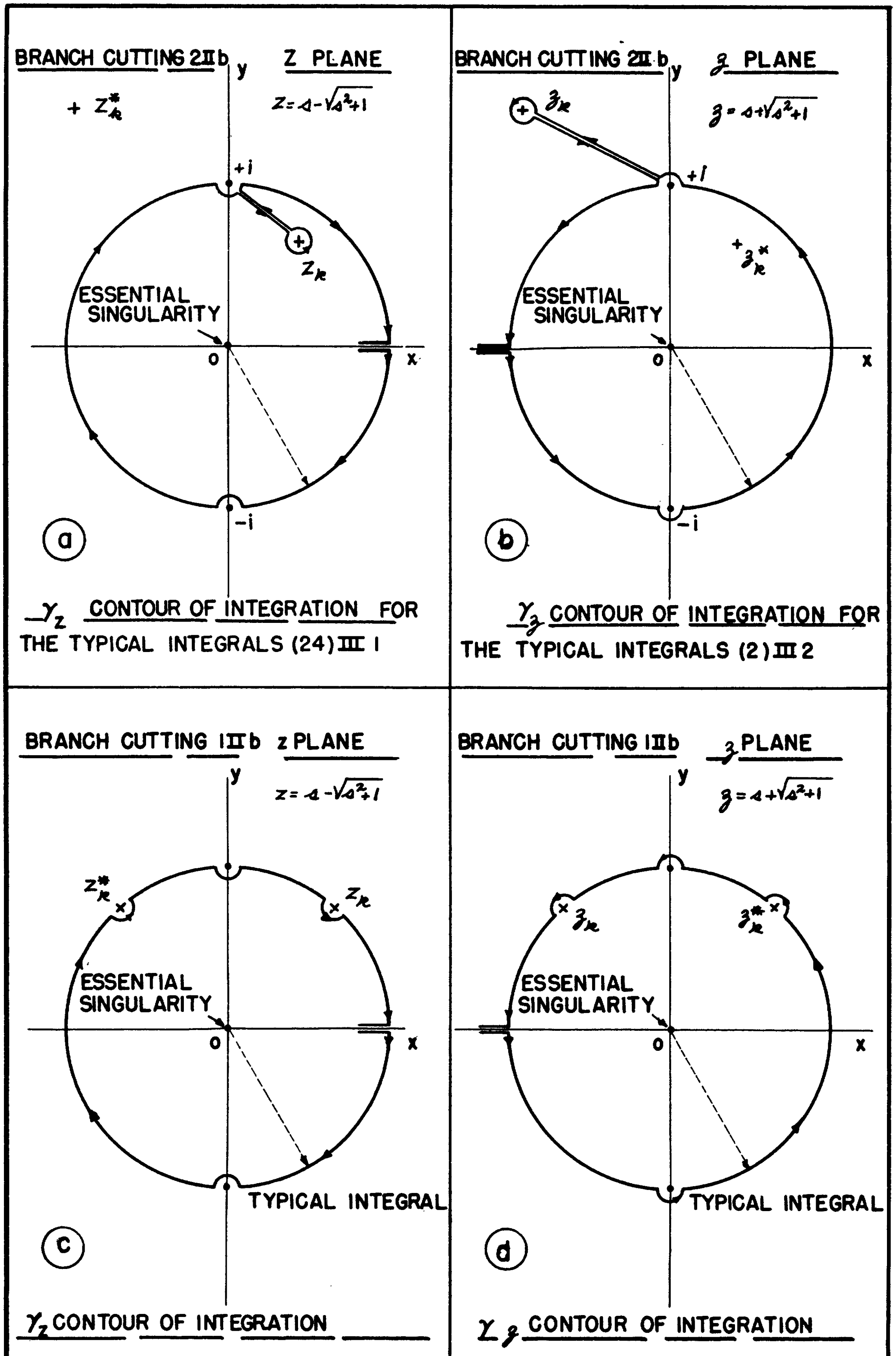
After a simple but careful discussion of the deformation of the contours, the γ contours which are going to be used in this investigation are finally obtained. They are indicated in Fig. 7III for the Z or the z complex transformations. The elementary steps required to pass from Figs. 4III, 5III, and 6III to the final γ_Z or γ_z will be omitted in Fig. 7III. It is not hard to prove their validity.

It is immediately noticeable in Fig. 7III that γ_Z contours are equivalent to integrating around the essential singularity at $Z=0$, while γ_z contours are equivalent to integrating around the essential singularity of the point at infinity.

Figures 7IIIc and 7IIId were drawn respectively for the case in which $|Z_k| = |Z_k^*| = 1$ or $|z_k| = |z_k^*| = 1$.

Section 3. - Integration around the poles. Series expansion of the typical integral and new subtype. Integration of the subtypes and complete solution of the typical integrals of Section 1.

FIG. 7 III


BRANCH CUTTING 2IIb
Z PLANE
z = s - √(s²+1)
+ z*k
zk
+i
-i
ESSENTIAL SINGULARITY
0
x
y
a
γz CONTOUR OF INTEGRATION FOR THE TYPICAL INTEGRALS (24) III 1
BRANCH CUTTING 2II b
ʒ PLANE
ʒ = s + √(s²+1)
ʒk
ʒ*k
+i
-i
ESSENTIAL SINGULARITY
0
x
y
b
γʒ CONTOUR OF INTEGRATION FOR THE TYPICAL INTEGRALS (2) III 2
BRANCH CUTTING 1IIb
z PLANE
z = s - √(s²+1)
z*k
zk
ESSENTIAL SINGULARITY
0
x
y
TYPICAL INTEGRAL
c
γz CONTOUR OF INTEGRATION
BRANCH CUTTING 1IIb
ʒ PLANE
ʒ = s + √(s²+1)
ʒk
ʒ*k
ESSENTIAL SINGULARITY
0
x
y
TYPICAL INTEGRAL
d
γʒ CONTOUR OF INTEGRATION

III-3.0 Section 3 will be devoted to integrating the typical integrals*

$$\left.\begin{aligned}
&\frac{1}{2\pi i}\int_{\gamma_Z}\frac{e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\kappa}{2}(Z+\frac{1}{Z})}}{(Z-Z_k)^{\alpha}}\,dZ\\
&\frac{1}{2\pi i}\int_{\gamma_Z}\frac{e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\kappa}{2}(Z+\frac{1}{Z})}}{(Z-Z_k^{*})^{\alpha}}\,dZ
\end{aligned}\right\}\qquad (1)\text{III}3$$

in which γ_Z is the contour given already in Figs. 7IIIa and 7IIIc.

In the future only the complex transformation

$$Z = s - \sqrt{s^2+1}$$

will be considered. The reciprocal transformation z leads us to similar results and is equivalent to working in the σ_{II} plane instead of in σ_{I}.

Both integrals given in (1)III3 appear simultaneously, since it has been proved that Z_k and Z_k^* correspond to one pole s_k in the s plane. Besides, they are connected by the relation

$$Z_k \times Z_k^* = -1 \;.$$

We will consider three cases.

Case a. - $0 < |Z_k| < 1$ in which case $|Z_k^*| > 1$

Case b. - $0 < |Z_k| > 1$ in which case $|Z_k^*| < 1$

Case c. - $|Z_k| = 1$ in which case $|Z_k^*| = 1$

Recall that $\alpha \geq 1$ is a positive integer. From Eqs. (24)III1 $\alpha = \alpha_k - j + 1$ in which α_k is the multiplicity of the pole Z_k or Z_k^*.

---

* See Appendix I

III-3.1 Consider the integration of the typical integrals in Case a. Here $0<|Z_k|<1$. The pole Z_k^* lies in the outside of the unit circle. The contour of integration is given in Fig. 7IIIa.

The typical integral can be written

$$\int_{\gamma_Z} = \int_{\circlearrowleft +i} + \int_{\circlearrowleft -i} + \int_{\odot Z_K} + \int_{\text{unit circle}} \qquad (2)\text{III}3$$

1st - It is simple to prove in Case a. that the first and second integrals of the second member are zero, by taking a small semicircle of small r and making $r \to 0$.

2nd - Now take the third integral of the second member of (2)III3. Since Z_k is not a pole in the second integral of (1)III3, the integration around Z_k is zero.

3rd - It will be proved that

$$I_{Z_k} = \frac{1}{2\pi i}\int_{\odot Z_K} \frac{e^{\frac{\bar{\tau}}{2}(Z-\frac{1}{Z})+\frac{\bar{\kappa}}{2}(Z+\frac{1}{Z})}}{(Z-Z_k)^{\alpha}}\,dz = \left\{\begin{array}{ll} 0 & \text{for } \alpha>1 \\ e^{\frac{\bar{\tau}}{2}(Z_K-\frac{1}{Z_K})+\frac{\bar{\kappa}}{2}(Z_K+\frac{1}{Z_K})} & \text{for } \alpha=1 \end{array}\right\} \qquad (3)\text{III}3$$

If one sets $Z - Z_k = re^{i\theta}$, $dZ = ire^{i\theta}d\theta$, and then

$$I_{Z_k} \cong \frac{e^{\frac{\bar{\tau}}{2}(Z_K-\frac{1}{Z_K})+\frac{\bar{\kappa}}{2}(Z_K+\frac{1}{Z_K})}}{2\pi r^{\alpha-1}}\int_0^{2\pi} e^{-i(\alpha-1)}d\theta = \frac{e^{\frac{\bar{\tau}}{2}(Z_K-\frac{1}{Z_K})+\frac{\bar{\kappa}}{2}(Z_K+\frac{1}{Z_K})}}{-2\pi i(\alpha-1)r^{\alpha-1}}\left[e^{-i(\alpha-1)2\pi}-1\right]=0$$

for $\alpha>1$,

independent of the magnitude of r.

But if $\alpha=1$

$$I_{Z_k} = \frac{e^{\frac{\bar{\tau}}{2}(Z_K-\frac{1}{Z_K})+\frac{\bar{\kappa}}{2}(Z_K+\frac{1}{Z_K})}}{2\pi}\int_0^{2\pi} d\theta = e^{\frac{\bar{\tau}}{2}(Z_K-\frac{1}{Z_K})+\frac{\bar{\kappa}}{2}(Z_K-\frac{1}{Z_K})} \quad .$$

4th - Introducing the last result one finally obtains:

$$\left.\begin{aligned}
&\frac{1}{2\pi i}\int_{\gamma_z}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})+\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k^*)^{\alpha}}\,dZ=\frac{1}{2\pi i}\int_{\text{unit circle}}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})+\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k^*)^{\alpha}}\,dZ\\
&\frac{1}{2\pi i}\int_{\gamma_z}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})+\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k)^{\alpha}}\,dZ=\frac{1}{2\pi i}\int_{\text{unit circle}}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})+\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k)^{\alpha}}\,dZ+\begin{cases}0 & \text{if } \alpha>1\\ e^{\frac{\bar{\varsigma}}{2}(z_k-\frac{1}{z_k})+\frac{\bar{\kappa}}{2}(z_k+\frac{1}{z_k})} & \text{if } \alpha=1\end{cases}
\end{aligned}\right\}\quad (4)\text{III}3$$

III-3.2 Consider here Case b in which $0<|Z_k^*|<1$.

In this case Z_k^* lies in the inside of the unit circle and Z_k lies on the outside. It is then obvious that one will obtain:

$$\left.\begin{aligned}
&\frac{1}{2\pi i}\int_{\gamma_z}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})+\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k^*)^{\alpha}}\,dZ=\frac{1}{2\pi i}\int_{\text{unit circle}}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})+\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k^*)^{\alpha}}\,dZ+\begin{cases}0 & \text{if } \alpha>1\\ e^{\frac{\bar{\varsigma}}{2}(z_k^*-\frac{1}{z_k^*})+\frac{\bar{\kappa}}{2}(z_k^*+\frac{1}{z_k^*})} & \text{if } \alpha=1\end{cases}\\
&\frac{1}{2\pi i}\int_{\gamma_z}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})+\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k)^{\alpha}}\,dZ=\frac{1}{2\pi i}\int_{\text{unit circle}}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})+\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k)^{\alpha}}\,dZ\,.
\end{aligned}\right\}\quad (5)\text{III}3$$

III-3.3 Consider here Case c in which $|Z_k|=|Z_k^*|=1$.

In this case the contour of Fig. 7IIIc must be used which has circular dents at Z_k and Z_k^*.

The corresponding integrals are given as follows:

$$\left.\begin{aligned}
&\frac{1}{2\pi i}\int_{\gamma_z}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})+\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k^*)^{\alpha}}\,dZ=\frac{1}{2\pi i}\int_{\text{unit circle}}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})+\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k^*)^{\alpha}}\,dZ+\begin{cases}0 & \text{if } \alpha>1\\ \frac{1}{2}e^{\frac{\bar{\varsigma}}{2}(z_k^*-\frac{1}{z_k^*})+\frac{\bar{\kappa}}{2}(z_k^*+\frac{1}{z_k^*})} & \text{if } \alpha=1\end{cases}\\
&\frac{1}{2\pi i}\int_{\gamma_z}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})-\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k)^{\alpha}}\,dZ=\frac{1}{2\pi i}\int_{\text{unit circle}}\frac{e^{\frac{\bar{\varsigma}}{2}(z-\frac{1}{z})-\frac{\bar{\kappa}}{2}(z+\frac{1}{z})}}{(Z-Z_k)^{\alpha}}\,dZ+\begin{cases}0 & \text{if } \alpha>1\\ \frac{1}{2}e^{\frac{\bar{\varsigma}}{2}(z_k-\frac{1}{z_k})+\frac{\bar{\kappa}}{2}(z_k+\frac{1}{z_k})} & \text{if } \alpha=1\end{cases}
\end{aligned}\right\}\quad (6)\text{III}3$$

III-3.4 The following may now be concluded.

"The problem is reduced to the integration around the unit circle."

For compactness in the notation, the contour along the unit circle will be indicated by c_1 when it is taken in the direction given in Figs. 7IIIa and 7IIIc.

Cases a and b will be studied first. The following analysis is based on the assumption that $0 < |Z_k| < 1$ (Case a). If this is not the case, Z_k and Z_k^* are interchanged. Since it must be integrated around the unit circle, it is necessary to have

$$|Z| = 1$$

and by the above hypothesis of Case a

$$\left|\frac{Z_k}{Z}\right| < 1 \ ; \ \left|\frac{Z}{Z_k^*}\right| < 1 \ .$$

Consider the expression

$$(1-u)^{-\alpha} = 1 + \alpha u + \frac{\alpha(\alpha+1)}{2!} u^2 + \cdots + \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{n!} u^n + \cdots$$

This series is <u>absolutely and uniformly convergent if $0 \leq u < 1$</u>.

Then, under the hypothesis of Case a,

$$\left.\begin{aligned} (Z-Z_k)^{-\alpha} &= Z^{-\alpha}\left(1-\frac{Z_k}{Z}\right)^{-\alpha} = Z^{-\alpha}\sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}\left(\frac{Z}{Z_k}\right)^n \\ (Z-Z_k^*)^{-\alpha} &= Z_k^{*-\alpha}\left(1-\frac{Z}{Z_k^*}\right) = (-1)^{\alpha} Z_k^{*-\alpha}\sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}\left(\frac{Z}{Z_k^*}\right)^n \end{aligned}\right\} \quad (7)\,\text{III}\,3$$

which are absolutely and uniformly convergent series so that term by term integration is justified.

Then:

$$\left.\begin{aligned}\frac{1}{2\pi i}\int_{C_1}\frac{e^{\frac{\tau}{2}(z-\frac{1}{z})+\frac{\kappa}{2}(z+\frac{1}{z})}}{(Z-Z_k)^{\alpha}}dZ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}Z_k^n\int_{C_1}\frac{e^{\frac{\tau}{2}(z-\frac{1}{z})+\frac{\kappa}{2}(z+\frac{1}{z})}}{Z^{n+\alpha}}dZ\\ \frac{1}{2\pi i}\int_{C_1}\frac{e^{\frac{\tau}{2}(z-\frac{1}{z})+\frac{\kappa}{2}(z+\frac{1}{z})}}{(Z-Z_k^*)^{\alpha}}dZ &= \frac{(-1)}{2\pi i}\sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}\frac{1}{Z_k^{*\alpha+n}}\int_{C_1}Z^n e^{\frac{\tau}{2}(z-\frac{1}{z})+\frac{\kappa}{2}(z+\frac{1}{z})}dZ\end{aligned}\right\} \quad (8)\text{III}3$$

From this last equation it can be observed that the problem is reduced to the integration of

$$\left.\begin{aligned}I_m &= \frac{1}{2\pi i}\int_{C_1}\frac{e^{\frac{\tau}{2}(z-\frac{1}{z})+\frac{\kappa}{2}(z+\frac{1}{z})}}{Z^m}dZ\ ;\ m=n+\alpha\\ I_n^* &= \frac{1}{2\pi i}\int_{C_1}Z^n e^{\frac{\tau}{2}(z-\frac{1}{z})+\frac{\kappa}{2}(z+\frac{1}{z})}dZ\ .\end{aligned}\right\} \quad (9)\text{III}3$$

III-3.5 The integral I_m, for $m=1$, plays an important role in this investigation. It will first be evaluated. Since the contour of integration C_1 is the unit circle in the direction indicated in Fig. 7IIIa, then one can write,

$Z = e^{i\varphi}$ and

$$I_1 = -\frac{1}{2\pi}\int_0^{2\pi}e^{\kappa\cos\varphi+i\tau\sin\varphi}\,d\varphi\ . \quad (10)\text{III}3$$

Now, it will be proved that

$$I_1 = -\frac{1}{\pi}\int_0^{\pi}e^{\kappa\cos\varphi}\cos(\tau\sin\varphi)\,d\varphi = -J_0\sqrt{\tau^2-\kappa^2}. \quad (11)\text{III}3$$

1st - It will be shown that the integral (10)III3 is purely real. Take the well-known Fourier series expansion.

$$e^{j\tau\sin\varphi} = J_0(\tau) + 2\sum_{n=1}^{\infty} J_{2n}(\tau)\cos 2n\varphi + 2i\sum_{n=1}^{\infty} J_{2n+1}(\tau)\sin(2n+1)\varphi$$

(See for example, "Theory of Bessel Functions", Watson, P. 22.)

Then (10)III3 becomes

$$-I_1 = +J_0(\tau)\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}d\varphi\right\} + \sum_{n=1}^{\infty} J_{2n}(\tau)\left\{\frac{1}{\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\cos 2n\varphi\, d\varphi\right\} + \\ + i\sum_{n=1}^{\infty} J_{2n+1}(\tau)\left\{\frac{1}{\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\sin(2n+1)\varphi\, d\varphi\right\}. \quad (12)\text{III}3$$

It is immediately seen that the integral expressions between brackets are the Fourier expansion coefficients of the function $e^{\kappa\cos\varphi}$. Since this is an even function of φ, then

$$\frac{1}{\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\sin(2n+1)\varphi\, d\varphi = 0 ,$$

and consequently, the imaginary part of (10)III 3 vanishes; that is

$$\frac{1}{\pi}\int_0^{\pi} e^{\kappa\cos\varphi}\sin(\tau\sin\varphi)\, d\varphi = 0 . \quad (13)\text{III}3$$

2nd - By a series expansion of the middle member of (11)III 3, it follows that

$$I_1 = -J_0\sqrt{\tau^2 - \kappa^2}.$$

The intermediate steps of this expansion are omitted because they can be found, for example, in "Theory of Bessel Functions", Watson, P. 21.

III-3.6 The integrals

$$\left.\begin{aligned} J_0\sqrt{\tau^2-\kappa^2} &= \frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\cos(\tau\sin\varphi)\,d\varphi \\ 0 &= \int_0^{2\pi} e^{\kappa\cos\varphi}\sin(\tau\sin\varphi)\,d\varphi \end{aligned}\right\} \qquad (14)\,\mathrm{III}\,3$$

play an important role in this investigation. Take the first and differentiate it with respect to τ and κ. One gets,

$$\frac{\tau}{\sqrt{\tau^2-\kappa^2}}J_0'\sqrt{\tau^2-\kappa^2} = -\frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\sin\varphi\sin(\tau\sin\varphi)\,d\varphi$$

$$\frac{-\kappa}{\sqrt{\tau^2-\kappa^2}}J_0'\sqrt{\tau^2-\kappa^2} = +\frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\cos\varphi\cos(\tau\sin\varphi)\,d\varphi\,.$$

By adding, substracting and using a well-known property of the Bessel functions, the following equations are obtained.

$$\left.\begin{aligned} \left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{\frac{1}{2}} J_1\sqrt{\tau^2-\kappa^2} &= -\frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\cos(\varphi+\tau\sin\varphi)\,d\varphi \\ \left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{-\frac{1}{2}} J_{-1}\sqrt{\tau^2-\kappa^2} &= -\frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\cos(-\varphi+\tau\sin\varphi)\,d\varphi \end{aligned}\right\}$$

Secondly, take the partial derivatives of (15) III 3 with respect to τ and κ. By adding the results corresponding to the first and subtracting those corresponding to the second equation the following is obtained.

$$\left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{\frac{2}{2}} J_2\sqrt{\tau^2-\kappa^2} = +\frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\cos(\varphi+\tau\sin\varphi)\,d\varphi$$

$$\left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{-\frac{2}{2}} J_{-2}\sqrt{\tau^2-\kappa^2} = +\frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\cos(-\varphi+\tau\sin\varphi)\,d\varphi\,.$$

Third: by continuing this process after p successive operations the following equations are found for p integer.

$$\left.\begin{aligned} (-1)^{p}\left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{\frac{p}{2}} J_p\sqrt{\tau^2-\kappa^2} &= \frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\cos(p\varphi+\tau\sin\varphi)\,d\varphi \\ (-1)^{-p}\left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{\frac{-p}{2}} J_{-p}\sqrt{\tau^2-\kappa^2} &= \frac{1}{2\pi}\int e^{\kappa\cos\varphi}\cos(-p\varphi+\tau\sin\varphi)\,d\varphi \end{aligned}\right\} \quad (16)\,\text{III}\,3$$

The above generalization can be justified directly by using the method of finite induction. That is: if (16) III 3 is true for p, then it will be true for $p+1$. This has already been shown for $p=1$. Take the partial derivatives of the first with respect to τ and κ.

$$\left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{\frac{p}{2}-1}\frac{\sqrt{\tau^2-\kappa^2}}{(\tau+\kappa)^2}\left\{p\kappa\frac{J_p\sqrt{\tau^2-\kappa^2}}{\sqrt{\tau^2-\kappa^2}}+\tau J_p'\sqrt{\tau^2-\kappa^2}\right\} = -\frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\sin\varphi\sin(p\varphi+\tau\sin\varphi)\,d\varphi$$

$$-\left(\frac{\tau-\kappa}{\tau-\kappa}\right)^{\frac{p}{2}-1}\frac{\sqrt{\tau^2-\kappa^2}}{(\tau+\kappa)^2}\left\{p\tau\frac{J_p\sqrt{\tau^2-\kappa^2}}{\sqrt{\tau^2-\kappa^2}}+\kappa J_p'\sqrt{\tau^2-\kappa^2}\right\} = +\frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\cos\varphi\cos(p\varphi+\tau\sin\varphi)\,d\varphi$$

By adding the above equation in accordance with one recursion formula of the Bessel functions, one gets

$$(-1)^{p+1}\left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{\frac{p+1}{2}} J_{p+1}\sqrt{\tau^2-\kappa^2} = \frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\varphi}\cos\left[(p+1)\varphi+\tau\sin\varphi\right]d\varphi$$

and therefore the result holds for all values of p. The second equation of (16) III 3 can be justified in the same manner.

Fourth: From the second integral of (14) III 3 it can be found that

$$0 = \int_0^{2\pi} e^{\kappa\cos\varphi}\sin(\pm p\varphi+\tau\sin\varphi)\,d\varphi \quad (17)\,\text{III}\,3$$

for all values of p.

III-3.7 Integrals (16)III3 and (17)III3 are the key to the solution in this research. The typical integrals (1)III3 can be expressed in terms of them. Since these integrals are to be taken along the unit circle, one can write

$$Z = e^{j\varphi}$$

and, therefore, by considering (17)III3, the following expressions are obtained.

$$I_{m+1} = -\frac{1}{2\pi}\int_0^{2\pi} e^{\varkappa\cos\tau} e^{i[-m\varphi+\tau\sin\varphi]}\,d\varphi = -\frac{1}{2\pi}\int_0^{2\pi} e^{\varkappa\cos\varphi}\cos[-m\varphi+\tau\sin\varphi]\,d\varphi$$

$$I^*_{n-1} = -\frac{1}{2\pi}\int_0^{2\pi} e^{\varkappa\cos\varphi} e^{i[n\varphi+\tau\sin\varphi]}\,d\varphi = -\frac{1}{2\pi}\int_0^{2\pi} e^{\varkappa\cos\varphi}\cos[n\varphi+\tau\sin\varphi]\,d\varphi .$$

By (16)III3 it can be seen that

$$\left.\begin{aligned} I_{(n+\alpha+1)} &= -(-1)^{-(n+\alpha)}\left(\frac{1-\theta}{1+\theta}\right)^{-\frac{n+\alpha}{2}} J_{-(n+\alpha)}(T) \\ I^*_{n-1} &= -(-1)^{n}\left(\frac{1-\theta}{1+\theta}\right)^{\frac{n}{2}} J_n(T) \\ &\text{in which} \\ &\theta = \frac{\varkappa}{\tau} \le 1 \text{ and } T = \sqrt{\tau^2 - \varkappa^2} \end{aligned}\right\} \qquad (18)\text{III}3$$

and finally, the solution of the integrals (1)III3 (Case a. $|Z_k|<1$; $|Z^*_k|>1$), is given by

$$\left.\begin{aligned} \frac{1}{2\pi i}\int_{\gamma_Z}\frac{e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\varkappa}{2}(Z+\frac{1}{Z})}}{(Z-Z_k)^{\alpha}}\,dZ &= -\sum_{n=0}^{\infty}(-1)^{-(n+\alpha-1)} Z_k^{n}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}\left(\frac{1-\theta}{1+\theta}\right)^{-\frac{n+\alpha-1}{2}} J_{-(n+\alpha-1)}(T) + \begin{cases} 0 & \text{for } \alpha>1 \\ e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\varkappa}{2}(Z+\frac{1}{Z})} & \text{for } \alpha=1 \end{cases} \\ \frac{1}{2\pi i}\int_{\gamma_Z}\frac{e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\varkappa}{2}(Z+\frac{1}{Z})}}{(Z-Z^*_k)^{\alpha}}\,dZ &= -\sum_{n=0}^{\infty}(-1)^{n+\alpha+1}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}\,\frac{1}{Z_k^{*\,\alpha+n}}\left(\frac{1-\theta}{1+\theta}\right)^{\frac{n+1}{2}} J_{n+1}(T) \end{aligned}\right\} \qquad (19)\text{III}3$$

in which $\theta = \frac{\varkappa}{\tau}$ and $T = \sqrt{\tau^2 - \varkappa^2}$.

Section 4 - Inverse transforms in terms of Lommel Functions. Solutions for simple poles. Generating functions. Conditions at $\tau = \kappa$ and at $\tau \to \infty$. Group velocity. Phase velocity. Solution for poles of higher multiplicity.

III-4.0 At the end of Section 1, the inverse Laplace transform $\phi(\tau,\kappa)$ was expressed in terms of three types of integrals: see Eqs. (24)III1, (25)III1 and (26)III1. At the end of Section 3 these integrals were evaluated and their values were given in Eq. (19)III3. From the formal point of view, the problem of finding the required inverse Laplace transformation was solved.

Nevertheless, these formal solutions came out in the form of uniformly convergent series expansions of the Neumann type, which have a rather involved structure and are difficult to both handle and discuss. The object of Section 4 is to obtain compact solutions for these series, which are suitable for a complete discussion.

III-4.1 The case of simple poles in G(Z), that is $\alpha = 1$, will be considered first in this investigation. This case has a fundamental character because:

1st - A solution for the case of poles with multiplicity $\alpha > 1$ can be easily derived from it.

2nd - This is quite a common case in practical applications.

Under the fundamental assumption that $\alpha = 1$, it can be seen that:

From (21) III 1

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_q = 1 \;;\; q = m \,. \qquad (1)\,\text{III}\,4$$

From (22) III 1

$$G(Z) = \sum_1^m \frac{\kappa_k}{(Z-Z_k)} + \sum_1^m \frac{\kappa_k^*}{(Z-Z_k^*)} + \begin{Bmatrix} \frac{\kappa_o}{Z} & \text{for } \gamma = 1 \\ 0 & \text{for } \gamma > 1 \end{Bmatrix} \qquad (2)\,\text{III}\,4$$

From (23) III 1

$$\kappa_k = \left\{(Z-Z_k)\,G(Z)\right\}_{Z=Z_k} ; \quad \kappa_k^* = \left\{(Z-Z_k^*)\,G(Z)\right\}_{Z=Z_k^*} ;$$

$$\kappa_o = \left\{Z\,G(Z)\right\}_{Z=0} \qquad (3)\,\text{III}\,4$$

From (24) III 1

$$\varphi(\tau,\kappa) = \begin{cases} 0 & \tau < \kappa \\ \displaystyle\sum_{k=1}^{\infty} \frac{\kappa_k}{2\pi i} \int_{\gamma_Z} \frac{e^{A(Z)}}{(Z-Z_k)}\,dZ + \begin{Bmatrix} 0 & \gamma > 1 \\ \displaystyle\frac{\kappa_o}{2\pi i} \int_{C_1} \frac{e^{A(Z)}}{Z}\,dZ ; & \gamma = 1 \end{Bmatrix} & \tau > \kappa \end{cases} \qquad (4)\,\text{III}\,4$$

From (25) III 1 consider the integral

$$\sum_{k=1}^{m} \frac{\kappa_k^*}{2\pi i} \int_{\gamma_Z} \frac{e^{A(Z)}}{Z-Z_k^*}\,dZ \,. \qquad (5)\,\text{III}\,4$$

Finally, from (19) III 3 one obtains,

$$\left.\begin{aligned} \frac{1}{2\pi i}\int_{\gamma_Z} \frac{e^{A(Z)}}{Z-Z_k}\,dZ &= -\sum_{n=0}^{\infty} (-1)^{-n} Z_k^n \left(\frac{1-\theta}{1+\theta}\right)^{-\frac{n}{2}} J_{-n}(T) + e^{A(Z_k)} \\ \frac{1}{2\pi i}\int_{\gamma_Z} \frac{e^{A(Z)}}{Z-Z_k^*}\,dZ &= +\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{Z_k^{*(n+1)}} \left(\frac{1-\theta}{1+\theta}\right)^{\frac{n+1}{2}} J_{n+1}(T) \\ \frac{1}{2\pi i}\int_{C_1} \frac{e^{A(Z)}}{Z}\,dZ &= -J_o(T) \,. \end{aligned}\right\} \qquad (6)\,\text{III}\,4$$

The notation used in the above equations is

$$\left.\begin{aligned}
&A(Z) = \frac{\tau}{2}\left(Z - \frac{1}{Z}\right) + \frac{\kappa}{2}\left(Z + \frac{1}{Z}\right) \\
&A(Z_k) = \frac{\tau}{2}\left(Z_k - \frac{1}{Z_k}\right) + \frac{\kappa}{2}\left(Z_k + \frac{1}{Z_k}\right) \\
&A(Z_k^*) = \frac{\tau}{2}\left(Z_k^* - \frac{1}{Z_k^*}\right) + \frac{\kappa}{2}\left(Z_k + \frac{1}{Z_k}\right) \\
&T = \sqrt{\tau^2 - \kappa^2} \\
&\theta = \frac{\kappa}{\tau} \\
&\tau = 2\pi\frac{t}{T_c} \; ; \; T_c = \frac{1}{f_c} \; ; \; f_c = \text{cut-off frequency} \\
&\kappa = 2\pi\frac{x_3}{\lambda_c} \; ; \qquad \lambda_c = \text{cut-off frequency} \\
&\text{and also to be introduced is} \\
&\Omega_k = \frac{\tau - \kappa}{iZ_k} \text{ complex quantity} \\
&\Omega_k^* = \frac{\tau - \kappa}{iZ_k^*} \text{ complex quantity}
\end{aligned}\right\} \qquad (7)\,\text{III}\,4$$

III-4.2 In this subsection the integral solutions (6) III 4 will be expressed in terms of compact expressions involving Lommel functions.

From the first, take the expression.

$$\sum_{n=0}^{\infty}(-1)^{n}Z_k^n\left(\frac{1-\theta}{1+\theta}\right)^{-\frac{n}{2}}J_{-n}(T) = \sum_{n=0}^{\infty}(-1)i^{-n}\left(\frac{\Omega_k}{T}\right)^{-n}J_{-n}(T)$$

$$\text{for } n=\text{odd integer} = \sum_{n=1}^{\infty} i^{-n}\left(\frac{\Omega_k}{T}\right)^{-n}J_{-n}(T)+$$

$$\text{for } n=\text{even integer} + \sum_{n=0}^{\infty} i^{-n}\left(\frac{\Omega_k}{T}\right)^{-n}J_{-n}(T) =$$

$$= \sum_{p=0}^{\infty} i^{-(2p+1)}\left(\frac{\Omega_k}{T}\right)^{-(2p+1)}J_{-(2p+1)}(T) + \tag{(8) III4}$$

$$+ \sum_{p=0}^{\infty} i^{-2p}\left(\frac{\Omega_k}{T}\right)^{-2p}J_{-2p}(T)$$

$$= + \sum_{p=0}^{\infty}(-1)^{p}\left(\frac{\Omega_k}{T}\right)^{-(2p+1)}J_{-(2p+1)}(T)$$

$$+ \sum_{p=0}^{\infty}(-1)^{p}\left(\frac{\Omega_k}{T}\right)^{-2p}J_{-2p}(T)$$

$$= V_0(\Omega_k, T) + iV(\Omega_k, T)$$

in which

$$V_0(\Omega_k, T),\ V_1(\Omega_k, T)$$

are the Lommel V functions of order zero and one. For more information on Lommel functions see the next subsection and "Theory of Bessel Functions", Watson, Pgs. 537-550.

From the second integral in (6)III4 one obtains

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{Z_k^{*n+1}}\left(\frac{1-\theta}{1+\theta}\right)^{\frac{n+1}{2}}J_{n+1}(T) = \sum_{m=1}^{\infty}(-1)^{m}i^{m}\left(\frac{\Omega_k^*}{T}\right)^{m}J_m(T) =$$

$$= \sum_{m=0}^{\infty}(-1)^{m}i^{m}\left(\frac{\Omega_k^*}{T}\right)^{m}J_m(T) - J_0(T) =$$

$$= -\sum_{p=0}^{\infty} i^{(2p+1)}\left(\frac{\Omega_k^*}{T}\right)^{2p+1}J_{2p+1}(T) - J_0(T) + \tag{(9) III4}$$

$$+ \sum_{p=0}^{\infty} i^{2p}\left(\frac{\Omega_k^*}{T}\right)^{2p}J_{2p}(T) =$$

$$= U_0(\Omega_k^*, T) - J_0(T) - iU_1(\Omega_k^*, T)$$

in which $U_0(\Omega^*_k, T)$ and $U_1(\Omega^*_k, T)$ are the Lommel U functions of order zero and one. (See Subsection III-4.3.)

With the aid of this result one can write*

$$\left.\begin{aligned} \frac{1}{2\pi i}\int_{\Gamma_Z}\frac{e^{A(Z)}}{Z-Z_k}\,dZ &= -\left[V_0(\Omega_k,T)+iV_1(\Omega_k,T)\right]+e^{A(Z_k)} \\ & \qquad |Z_k|<1 \\ \frac{1}{2\pi i}\int_{\Gamma_Z}\frac{e^{A(Z)}}{Z-Z^*_k}\,dZ &= \left[U_0(\Omega^*_k,T)-J_0(T)\right]-iU_1(\Omega^*_k,T) \\ & \qquad |Z^*_k|>1 \end{aligned}\right\} \qquad (10)\ \text{III}\ 5$$

It can be concluded that the typical integrals can be expressed in terms of Lommel functions. A further discussion of these solutions will be made in the next Subsection III-4.3.

III-4.3 Lommel functions are defined by the uniformly convergent series

$$\left.\begin{aligned} U_n(\Omega,T) &= \sum_{m=0}^{\infty}(-1)^m\left(\frac{\Omega}{T}\right)^{n+2m}J_{n+2m}(T) \\ V_n(\Omega,T) &= \sum_{m=0}^{\infty}(-1)^m\left(\frac{\Omega}{T}\right)^{-n-2m}J_{-n-2m}(T) \end{aligned}\right\} \qquad (11)\ \text{III}\ 4$$

The U and V functions are connected by

$$\left.\begin{aligned} U_n(\Omega,T) - V_{-n+2}(\Omega,T) &= \cos\left(\frac{\Omega}{2}+\frac{T^2}{2\Omega}-\frac{n\pi}{2}\right) \\ U_{n+1}(\Omega,T) - V_{-n+1}(\Omega,T) &= \sin\left(\frac{\Omega}{2}+\frac{T^2}{2\Omega}-\frac{n\pi}{2}\right) \end{aligned}\right\} \qquad (12)\ \text{III}\ 4$$

$$\left.\begin{aligned} U_n(\Omega,T)+U_{n+2}(\Omega,T) &= \left(\frac{\Omega}{T}\right)^n J_n(T) \\ V_n(\Omega,T)+V_{n+2}(\Omega,T) &= \left(\frac{\Omega}{T}\right)^{-n} J_{-n}(T) \end{aligned}\right\} \qquad (13)\ \text{III}\ 4$$

$$V_n(\Omega,T) = (-1)^n U_n\left(\frac{T^2}{\Omega},T\right) \qquad (14)\ \text{III}\ 4$$

It is not hard to prove these properties, and the proof is given in the reference mentioned earlier.

---

* See Appendix I.

These relations allow the following expressions to be written

$$\left.\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_Z}\frac{e^{A(Z)}}{(Z-Z_k)}\,dZ &= -\left[V_0(\Omega_k,T)+iV_1(\Omega_k,T)\right]+e^{A(Z_k)}=\\ &=\left[U_0(\Omega_k,T)-J_0(T)\right]-iU_1(\Omega_k,T)\\ &=-\left[U_2(\Omega_k,T)+iU_1(\Omega_k,T)\right];\\ &|Z_k|<1\end{aligned}\right\}\quad (15)\,\text{III}\,4$$

and

$$\left.\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_Z}\frac{e^{A(Z)}}{(Z-Z_k^*)}\,dZ &= -\left[U_2(\Omega_k^*,T)+iU_1(\Omega_k^*,T)\right]=\\ &=\left[U_0(\Omega_k^*,T)-J_0(T)\right]-iU_1(\Omega_k^*,T)\\ &=-\left[V_0(\Omega_k^*,T)+iV_1(\Omega_k^*,T)\right]+e^{A(Z_k^*)};\\ &|Z_k^*|>1\,.\end{aligned}\right\}\quad (16)\,\text{III}\,4$$

By simple observation of (15) III 4 and (16) III 4 it can be seen that the two integrals bear complete resemblance to each other. Thinking in terms of the variable s, one can notice that the first integral (15) III 4, is connected with $\mathcal{S}_I$ (Riemann surface $\mathcal{S}_I$) and the second, (16) III 4, with $\mathcal{S}_{II}$ in the s plane.

For compactness in the notation, write

$$\left.\begin{aligned}\mathcal{V}(\Omega,T) &= V_0(\Omega,T)+iV_1(\Omega,T)\\ \mathcal{U}(\Omega,T) &= U_2(\Omega,T)+iU_1(\Omega,T)\end{aligned}\right\}\quad (17)\,\text{III}\,4$$

and they will be referred to as <u>Transient Generating Functions</u>.

With the aid of these new generating functions, the inverse Laplace transform of the basic transform will be given by

$$\mathcal{L}_{\tau}^{-1} F(s,\sqrt{s^2+1})\, e^{-\kappa\sqrt{s^2+1}} = \varphi(\tau,\kappa)$$

$$= u_{-1}(\tau-\kappa)\left\{\left[\sum_{k=1}^{m} K_k\, e^{A(Z_k)} - \mathcal{V}(\Omega_k,T) - \begin{bmatrix} 0 & \gamma>1 \\ K_0 J_0(T) & \gamma=1 \end{bmatrix}\right] = -\sum_{k=1}^{m} K_k \mathcal{U}(\Omega_k,T) - \begin{bmatrix} 0 & \gamma>1 \\ K_0 J_0(T) & \gamma=1 \end{bmatrix}\right\} \quad (18)\,\text{III}\,4$$

For simple poles of F
$u_{-1}(\tau-\kappa)$ = unit step function

Therefore, the problem of inversion from the s domain into the t domain is completely solved for the case of simple poles of F. For the solution in case of multiplicity greater than one see Subsection III-4.9.

III-4.4 In this subsection the behavior of the generating functions $\mathcal{V}(\Omega,T)$ and $\mathcal{U}(\Omega,T)$ at $\tau=\kappa$, (wave front) will be investigated.

The simplest way to evaluate this function at $\tau=\kappa$ is by means of the integral

$$\frac{1}{2\pi i}\int_{\gamma_Z} \frac{e^{\frac{\tau}{2}(Z-\frac{1}{Z})+\frac{\kappa}{2}(Z+\frac{1}{Z})}}{(Z-Z_k)}\, dZ = e^{A(Z_k)} - \mathcal{V}(\Omega_k,T) \ .$$

Now, when $\tau=\kappa$ then $A(Z)=\kappa Z$ and the essential singularity at $Z=0$ disappears. Therefore, for $\tau=\kappa$, the above integrand is analytic inside and on the contour γ_Z. Since the integral along γ_Z must vanish, then

$$\left.\begin{aligned} \mathcal{V}(\Omega_k,T)_{\tau=\kappa} &= e^{\kappa Z_k} \\ \mathcal{U}(\Omega_k,T)_{\tau=\kappa} &= 0 \ . \end{aligned}\right\} \quad (19)\,\text{III}\,4$$

It can be shown that

$$\left.\begin{aligned} \mathcal{V}(\Omega_k^*,T)_{\tau=\kappa} &= e^{\kappa Z_k^*} \\ \mathcal{U}(\Omega_k^*,T)_{\tau=\kappa} &= 0 \ . \end{aligned}\right\} \quad (20)\,\text{III}\,4$$

As a consequence:

$$\varphi(\tau,\varkappa)_{\tau\to\varkappa} = \begin{Bmatrix} 0 & \text{when } \gamma > 1 \\ K_0 & \text{when } \gamma = 1 \end{Bmatrix} \qquad (21)\text{III } 4$$

Equations (21)III 4 corroborate the statement given in Ch. I about the behavior of the function $F(s,\sqrt{s^2+1})$ when $s \to \infty$.

It has been said previously in Ch. I, Section 9, that the solution of the vector field, in the t domain, is electromagnetic if $F(s,\sqrt{s^2+1})_{s\to\infty} \longrightarrow 0(\frac{M}{s^\gamma})$; $\gamma > 1$. It can be concluded, therefore, that the propagation of the TE and TH fields in hollow cylinders is such that the wave front $(\tau=\varkappa)$ vanishes.

III-4.5 In this subsection the behavior of $\mathcal{V}(\Omega_k,T)$ and $\mathcal{U}(\Omega_k,T)$ when $\tau\to\infty$ will be investigated. (Permanent state.) Take first $V_n(\Omega_k,T)$ and prove that

$$V_n(\Omega_k,T)_{T\to 0} \longrightarrow 0 \qquad (22)\text{III } 4$$

for all values of n, independently of Ω_k.

No simple and direct analysis of this situation can be made by means of the series which define the Lommel functions. The same happens with the other formulas already given. If one of the integral representations of these functions is used, then the proof is simple.

$$V_n(\Omega,T) = \frac{1}{2\pi i}\int_{-\infty}^{0+} \frac{(\frac{1}{2}\frac{\Omega}{t})^n}{1+\frac{1}{4}(\frac{\Omega}{t})^2}\, e^{\frac{t}{2}} e^{-\frac{T^2}{4t}}\, \frac{dt}{t} \qquad (23)\text{III } 4$$

in which t is a positive real variable. Since $\tau > \varkappa$, then $\sqrt{\tau^2-\varkappa^2}$ is always real and $T\to\infty$ with $\tau\to\infty$, and the integrand goes to zero for all values of n and Ω. Therefore,

$$V_n(\Omega,T)_{\tau\to\infty} \longrightarrow 0$$

for all values of n and Ω, only if $\tau > \kappa$.

Since this is independent of Ω,

$$\left.\begin{array}{l} V_n(\Omega_k, T)\underset{\tau\to\infty}{\longrightarrow} 0 \\ V_n(\Omega_k^*, T)\underset{\tau\to\infty}{\longrightarrow} 0 \ . \end{array}\right\} \qquad (24)\text{III } 4$$

By considering the above results as well as (15)III 4 and (16)III 4 it can be concluded that

$$\left.\begin{array}{l} \mathcal{V}(\Omega_k, T)\underset{\tau\to\infty}{\longrightarrow} 0 \\ \mathcal{U}(\Omega_k, T)\underset{\tau\to\infty}{\longrightarrow} e^{A(Z_k)} \\ \mathcal{L}^{-1} F(s, \sqrt{s^2+1})\, e^{-\kappa\sqrt{s^2+1}} = \varphi(\tau, \kappa)\underset{\tau\to\infty}{\longrightarrow} \sum_{k=1}^{m} K_k e^{A(Z_k)} \\ \quad \text{when } s \to 0 \end{array}\right\} \qquad (25)\text{III } 4$$

and therefore, the well-known solutions for the permanent state are obtained.

Also, as an extension

$$\left.\begin{array}{l} \mathcal{V}(\Omega_k^*, T)\underset{\tau\to\infty}{\longrightarrow} 0 \\ \mathcal{U}(\Omega_k^*, T)\underset{\tau\to\infty}{\longrightarrow} e^{A(Z_k^*)} \ . \end{array}\right\} \qquad (26)\text{III } 4$$

III-4.6 Next the concept of group velocity of waves represented by the generating functions $\mathcal{V}$ and $\mathcal{U}$ will be introduced. In order to do so, go back to the s plane:

$$\varphi(\tau, \kappa) = \frac{1}{2\pi i}\int_{\gamma_z} F(s, \sqrt{s^2+1})\, e^{-\kappa\sqrt{s^2+1}}\, ds \ . \qquad (27)\text{III } 4$$

In accordance with Brillouin, the complex group velocity is defined by

$$\frac{d}{ds}(s\tau - \kappa\sqrt{s^2+1}) = 0 \qquad (28)\text{III } 4$$

from which

$$\nu_g = \left.\frac{\kappa}{\tau}\right)_g = \frac{\sqrt{s^2+1}}{s} = -\frac{Z + \frac{1}{Z}}{Z - \frac{1}{Z}} \ . \qquad (29)\text{III } 4$$

The above expression is called the normalized group velocity, that is, the ratio of the group velocity c_g to the velocity of light c. To check, write

$$\nu_g = \frac{2\pi \frac{x_3}{\lambda c}}{2\pi \frac{t}{Tc}} = \frac{\left(\frac{x_3}{t}\right)_g}{c} = \frac{c_g}{c} \quad .$$

As an extension of the above concept, the expression

$$\nu_{g_k} = \frac{\sqrt{\mathit{s}_k^2+1}}{\mathit{s}_k} = -\frac{Z_k + \frac{1}{Z_k}}{Z_k - \frac{1}{Z_k}} = -\frac{Z_k^2+1}{Z_k^2-1} \qquad (30)\text{III } 4$$

will be called the group velocity corresponding to the pole at s_k or Z_k.

Two simple, but important, theorems will be given:

Theorem A - "The poles Z_k and Z_k^* have equal but opposite complex group velocity."

The theorem follows from

$$Z_k Z_k^* = -1 \quad .$$

Theorem B - "If $\overline{Z}_k$ (conjugate of Z_k) is also a pole, then Z_k and $\overline{Z}_k$ have the same complex group velocity."

The generating functions $\mathcal{U}(\Omega_k,T)$ and $\mathcal{V}(\Omega_k,T)$ are closely associated with the pole at Z_k (see Eqs. (15)III 4, (16)III 4 and (17)III 4.) In the future ν_{g_k} will be referred to as the group velocity of the waves represented by these generating functions. The next theorem is of primary importance.

Theorem C - "The values of the generating functions at the time corresponding to the arrival with group velocity are given by

$$\left.\begin{array}{l}\mathcal{U}(\Omega_k,\Omega_k) \text{ and} \\ \mathcal{V}(\Omega_k,\Omega_k)\text{."}\end{array}\right\} \qquad (31)\text{III } 4$$

The proof is simple: since $T=\Omega_k$, then

$$\frac{\tau-\kappa}{iZ_k} = \sqrt{\tau^2-\kappa^2}$$

$$\left(\frac{\kappa}{\tau}\right)_g = -\frac{Z_k^2+1}{Z_k^2-1}$$

and the theorem is proved.

Now, the values of the generating functions $\mathcal{U}$ and $\mathcal{V}$ when the ratio $\frac{\kappa}{\tau}$ is equal to the group velocity of the corresponding pole will be found.

From (12)III 4 and (13)III 4, when $\Omega_k = T$,

$$\left.\begin{array}{l} U_0(T,T) = V_0(T,T) = \frac{1}{2}\left[J_0(T) + \cos T\right] \\ U_1(T,T) = -V_1(T,T) = \frac{1}{2}\sin T \\ U_2(T,T) = \frac{1}{2} J_0(T) - \frac{1}{2}\cos T \end{array}\right\} \qquad (32)\text{III } 4$$

so that:

$$\left.\begin{array}{l} \mathcal{U}(T,T) = \mathcal{U}(\Omega_k,\Omega_k) = \frac{1}{2} J_0(T) - \frac{1}{2} e^{-iT} \\ \mathcal{V}(T,T) = \mathcal{V}(\Omega_k,\Omega_k) = \frac{1}{2} J_0(T) + \frac{1}{2} e^{-iT}\ . \end{array}\right\} \qquad (33)\text{III } 4$$

To make the proper interpretation of these results, the permanent state term $e^{A(Z_k)}$ corresponding to the pole Z_k will be considered. Since $T=\Omega_k$, from (7)III 4

$$A(Z_k) = \frac{\tau}{2}\left(Z_k - \frac{1}{Z_k}\right) + \frac{\kappa}{2}\left(Z_k + \frac{1}{Z_k}\right) = \frac{T}{2}\left(Z_k\sqrt{\frac{\tau-\kappa}{\tau+\kappa}} - \frac{1}{Z_k}\sqrt{\frac{\tau+\kappa}{\tau-\kappa}}\right) = -iT \qquad (34)\text{III } 4$$

since, for the group velocity

$$iZ_k = \sqrt{\frac{1-\theta}{1+\theta}}\ . \qquad (35)\text{III } 4$$

Then the permanent state term, corresponding to the pole Z_k, becomes at group velocity time

$$e^{-iT} . \tag{36)III 4}$$

From (33)III4 and (36)III4 the following theorem is obtained: "The generating function corresponding to the pole Z_k has, at the time corresponding to the arrival with group velocity, a value which is equal to one half the permanent state plus (or minus) one half the Bessel function of order zero." At large distances from the origin of excitation of the wave guide, the value of T corresponding to the group velocity becomes large. In this case $J_0(T)$ can be neglected and then it can be said: "At large distances from the origin of excitation and at the time corresponding to the group velocity, the generating functions have a magnitude approximately equal to one half of the permanent state."

The equivalent relations for the (*) function can be derived. The results are quite similar to those obtained in this section.

III-4.7 In this subsection the concept of phase velocity will be introduced.

The complex phase velocity is obtained from (27)III4 when

$$s\tau - \varkappa\sqrt{s^2+1} = 0 \qquad (37)\text{III } 4$$

The ratio given by

$$\left.\frac{\varkappa}{\tau}\right)_{ph} = \nu_{ph} = \frac{s}{\sqrt{s^2+1}} \qquad (38)\text{III } 4$$

will define the normalized phase velocity, that is, the ratio of the actual phase velocity to the velocity of light.

For the pole at Z_k, the corresponding phase velocity has the form

$$\left.\frac{\varkappa}{\tau}\right)_{ph} = -\frac{Z_k^2 - 1}{Z_k^2 + 1} \; . \qquad (39)\,\text{III}\,4$$

The next three theorems follow:

"For the same pole Z_k, the corresponding group and phase velocities are reciprocal."

"The phase velocities corresponding to the poles Z_k and Z_k^* are equal but opposite in sign."

"If $\bar{Z}_k$ is also a pole of G(Z), then the poles Z_k and $\bar{Z}_k$ have phase velocities which are equal."

Now, the way in which the variables Ω_k and T are related will be studied, that is, at the time corresponding to the phase velocity.

From (7)III4 and (39)III4,

$$\Omega_k = \frac{\tau - \varkappa}{iZ_k} = \frac{\tau}{iZ_k}\left(1 - \frac{\varkappa}{\tau}\right) = \tau\,\frac{2Z_k}{i(Z_k^2+1)}$$

$$T = \sqrt{\tau^2 - \varkappa^2} = \tau\sqrt{1 - \frac{\varkappa}{\tau}} = \tau\,\frac{2\sqrt{Z_k}}{Z_k^2+1} \qquad (40)\,\text{III}\,4$$

from which

$$\left.\Omega_k\right)_{ph} = \frac{\sqrt{Z_k}}{i}\left. T\right)_{ph} \; . \qquad (41)\,\text{III}\,4$$

Then:

"At the planes of equal phase, the variables Ω_k and T are proportional."

It was not possible to obtain exact expressions which give the values of the generating functions corresponding to the phase velocity value of $(\frac{\gamma}{\tau})$. Asymptotic expressions can be derived for large values of T; see Ch. IV.

III-4.8 The signal velocity and the time of formation of the generating functions $\mathcal{U}$ and $\mathcal{V}$ will be discussed in Ch. IV.

III-4.9 To close Section 4, the solution of the inverse transforms when there are poles with multiplicity greater than one will be given.

In this case the typical integrals are:

$$\frac{1}{2\pi i}\int_{\mathcal{L}_Z}\frac{e^{A(Z)}}{(Z-Z_k)^{\alpha}}\,dZ = \frac{1}{2\pi i}\int_{C_1}\frac{e^{A(Z)}}{(Z-Z_k)^{\alpha}}\,dZ \qquad \text{for } \alpha > 1 \qquad (42)\text{III}\,4$$

since the integration vanishes around the pole.

Integral (42)III4 can be expressed in terms of the derivatives of the function $\mathcal{V}_1$, since

$$\frac{1}{2\pi i}\int_{C_1}\frac{e^{A(Z)}}{(Z-Z_k)^{\alpha}}\,dZ = -\frac{1}{(\alpha-1)!}\,\frac{\partial^{(\alpha-1)}}{\partial Z_k^{(\alpha-1)}}\mathcal{V}(\Omega_k, T) \; . \qquad (43)\text{III}\,4$$

Now it can be shown that: (see "Bessel's Functions", Watson, P. 539)

$$2\frac{\partial}{\partial\Omega_k}\,V_n = V_{n+1} + \left(\frac{T}{\Omega_k}\right)^2 V_{n-1} \qquad (44)\text{III}\,4$$

from which

$$\frac{\partial}{\partial\Omega_k}V_0 = \frac{1}{2}V_1 + \frac{1}{2}\left(\frac{T}{\Omega_k}\right)^2 V_{-1}$$

$$\frac{\partial}{\partial\Omega_k}\,V_1 = \frac{1}{2}V_2 + \frac{1}{2}\left(\frac{T}{\Omega_k}\right)^2 V_0$$

so that

$$\frac{\partial \mathcal{V}}{\partial \Omega_k} = \frac{1}{2}(V_1 + iV_2) + \frac{1}{2}\left(\frac{T}{\Omega_k}\right)^2 (V_{-1} + iV_0) \quad . \tag{(45) III 4}$$

By (14) III 4

$$V_2 = -V_0 + J_0(T); \quad V_{-1} = -V_1 + \frac{\Omega_k}{T} J_1(T)$$

and therefore,

$$\frac{\partial \mathcal{V}}{\partial Z_k} = \frac{iT}{2Z_k}\left\{\left(\frac{\Omega_k}{T} - \frac{T}{\Omega_k}\right)\mathcal{V} - \left[\frac{\Omega_k}{T} J_0(T) - iJ_1(T)\right]\right\} \tag{(46) III 4}$$

and finally

$$\frac{1}{2\pi i}\int_{\gamma_Z} \frac{e^{A(Z)}}{(Z-Z_k)^2} dZ = \frac{iT}{2Z_k}\left\{\left(\frac{\Omega_k}{T} - \frac{T}{\Omega_k}\right)\mathcal{V}(\Omega_k, T) - \left[\frac{\Omega_k}{T} J_0(T) - iJ_1(T)\right]\right\} \tag{(47) III 4}$$

$$= -\frac{1}{2}\sqrt{\frac{\tau+\kappa}{\tau-\kappa}}\,\Omega_k\left\{\left(\frac{\Omega_k}{T} - \frac{T}{\Omega_k}\right)\mathcal{V}(\Omega_k, T) - \left[\frac{\Omega_k}{T} J_0(T) - iJ_1(T)\right]\right\} \tag{(48) III 4}$$

It can be noticed that this integral can be expressed in terms of the generating function $\mathcal{V}$. By continuing this process of differentiation, the values of the integral (43)III4 can be obtained, and therefore, the whole problem of the inversion is completely solved. The computations of some transforms are given in Section 6.

Section 7 will be devoted to the application of this theory to wave guides.

Section 5 - Computation of the inverse Laplace transform of some useful transforms. General solution of the inverse Laplace transform.

III-5.0 This section is devoted to illustrating the method of inversion developed in this chapter. Some simple and common transforms will be considered. They will satisfy the condition that F must be of the order $\frac{M}{s^{\gamma}}$; $\gamma \geq 1$ when $s \to \infty$. One has to recall that $\gamma \geq 2$ for those transforms which find application in wave guides. This last condition will be required in Section 6, in which this theory is applied to some specific examples of wave propagation in wave guides.

III-5.1 The steps required to obtain

$$\mathcal{L}^{-1}F(s,\sqrt{s^2+1})\,e^{-x\sqrt{s^2+1}} \qquad (1)\,\text{III 5}$$

will be summarized in which $F(s,\sqrt{s^2+1})$ is the ratio of two polynomials in s and $\sqrt{s^2+1}$. Besides $F(s,\sqrt{s^2+1}) \underset{s\to\infty}{\to} (\frac{M}{s^\gamma})$; with $\gamma \geq 1$.

The procedure must be as follows:

1st - Use the complex transformation

$$Z = s - \sqrt{s^2+1} \qquad (2)\,\text{III 5}$$

and set

$$s = \frac{1}{2}(Z - \frac{1}{Z}) \;;\; \sqrt{s^2+1} = \frac{1}{2}(Z + \frac{1}{Z}) \;. \qquad (3)\,\text{III 5}$$

2nd - Substitute the above expressions in $F(s,\sqrt{s^2+1})$ and so obtain $F(Z)$.

3rd - Find the function

$$G(Z) = \frac{1}{2}\,\frac{1+Z^2}{Z^2}\,(F(Z) \qquad (4)\,\text{III 5}$$

which is the ratio of two polynomials in Z. The corresponding degree of this polynomial was already discussed in Section 1. If m is the degree of the denominator of the polynomial in $F(s,\sqrt{s^2+1})$, then, the degree of the polynomial in the denominator of G(Z) is

$$\begin{array}{ll} 2m+1 & \text{if } \gamma = 1 \\ 2m & \text{if } \gamma > 1 \;. \end{array} \qquad (5)\,\text{III 5}$$

(See the theorem in Section 1 of this chapter.)

<u>4th</u> - Find the poles of G(Z). Recall Theorem 1 Section 1. If s_k is a pole of $F(s,\sqrt{s^2+1})$, then

$$\left.\begin{aligned} Z_k &= s_k - \sqrt{s_k^2+1} \\ Z_k^* &= s_k + \sqrt{s^2+1}, \text{ so that} \\ Z_k Z_k^* &= -1 \end{aligned}\right\} \qquad (6)\text{III }5$$

are poles of G(Z).

If pole s_k has a multiplicity α, then Z_k and Z_k^* have the same multiplicity. (See other theorems in Section 1 of this chapter.)

<u>5th</u> - In what follows always suppose that

$$\left.\begin{aligned} &|Z_k^*| \geq 1 \\ \text{and } &|Z_k| \leq 1 \end{aligned}\right\}. \qquad (7)\text{III }5$$

The asterisk in (6)III5 must be associated with the poles of G(Z) in accordance with the convention given in (7)III5. This is always possible.

<u>6th</u> - Expand G(Z) in partial fractions. Use the notation given in Section 3 of this chapter. For the convenience of the reader the following expressions will be repeated.

$$G(Z) = \sum_{k=1}^{m} \sum_{j=1}^{\alpha_k} \frac{K_{kj}}{(Z-Z_k)^{\alpha_k-j+1}} + \sum_{k=1}^{m} \sum_{j=1}^{\alpha_k} \frac{K_{kj}^*}{(Z-Z_k^*)^{\alpha_k-j+1}} + \begin{Bmatrix} \frac{K_o}{Z} & \text{for } \gamma = 1 \\ 0 & \text{for } \gamma > 1 \end{Bmatrix} \qquad (8)\text{III }5$$

in which

$$\left.\begin{aligned} K_{kj} &= \frac{1}{(j-1)!}\left[\frac{d^{j-1}}{dZ^{j-1}}(Z-Z_k)^{\alpha_k}\ G(Z)\right]_{Z=Z_k} \\ K_{kj}^* &= \frac{1}{(j-1)!}\left[\frac{d^{j-1}}{dZ^{j-1}}(Z-Z_k^*)^{\alpha_k} G(Z)\right]_{Z=Z_k^*} \\ K_o &= \Big[Z\ G(Z)\Big]_{Z=0} \end{aligned}\right\} \qquad (9)\text{III }5$$

and

Z_1 has a multiplicity α_1

Z_2 has a multiplicity α_2

- - - - - - - - - - - - -

Z_k has a multiplicity α_k (10) III 5

- - - - - - - - - - - - -

Z_q has a multiplicity α_q

such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k + \cdots + \alpha_q = m$.

6th - To obtain

$$\varphi(\tau,\kappa) = \mathcal{L}^{-1} F(s, \sqrt{s^2+1})\, e^{-\kappa\sqrt{s^2+1}} \qquad (11)\,\text{III } 5$$

use only the terms without asterisks. The contribution of the terms with asterisks must not be added since they represent the contribution of the integration around the branch cut in the Riemann surface $\mathcal{S}_{II}$. This statement is easily proved.

7th - Write the following expression as

$$\sum_{k=1}^{m} \sum_{j=1}^{j=\alpha_k} \frac{K_{kj}}{(Z-Z_k)^{\alpha_k - j - 1}} = \sum_{k=1}^{m} \frac{K_{k\alpha_k}}{(Z-Z_k)} + \sum_{k=1}^{m} \sum_{j=1}^{\alpha_k - 1} \frac{K_{kj}}{(Z-Z_k)^{\alpha_k - j - 1}} \qquad (12)\,\text{III } 5$$

For each term of the form

$$\frac{K_{k\alpha_k}}{Z-Z_k} \text{ write } K_{k\alpha_k}\left\{e^{A(Z_k)} - \mathcal{V}(\Omega_k, T)\right\} = K_{k\alpha_k}\mathcal{U}(\Omega_k, T) \qquad (13)\,\text{III } 5$$

For each term of the form

$$\frac{K_{kj}}{(Z-Z_k)^{\alpha_k - j - 1}} \text{ write } -K_{kj}\frac{1}{(\alpha_k - j)!}\frac{\partial^{(\alpha_k - 1)}}{\partial Z_k^{(\alpha_k - 1)}}\mathcal{V}(\Omega_k, T) \qquad (14)\,\text{III } 5$$

For the term of the form ($\delta = 1$)

$$\frac{K_o}{Z} \text{ write } -J_o(T)\,. \qquad (15)\,\text{III } 5$$

Then, the inverse Laplace transform is given by

$$\mathcal{L}^{-1} F(s, \sqrt{s^2+1})\, e^{-\varkappa\sqrt{s^2+1}}$$

$$\left.\begin{aligned}\varphi(\tau,\varkappa) = u_{-1}(\tau-\varkappa)&\left\{\left[\sum_{k=1}^{m} K_{k\alpha_k}\left[e^{A(Z_k)} - \mathcal{V}(\Omega_k, T)\right] - \right.\right.\\ &-\sum_{k=1}^{m}\sum_{j=1}^{\alpha_k-1}\frac{K_{kj}}{(\alpha_k-j)!}\frac{\partial^{\alpha_k-j}}{\partial Z_k^{\alpha_k-j}}\mathcal{V}(\Omega_k,T) - \begin{cases}0 & \gamma>1\\ K_o J_0(T) & \gamma=1\end{cases}\end{aligned}\right\} \quad (16)\,\text{III}\,5$$

$$\left.\begin{aligned}= u_{-1}(\tau-\varkappa)&\left\{-\sum_{k=1}^{m} K_{k\alpha_k}\mathcal{U}(\Omega_k, T) - \right.\\ &-\sum_{k=1}^{m}\sum_{j=1}^{\alpha_k-1}\frac{K_{kj}}{(\alpha_k-j)!}\frac{\partial^{\alpha_k-j}}{\partial Z_k^{\alpha_k-j}}\mathcal{V}(\Omega_k,T) - \begin{cases}0 & \gamma>1\\ K_o J_0(T) & \gamma=1\end{cases}\end{aligned}\right\} \quad (17)\,\text{III}\,5$$

The derivatives of $\mathcal{V}(\Omega_k, T)$ can be computed as in Subsection III-4.9.

III-5.2 In the computation of the inverse transforms, it is convenient to use the following properties:

1st -

$$\left.\begin{aligned}V_0(-\Omega,T) &= V_0(\Omega,T)\\ V_1(-\Omega,T) &= -V_1(\Omega,T)\\ U_2(-\Omega,T) &= U_2(\Omega,T)\\ U_1(-\Omega,T) &= -U_1(\Omega,T)\ .\end{aligned}\right\} \quad (18)\,\text{III}\,5$$

2nd - The arguments Ω_k and T of the Lommel functions have the value

$$\left.\begin{aligned}T &= \sqrt{\tau^2-\varkappa^2}\\ \Omega_k &= \frac{\tau-\varkappa}{iZ_k}\ .\end{aligned}\right\} \quad (19)\,\text{III}\,5$$

Notice that T is always positive and real, since $\tau > \varkappa$.

Ω_k is, in general, a complex quantity. It is real only when Z_k is pure imaginary, in which case

$Z_k = i|Z_k|$ or $= -i|Z_k|$

and therefore,

$$\Omega_k = -\frac{\tau-\kappa}{|Z_k|} \text{ or } \frac{\tau-\kappa}{|Z_k|} \quad .$$

3rd - When τ or κ change in value, then the complex argument Ω_k represents a moving point in the Z plane. For a given pole Z_k, the lines described by the variable point Ω_k are straight lines through the origin. This property can be visualized by writing

$$Z_k = |Z_k| e^{j\varphi_k}$$

in which $|Z_k|$ and φ_k are constant quantities.

Then

$$\Omega_k = \frac{\tau-\kappa}{|Z_k|} e^{-i(\varphi_k - \frac{\pi}{2})} . \qquad (20)\text{ III 5}$$

III-5.3 In this subsection some simple and useful inverse transforms will be computed.

1st - $\dfrac{e^{-\kappa\sqrt{s^2+1}}}{s-i\nu_o}$; $\nu_o > 1$.

a. Notice that $\gamma = 1$.

b. $F(s) = \dfrac{1}{s-i\nu_o}$ then $s_1 = i\nu_o$ is a root

from which

$$Z_1 = i(\nu_o - \sqrt{\nu_o^2 - 1})$$
$$Z_1^* = i(\nu_o + \sqrt{\nu_o^2 - 1}) \quad .$$

Notice that

$$|Z_1| < 1 \text{ and } |Z_1^*| > 1.$$

c. $$G(Z) = \frac{1+Z^2}{Z(Z^2 - 2s_1 Z - 1)}$$

$$\therefore\; K_o = -1 \;;\; K_1 = 1 \;;\; K_1^* = 1$$

$$\Omega_k = -\frac{\tau - \kappa}{\nu_o - \sqrt{\nu_o^2 - 1}}$$

$$A(Z_1) = \frac{\tau}{2}\left(Z_1 - \frac{1}{Z_1}\right) + \frac{\kappa}{2}\left(Z_1 + \frac{1}{Z_1}\right) = i(\tau\nu_o - \kappa\sqrt{\nu_o^2 - 1}\;.$$

Finally

$$\mathcal{L}^{-1}\,\frac{e^{-\kappa\sqrt{s^2+1}}}{s - j\nu_o} = u_1(\tau-\kappa)\left\{e^{i(\nu_o\tau - \kappa\sqrt{\nu_o^2-1})} - \mathcal{V}\left(-\frac{\tau-\kappa}{\nu_o - \sqrt{\nu_o^2-1}}, \sqrt{\tau^2 - \kappa}\right) + J_0(\sqrt{\tau^2 - \kappa^2})\right\} =$$

$$= u_{-1}(\tau-\kappa)\left\{-\mathcal{U}\left(-\frac{\tau-\kappa}{\nu_o - \sqrt{\nu_o^2-1}}, \sqrt{\tau^2-\kappa^2}\right) + J_0(\sqrt{\tau^2-\kappa^2})\right\} \qquad (21)\,\text{III}\,5$$

For the values of the generating functions $\mathcal{V}$ and $\mathcal{U}$ see Eq. (18)III4.

<u>2nd</u> - $\dfrac{e^{-\kappa\sqrt{s^2+1}}}{(s - j\nu_o)\sqrt{s^2+1}}$; $\nu_o > 1$

Here $\gamma = 2 \;\therefore K_o = 0$

$$G(Z) = -\frac{2}{Z^2 - 2i\nu_o Z - 1}$$

$$Z_1 = i(\nu_o - \sqrt{\nu_o^2 - 1}) \;;\; |Z_1| < 1$$

$$Z_1^* = i(\nu_o + \sqrt{\nu_o^2 - 1}) \;;\; |Z_1^*| > 1$$

$$K_1 = \frac{-i}{\sqrt{\nu_o^2 - 1}} \;;\; K_1^* = \frac{i}{\sqrt{\nu_o^2-1}}$$

$$\therefore\; \mathcal{L}^{-1}\,\frac{e^{-\kappa\sqrt{s^2+1}}}{(s - j\nu_o)\sqrt{s^2+1}} = u_{-1}(\tau-\kappa)\left\{-\frac{i}{\sqrt{\nu_o^2-1}}\left[e^{i(\nu_o\tau - \kappa\sqrt{\nu_o^2-1})} - \mathcal{V}\left(-\frac{\tau-\kappa}{\nu_o - \sqrt{\nu_o^2-1}}, T\right)\right]\right\} = \qquad (22)\,\text{III}\,5$$

$$= +u_{-1}(\tau-\kappa)\,\frac{i}{\sqrt{\nu_o^2-1}}\,\mathcal{U}\left(-\frac{\tau-\kappa}{\nu_o^2 - \sqrt{\nu_o^2-1}}, T\right)$$

in which $u_{-1}(\tau-\kappa)$ is the unit step function shifted to $\tau = \kappa$.

3rd $- \frac{e^{-\varkappa\sqrt{s^2+1}}}{s^2+\nu_o^2}$ $\nu_o > 1$

One has: $\gamma = 2$; $K_o = 0$; $s_1 = i\nu_o$; $s_2 = -i\nu_o$

$Z_1 = i(\nu_o - \sqrt{\nu_o^2-1})$; $|Z_1| < 1$; $Z_2 = -i(\nu_o - \sqrt{\nu_o^2-1})$; $|Z_2| < 1$

$Z_1^* = i(\nu_o + \sqrt{\nu_o^2-1})$; $|Z_1^*| > 1$; $Z_2^* = -i(\nu_o + \sqrt{\nu_o^2+1})$; $|Z_2^*| > 1$

$$G(Z) = \frac{Z_o^2+1}{\left[Z^2-2i\nu_o Z-1\right]\left[Z^2+2i\nu_o Z-1\right]}$$

$K_1 = -\frac{i}{2\nu_o}$; $K_2 = +\frac{i}{2\nu_o}$; $K_1^* = -\frac{i}{2\nu_o}$; $K_2^* = +\frac{i}{2\nu_o}$

$A(Z_1) = i(\tau\nu_o - \varkappa\sqrt{\nu_o^2-1})$; $A(Z_2) = -i(\tau\nu_o - \varkappa\sqrt{\nu_o^2-1})$

$\Omega_1 = -\frac{\tau-\varkappa}{\nu_o-\sqrt{\nu_o^2-1}}$; $\Omega_2 = +\frac{\tau-\varkappa}{\nu_o^2-\sqrt{\nu_o^2-1}}$; $T = \sqrt{\tau^2-\varkappa^2}$

and finally

$$\mathcal{L}^{-1}\frac{e^{-\varkappa\sqrt{s^2+1}}}{s^2+\nu_o^2} = u_{-1}(\tau-\varkappa)\times\frac{1}{\nu_o}\left\{\sin(\nu_o\tau-\varkappa\sqrt{\nu_o^2-1}) - \frac{i}{2}\left[\mathcal{V}\left(-\frac{\tau-\varkappa}{\nu_o-\sqrt{\nu_o^2-1}}; T\right) - \mathcal{V}\left(+\frac{\tau-\varkappa}{\nu_o^2-\sqrt{\nu_o^2-1}}, T\right)\right]\right\}$$

$$= u_{-1}(\tau-\varkappa)\times\frac{1}{\nu_o}\left\{\sin(\nu_o\tau-\varkappa\sqrt{\nu_o^2-1}) - V_1\left(\frac{\tau-\varkappa}{\nu_o-\sqrt{\nu_o^2-1}}, T\right)\right\}. \qquad (23)\,\text{III}\,5$$

In a similar way one can obtain:

4th $- \frac{s\,e^{-\varkappa\sqrt{s^2+1}}}{(s-i\nu_o)\sqrt{s^2+1}}$; $\nu_o > 1$ and $\gamma = 1$

$$\mathcal{L}^{-1}\frac{s\,e^{-\varkappa\sqrt{s^2+1}}}{(s-i\nu_o)\sqrt{s^2+1}} = u_{-1}(\tau-\varkappa)\times\frac{\nu_o}{\sqrt{\nu_o^2-1}}\left\{e^{j(\nu_o\tau-\varkappa\sqrt{\nu_o^2-1})} - \mathcal{V}\left(-\frac{\tau-\varkappa}{\nu_o-\sqrt{\nu_o^2-1}}, T\right) + J_o(T)\right\} \qquad (24)\,\text{III}\,5$$

5th - $\frac{\sqrt{s^2+1}\; e^{-\kappa\sqrt{s^2+1}}}{(s^2+\nu_o^2)}$; $\nu_o > 1 \quad \gamma = 1$

$$\mathcal{L}^{-1} \frac{\sqrt{s^2+1}\; e^{-\kappa\sqrt{s^2+1}}}{s^2+\nu_o^2} = \tag{25 III 5}$$

$$= u_{-1}(\tau-\kappa)\left\{J_o(T) + \frac{\sqrt{\nu_o^2-1}}{\nu_o}\left[\cos(\nu_o\tau-\kappa\sqrt{\nu_o^2-1}) + V_o\left(\frac{\tau-\kappa}{\nu_o-\sqrt{\nu_o^2-1}}, T\right)\right]\right\}$$

6th - $\mathcal{L}^{-1} \frac{s e^{-\kappa\sqrt{s^2+1}}}{(s^2+\nu_o^2)\sqrt{s^2+1}} =$

$$= \frac{u_{-1}(\tau-\kappa)}{\sqrt{\nu_o^2-1}}\left\{\sin(\nu_o\tau-\kappa\sqrt{\nu_o^2-1}) + V_1\left(\frac{\tau-\kappa}{\nu_o-\sqrt{\nu_o^2-1}}, T\right)\right\} \text{ when } \nu_o > 1 \tag{26 III 5}$$

7th - $\mathcal{L}^{-1} \frac{e^{-\kappa\sqrt{s^2+1}}}{(s^2+\nu_o^2)\sqrt{s^2+1}} =$

$$= \frac{u_{-1}(\tau-\kappa)}{\nu_o\sqrt{\nu_o^2-1}}\left\{-\cos(\nu_o\tau-\kappa\sqrt{\nu_o^2-1}) + V_o\left(\frac{\tau-\kappa}{\nu_o-\sqrt{\nu_o^2-1}}, T\right)\right\} \quad \nu_o > 1 \tag{27 III 5}$$

8th - $\mathcal{L}^{-1} \frac{e^{-\kappa\sqrt{s^2+1}}}{s^2+\nu_o^2} =$

$$= u_{-1}(\tau-\kappa)\left\{\cos(\nu_o\tau-\kappa\sqrt{\nu_o^2-1}) - V_o\left(\frac{\tau-\kappa}{\nu_o-\sqrt{\nu_o^2-1}}, T\right)\right\} \quad \nu_o > 1 \tag{28 III 5}$$

Section 6 - The transient phenomena in wave guides. Formation of envelopes with elementary wave forms and with some orthogonal polynomial. Example of the transient field in wave guides.

III-6.0 In Ch. I, Sections 8 and 9, it was shown that the solution of the transient phenomena in wave guides was closely associated with the solution of the inverse Laplace transform of the prototype transform

$$F(s,\sqrt{s^2+1})\,e^{-x\sqrt{s^2+1}}$$

with the restrictions given in the above-mentioned sections. Now, since the solution of the corresponding Laplace transformation was already obtained and given in equations (16)III5 and (17)III5, it can be concluded that the problem of finding the transient response of a wave is solved, when the excitation of the guide is such that it leads to a function of the type of $F(s,\sqrt{s^2+1})$.

It was also shown in Ch. I that the function $F(s,\sqrt{s^2+1})$ can cover a practically unlimited number of cases of excitation. Therefore, the solutions already obtained have a general character.

The object of this section is to indicate the intermediate steps for passing from the inverse transform to the solution of the components of the electromagnetic vector. The required relations between them are given in Table I, Ch. I. It will be convenient to consult Sections 8 and 9 of Ch. I in which some important relations are given.

III-6.1 It is convenient to give a brief summary of the present situation.

1st - Inside wave guides two fundamental types of wave, TE and TH, can be excited.

2nd - In the case of TE waves at $x_3 = 0$, the component $\mathcal{H}_3$ or the space derivative of $\mathcal{H}_3$ with respect to x_3 can be prescribed (see Table I, Ch. I). If $\left(\frac{\partial \mathcal{H}_3}{\partial x_3}\right)_{x_3=0}$ is prescribed as an initial condition, it is equivalent to specifying the transverse components of the magnetic field. See Eqs. (3)I5 on P. 31.

3rd - In the case of TH waves at $x_3 = 0$, $\mathcal{E}_3$ or $\frac{\partial \mathcal{E}_3}{\partial x_3}$ can be prescribed. If this derivative is given as the initial condition, it is equivalent to specifying the transverse components of the electric field. See Eq. (2)I5, P. 31.

4th - In Section 8, Paragraph I-8.1, the manner of obtaining the corresponding initial condition in the **S** domain was indicated. In the same section, some theorems and methods are given which facilitate this purpose.

5th - In Section I-3.1, Eqs. (8)I3, the differential equations which yield the functions Ψ_3 or Θ_3 will be found as will the separation constant p. The solutions of these equations are well known for typical cross sections of the wave guides. The solutions can be found in any text book on wave guides and are omitted in this investigation.

6th - The Laplace transforms given in Table I, Ch. I, are functions of the complex frequency s. The transforms considered in this chapter are given in terms of the normalized complex frequency

$$\mathfrak{s} = \frac{s}{\omega_c} \; ; \quad \omega_c = pc \quad . \text{(See p. 19)}$$

It is necessary therefore to use the relations

$$\mathcal{L}_t^{-1} F(s, \sqrt{s^2+\omega_c^2})\, e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}} \equiv \omega_c \mathcal{L}_t^{-1} F(\mathfrak{s}, \sqrt{\mathfrak{s}^2+1})\, e^{-k\sqrt{\mathfrak{s}^2+1}} . \qquad (1)\text{III } 6$$

7th - The solution of a transient problem is now easily obtained by using the inverse Laplace transforms in the manner indicated.

The manipulation of a transient problem will become simpler by using some of the additional theorems or methods given in the following subsections.

III-6.1 Once an initial condition is given, it must be transformed from the t domain into the S domain. This initial condition is a function of time which oscillates rapidly and in general changes in amplitude or phase. In such cases, it is convenient to express this time function as a product of two factors. One of these factors gives the amplitude of the oscillations and the other corresponds to the period of the oscillations. The first factor alone represents the envelope of the oscillation. When this separation into factors is possible, then the Laplace transform corresponding to each of these factors can be found. The Laplace transform of the whole signal can be obtained by complex convolution. If one of these factors has a transform which is equal to the ratio of two polynomials in s, then the complete Laplace transform can be computed by means of the theorems given in Section I-8.5, P. 41, Ch. I.

In the case of simple wave forms of excitation, the Laplace transform of the incoming signal can be found directly without much labor. When this initial time function represents complicated wave forms, then some difficulties may arise, mainly when the initial condition is given in the form of a graph, which is often the case. It is, therefore, convenient to develop a practical and simple method which yields the required transforms of rather complicated wave forms of excitation.

In what follows use will be made of the notation

$$f(t) = m(t)g(t) \qquad (2)\,\text{III 6}$$

in which m(t) represents the envelope function and g(t) indicates the corresponding highly oscillating function.

III-6.2 Attention will be confined to the envelope function m(t). Complicated wave envelopes can be obtained by compounding elementary wave forms. By elementary forms the meaning is

Infinite rectangular
Simple exponential
Uniformly rising front
Sinusoidal
Damped sinusoidal
Difference of two exponential .

All these elementary forms can be reproduced by giving particular values to the paramenter a,b,A,B of the two exponentials

$$Ae^{-at} + Be^{-bt} \; . \qquad (3)\,\text{III 6}$$

Another family of simple wave forms can be contained in

$At^{n}e^{\alpha t}$ (4)III6

by giving suitable values to the parameters A, n, α. Figure 8IIIa illustrates some examples of the composition of elementary wave forms for producing more complicated forms. Immediately it can be visualized that an unlimited number of complicated wave forms can be synthesized. It is evident that complicated envelopes can be approximated and, in the case of graphs, simple analytic expressions for the curves which represent the envelope function can be produced. The expansion of the envelope function in terms of these elementary forms yield transforms which are the ratio of two polynomials in s.

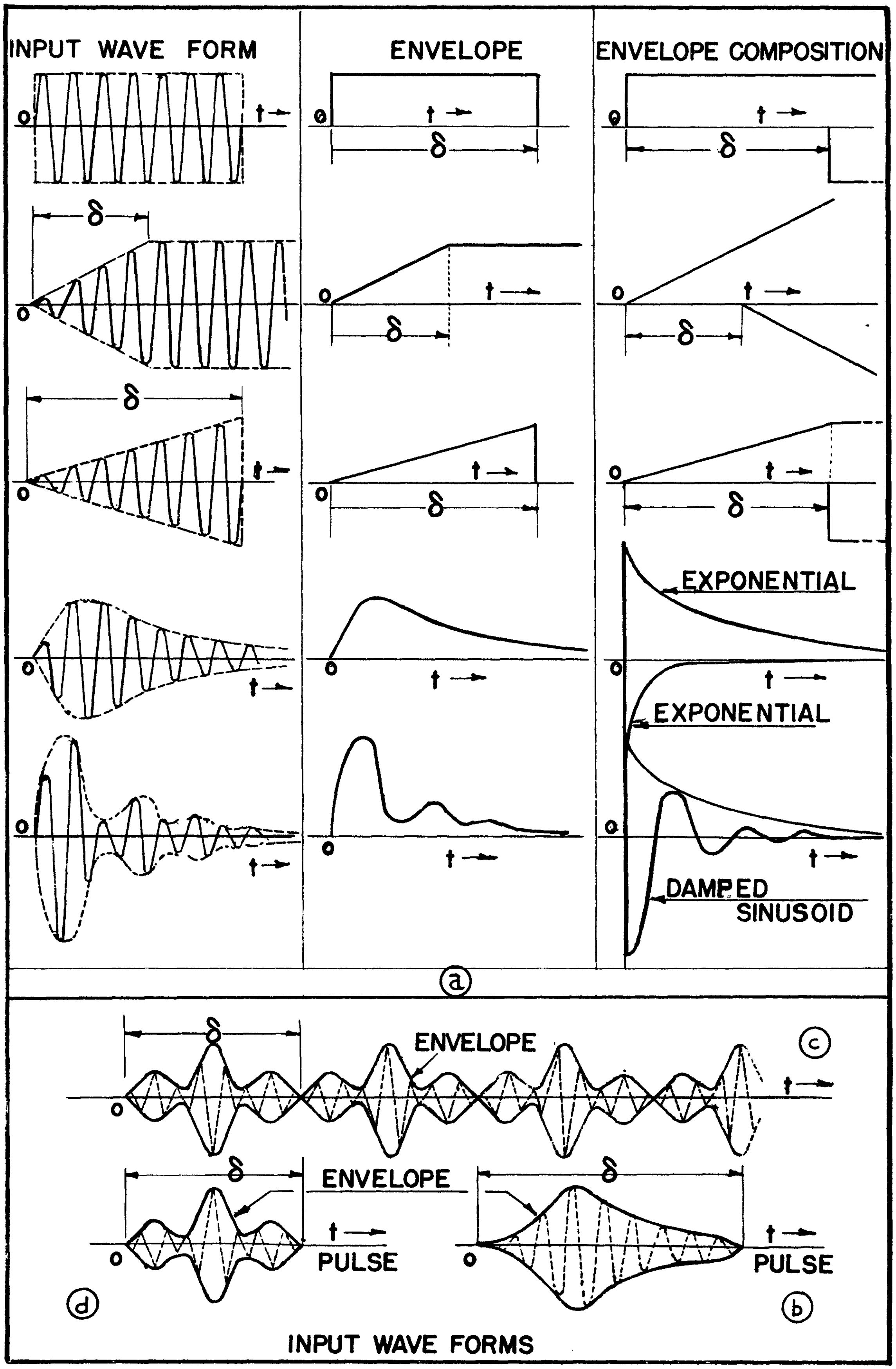
III-6.3 The application of the above method to the analysis of waves presupposes two things:

1st - That the type and position of the elementary wave components can be recognized by inspection.

2nd - That the proper values of parameters a,b,A,B in (3)III6 or A, α, n in (4)III6 can be computed, without much labor, in each elementary wave.

If this is not the case, then the above method is worthless and one has to introduce orthogonal polynomials. The object of this Subsection III-6.3 is to expand the envelope function in orthogonal polynomials which are suitable to the solution of the problem. The orthogonal

FIG. N° 8 III


INPUT WAVE FORM
ENVELOPE
ENVELOPE COMPOSITION
EXPONENTIAL
EXPONENTIAL
DAMPED SINUSOID
ⓐ
ENVELOPE
ⓒ
ENVELOPE
PULSE
PULSE
ⓓ
ⓑ
INPUT WAVE FORMS

polynomial must be selected by considering two things:

1st - That the given envelope with a few of the elements of the expansion can be approximated.

2nd - That the terms of the expansion lead to transforms which can be contained in the prototype $F(s, \sqrt{s^2+1})$.

In all practical cases, these two conditions can be satisfied with ease. When the envelope function repeats at equal intervals of time, then the Fourier series approximation is indicated. When the corresponding envelope function has an accentuated monotonic character as represents a pulse, then the Laguerre polynomials can be used. In the case of frequency modulation, it is convenient to use expansions of the Neumann type.

In Fig. 8IIIb, an AM pulse of duration δ is shown. The amplitude of the oscillation follows an envelope which has a pronounced monotonic character. The envelope function m(t) can be expanded in a series of Laguerre polynomials as follows

$$\left.\begin{aligned} m(t) &= e^{-\frac{\alpha t}{2}}\left\{a_0L_0(\alpha t) + a_1L_1(\alpha t) + \cdots + a_nL_n(\alpha t) + \cdots\right\} \\ &= e^{-\frac{\alpha t}{2}} \sum_{n=0}^{\infty} a_nL_n(\alpha t) \end{aligned}\right\} \quad (5)\text{III6}$$

in which the Laguerre polynomial of the order n is defined by,*

* See, for example, "Methoden Der Mathematischen Physik", Courrant and Hilbert, Volumen I, P. 79. A slightly modified definition was used which is more suitable for this problem.

$$L_n(\alpha t) = \frac{1}{\lfloor n} \frac{e^{\alpha t}}{\sqrt{\alpha}} \frac{d^n}{dt^n} (t^n e^{-\alpha t}), \tag{6)III6}$$

α = constant.

These polynomials form a complete system of orthogonal functions, and they have the property

$$\left.\begin{aligned} &\int_0^\infty e^{-\alpha t} L_n(\alpha t) L_m(\alpha t)\,dt = 0 \quad \text{for } m \neq n \\ &\int_0^\infty e^{-\alpha t} L_n^2(\alpha t)\,dt = 1 \qquad\quad \text{for } m = n \,. \end{aligned}\right\} \tag{7)III 6}$$

The coefficients in the expansion (5)III6 are determined by

$$a_n = \int_0^\infty e^{\frac{\alpha t}{2}} m(t) L_n(\alpha t)\,dt \,. \tag{8)III 6}$$

Since m(t) is the envelope of a pulse of duration δ, then

$$a_n = \int_0^\delta e^{-\frac{\alpha t}{2}} m(t) L_n(\alpha t)\,dt \,. \tag{9)III 6}$$

The explicit expressions of the first Laguerre polynomials as they are defined in (6)III6 are

$$\left.\begin{aligned} &L_0(\alpha t) = \frac{1}{\sqrt{\alpha}} \;; \quad L_3(\alpha t) = \frac{1}{\sqrt{\alpha}}\left(-\frac{\alpha^3 t^3}{6} + \frac{3}{2}\alpha^2 t^2 - 3\alpha t + 1\right) \\ &L_1(\alpha t) = -\frac{1}{\sqrt{\alpha}}(\alpha t - 1)\,; \quad L_4(\alpha t) = \frac{1}{\sqrt{\alpha}}\left(\frac{\alpha^4 t^4}{24} - \frac{2}{3}\alpha^3 t^3 + 3\alpha^2 t^2 - 4\alpha t + 1\right) \\ &L_2(\alpha t) = \frac{1}{\sqrt{\alpha}}\left(\frac{\alpha^2 t^2}{2} - 2\alpha t + 1\right) \end{aligned}\right\} \tag{10)III 6}$$

and they satisfy the recursion formulas

$$L_{n+1}(\alpha t) - (2n+1 - \alpha t) L_n(\alpha t) + n^2 L_{n-1}(\alpha t) = 0 \tag{11)III 6}$$

In the case m(t) given by a graph, the values of a_n can be computed by the approximate formula

$$a_n = \frac{\delta}{\nu} \sum_{k=1}^{k=\nu} e^{-\frac{\alpha k \Delta}{2}} m(k\Delta) L_n(\alpha k \Delta) \tag{12)III6}$$

in which $\Delta = \frac{\delta}{\nu}$ and ν = number of parts in which the interval δ is divided.

Once the coefficients $a_0, \cdots, a_n, \cdots$ are computed the corresponding Laplace transform of m(t) can be computed as follows:

$$\left.\begin{aligned} \mathcal{L}[m(t)] &= \sum_0^{\infty} a_n \mathcal{L} e^{-\frac{\alpha t}{2}} L_n(\alpha t) = \\ &= \sum_0^{\infty} a_n \frac{2}{\sqrt{\alpha}} \frac{(2s-\alpha)^n}{(2s+\alpha)^{n+1}} \end{aligned}\right\} \qquad (13)\,\text{III}\,6$$

since it can be proved without difficulty that

$$\mathcal{L} e^{-\frac{\alpha t}{2}} L_n(\alpha t) = \frac{2}{\sqrt{\alpha}} \frac{(2s-\alpha)^n}{(2s+\alpha)^{n+1}} \ . \qquad (14)\,\text{III}\,6$$

Once the Laplace transform of m(t) is computed, then the Laplace transform of the complete initial signal can be obtained by means of the theorems given in Section 8, Ch. I.

This method of approximation with Laguerre polynomials is recommended when the function m(t) has only one maximum, in which case one or two terms of the expansion give enough accuracy from the practical point of view.

Other types of orthogonal functions can be used in a similar manner.

III-6.4 If the input function has an envelope m(t) which repeats itself at equal intervals of t, then a Fourier expansion is convenient, with the further requirement $m(t) = 0$ for $t < 0$. Then

$$m(t) = \left\{\begin{matrix} 0 & \text{for } t < 0 \\ \sum_0^{\infty} a_n \cos n\vartheta t + \sum_1^{\infty} b_n \sin n\vartheta t & \vartheta = \frac{2\pi}{\delta} \end{matrix}\right\} \qquad (15)\,\text{III}\,6$$

In the case of input pulses of duration δ, which shows a few oscillations in the envelope m(t), then a Fourier expansion can also be used as

$$m(t)=\begin{cases} 0 & t<0 \\ \sum_{0}^{\infty} a_n \cos n\vartheta t + \sum_{1}^{\infty} b_n \sin n\vartheta t & 0<t<\delta \\ 0 & \delta = 0 \end{cases} \qquad (16)\text{III } 6$$

(See Fig. 8IIIc.)

The coefficients are determined by the well-known formulas when m(t) is given in an analytical or graphical form. The Laplace transform of m(t) is given by

$$\mathcal{L}m(t) = M(s) = \sum_{0}^{\infty} a_n \frac{s}{s^2+n^2\vartheta^2} + \sum_{1}^{\infty} b_n \frac{\vartheta n}{s^2+n^2\vartheta^2} \qquad (17)\text{III}6$$

for the semi-infinite envelope

$$\mathcal{L}m(t) = M(s) = \sum_{0}^{\infty} a_n \frac{s}{s^2+n^2\vartheta^2}(1-e^{-s\delta}) + \sum_{1}^{\infty} b_n \frac{n\vartheta}{s^2+n^2\vartheta^2}(1-e^{-s\delta}) \,. \qquad (18)\text{III } 6$$

In practical cases, only a few terms are required to obtain a good approximation.

Once M(s) is computed, the transform of the whole input signal can be obtained by the method given in Section 8 of Ch. I. The factor $e^{-s\delta}$ simply means a time shift.

III-6.5 In this subsection, one specific example of the instantaneous propagation of the electromagnetic field inside the wave guide will be worked out.

Suppose that, at $x_3 = 0$, the partial derivative of $\mathcal{E}_3$ with respect to x_3 is specified. (This assumption is equivalent to specifying a transverse component of the electric field. (See Eqs. (3)I5, Ch. I.)

$$\left.\frac{\partial \mathcal{E}_3}{\partial x_3}\right)_{x_3=0} = u_{-1}(t)\mathcal{V}_3 \sin \omega_0 t \,. \qquad (19)\text{III}6$$

ψ_3 is a function of the transverse coordinates of the wave guide. It must satisfy the differential equation

$$\nabla^2_{(x_1,x_2)}\psi_3 + p^2\psi_3 = 0 \qquad (20)\text{III}6$$

(See Eq. (8)I1, P. 20, Ch. I.) For the moment, the form of the cross section of the wave guide is not specified. It will simply be assumed that ψ_3 and p are so determined that the boundary conditions are satisfied.

The corresponding Laplace transform of the initial condition is given by

$$\mathcal{L}\left[\frac{\partial \mathcal{E}_3}{\partial x_3}\right]_{x_3=0} = \psi_3 \frac{\omega_0}{s^2+\omega_0^2} = \psi_3 \frac{\nu_0}{\sigma^2+\nu_0^2}\frac{1}{\omega_c} \qquad (21)\text{III}6$$

in which

$$\nu_0 = \frac{\omega_0}{\omega_c} \text{ and } \sigma = \frac{s}{\omega_c} .$$

The function $A_3'(0,s)$ is obtained from (21)III6. (See Table I, Ch. I and also Eqs. (1)I3, Section 3, Ch. I.)

$$A_3'(0,s) = \frac{\omega_0}{s^2+\omega_0^2} \qquad (22)\text{III}6$$

From Table I, Ch. I, the corresponding values of the electric and magnetic components of the field can be picked up.

In order to obtain the instantaneous field in the t domain, the following inverse transforms must be found.

$$\left.\begin{aligned}
&\text{For } E_1 \text{ and } E_2; \quad \frac{1}{\omega_0^2+s^2}e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}} \quad \text{or} \quad \frac{1}{\sigma^2+\nu_0^2}e^{-\kappa\sqrt{\sigma^2+1}}\\
&\text{For } E_3 \quad ; \quad \frac{e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{(s^2+\omega_0^2)\sqrt{s^2+\omega_c^2}} \quad \text{or} \quad \frac{e^{-\kappa\sqrt{\sigma^2+1}}}{(\sigma^2+\nu_0^2)\sqrt{\sigma^2+1}}\\
&\text{For } H_1 \text{ and } H_2; \quad \frac{s e^{-\frac{x_3}{c}\sqrt{s^2+\omega_c^2}}}{(s^2+\omega_0^2)\sqrt{s^2+\omega_c^2}} \quad \text{or} \quad \frac{\sigma e^{-\kappa\sqrt{\sigma^2+1}}}{(\sigma^2+\nu_0^2)\sqrt{\sigma^2+1}}
\end{aligned}\right\} \qquad (23)\text{III}6$$

except for constants of geometrical factors which are independent of s.

The inverse Laplace transforms of (23)III6 are given in (23)III5, (26)III5 and (27)III5 of Section 5. The field vectors have the expressions

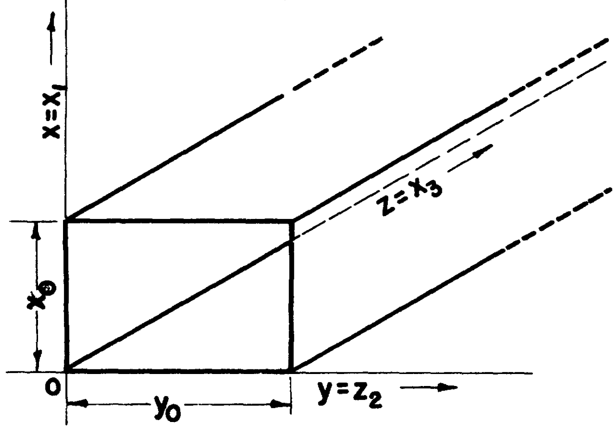
$$\left.\begin{aligned}
&\left.\begin{aligned}\mathcal{E}_1&=+\left(\frac{1}{h_1}\frac{\partial\psi_3}{\partial x_1}\right)\\ \mathcal{E}_2&=+\left(\frac{1}{h_2}\frac{\partial\psi_3}{\partial x_2}\right)\end{aligned}\right\} u_{-1}(\tau-\kappa)\frac{1}{p^2}\left\{\sin(\nu_0\tau-\kappa\sqrt{\nu_0^2-1})-V_1\left[\frac{\tau-\kappa}{\nu_0-\sqrt{\nu_0^2-1}},\sqrt{\tau^2-\kappa^2}\right]\right\}\\
&\mathcal{E}_3=-\frac{\psi_3}{p\sqrt{\nu_0^2-1}}u_{-1}(\tau-\kappa)\left\{-\cos(\nu_0\tau-\kappa\sqrt{\nu_0^2-1}+V_0\left[\frac{\tau-\kappa}{\nu_0-\sqrt{\nu_0^2-1}},\sqrt{\tau^2-\kappa^2}\right]\right\}\\
&\left.\begin{aligned}\mathcal{H}_1&=-\left(\frac{1}{h_2}\frac{\partial\psi_3}{\partial x_2}\right)\\ \mathcal{H}_2&=+\left(\frac{1}{h_1}\frac{\partial\psi_3}{\partial x_1}\right)\end{aligned}\right\} u_{-1}(\tau-\kappa)\frac{\varepsilon c\nu_0}{p^2\sqrt{\nu_0^2-1}}\left\{\sin(\nu_0\tau-\kappa\sqrt{\nu_0^2-1})+V_1\left[\frac{\tau-\kappa}{\nu_0-\sqrt{\nu_0^2-1}},\sqrt{\tau^2-\kappa^2}\right]\right\}\\
&\mathcal{H}_3=0
\end{aligned}\right\}(24)\text{III }6$$

in which: x_1, x_2, x_3 = cylindrical coordinates; $h_1, h_2, h_3 = 1$, metric coefficients; $\nu_0 = \frac{\omega_0}{\omega_c}$; $\tau = 2\pi\frac{t}{Tc}$; $\kappa = 2\pi\frac{x_3}{\Lambda_c}$; p = separation constant (see p. 19).

By simple inspection of (24)III6 it can be observed that: "At equal cut-off frequency values and at equal applied frequency, the time propagation of the electromagnetic field inside wave guides is independent of the geometry of the cross section. The form of the cross section produces only geometrical factors, independent of the time, which only changes the value of the final vectors." A generalization of this principle can be made by inspection of Table I, Ch. I.

In the solutions (24)III6 a definite cross section of the guide has not been specified. Let us specify the cross section. Take for example, a rectangular wave guide whose dimensions are indicated in Fig. 9III.

FIG. N° 9 III



The value of ψ_3 is given by

$$\left.\begin{aligned} &\psi_3 = \mathcal{E}'_0 \sin \frac{m\pi}{x_0} x \sin \frac{n\pi}{y_0} y \\ &\mathcal{E}'_3 = \text{constant} \\ \text{and} \quad &p^2 = \left(\frac{m\pi}{x_0}\right)^2 + \left(\frac{n\pi}{y_0}\right)^2 \end{aligned}\right\} \qquad (25)\text{III } 6$$

in which m and n are integers.

In order to introduce the standard notation employed in some texts on wave guides, set

$$\left.\begin{aligned} &\mathcal{E}_0 = \frac{\mathcal{E}'_0 c}{\beta_{m,n}} = \text{constant} \\ &\beta_{m,n} = \sqrt{\omega_0^2 - \omega_c^2} = \sqrt{\omega_0^2 - c^2\left[\left(\frac{\pi m}{x_0}\right)^2 + \left(\frac{\pi n}{y_0}\right)^2\right]} \end{aligned}\right\} \qquad (26)\text{III } 6$$

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\mathcal{E}_1 = +\frac{m\pi}{x_o}\cos\left(\frac{m\pi}{x_o}x\right)\sin\left(\frac{n\pi}{y_o}y\right) \\
\mathcal{E}_2 = +\frac{n\pi}{y_o}\sin\left(\frac{m\pi}{x_o}x\right)\cos\left(\frac{n\pi}{y_o}y\right)
\end{array}\right\} \mathcal{E}_o \frac{\beta_{m,n}}{p^2}\left[\sin\left(\nu_o\tau - \kappa\sqrt{\nu_o^2-1}\right) - V_1(\Omega_k, T)\right] u_{-1}(\tau-\kappa) \\
\mathcal{E}_3 = \mathcal{E}_o \sin\left(\frac{m\pi}{x_o}x\right)\sin\left(\frac{n\pi}{y_o}y\right)\left[\cos\left(\nu_o\tau - \kappa\sqrt{\nu_o^2-1}\right) - V_o(\Omega_k, T)\right] u_{-1}(\tau-\kappa) \\
\left.\begin{array}{l}
\mathcal{H}_1 = -\frac{n\pi}{y_o}\sin\left(\frac{m\pi}{x_o}x\right)\cos\left(\frac{n\pi}{y_o}y\right) \\
\mathcal{H}_2 = +\frac{m\pi}{x_o}\cos\left(\frac{m\pi}{x_o}x\right)\sin\left(\frac{n\pi}{y_o}y\right)
\end{array}\right\} \mathcal{E}_o \frac{\varepsilon\omega_o}{p^2}\left[\sin\left(\nu_o\tau - \kappa\sqrt{\nu_o^2-1}\right) - V_1(\Omega_k, T)\right] u_{-1}(\tau-\kappa) \\
\mathcal{H}_3 = 0
\end{array}\right\} \quad (27)\ \text{III 6}$$

The transient for other initial conditions and different cross sections can be computed in a similar manner.

CHAPTER IV

The asymptotic solutions. Envelopes and phase-generating functions. Master curves and graphical methods of solution. Group and signal velocities. Time, distance, and slope of signal formation.

Section 0 - Object and contents of this chapter.(*)

IV-0.0 The formal and compact solutions obtained in the last chapter are not suitable for numerical computation for the following reasons.

1st - Few Lommel functions are tabulated and they do not cover these cases.

2nd - The series expansions which serve as definitions of the Lommel functions, although absolute and uniformly convergent, are not suitable for numerical work because they converge very slowly.

3rd - It is rather difficult to visualize through them the wave forms they represent, except for particular values of the corresponding arguments Ω and T.

4th - The Lommel functions represent highly oscillatory functions. In practice it is much better to deal with the corresponding envelope and phase functions of these oscillations.

5th - Although it is rather simple to define the group and phase velocity of these generating functions, it is not easy to find the corresponding expression of the signal velocity and the time of signal formation.

(*) A complete and detailed discussion of the method and procedure of integration used in this chapter will be found in a future report (No. 55) of RLE, where the general theory of

The object of this chapter is to find a type of solution which is very well adapted for numerical work and is such that the above handicaps are avoided.

The solutions given in this chapter not only are useful for numerical computation but provide simple methods for the discussion of the signal, phase and group velocities and are also appropriate for developing graphical methods of solution.

It is of primary importance to obtain asymptotic solutions which hold good when the applied frequency is close to the cut-off frequency of the excited mode. All the asymptotic solutions given here are obtained by the saddle-point method of integration.

IV-0.1 In Section 1 a new complex transformation is introduced from which asymptotic solutions can be easily derived. This transformation removes the possibility of having a pole at the branch points, so that the solutions hold good when the applied frequencies lie in this branch point .

IV-0.2 In Section 2, the appropriate intervals of the variables for the different type of solutions are presented. They are: Silenced Region, Precursory Region, Main Formation Region, and the Coda Region.

IV-0.3 In Section 3, the appropriate solutions for the Precursory and Coda Regions are given.

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asymptotic solutions of integrals of the type $\int_{\gamma_s} F(s)e^{w(s)}ds$ is given. The reader is also referred to the paper "Uber die Fortpflanzung von Signalen in Dispergierenden Systemen", Hans Georg Bearwald, Annalen der Physik, 5 folge Band 6, p.295.

IV-0.4 In Section 4, the appropriate solutions for the moving signal formation region, are given. Two cases are considered.

<u>1st</u> - The corresponding formation per pole when the pole is on the imaginary axis of the s plane.

<u>2nd</u> - The corresponding formation per pole when the pole lies outside the imaginary axis of the plane.

IV-0.5 In Section 5, the expressions for the envelope and phase-generating functions are obtained. By means of a functional transformation all transients can be expressed in terms of a master. The definitions of the signal and group velocity and the time or distance of formation are given in terms of a new variable.

IV-0.6 In Section 6, the graphical methods of transient computation in the main signal formation region are given.

IV-0.7 In Section 7, the way to connect the solutions in these four regions is shown.

<u>Section 1</u> - <u>The complex transformation $s = \sinh \xi$. Mapping properties. Contour transformation lines of steepest descent</u>.

IV-1.0 The transformation

$$s = \sinh \xi$$

in which (1)IV 1

$$\xi = \eta + i\zeta$$

leads to a simple procedure for obtaining approximate solutions of the inverse transform of the prototype Laplace transform

$$F(s,\sqrt{s^2+1})e^{-\mathcal{X}\sqrt{s^2+1}} \quad . \qquad (2)\text{IV }1$$

The solutions obtained converge very rapidly or they will have a compact form. The solutionsobtained by this transformation are so close to the true solutions given in Ch. III, that there is no practical difference between them. The expressions for the envelope and phase functions of the corresponding wave forms will be given by simple and illuminating mathematical expressions.

IV-1.1 The transformation (1) has a multivalued character. The s plane must be composed of an infinite number of leaves which map in horizontal strips of the ξ plane. These strips of the ξ plane repeat periodically along the direction of the imaginary ζ axis. A new cut must be introduced in the s plane to connect these leaves. This new cut shall be so placed that it does not violate the conditions for the Laplace inversion. Besides, this new cut must not interfere with the contours of integration already studied.

At this point it is necessary to recall that the plane was already composed of two Riemann surfaces. The introduction of the new transformation requires an infinite number of sheets, which means that the primitive Riemann surfaces $\mathcal{S}_I$ and $\mathcal{S}_{II}$ must each break up into a -

manifold of an infinite number of leaves. These manifolds will be denoted respectively by $\mathcal{J}_{I}M$ and $\mathcal{J}_{II}M$.

The new cut is as indicated, for example, in Fig. 1IVa. The leaves of each manifold must be connected by this new cut. The manifolds themselves must be connected by the old cut, which joins the points +i and -i.

The following is very important. The integration along the bank of the new cut does not cancel out as before. That is: <u>The true inverse transform must not contain this new contribution</u>.

IV-1.2 The new Riemann surfaces as well as the manifolds must be defined, in such a way, that the definitions are consistent with those already given in Ch. II.

a. - The leaves of each infinite manifold must be defined with respect to the branch point at $\xi=0$. The leaf of index zero, which will be used most, will be defined by

$$-\pi \leq \varphi < \pi. \tag{3)IV 1}$$

The general sheet is given by

$$(M-1)\pi \leq \varphi < (M+1)\pi \tag{4)IV 1}$$

where M is an even integer.

b. - To define each manifold one uses the expression

$$\pm\sqrt{s^2+1} = \cosh \xi \tag{5)IV 1}$$

In Ch. II the sign distribution was studied of the function

$$w = u + iv = -\sqrt{s^2+1} \tag{6)IV 1}$$

and therefore,

$$u = -\cosh \eta \cos \zeta$$
$$v = -\sinh \eta \sin \zeta . \tag{7)IV 1}$$

Now, in order to have a definition of the manifolds which is consistent with the definitions adopted in (1)II2 of Ch. I, η and ζ must be chosen in such a way that (7)IV1 has the sign distribution of Fig. 3IIa.

IV-1.3 Once the manifolds $\mathcal{S}_I M$ and $\mathcal{S}_{II} M$ are properly defined then it is easy to show that:

- 1st - The manifold $\mathcal{S}_I M$ maps into the right half of the ξ plane.
- 2nd - The manifold $\mathcal{S}_{II} M$ maps into the left half of the ξ plane.
- 3rd - Each sheet of the $\mathcal{S}_I M$ manifold will map into a horizontal strip of the right half ξ plane.
- 4th - Each sheet of the $\mathcal{S}_{II} M$ manifold will map into a horizontal strip of the left half ξ plane.

This situation is indicated in Figs. 1IVb,c,d.

Since the transformation (1)IV1 is very well known, further details of its mapping properties will not be given.

IV-1.4 In this subsection, the transformation of the integral which furnishes the inverse Laplace transform of the prototype will be studied. After elementary algebraic manipulations, one obtains

$$\frac{1}{2\pi i}\int_{\gamma_s} F(s,\sqrt{s^2+1})\, e^{s\tau - \varkappa\sqrt{s^2+1}}\, ds = \frac{1}{2\pi i}\int_{\gamma_\xi} H(\xi)\, e^{iT\cosh(\xi-\xi_s)}\, d\xi \qquad (8)\mathrm{IV}1$$

in which

$$H(\xi) = F(\xi)\ \cosh\xi \qquad (9)\mathrm{IV}1$$

γ_ξ indicates the appropriate contour of integration in the ξ plane

$T = \sqrt{\tau^2 - \varkappa^2}$; as before

$\frac{\varkappa}{\tau} = \text{ctgh}\ \xi_s$; $\sinh \xi_s = \frac{i\tau}{T}$. (10)IV 1

IV-1.41 It is at once clear that ξ_s is a saddle point of the exponent of e in the second integral (8)IV1. Due to the multivalued character of the transformation, an infinite number of saddle points are distributed in the ξ plane.

The saddle points corresponding to the manifolds $\mathcal{J}_I M$ and $\mathcal{J}_{II} M$ are respectively given by

$$\xi_s = L_n\left(\frac{\tau+\varkappa}{\tau-\varkappa}\right)^{\frac{1}{2}} + i\left(\frac{\pi}{2} + M\pi\right); \text{ for } \mathcal{J}_I M$$
$$\xi_s = L_n\left(\frac{\tau-\varkappa}{\tau+\varkappa}\right)^{\frac{1}{2}} + i\left(\frac{\pi}{2} + M\pi\right); \text{ for } \mathcal{J}_{II} M \ . \qquad (11)\text{IV 1}$$

The corresponding position of these saddle points is given in Fig. 1IV.

IV-1.42 In this subsection the exponent

$$W = iT \cosh(\xi - \xi_s) \qquad (12)\text{IV 1}$$

will be considered. The corresponding sign distribution regions for the real and imaginary part of this exponent, are given in Figs. 1IVf and 1IVg.

IV-1.43 The corresponding lines of steepest descent through the saddle points are indicated in Fig. 2IVa. These lines repeat periodically for the different sheets of the manifold $\mathcal{J}_I M$. In the above-mentioned figure the strip of $\mathcal{J}_I M$ was chosen for the particular value $M = 0$.

It is interesting to notice the form of this line in the s plane. See Fig. 2IVb.

IV-1.5 Without loss of generality, the strip $M=0$ can be selected in the right half ξ plane to obtain the inverse transformation. This is due to the fact that the transformation has a periodic character in the direction of the ζ axis. In the rest of this work, the strip $M=0$ of $\mathcal{S}_I M$ will be consistently used.

Suppose now that s_k is a pole of $F(s,\sqrt{s^2+1})$. The point transforms into the ξ plane as

$$\sinh\xi_k = s_k \tag{13)IV 1}$$

or, by using the notation of the last chapter,

$$\xi_k = L_n \frac{1}{|Z_k|} + i(\varphi + M\pi) \tag{14)IV 1}$$

if s_k is in the M sheet of the $\mathcal{S}_I M$ manifold

and

$$\xi_k = L_n |Z_k| + i(\varphi + M\pi) \tag{15)IV 1}$$

if s_k is in the M sheet of the manifold $\mathcal{S}_{II} M$.

For $M=0$ and on $\mathcal{S}_I M$ one has

$$\xi_k = L_n \frac{1}{|Z_k|} + i\varphi \tag{16)IV 1}$$

so that

$$\eta_k = L_n \frac{1}{|Z_k|};\ \zeta_k = \varphi\ . \tag{17)IV 1}$$

In the above equation

$$Z_k = s_k - \sqrt{s_k^2+1} = |Z_k|\, e^{j\varphi}\ . \tag{18)IV 1}$$

The Fig. 2IVc illustrates the corresponding position of the poles of $H(\xi)$ for different positions of the s_k pole in $\mathcal{S}_I M$; $M=0$.

Section 2 - Principal subintervals or regions in the transient solutions.

IV-2.0 The saddle-point method of integration will be used to obtain the asymptotic solution corresponding to the inversion of our prototype transform. It is not possible to obtain one single asymptotic solution which holds good in the complete interval of variation of τ and κ, that is

$$\left.\begin{matrix} 0 < \tau < \infty \\ 0 < \kappa < \infty \end{matrix}\right\} \tau > \kappa$$

In order to obtain appropriate asymptotic solutions four regions will be defined.

1st - Silence region: This region is given by the interval $0 < \tau < \kappa$. As was shown in Ch. II, the inverse transform is equal to zero. This property justifies the name given to this first region.

2nd - Precursory region: This region is given by the interval

$$+\kappa \leq \tau < \tau_{sig}$$

in which τ_{sig} means the corresponding normalized time for the arrival with signal velocity.

3rd - Main signal formation region: This region can be defined as follows: Suppose that among the m poles ξ_k, there is one, say ξ_f, which has the property

$$|Z_f| < |Z_1|, \cdots, |Z_{f-1}|, |Z_{f+1}|, \cdots, |Z_m|.$$

Then, the main signal formation region will be defined by the interval

$$\tau_{sig} < \tau < \kappa \frac{1+|Z_f|^2}{1-|Z_f|^2} \quad .$$

4th - The Coda region will be defined by

$$\kappa \frac{1+|Z_f|^2}{1-|Z_f|^2} < \tau < \infty .$$

The justification of selection of these intervals will be found in the discussion of each case. In this region the transient terms vanish rapidly.

IV-2.1 Inside the main signal formation region there are the times of arrival with group velocity of each pole. This statement is justified if the definition is recalled of the group velocity of a particular pole which was given in Ch. III. That is

$$v_{g_k} = \frac{\kappa}{\tau}\Big)_k = \frac{1+|Z_k|^2}{1-|Z_k|^2} \quad .$$

At the instant when the saddle point coincides with a given pole, then the arrival with group velocity is obtained corresponding to this particular pole. If the given pole lies on the horizontal lines $i\pi/2$ or $-i\pi/2$ in the ξ plane, then the saddle point touches the pole for real values of the time. If the given pole lies on the outside of the above lines, then the instant of group velocity corresponding to this pole is obtained approximately when the line of steepest descent touches the pole. This is illustrated in Figs. 2IVe,g.

IV-2.2 As is shown in Fig. 1IVe, there are two saddle points in the strip $M = 0$. These saddle points are conjugate ones. They are given by

$$\xi_s = L_n\left(\frac{\tau+\kappa}{\tau-\kappa}\right)^{\frac{1}{2}} + i\frac{\pi}{2}$$
$$\bar{\xi}_s = L_n\left(\frac{\tau+\kappa}{\tau-\kappa}\right)^{\frac{1}{2}} - i\frac{\pi}{2} \qquad (1)\text{IV2}$$

When the time changes, the saddle points move along the horizontal line as $i\pi/2$ and $-i\pi/2$. At the beginning of the transient, the saddle points are at $+\infty$. When the permanent state is reached, then the saddle points coincide with $i\pi/2$ and $-i\pi/2$. It can, therefore, be said that in the saddle-point method of integration, the contour γ_ξ changes its position with the time.

Section 3 - The asymptotic solutions for the precursory and coda regions. The corresponding envelope and phase functions.

IV-3.0 The contour of integration which must be used for the precursory region is indicated in Fig. 2IVd. Here all the poles lie to the left of the saddle points. Since most of the value of the integral is given by the integration in the vicinity of these saddle points, the contribution of the rest of the contour can be neglected. In what follows all the intermediate steps will be omitted in the process of integration and only the final result will be given. The computations are rather long but not hard to obtain. The method is only valid for $\gamma \geq 2$.

IV-3.1 The asymptotic solution of the inverse Laplace transform in the precursory region is given by

$$\mathcal{L}^{-1}F(s,\sqrt{s^2+1})\,e^{-\kappa\sqrt{s^2+1}} = \varphi(\tau,\kappa) \approx \frac{1}{\sqrt{\frac{\pi T}{2}}}\sqrt{M^2+N^2}\,\sin(T+\frac{\pi}{4}+\theta_s) \qquad (1)\text{IV3}$$

in which

$$\left.\begin{array}{l} M=\text{Real} \\ N=\text{Im} \end{array}\right\}\left\{H(\xi_s)-\frac{H^{IV}(\xi_s)}{2.4T^2}+\frac{H^{VIII}(\xi_s)}{2\cdot4\cdot6\cdot8T^4}+\cdots+i\left[\frac{H^{II}(\xi_s)}{2T}-\frac{H^{VI}(\xi_s)}{2\cdot4\cdot6\cdot T^3}+\cdots\right]\right\} \qquad (2)\text{IV3}$$

and

$$\xi_s = L_n\left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{\frac{1}{2}} + i\frac{\pi}{2}$$

$$.\,tg\,\theta_s = \frac{N}{M} \qquad (3)\text{IV3}$$

and

$$H^N(\xi_s) = \left(\frac{d^N H}{d\xi^N}\right)_{\xi=\xi_s}. \qquad (4)\text{IV3}$$

The asymptotic series given in (2)IV3 converges very rapidly for large values of τ. In almost all cases of practical application the first term $H(\xi_s)$ is enough to obtain a high degree of approximation.

From (1)IV3 it is clear that the corresponding envelope function is given by

$$\Pi = \frac{\sqrt{2}}{\sqrt{\pi T}}\sqrt{M^2+N^2} \qquad (5)\text{IV3}$$

and the phase function is given by

$$\theta_s = tg^{-1}\,\frac{N}{N}. \qquad (6)\text{IV3}$$

IV-3.2 The contour of integration which must be used to obtain the asymptotic solution for the coda region is given in the Fig. 2IVe. The contribution around each pole must be added here.

Similarly, for the coda region, one obtains

$$\mathcal{L}^{-1}F(s,\sqrt{s^2+1})\,e^{-\varkappa\sqrt{s^2+1}}$$
$$=\varphi(\tau,\varkappa)\approx\sum_{k=1}^{m}R_k e^{iT\cdot\cosh(\xi_k-\xi_s)}+\frac{\sqrt{2}}{\sqrt{\pi T}}\sqrt{M^2+N^2}\;\sin\left(T+\Theta_s+\frac{\pi}{4}\right) \qquad (7)\text{IV3}$$

in which

m = total number of poles of $H(\xi_k)$

R_k = residue of H at each pole k

ξ_k is given by (16)IV1.

The other letters have the same meaning as in IV-3.1.

Section 4 - The asymptotic solutions valid in the main signal formation per pole. Envelope and phase functions.

IV-4.0 When the saddle point enters in the interval corresponding to the main signal formation region, the amplitude of the oscillations increases suddenly and the signal represented by the transform begins to form. In this region the oscillations acquire almost the final values.

The classical method of the saddle-point integration fails to render adequate values for the inversion integral. It was necessary in this investigation to develop a method of integration suitable for this region. This can be done by making a further elementary transformation such that the exponent of the second integral in (8)IV1 can be conveniently split. The saddle-point method of integration is then applied to the new exponent.

To be precise, this method is rather connected with the poles of $H(\xi)$ and not the whole wave formation. In what follows the formation of the signal in the vicinity of a pole will be discussed. A pair of conjugate poles can be associated in such a way that they render a simple expression for the wave formation in its vicinity.

This method of integration is delicate to handle and requires rather involved algebraic developments. For this reason, many intermediate steps of computation will be omitted and attention will be concentrated on the main idea of this method.

IV-4.1 Let ξ_k be a pole of $H(\xi)$ and R_k its residue. $H(\xi)$ can be expanded in a Laurent series. In the neighborhood of ξ_k, the function $H(\xi)$ behaves as

$$H(\xi) \approx \frac{R_k}{\xi - \xi_k} \quad . \tag{1)IV4}$$

Also it can be proved that in the vicinity of the saddle point

$$e^{iT \cosh (\xi - \xi_s)} \approx e^{iT \cosh u_k} \times e^{iT\left[\frac{(\xi-\xi_k)^2}{2} + u_k(\xi - \xi_k)\right]} \tag{2)IV4}$$

in which

$$u_k = \xi_k - \xi_s \quad . \tag{3)IV4}$$

In the vicinity of the pole ξ_k the integral has the form

$$\frac{1}{2\pi i}\int_{\gamma_\xi} H(\xi)\, e^{iT \cosh(\xi - \xi_s)}\, d\xi = \frac{R_k e^{iT \cosh u_k}}{2\pi i}\int_{\gamma_{\xi_{prox\ of\ \xi_k}}} \frac{e^{i\frac{Tr^2}{2}e^{2i\theta} + iTu_k r e^{i\theta}}}{re^{i\theta}}\, d(re^{i\theta}) \tag{4)IV4}$$

in which

$$\xi - \xi_k = re^{i\theta} \quad . \tag{5)IV4}$$

It is not hard to show that the sign distribution of the new exponent under the integral sign is given by the Fig. 3IVa. If $\bar{\xi}_k$ is also a pole of $H(\xi)$, then the corresponding sign distribution diagram of the new exponent is given by the Fig. 3IVb. The modified new contour of integration is indicated also in the same figures.

After a painfully long process of integration the results can be obtained.

In the main signal formation region, the required inverse transform is given by

$$\mathcal{L}^{-1}F(s,\sqrt{s^2+1})\,e^{-\kappa\sqrt{s^2+1}}$$
$$\approx \sum_{k=1}^{m} R_k e^{iT\cosh u_k}\frac{1}{2}\left\{\left[1+C(v_k)+S(v_k)\right]+i\left[C(v_k)-S(v_k)\right]\right\} \qquad (6)\text{IV4}$$

in which

$$\left.\begin{aligned}
&T=\sqrt{\tau^2-\kappa^2}\\
&R_k = \text{residue of } H(\xi) \text{ at pole } \xi_k\\
&u_k=\xi_k-\xi_s=L_n\left\{\frac{1}{|Z_k|}\left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{\frac{1}{2}}\right\}-i\left(\varphi_k-\frac{\pi}{2}\right)\ \text{complex}\\
&v_k=\sqrt{\frac{T}{\pi}}\left\{L_n\left[\frac{1}{|Z_k|}\left(\frac{\tau-\kappa}{\tau+\kappa}\right)^{\frac{1}{2}}\right]-i\left(\varphi_k-\frac{\pi}{2}\right)\right\}\ \text{complex}\\
&C(v_k)=\text{Fresnel C function}\\
&S(v_k)=\text{Fresnel S function}\\
&F(v_k)=C(v_k)+iS(v_k)=\int_0^{v_k} e^{i\frac{\pi}{2}v^2}dv\\
&Z_k=s_k-\sqrt{s_k^2+1}=|Z_k|e^{i\varphi_k};\ m=\text{number of poles.}
\end{aligned}\right\} \qquad (7)\text{IV4}$$

The envelope and phase functions will be introduced as follows:

$$\Pi_k=\frac{1}{2}\left|\left[1+C(v_k)+S(v_k)\right]+i\left[C(v_k)-S(v_k)\right]\right|e^{i\Phi(v_k)}=\Psi(v_k)\,e^{i\Phi(v_k)} \qquad (8)\text{IV4}$$

in which:

$$\left.\begin{aligned}
&\Psi(v_k)=|\Pi_k|\\
&\operatorname{tg}\Phi(v_k)=\frac{\text{Im}\ \Pi_k}{\text{Real}\ \Pi_k}\ .
\end{aligned}\right\} \qquad (9)\text{IV4}$$

So that finally, for the main signal formation region the asymptotic solution is ($\gamma \geqslant 2$).

$$\mathcal{L}^{-1}F(s,\sqrt{s^2+1})\,e^{-\kappa\sqrt{s^2+1}} \approx \sum_{k=1}^{m} R_k\Psi(v_k)\,e^{i\left[T\cosh u_k+\Phi(v_k)\right]} \qquad (10)\text{IV4}$$

Note that if s_k is pure imaginary and $|s_k|>1$ then u_k and v_k are real quantities (see (7)IV4.

Section 5 - The envelope and phase generating function. Group and signal velocities. Time and space of signal formation.

IV-5.0 The solution given in (10)IV4 is the sumation of waves corresponding to each pole of $H(\xi)$. The function Π_k given in (8)IV2.produces the transient envelope and the transient phase function per pole, during the main signal formation region. It can be proved that

$$e^{iT \cosh u_k} = e^{A(Z_k)} \qquad (1)IV5$$

and, therefore, $\Phi(v_k)$ represents the phase deviation from the permanent one, during the transient state of the waves inside the guide. If one lets $\tau \to \infty$, then $v_k \to \infty$ and it can be shown that

$$\left.\begin{array}{l} C(v_k) \to 1 \\ S(v_k) \to 1 \end{array}\right\} \text{ when } v_k \to \infty$$

so that

$$\phi(v_k) \xrightarrow[v_k \to \infty]{} 0$$

$$II(v_k) \xrightarrow[v_k \to \infty]{} 1 \; .$$

This means that the solution (10)IV4 goes into the permanent state.

Since the function $\Pi(v_k)$ produces the envelope and the phase functions per pole, it will be called the "Generating function". The object of this section is to study this function with some detail.

IV-5.1 Consider first the case when the corresponding pole in the s plane is pure imaginary and $|s_k| > 1$. In this case $\varphi_k = \frac{\pi}{2}$ and

$$v_k = \sqrt{\frac{T}{\pi}}\left\{L_n \frac{1}{(v_k - \sqrt{\nu_k^2 - 1})} \left(\frac{1-\theta}{1+\theta}\right)^{\frac{1}{2}}\right\} \qquad (2)\text{IV5}$$

is real. In this case $C(v_k)$ and $S(v_k)$ are also real and one can write:

$$\Psi(v_k) = \frac{1}{\sqrt{2}}\sqrt{\left[0.5 + C(v_k)\right]^2 + \left[0.5 + S(v_k)\right]^2}$$

$$\tan \Phi(v_k) = \frac{\left[0.5 + C(v_k)\right] - \left[0.5 + S(v_k)\right]}{\left[0.5 + C(v_k)\right] + \left[0.5 + S(v_k)\right]} \qquad (3)\text{IV5}$$

Both functions, $\Psi(v_k)$ and $\Phi(v_k)$ can be computed graphically by means of Cornu's spiral as is indicated in Fig. 3IVc.

IV-5.2 A similar graphical procedure of computation can be followed when v_k is a complex quantity. In this case, a new spiral can be formed as indicated in Fig. 3IVd. The form of this spiral changes with the value of the difference $\varphi_k - \frac{\pi}{2}$. This spiral must be applied when the poles s_k lie outside the imaginary axis of the s plane.

IV-5.3 Here the concept of group velocity of a given pole will be introduced. It was proved in Ch. III that the group velocity of a pole can be defined by the condition

$$\Omega_k = T \qquad (4)\text{IV5}$$

from which

$$Z_k = i\left(\frac{\tau+\kappa}{\tau-\kappa}\right)^{\frac{1}{2}} \qquad (5)\text{IV5}$$

so that

$$\frac{\left(\frac{\tau+\kappa}{\tau-\kappa}\right)^{\frac{1}{2}}}{|Z_k|} = 1 \qquad (6)\text{IV5}$$

and consequently, at group velocity

$$v_k = \sqrt{\frac{T}{\pi}}\left\{L_n 1 - i\left(\varphi_k - \frac{\pi}{2}\right)\right\} = -i\left(\varphi_k - \frac{\pi}{2}\right)\sqrt{\frac{T}{\pi}} \qquad (7)\text{IV5}$$

and one obtains the important theorem:

"Let s_k be a pole in the imaginary axis of the s plane and such that $|s_k| > 1$. Then:

a. - The group velocity is characterized by $v_k = 0$;

b. - The generating envelope function has the value $\frac{1}{2}$;

c. - The generating phase function has the value 0."

The theorem follows from the fact that $C(0) = 0$ and $S(0) = 0$.

In the case of a pole s_k which lies outside the imaginary axis, it can be said:

"Let s_k be a pole outside the imaginary axis of the plane. Then: The group velocity is characterized by the instant at which the saddle points occur at the minimum distance from the pole ξ_k, in the ξ plane."

IV-5.4 In this section the slope of formation of the envelope of a given pole will be considered at the instant of group velocity.

Consider first the case $s_k = i\nu_k$ and $|\nu_k| > 1$. By a simple process of differentiation it can be found that

$$\left.\frac{\partial \Psi(v_k)}{\partial v_k}\right)_{v_k = 0} = \frac{1}{2} \ .$$

This means "that the tangent to the envelope function at $v_k = 0$ touches the axis v_k at a point $v_k = -1$, when s_k is purely imaginary."

The following important theorem is also true.

"The tangent to the envelope function at $v_k = 0$ touches the line $\psi(v_k) = 1$ at $v_k = +1$, when s_k is purely imaginary."

Now the definition of signal velocity, valid when s_k is purely imaginary, will be introduced.

"The signal velocity of a pole s_k, when s_k is purely imaginary, is characterized by $v_k = -1$."

The interval $-1 \leq v_k \leq 1$ will be defined as "interval of formation of the signal", corresponding to a pure imaginary pole.

IV-5.5 The definition of signal velocity and interval of formation corresponding to a complex pole s_k can be obtained, in a similar way, by computing the derivative

$$\left.\frac{d\Psi(v_k)}{dv_k}\right)_{v_k = v_{kg}}$$

IV-5.6 It is now possible to give a definition of the signal velocity and interval of formation of a multipole wave.

"The signal velocity of the complete signal is equal to the largest signal velocity of its poles."

"The interval of formation of the complete signal may be defined as the interval between signal velocity of the wave and the velocity at which the last pole component is formed."

Section 6 - Graphical method for the construction of $\Psi(v_k)$ and $\Phi(v_k)$ for pure imaginary poles. Main signal formation region.

IV-6.0 It is rather simple to construct graphically the envelope $\Psi(v_k)$ and phase $\Phi(v_k)$ corresponding to a given pure imaginary pole, for the region in which the signal is formed. The object of Section 6 is to describe this method of construction.

IV-6.1 The spiral which generates the envelope and phase functions for a pole of the type $s_k = i\nu_k$; $|\nu_k| > 1$ is given in the Fig. 4IV.

The envelope function $\Psi(v_k)$ is given as a function of v_k in Fig. 5IV. The points corresponding to the group and signal velocity as well as signal formation interval are indicated in this figure.

Figure 6IV shows the variation of the phase function $\phi(v_k)$ as a function of v_k.

IV-6.2 In the practical application of this theory it is necessary to deal with two principal problems:

Problem A: At a fixed cross section x_3 (or $\kappa = 2\pi \frac{x_3}{\lambda_c}$) the problem is to know how the signal if formed as a function of t (or $\tau = 2\pi \frac{t}{T_c}$).

Problem B: At a fixed instant t (or $\tau = 2\pi \frac{t}{T_c}$) the problem is to know the distribution of the signal inside the wave guide, when x_3 (or $\kappa = 2\pi \frac{t}{T_c}$) changes.

The separation of these problems can be accomplished by considering κ = constant and τ variable for Problem A or τ = constant and κ variable in Problem B.

Figure 7IV furnishes a family of curves

$$v_k = v\left(\frac{t}{T_c}\right); \quad \frac{x_3}{\lambda_c} = \text{constant}$$

for different values of the parameter ν_k. To apply these curves in a concrete case the procedure is as follows: 1. Take the curve which corresponds to the given value of ν_k. 2. Multiply the abscissa by the constant $\frac{x_3}{\lambda_c}$.

3. Multiply the ordinates by the factor $\sqrt{\frac{x_3}{\lambda_c}}$. The result is the corresponding function $v_k(\frac{t}{T_c})$ for $\frac{x_3}{\lambda_c} =$ constant and $\vartheta_k =$ constant.

Figure 8IV gives the family of curves $v_k(\frac{x_3}{\lambda_c})$; $\frac{t}{T_c}$ constant for different values of the parameter ϑ_k. To apply these curves to a particular case, proceed as follows: 1. Take the curve which corresponds to the given value of ϑ_k. 2. Multiply the absċissa by the factor $\frac{t}{T_c}$. 3. Multiply the ordinate by the factor $\sqrt{\frac{t}{T_c}}$.

The result is the corresponding function $v_k(\frac{x_3}{\lambda_c})$ for $\frac{t}{T_c} =$ constant and $\vartheta_k =$ constant.

<u>Problem A</u>: Figure 9IV indicates the graphical process by means of which one time envelope can be obtained from the master envelope given in Fig. 5IV.

Figure 14IVa illustrates one example of how to obtain graphically the corresponding phase function in terms of the variable $\frac{t}{T_c}$.

<u>Problem B</u>: Figure 10IV indicates the graphical process by means of which one space envelope can be obtained from the master envelope function given in Fig. 5IV.

IV-6.3 The effect of the pole frequency on the formation of the signal can be studied with ease by means of this graphical method of construction. Figures 11IV, 12IV, and 13IV give the corresponding time envelopes for $\vartheta_k = 1.1, 1.5, 2.0$ at distances $\frac{x_3}{\lambda_c} = 10, 100, 1000$. It can be noticed how the frequency controls the shape of the signals.

IV-6.4 Figure 15IV shows a family of curves which give the time of formation of time envelopes (Problem A) at different frequencies and at different distances from the origin.

Figure 16IV shows a family of curves which give the distance of formation for space envelopes (Problem B) at different time of the penetration of the wave front.

Section 7 - Complete formation of transient wave

IV-7.0 In this section it will be shown how to combine the different asymptotic solutions in order to obtain the complete construction of the formation of a signal.

IV-7.1 A convenient method to follow can be indicated as:

1st - Take the function $F(s,\sqrt{s^2+1})$ of the corresponding transform. Substitute $s = \sinh\xi$; $\sqrt{s^2+1} = \cosh\xi$ and from

$$H(\xi) = F(\xi)\ \cosh\xi .$$

2nd - Find the poles (ξ_k) of H and the corresponding residues.

3rd - Compute the signal and group velocity for each pole. Determine the wave signal velocity.

4th - Compute the signal in the precursory region by using formula (1)IV3. Stop the computation around the signal velocity of the complete wave.

5th - By graphical or analytical methods find the main formation of each pole and obtain the instantaneous oscillations for each pole.

Place the solution for each pole in accordance with the corresponding group velocity and sum up the results as indicated in Eq. (6)IV4.

It must be carefully noted that the envelope of the complete signal is not necessarily equal to the sum of the partial envelopes which correspond to each separate pole.

6th - After the signal is formed, correct the final envelope by means of the transitory term given in Eq. (7)IV3. In the coda region the signal is practically formed.

IV-7.2 Figure 17IV shows the above procedure in the formation of a complete signal. In the construction of the figure the function

$$\mathcal{L}^{-1}\ \frac{e^{-\mathcal{K}\sqrt{s^2+1}}}{(s^2+\nu_0^2)\sqrt{s^2+1}}$$

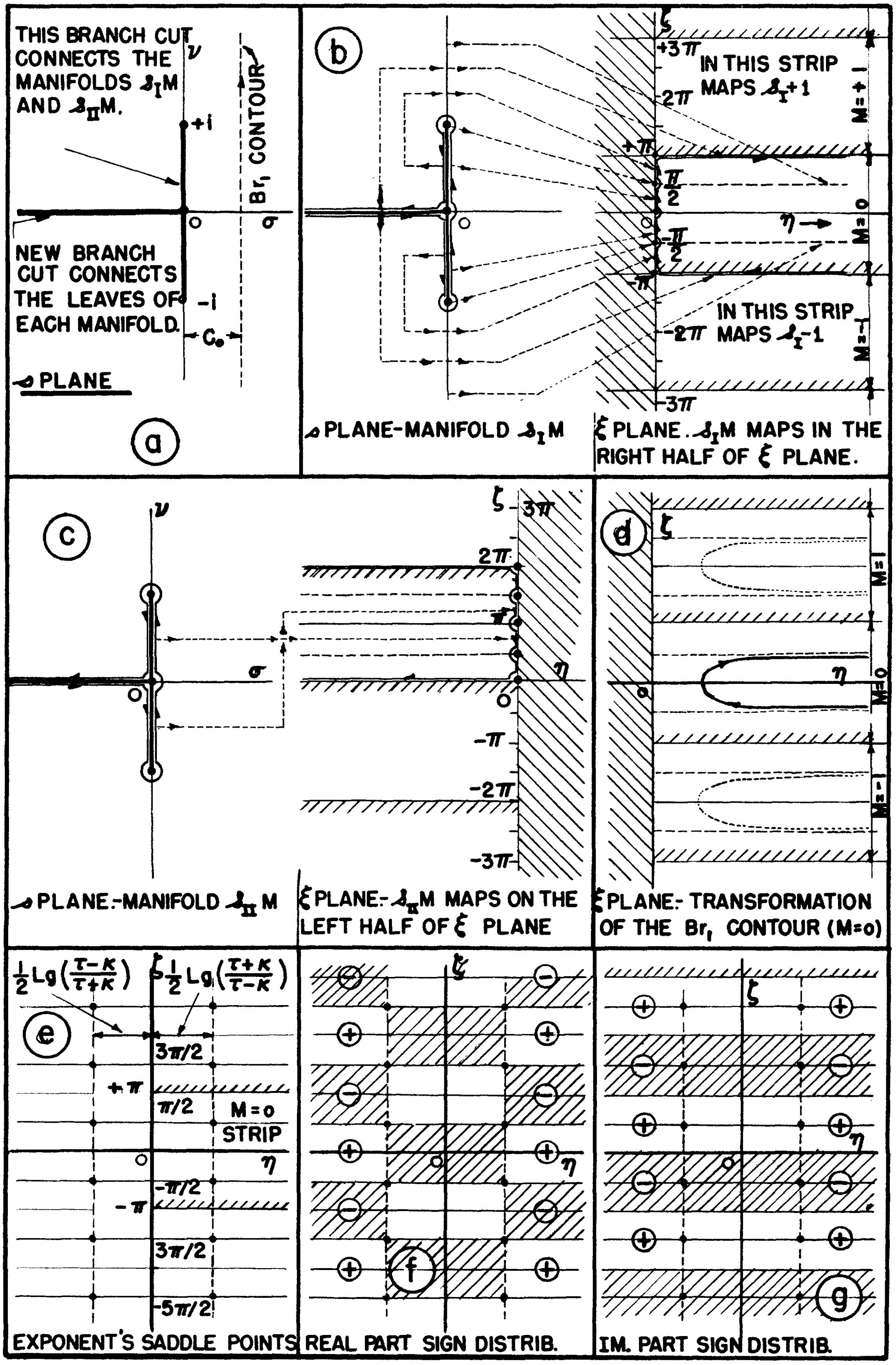
is used. This transform corresponds to the $\mathcal{E}_3$ vector in the example given in the last section of Ch. III.

Figure 18IV shows the corresponding time envelope (Problem A), and Fig. 19IV shows the corresponding space envelope (Problem B). The rapid instantaneous oscillations are not shown in the figures. More comments are not added as these figures are self-explanatory.

IV-7.3 Experiments were conducted by David Winter in the Research Laboratory of Electronics in order to verify correctness of the theoretical results obtained in this investigation. The description of these experiments will be found in Winter's report* on this subject. Only included here are some of the oscillograms which show the complete agreement of the experimental with the predicted results.

*(See Quarterly Progress Report, April 15, 1947, p. 84)

FIG. Nº 1 IV.



THIS BRANCH CUT CONNECTS THE MANIFOLDS s_IM AND s_{II}M.
NEW BRANCH CUT CONNECTS THE LEAVES OF EACH MANIFOLD.
Br₁ CONTOUR
s PLANE
(a)
(b)
s PLANE-MANIFOLD s_IM
IN THIS STRIP MAPS s_I+1
IN THIS STRIP MAPS s_I-1
ξ PLANE. s_IM MAPS IN THE RIGHT HALF OF ξ PLANE.
(c)
s PLANE-MANIFOLD s_{II}M
ξ PLANE- s_{II}M MAPS ON THE LEFT HALF OF ξ PLANE
(d)
ξ PLANE- TRANSFORMATION OF THE Br₁ CONTOUR (M=o)
(e)
$\frac{1}{2}Lg\left(\frac{\tau-K}{\tau+K}\right)$
$\frac{1}{2}Lg\left(\frac{\tau+K}{\tau-K}\right)$
M=o STRIP
EXPONENT'S SADDLE POINTS
(f)
REAL PART SIGN DISTRIB.
(g)
IM. PART SIGN DISTRIB.

FIG. Nº 2IV.

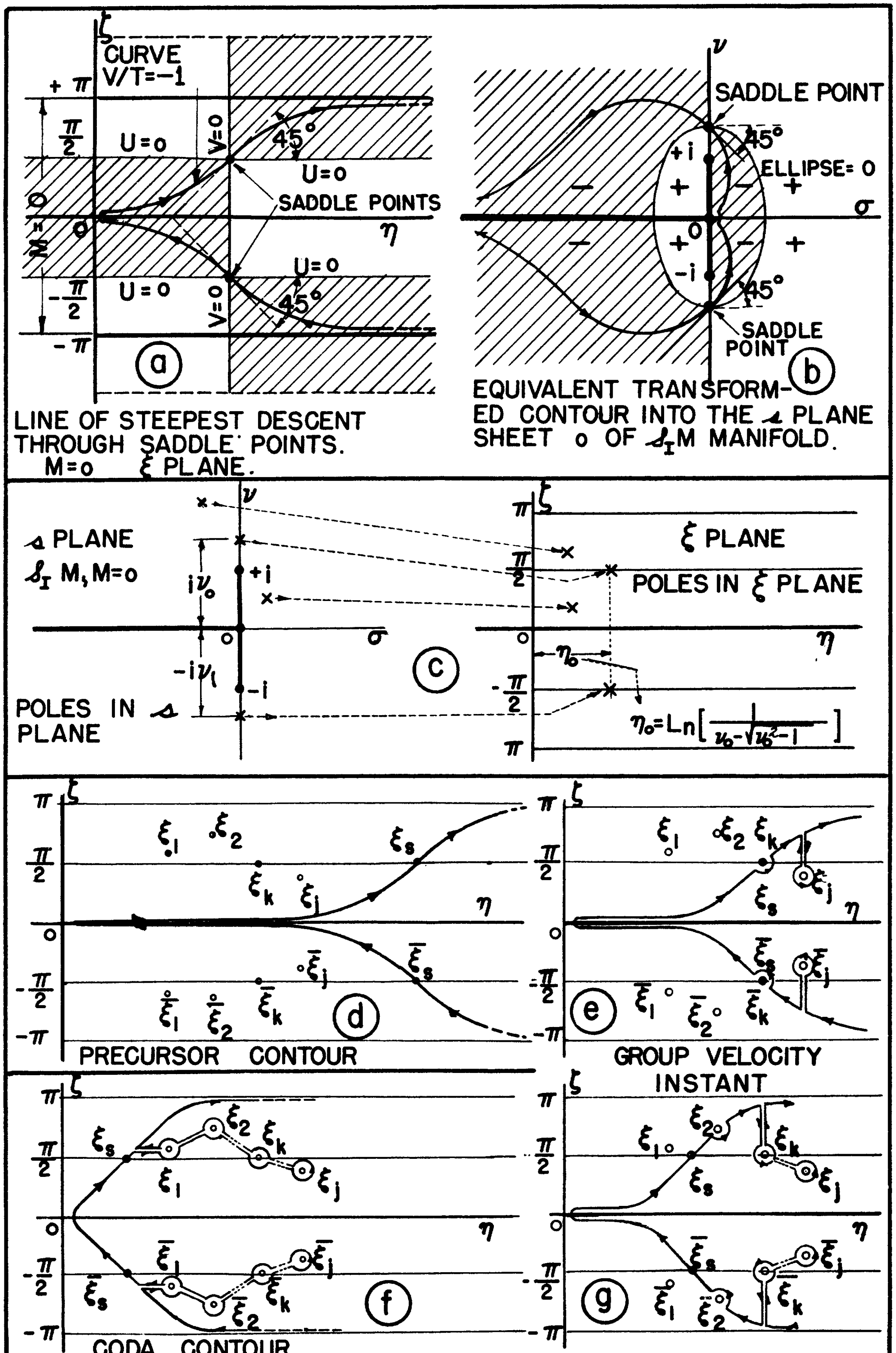
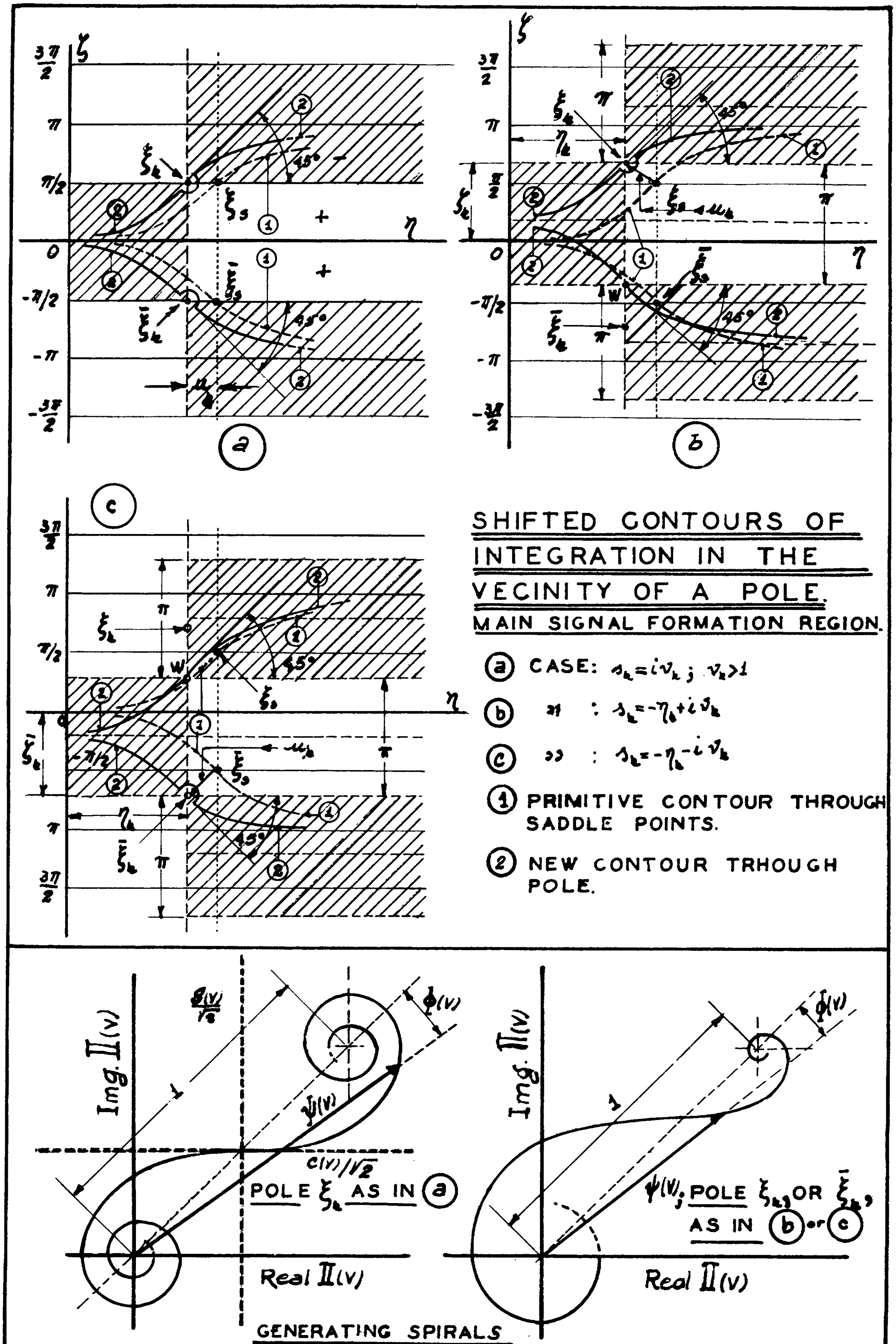
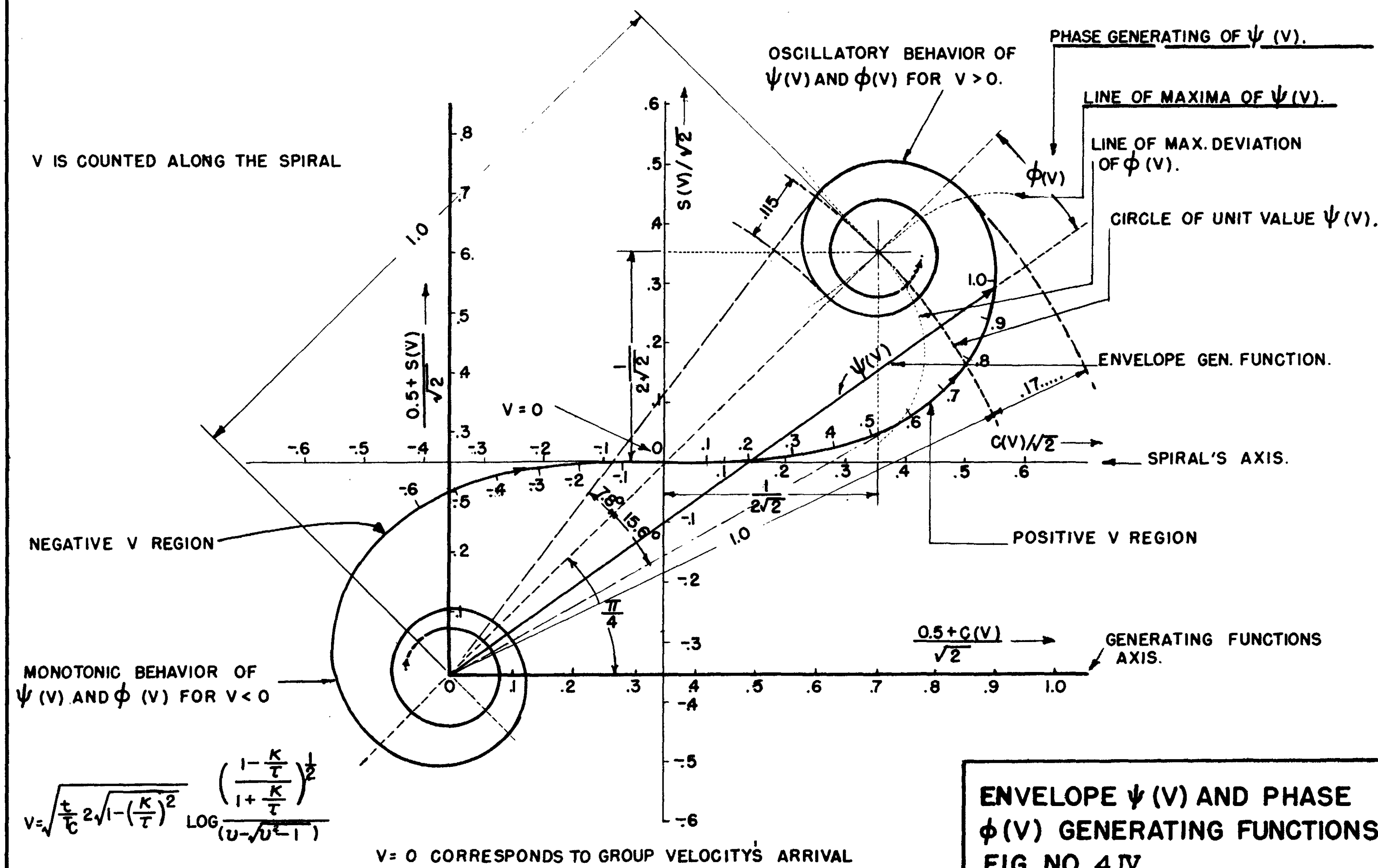
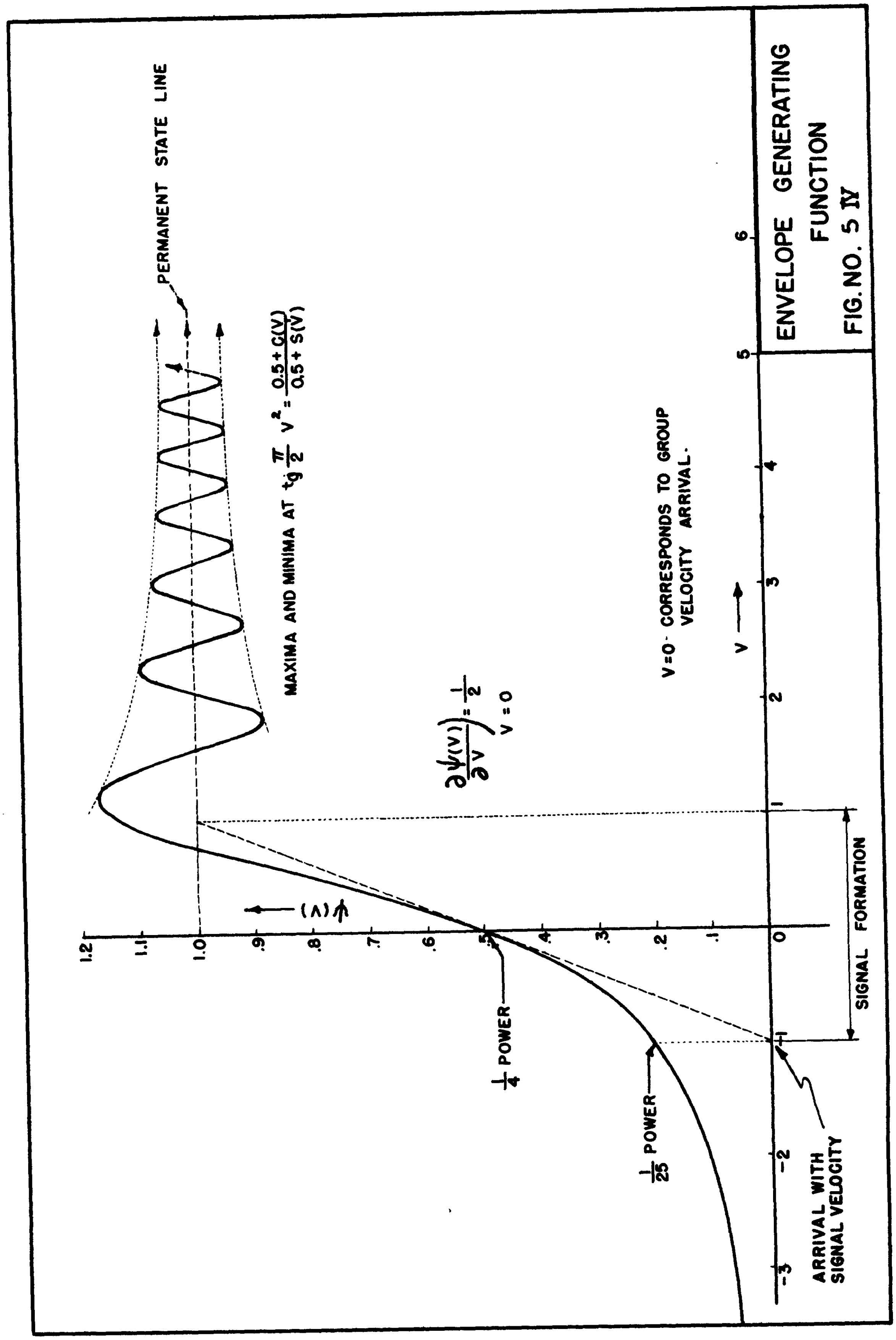

CURVE V/T=-1
U=o
V=0
45°
SADDLE POINTS
M=O
LINE OF STEEPEST DESCENT THROUGH SADDLE POINTS. M=o ξ PLANE.
SADDLE POINT
ELLIPSE= 0
EQUIVALENT TRANSFORMED CONTOUR INTO THE s PLANE SHEET o OF $\mathcal{S}_I$M MANIFOLD.
s PLANE $\mathcal{S}_I$ M, M=o
POLES IN s PLANE
ξ PLANE
POLES IN ξ PLANE
$\eta_0 = Ln\left[\frac{1}{\nu_0 - \sqrt{\nu_0^2 - 1}}\right]$
PRECURSOR CONTOUR
GROUP VELOCITY INSTANT
CODA CONTOUR

FIG._ Nº 3 IV._


SHIFTED CONTOURS OF INTEGRATION IN THE VECINITY OF A POLE.
MAIN SIGNAL FORMATION REGION.
(a) CASE: $s_k = i\nu_k$; $\nu_k > 1$
(b) " : $s_k = -\eta_k + i\nu_k$
(c) " : $s_k = -\eta_k - i\nu_k$
(1) PRIMITIVE CONTOUR THROUGH SADDLE POINTS.
(2) NEW CONTOUR TRHOUGH POLE.
POLE ξ_k AS IN (a)
$\psi(V)$, POLE ξ_k, OR $\bar{\xi}_k$, AS IN (b) or (c)
Img. II(v)
Real II(v)
GENERATING SPIRALS

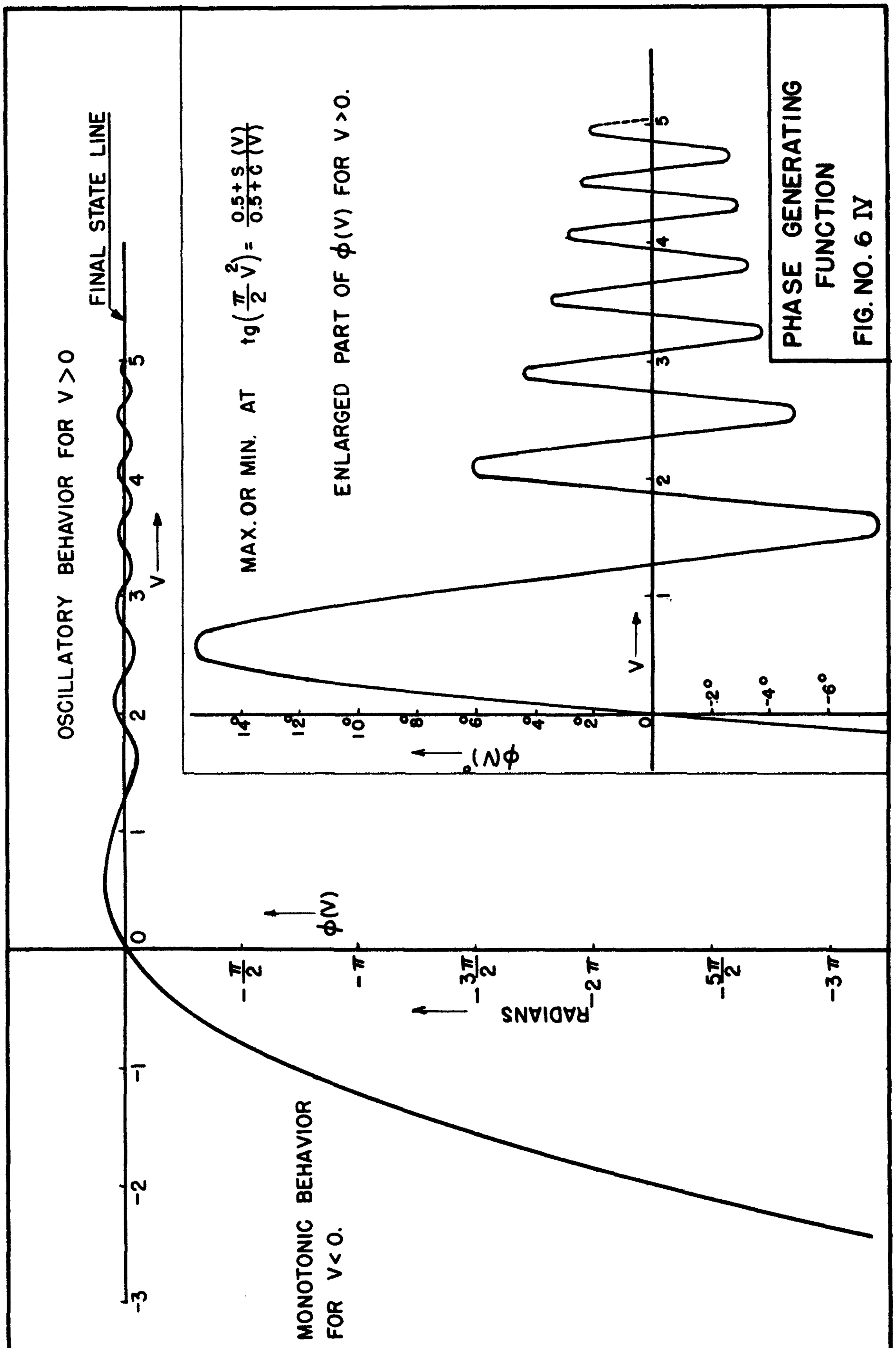


ENVELOPE ψ (V) AND PHASE φ (V) GENERATING FUNCTIONS.
FIG. NO. 4 IV



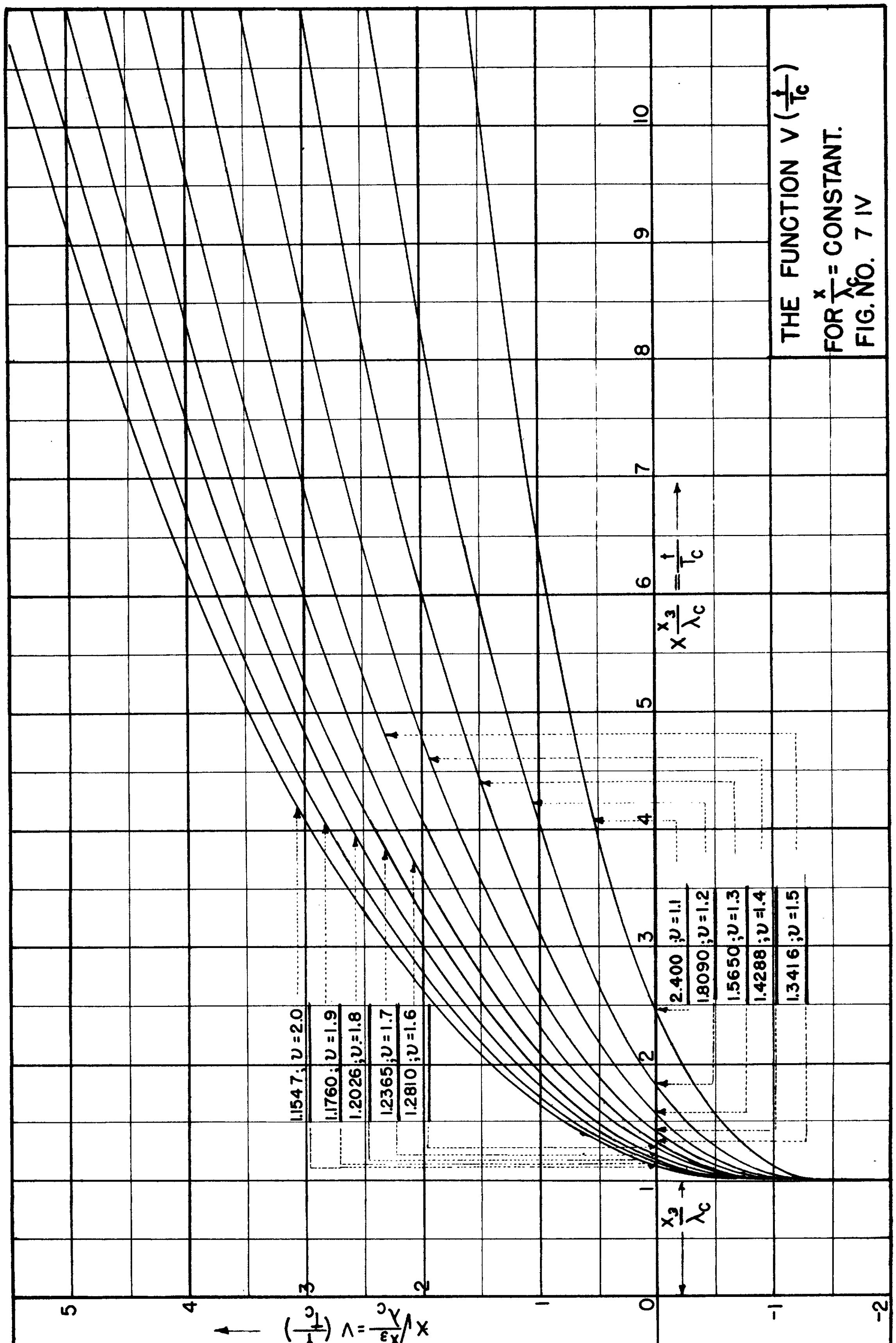
ENVELOPE GENERATING FUNCTION

FIG. NO. 5 IV

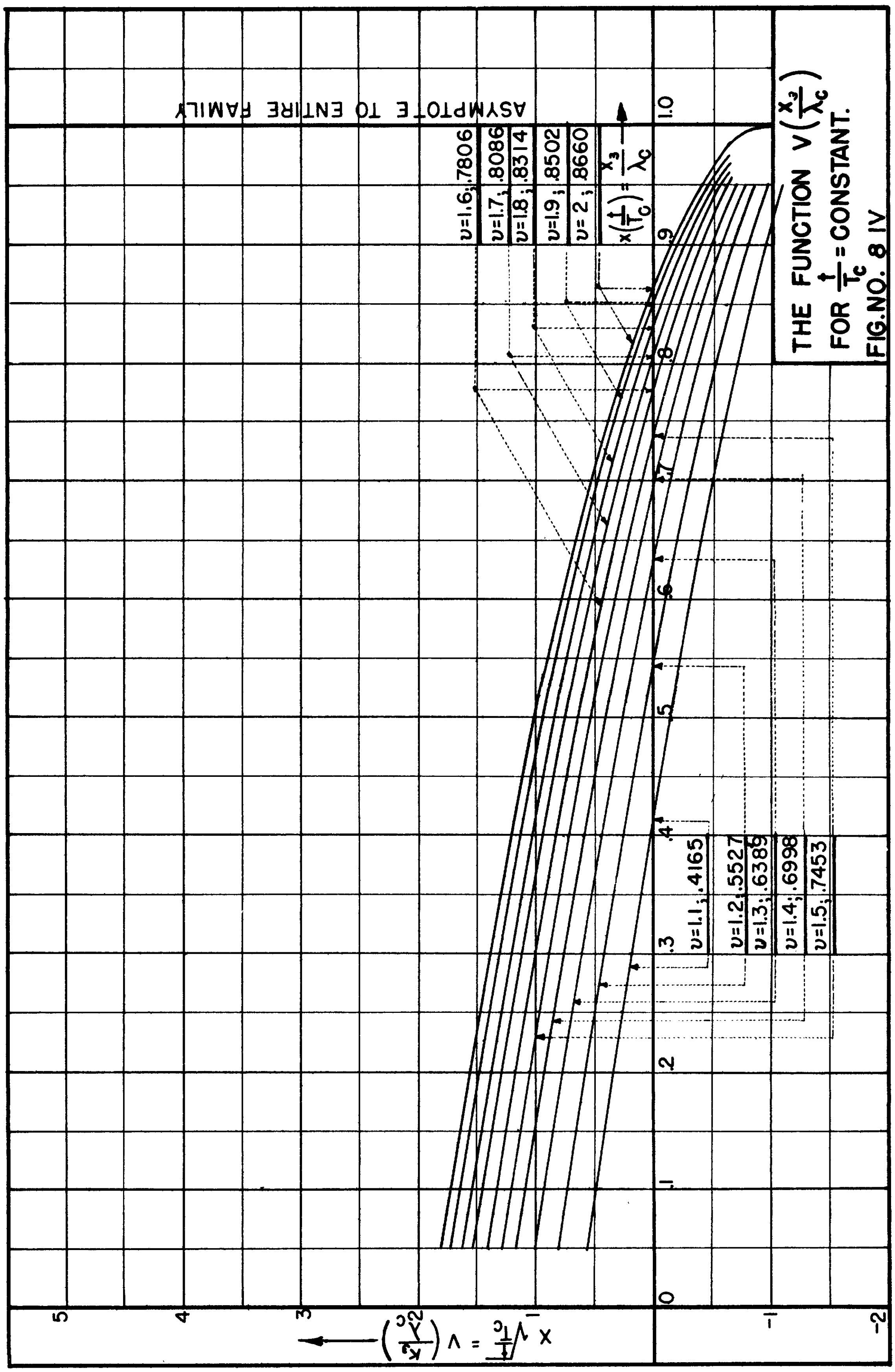


PHASE GENERATING FUNCTION

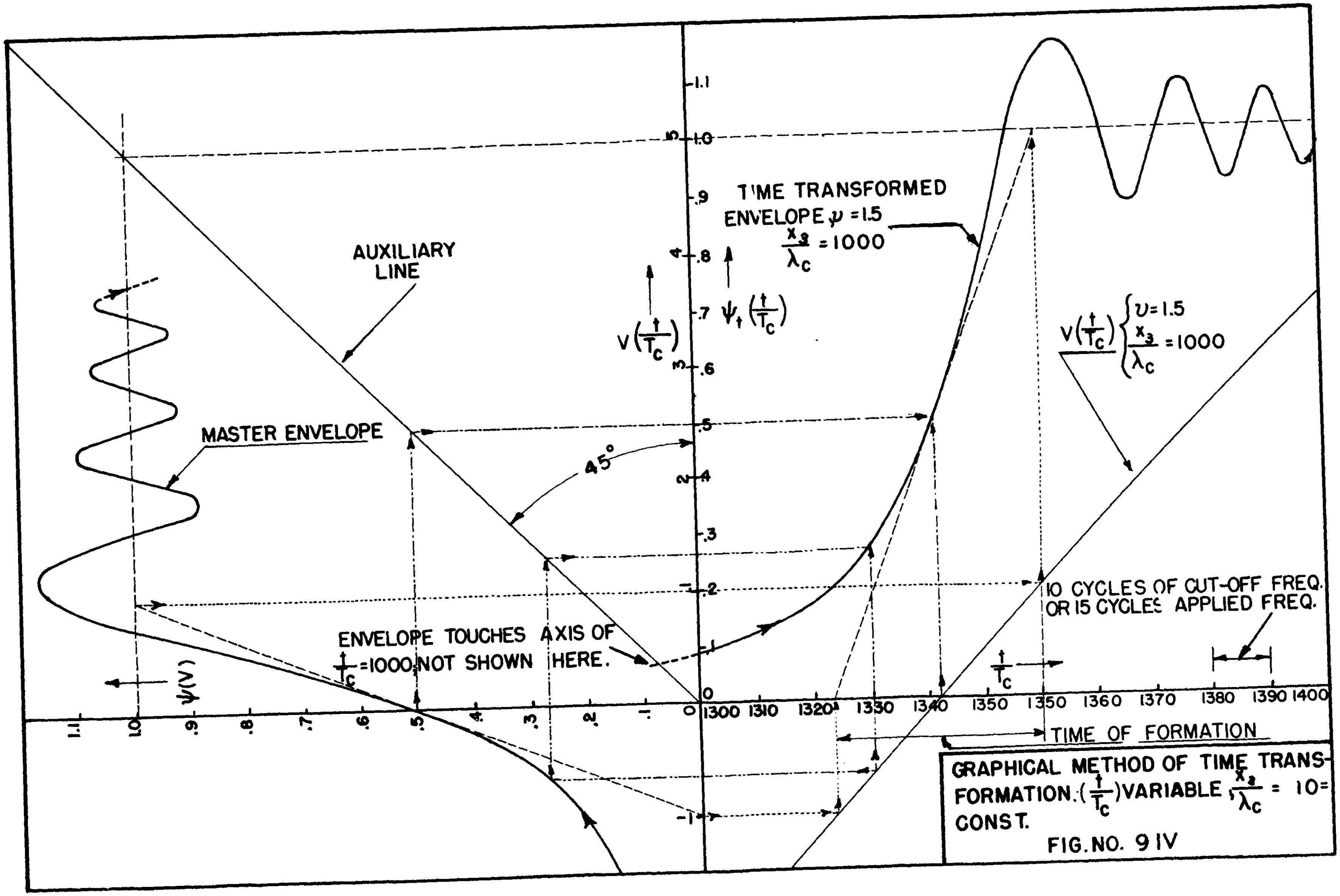
FIG. NO. 6 IV



THE FUNCTION $V\left(\frac{t}{T_c}\right)$
FOR $\frac{x}{\lambda_c}$ = CONSTANT.
FIG. NO. 7 IV

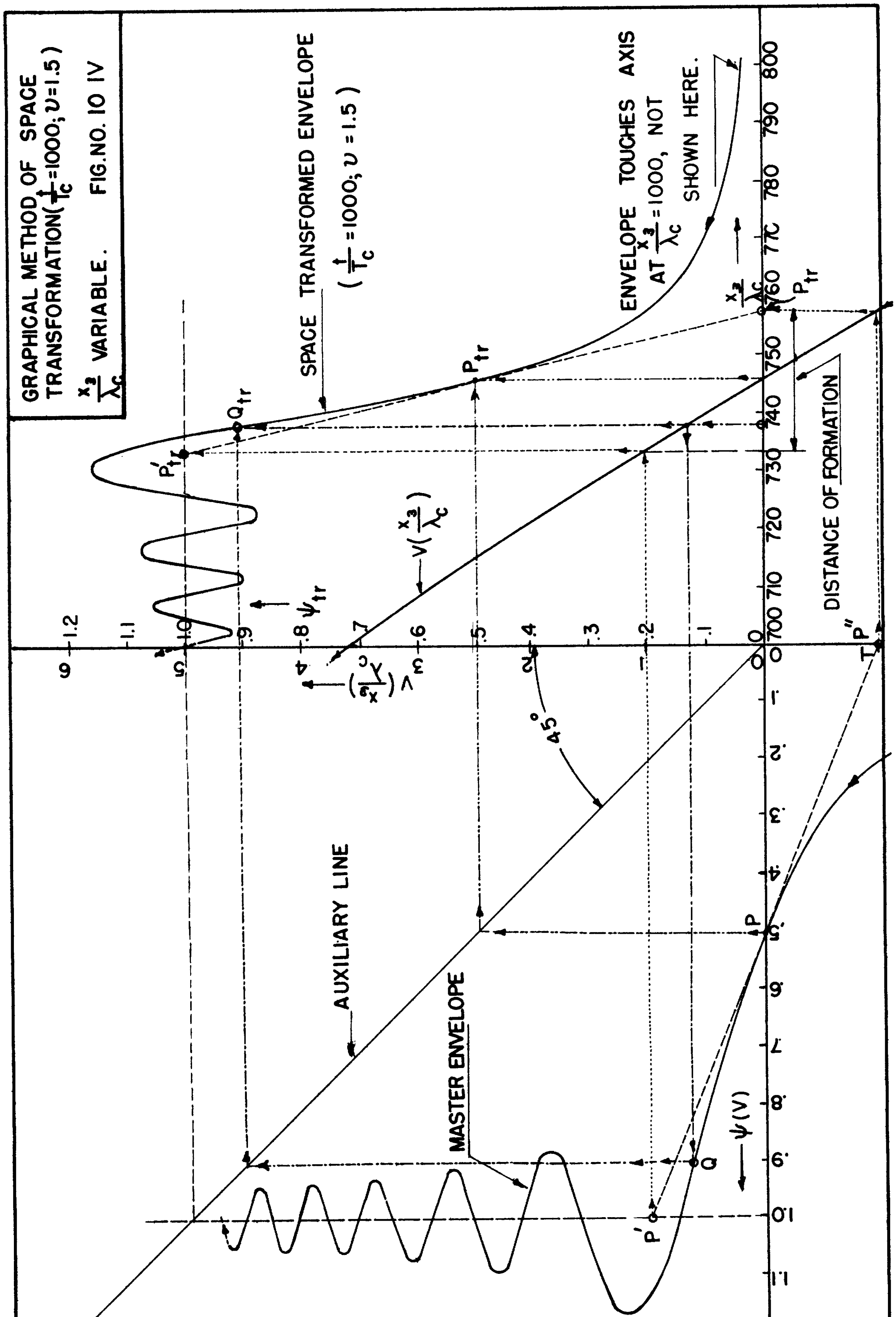


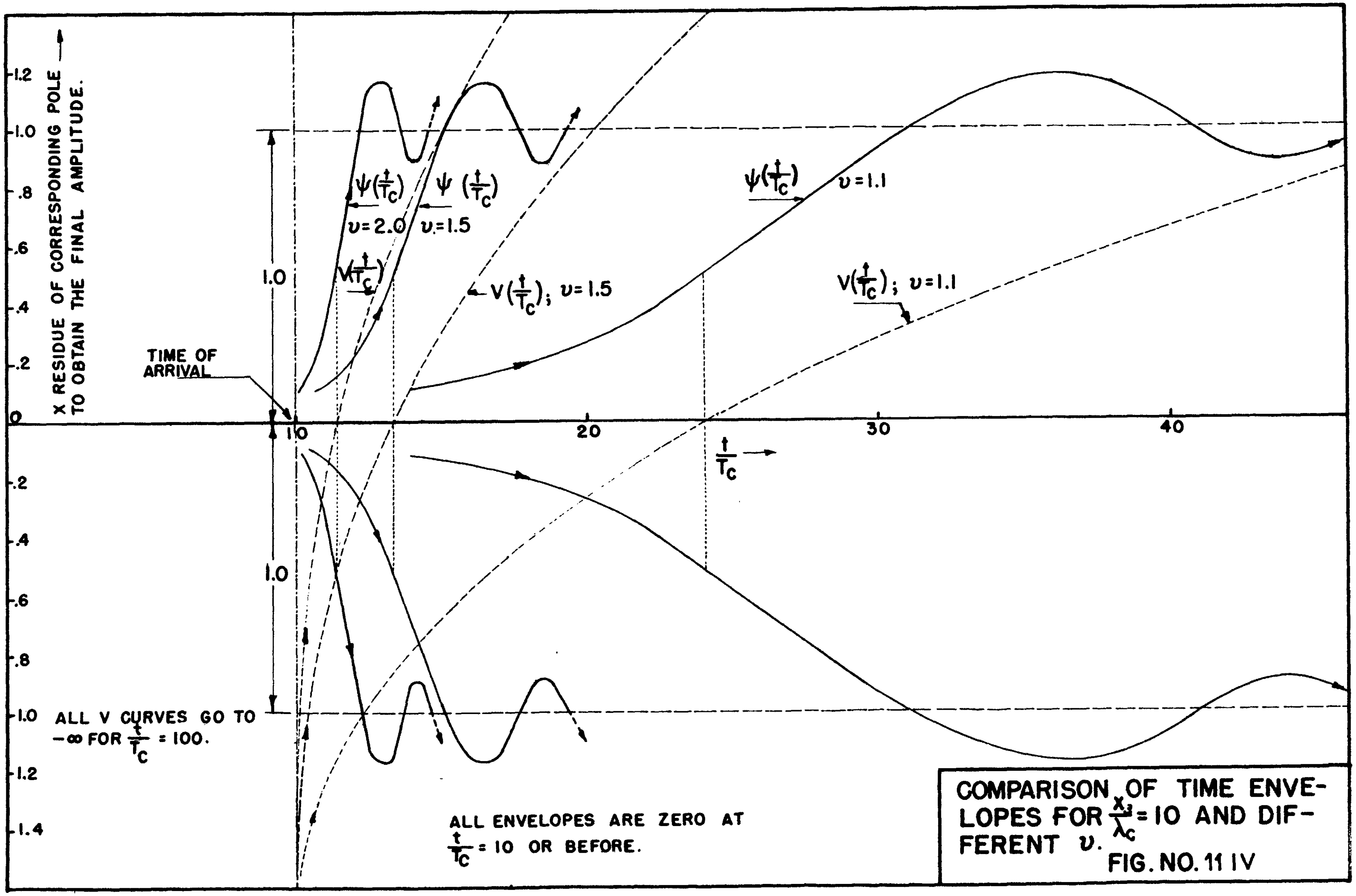
THE FUNCTION $V\left(\frac{x_3}{\lambda_c}\right)$
FOR $\frac{t}{T_c}$ = CONSTANT.
FIG. NO. 8 IV



GRAPHICAL METHOD OF TIME TRANSFORMATION. ($\frac{t}{T_c}$) VARIABLE, $\frac{x_3}{\lambda_c} = 10 =$ CONST.

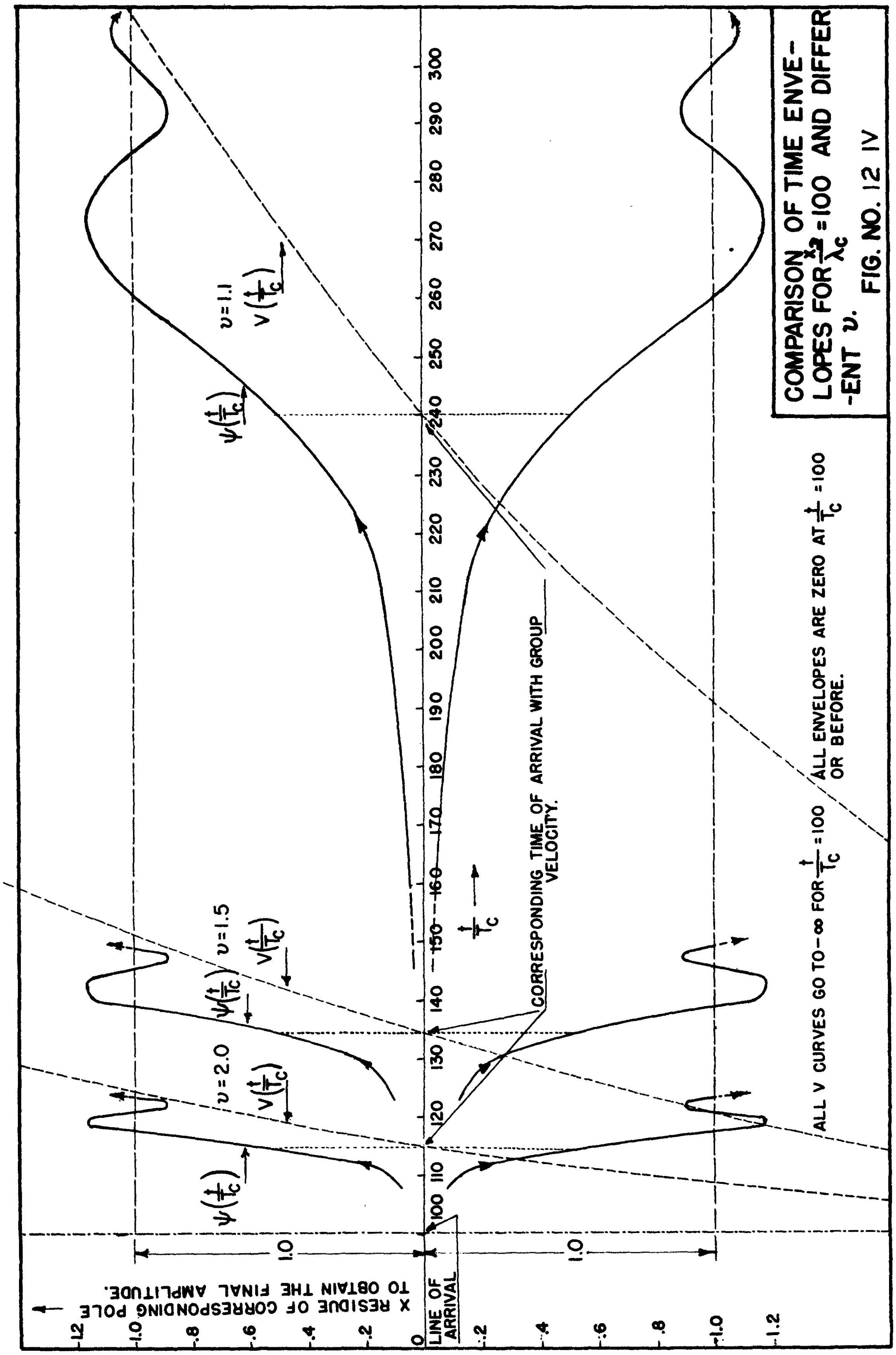
FIG. NO. 9 IV


GRAPHICAL METHOD OF SPACE TRANSFORMATION($\frac{t}{T_c}$=1000; ν=1.5)
$\frac{x_3}{\lambda_c}$ VARIABLE. FIG.NO. 10 IV
SPACE TRANSFORMED ENVELOPE
($\frac{t}{T_c}$ =1000; ν =1.5)
ENVELOPE TOUCHES AXIS AT $\frac{x_3}{\lambda_c}$ =1000, NOT SHOWN HERE.
DISTANCE OF FORMATION
AUXILIARY LINE
MASTER ENVELOPE
45°
ψ_{tr}
$V(\frac{x_3}{\lambda_c})$
$\psi(V)$
P'_{tr}
Q_{tr}
P_{tr}
P
Q
P'
T
P''
$\frac{x_3}{\lambda_c}$



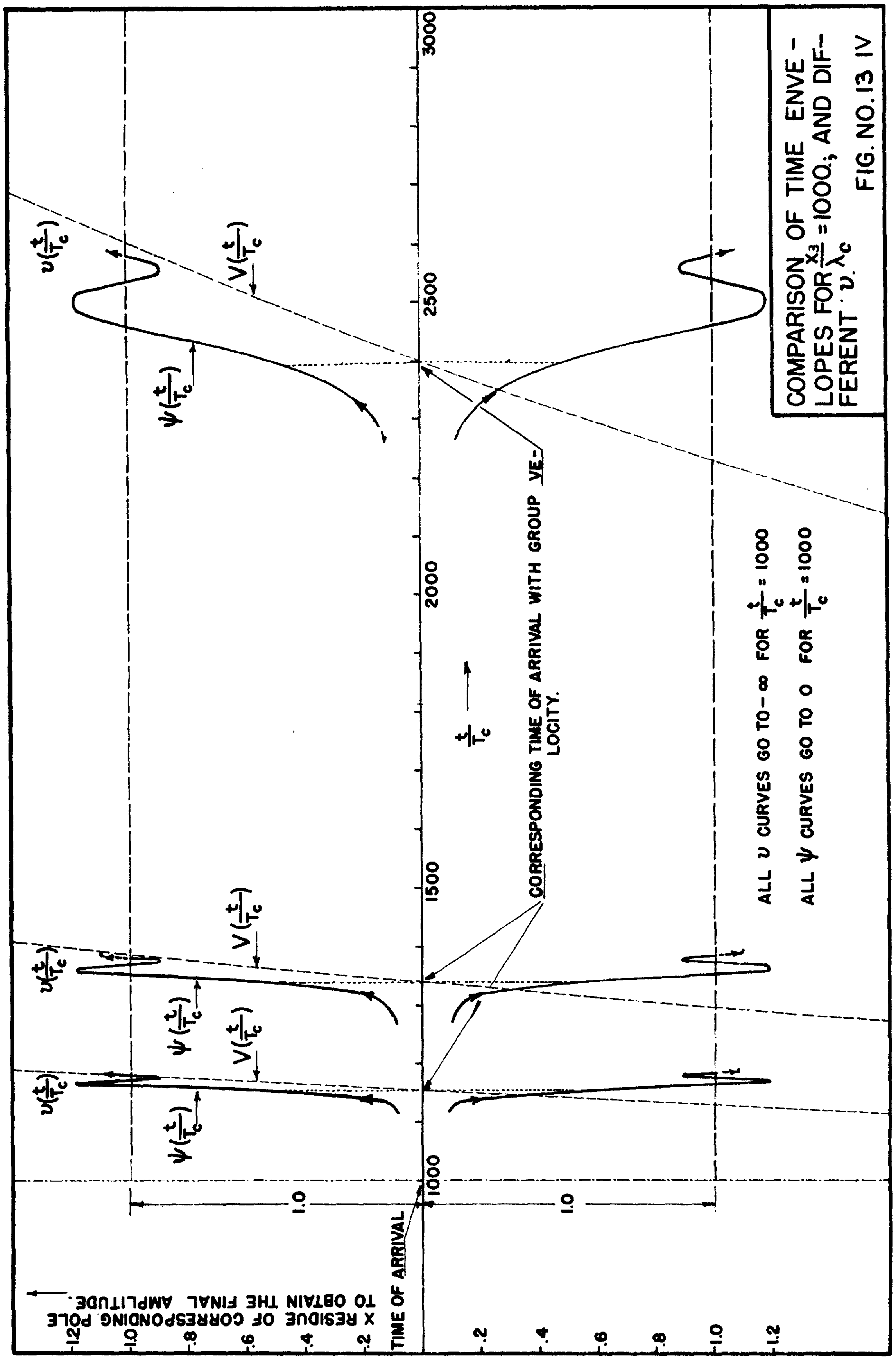
COMPARISON OF TIME ENVELOPES FOR $\frac{x_3}{\lambda_c}=10$ AND DIFFERENT υ.

FIG. NO. 11 IV



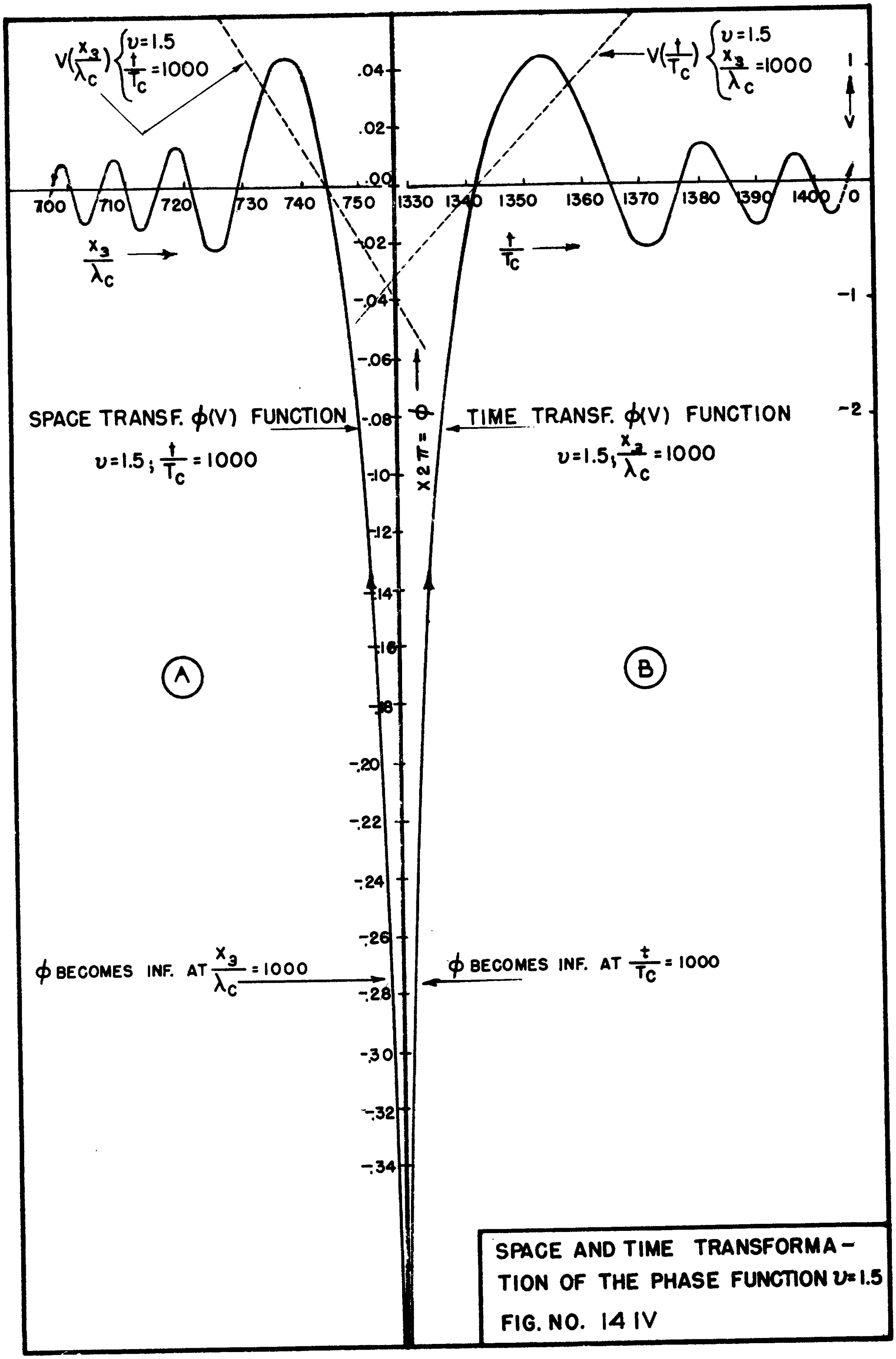
COMPARISON OF TIME ENVELOPES FOR $\frac{x}{\lambda_c} = 100$ AND DIFFERENT υ.

FIG. NO. 12 IV



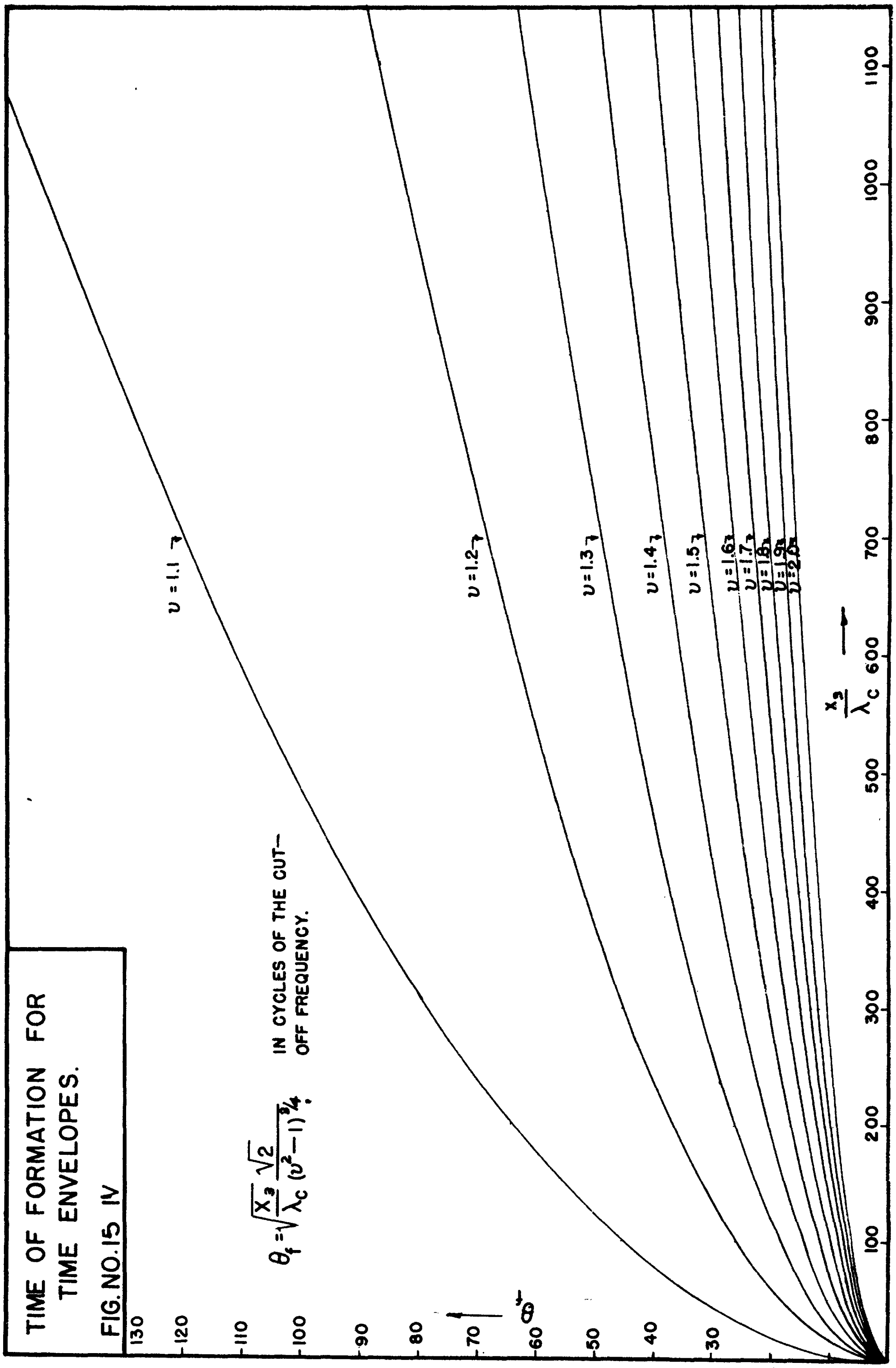
COMPARISON OF TIME ENVELOPES FOR $\frac{x_3}{\lambda_c}$ = 1000; AND DIFFERENT υ.

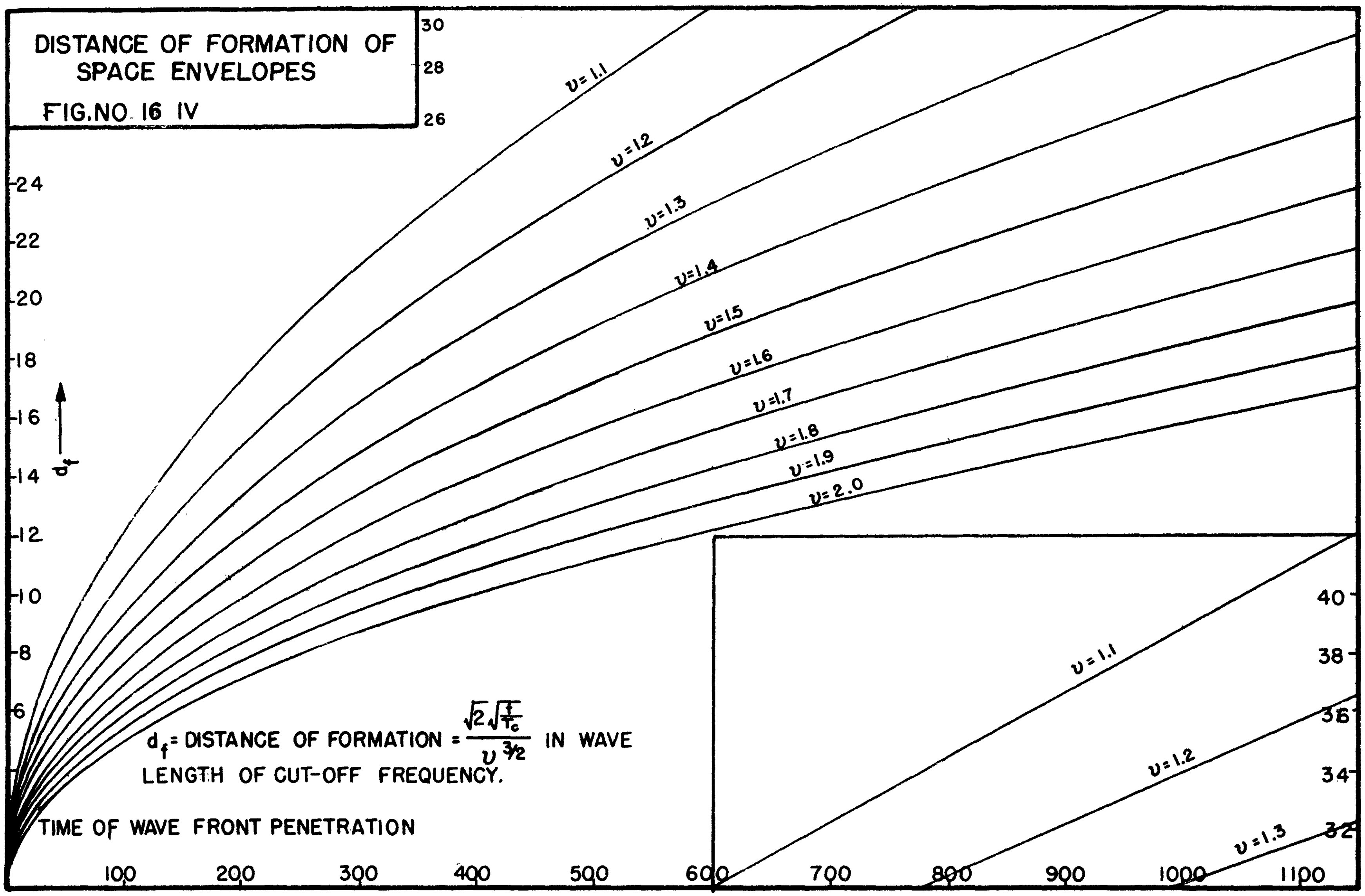
FIG. NO. 13 IV

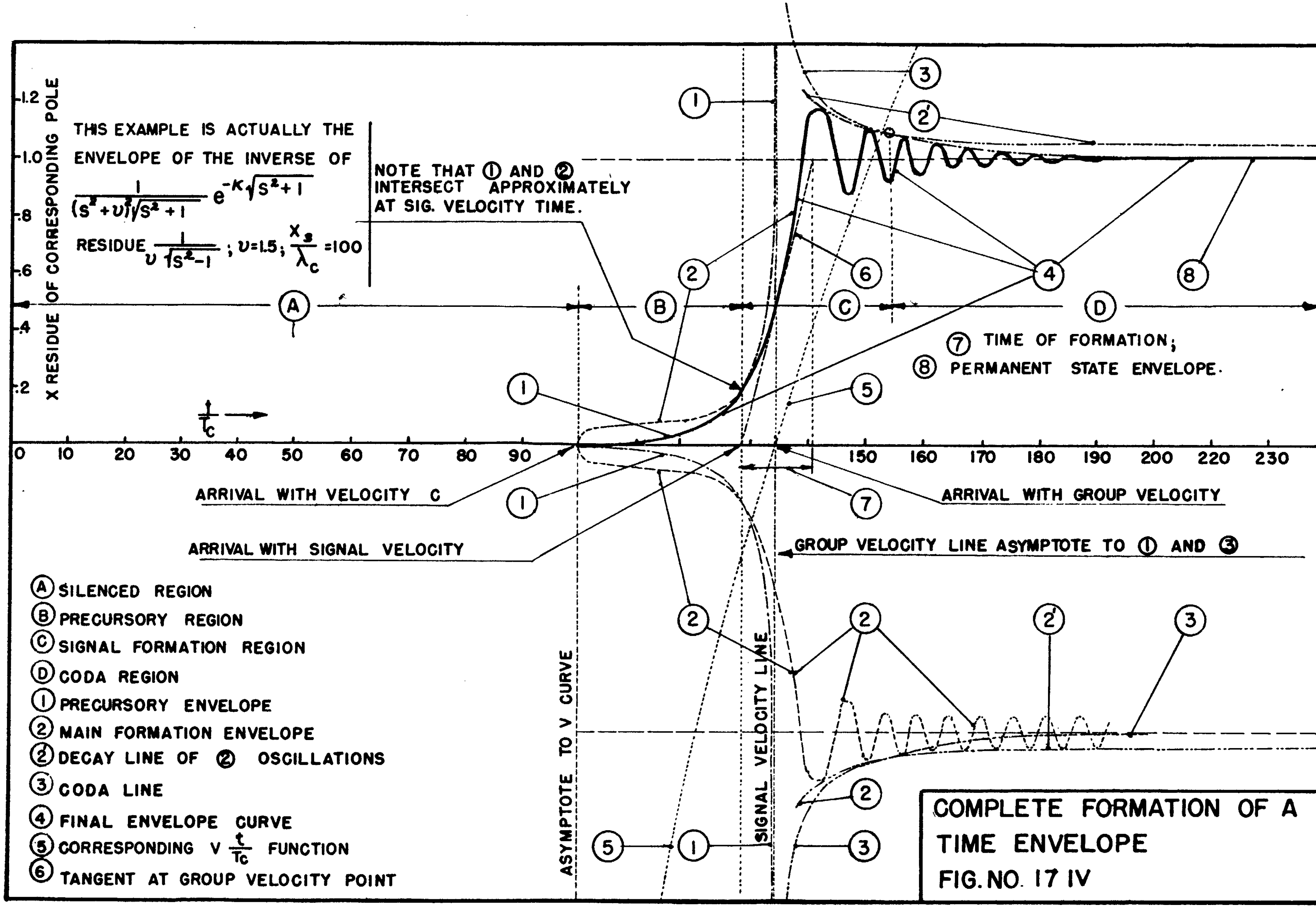


SPACE AND TIME TRANSFORMATION OF THE PHASE FUNCTION υ=1.5

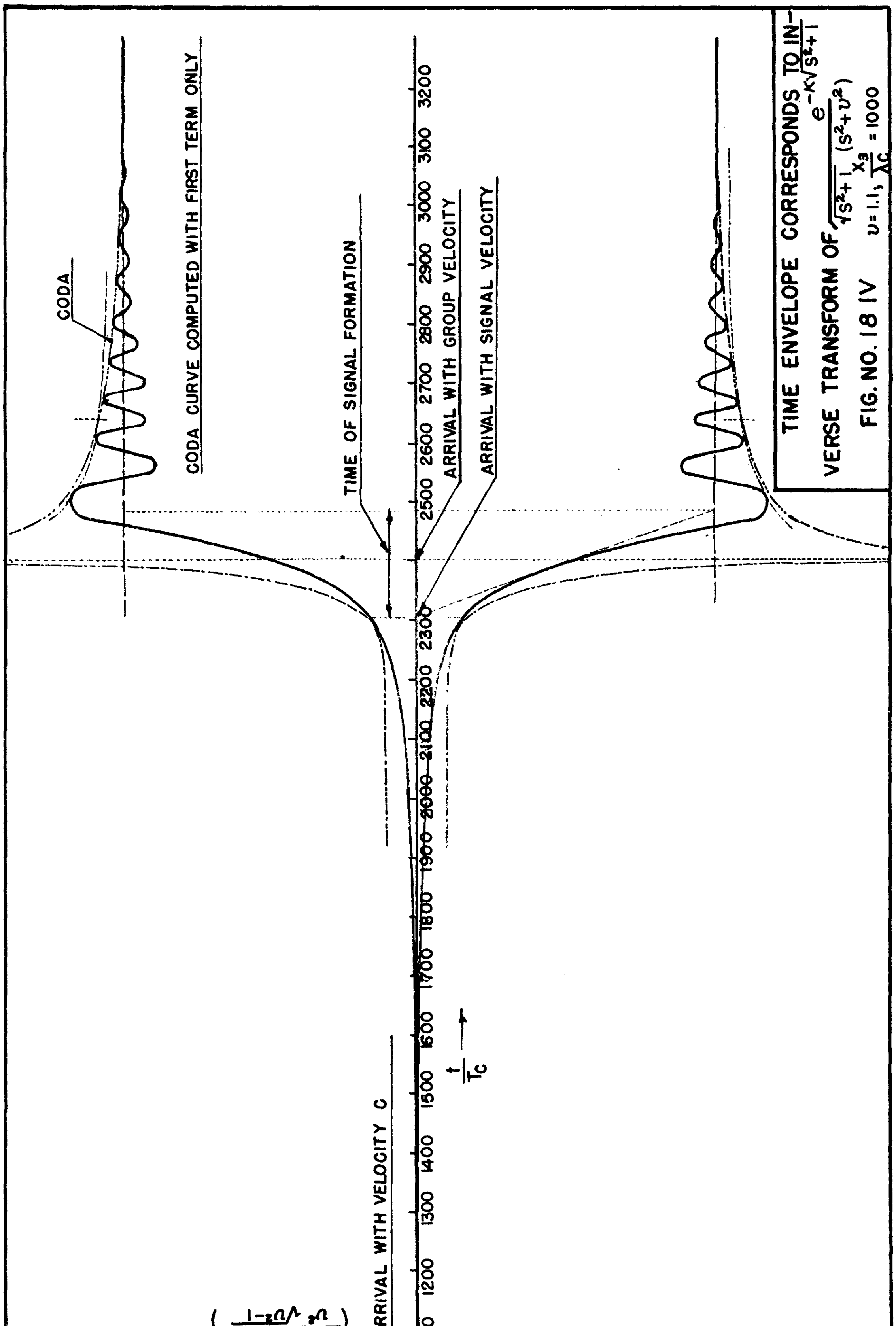
FIG. NO. 14 IV


TIME OF FORMATION FOR TIME ENVELOPES.
FIG. NO. 15 IV
$\theta_f = \sqrt{\frac{X_3}{\lambda_c}\frac{\sqrt{2}}{(\nu^2-1)^{3/4}}}$
IN CYCLES OF THE CUT-OFF FREQUENCY.
θ_f
$\nu = 1.1$
$\nu = 1.2$
$\nu = 1.3$
$\nu = 1.4$
$\nu = 1.5$
$\nu = 1.6$
$\nu = 1.7$
$\nu = 1.8$
$\nu = 1.9$
$\nu = 2.0$
$\frac{X_3}{\lambda_c}$
30 40 50 60 70 80 90 100 110 120 130
100 200 300 400 500 600 700 800 900 1000 1100


DISTANCE OF FORMATION OF SPACE ENVELOPES
FIG. NO. 16 IV
$\nu = 1.1$
$\nu = 1.2$
$\nu = 1.3$
$\nu = 1.4$
$\nu = 1.5$
$\nu = 1.6$
$\nu = 1.7$
$\nu = 1.8$
$\nu = 1.9$
$\nu = 2.0$
d_f
d_f = DISTANCE OF FORMATION = $\frac{\sqrt{2}\sqrt{\frac{t}{T_c}}}{\nu^{3/2}}$ IN WAVE LENGTH OF CUT-OFF FREQUENCY.
TIME OF WAVE FRONT PENETRATION
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
100
200
300
400
500
600
700
800
900
1000
1100

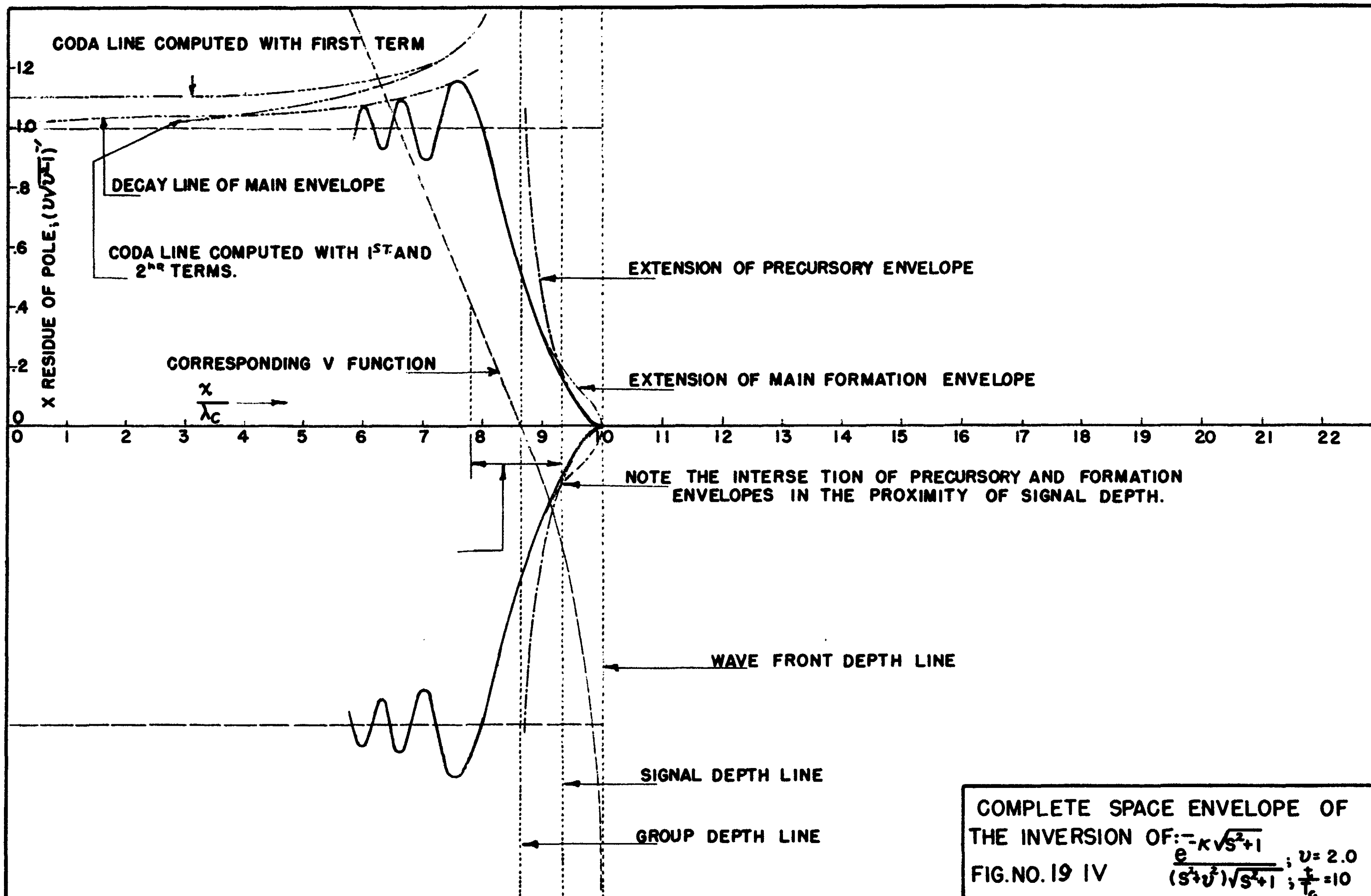


COMPLETE FORMATION OF A TIME ENVELOPE
FIG. NO. 17 IV



TIME ENVELOPE CORRESPONDS TO INVERSE TRANSFORM OF $\frac{e^{-K\sqrt{S^2+1}}}{\sqrt{S^2+1}\,(S^2+\upsilon^2)}$

FIG. NO. 18 IV $\upsilon = 1.1$, $\frac{X_3}{\lambda_c} = 1000$



COMPLETE SPACE ENVELOPE OF THE INVERSION OF: $\frac{e^{-\kappa\sqrt{s^2+1}}}{(s^2+\upsilon^2)\sqrt{s^2+1}}$; $\upsilon = 2.0$; $\frac{t}{t_c} = 10$

FIG. NO. 19 IV

APPENDIX I

The integrals (1)III3, page 110, can be evaluated directly without the necessity of series expansions, as it is done in the text. For a reader who is acquainted with the functions of Lommel, the following short-cut is preferable:

Take first: $\alpha = 1$.

a. If $Z_k = 0$, then go back to the s plane. It can be written

$$\frac{1}{2\pi i}\int_{\gamma_Z} \frac{e^{A(Z)}}{Z}\, dZ = -\frac{1}{2\pi i}\int_{\gamma_s} \frac{e^{s\tau - \mathcal{K}\sqrt{s^2+1}}}{\sqrt{s^2+1}}\, ds = -J_o\sqrt{\tau^2-\mathcal{K}^2} \quad (1)\text{AI}$$

in which $A(Z) = \frac{\tau}{2}\left(Z - \frac{1}{Z}\right) + \frac{\mathcal{K}}{2}\left(Z + \frac{1}{Z}\right)$ (2)AI

since it is well known that (see also the text)

$$\mathcal{L}^{-1}\, \frac{e^{-\mathcal{K}\sqrt{s^2+1}}}{\sqrt{s^2+1}} = J_o\sqrt{\tau^2 - \mathcal{K}^2} \quad (3)\text{AI}$$

b. If $0 < |z_k| < 1$, then introduce the complex transformation

$$u = -\frac{\tau - \mathcal{K}}{2Z} \qquad u = \text{complex} \quad (4)\text{AI}$$

$$\text{then: } \frac{1}{2\pi i}\int_{\gamma_Z} \frac{e^{A(Z)}}{(Z - Z_k)}\, dZ = \frac{1}{2\pi i}\int_{\gamma_u} \frac{e^{u - \frac{T^2}{4u}}}{\left(u + i\,\frac{\Omega_k}{2}\right)}\; \frac{i\Omega_k}{2u}\; du \quad (5)\text{AI}$$

where $T = \sqrt{\tau^2 - \mathcal{K}^2}\,;\quad \Omega_k = \frac{\tau - \mathcal{K}}{iZ_k}$. (6)AI

The corresponding contour of integration is indicated in the figure. Notice that this contour is a slight modification of γ_Z given in Fig. 7 III b.

Now, consider the Gilbert integral representation of the Lommel's U function. See "Theory of Bessel Functions" Watson, page 548, Equation 1.

$$U_\nu\left(\Omega_k, T\right) = \frac{1}{2\pi i}\int_{\gamma_u} \frac{\left(\frac{1}{2}\,\frac{\Omega_k}{u}\right)^{\nu}}{\left(1 + \frac{\Omega_k^2}{4u^2}\right)}\, e^{u - \frac{T^2}{4u}}\, \frac{du}{u} \quad . \quad (7)\text{AI}$$

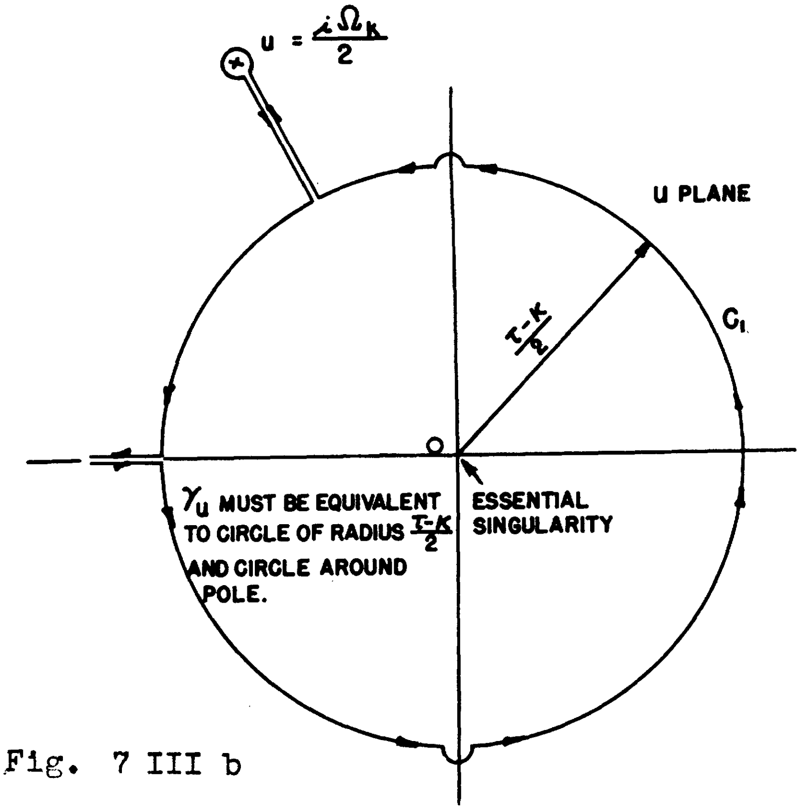


Fig. 7 III b

It follows immediately that

$$U_2(\Omega_k, T) + i\, U_1(\Omega_k, T) = \frac{1}{2\pi i} \int_{\gamma_Z} \frac{e^{A(Z)}}{Z - Z_k}\, dZ \qquad (8)AI$$

which is the result given in (15)III4 on page 124.

c. For $1 < |Z_k^*|$, a similar procedure can be followed by using the inverse function of (4)AI as a new variable.

Take second: $\gamma = 1$.

This is the general case and will not be considered here since the procedure given in the text is the simplest one.