SQUARE-WAVE MODULATION OF THE POUND FREQUENCY STABILIZER

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by

C. G. Aurell

Abstract

The problem of using the Pound circuit for obtaining a voltage with a frequency which is stabilized but has a certain type of predetermined time variation is considered. It is desired to shift the frequency periodically between two fixed values, or to have the generated voltage shifted between on and off, and frequency stabilized when on. A square-wave voltage is introduced in series between the output of the stabilizer and the repeller of the klystron oscillator. The R-C-circuit at the output of the stabilizer is the principal factor governing the transient behavior. A theory is worked out on this assumption. Experimental results are shown that verify the theory, but also show its limitations.
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I. Introduction

The Pound frequency-stabilizing circuit for microwaves has proved to have excellent frequency-stabilizing properties when used to produce continuous waves at constant frequency. The question, therefore, naturally arose as to whether it could be used to produce a voltage with a frequency which would be time-varying, but also continuously stabilized. The question suggested to the author of this paper was whether it could be utilized to generate a voltage with a frequency which is periodically shifted between two different fixed values or periodically switched on and off, being frequency stabilized when on.

A thorough treatment of this problem must necessarily be extremely complicated because of the large number of components involved. The essential properties, however, can be easily demonstrated. The reason for this is that the transient behavior is mainly governed by the RC-circuit in the output of the stabilizer. The circuit has been introduced in order to prevent self-oscillation of the stabilizing circuit by cutting down the loop gain to below one at the frequencies where the feedback is positive. From the theory of feedback amplifiers it is known that at high loop gains, such a circuit must have a bandwidth which is very much smaller than the rest of the circuit. Thus the transient build-up time in the rest of the circuit can be neglected to a first approximation in comparison with the build-up time in the RC-circuit. The following mathematical treatment is based on this assumption.

2. Theory

Figure 1 shows the manner in which the square-wave voltage $e_4$ is introduced between the stabilizer output voltage $e_3$ and the repeller of the klystron. The resulting voltage at the repeller of the klystron is voltage $e_1$. No attention has been paid to d-c...
levels here. The equations governing the stabilization process are

\[ e_2 = D(e_1) \]  
(1) \[ e_2 = (1 - \frac{d}{dt}T) e_3 \] with \( T = RC \)  
(2) \[ e_1 = e_3 + e_4 \]  
(3)

In Eq. (1), \( D \) symbolizes the discriminator characteristic, and it is assumed that \( e_1 \) gives an instantaneous response \( e_2 \); thus Eq. (1) is not a function of time.

Eliminating \( e_2 \) and \( e_3 \) by using Eqs. (1), (2), and (3) gives:

\[ (1 + T \frac{d}{dt}) (e_1 - e_4) = D(e_1) \]  
(4)

Here \( t \) is the independent and \( e_1 \) the dependent variable, while \( e_4 \), the impressed voltage, is a known function of time. The resulting frequency as a function of time is the quantity desired, but as the frequency deviation is proportional to \( e_1 \) for small variations in voltage, we can restrict our studies to \( e_1 \) as a function of time.

Let us assume that \( e_4 \) is an ideal square wave, where the voltage is instantaneously shifted between the levels \( e_{40} \) and \( e_{41} \) (Fig. 2).

![Figure 2](image)

Except at the discontinuities, we have \( \frac{d e_4}{dt} = 0 \). Thus, the corresponding differential equation obtained from Eq. (4a) is

\[ T \frac{d e_1}{dt} = e_4 - [e_1 - D(e_1)] \]  
(5)

where \( e_4 \) is either \( e_{40} \) or \( e_{41} \). Equation (5) can be integrated for constant \( e_4 \) by separating the variables:

\[ \frac{t + t_0}{T} = \int \frac{d e_1}{e_4 - [e_1 - D(e_1)]} \]  
(6)
where \( t_0 \) is a constant of integration. This gives \( t \) as a function of \( e_1 \). The value of \( t_0 \) is determined by the initial conditions. As the function \( D(e_1) \) can be measured, the integral can always be solved graphically. Let us consider the function, \( e_1 = e_1(t) \), when the discriminator curve \( e_2 = D(e_1) \) has the general shape depicted in Fig. 3.

![Figure 3](image)

A positive value \( e_{41} \) will be chosen for the impressed voltage \( e_4 \). Thus Eq. (5) becomes

\[
\frac{de_1}{dt} = e_{41} - [e_1 - D(e_1)]
\]  
\( \text{(5a)} \)

By studying Eq. (5a) the main properties of \( e_1 = e_1(t) \) can be understood. First, it must be noted that the equilibrium value, i.e., the value \( e_{11} \), which \( e_1 \) approaches when \( e_{41} \) is kept constant for a very long time, is obtained by putting \( \frac{de_1}{dt} = 0 \) in Eq. (5a). Thus,

\[
e_{11} - D(e_{11}) = e_{41}
\]  
\( \text{(5b)} \)

As shown in Figs. 4a, b, and c, we must distinguish among three different cases, depending upon the magnitude of the positive voltage \( e_{41} \).

![Figure 4](image)
(a): $e_{41}$ is less than the minimum in the $e_1 - D(e_1)$ curve. Here an equilibrium point is obtained on the steep part of the discriminator characteristic. For practical applications this is the most important case.

(b): $e_{41}$ has a value between the maximum and the minimum of the $e_1 - D(e_1)$ curve. Three equilibrium points are obtained in this case. By studying the sign of $\frac{de_1}{dt}$ the two outer equilibrium points are found to be stable, while the intermediary one is unstable. The left-hand equilibrium point still occurs on the steep portion of the discriminator curve.

(c): $e_{41}$ is greater than the maximum in the $e_1 - D(e_1)$ curve. As in case (a) there is only one stable equilibrium point, but it is no longer on the steep part. No stabilization is obtained in this case, but rather a slight decrease in stability compared with the case of no stabilization at all.

The arrows along the curves show how the equilibrium points are reached for different initial values $e_{10}$ of $e_1$. Initial value: When switching from one level $e_4 = e_{40}$ to another, $e_4 = e_{41}$, because of the RC-network, $e_3$ cannot change instantaneously. Assume that $e_3 = e_{30}$ just before switching $e_4$ from $e_{40}$ to $e_{41}$. Then from Eq. (3) the initial value $e_{10}$ of $e_1$ is determined, since

$$e_{10} = e_{30} + e_{41}. \quad (3a)$$

The initial value $e_{10}$ is known and the derivative $\frac{de_1}{dt}$, with reversed sign, is proportional to the height of the curve $e_1 - D(e_1)$ above a horizontal line through $e_{41}$. Thus we can by inspection of the curves in Fig. 4 get a good idea of how $e_1$ varies with time.

To make a more rigorous treatment we must make certain assumptions about the function $e_2 = D(e_1)$. The discriminator characteristic can be written as

$$e_2 = D(e_1) = -\frac{e_2}{\frac{e_1}{E_1} + \frac{1}{e_1}}. \quad (7)$$

where $E_1$ and $E_2$ are defined in Fig. 5. This expression lends itself well to mathematical treatment. In the practical case several factors enter which tend to distort Eq. (7); one such factor is the variation of oscillator power and thus the input voltage to the stabilizer when the repeller voltage $e_1$ is varied. It can be expected, however, that for the case when $e_1$ is small, Eq. (7) will give a good approximation for the actual function $e_2 = D(e_1)$. Substituting Eq. (7) into (3a) gives

$$T \frac{de_1}{dt} = e_{41} - e_1 - \frac{e_2}{\frac{e_1}{E_1} + \frac{1}{e_1}}. \quad (8)$$
Separating variables and integrating, we obtain

\[
\frac{t + t_0}{T} = \int \frac{e_1^2 + E_1^2}{e_1^3 - e_4e_1^2 + E_1(E_1 + E_2)e_1 - e_4E_1} \, de_1. \tag{9}
\]

Expanding the integrand into partial fractions and integrating term by term gives the solution of the integral. As the denominator is of the third degree in \(e_1\), its roots can always be exactly determined.

![Figure 5.](image)

Figure 5.

Without losing much in accuracy the following approximation can be made:

\[
D(e_1) = -\frac{E_2}{E_1} \frac{e_1}{0 + \frac{e_1}{e_1}} = \frac{E_2e_1}{E_1} \text{ for } |e_1| \leq E_1 \tag{10}
\]

\[
D(e_1) = -\frac{E_2}{e_1} \frac{e_1}{0 + \frac{e_1}{E_1}} = \frac{E_2e_1}{E_1} \text{ for } |e_1| \geq E_1. \tag{11}
\]

Figure 6 shows this approximation, with the discriminator curve according to Eq. (7) dotted.

Case A. \(|e_1| \leq E_1\). When Eq. (10) is substituted into Eq. (5a),

\[
T \frac{de_1}{dt} = e_1 - (1 + \frac{E_2}{E_1}) e_1. \tag{12}
\]

The solution of Eq. (12) is
where A is an integration constant. Here $\frac{E_2}{E_1}$ is equal to the loop gain $G$, and Eq. (13) can be written:

$$e_1 = A \alpha - \frac{t}{T/(1+G)} + \frac{e_{41}}{1+G}.$$  \hspace{1cm} (13a)

For appreciable loop gains the time constant for $e_1$ is greatly reduced compared with the time constant $T$ for the RC-network.

**Figure 6.**

*Case B. $|e_1| \geq E_1.* When Eq. (11) is substituted into (5a)

$$\frac{d}{dt}e_1 = e_{41} - (e_1 + \frac{E_2}{e_1});$$  \hspace{1cm} (14)

and integrating gives

$$\frac{t + t_0}{T} = \int \frac{e_1 \, de_1}{e_1^2 - e_{41} e_1 + E_1 E_2}.$$  \hspace{1cm} (15)
Figure 7 shows the curve

\[ e_1 - D(e_1) = e_1 + \frac{E_1 E_2}{e_1} \]

The properties of the integral:

\[ I = \int \frac{x \, dx}{x^2 - 2ax + b^2} \quad (16) \]

where \( x = e_1 \), \( a = \frac{e_1}{2} \), and \( b^2 = E_1 E_2 \), are well known. For the reader's convenience, its derivation is carried out in the Appendix.

We will now consider the cases where \( b \neq 0 \). When \( a \neq b \), i.e., \( e_1 < 2\sqrt{E_1 E_2} \), we have the conditions represented in Fig. 4a. The solution of Eq. (16) is

\[ \frac{t + t_0}{T} = -\frac{1}{2} \ln \left[ (x - x_r)^2 + x_1^2 \right] + \frac{x}{x_1} \, \text{ctn}^{-1} \frac{x - x_r}{x_1} \quad (17) \]

where \( x = e_1 \), \( x_r = a \), and \( x_1 = \sqrt{b^2 - a^2} \).

Let the initial condition \( x = x_0 \) (or \( e_1 = e_{10} \)) correspond to \( t = 0 \). Then Eq. (17) may be rewritten as

\[ \frac{1}{T} = \frac{1}{2} \ln \left[ \frac{(x_0 - x_r)^2 + x_1^2}{(x - x_r)^2 + x_1^2} \right] + \frac{x}{x_1} \left( \text{ctn}^{-1} \frac{x - x_r}{x_1} - \text{ctn}^{-1} \frac{x_0 - x_r}{x_1} \right) \quad (18) \]
Both the $\text{ctn}^{-1}$ angles must be in the first or second quadrant. We have here $t = t(x)$.

Let us study $t(0)$ for $x = 0$.

\[
\frac{t(0)}{T} = \frac{1}{2} \ln \left( \frac{\left(\frac{x_0^2 - x_0^2}{b^2} + 1\right) + \frac{a}{\sqrt{b^2 - a^2}} \left[ \text{ctn}^{-1} \left( -\frac{a}{b} \right) - \text{ctn}^{-1} \left( \frac{x_0}{x_1} \right) \right]}{\sqrt{b^2 - a^2}} \right) \]

\[
\frac{1}{2} \ln \left[ 1 + \frac{x_0^2 - 2a}{b^2} \right] + \frac{a}{\sqrt{b^2 - a^2}} \left[ \text{ctn}^{-1} \left( -\frac{a}{b} \right) - \text{ctn}^{-1} \left( \frac{x_0}{x_1} \right) \right]}
\]

Thus the point $x = 0 \ (e_1 = 0)$ is reached in finite time. This can also be taken as a satisfactory value for the time $t = t_1$ for reaching the point $e_1 = E_1$ because the derivative

\[
\frac{de_1}{dt} = \frac{E_1 E_2}{e_1} \rightarrow \infty \text{ when } e_1 \rightarrow 0.
\]

(Also $e_2 \rightarrow \infty$ and from Eq. (3) $e_3 = e_4 - e_1 - e_2$).

Let us now combine this result with the approximation for $|e_1| < E_1$, Eq. (13a).

At the peak of the simplified discriminator curve $e_1 = E_1$ and the corresponding time $t = t_1$.

Thus we can solve for the constant $A$ and substitute it into Eq. (13a) as,

\[
E_1 = A x + e_{41}
\]

\[
-\frac{1 + G}{T} t_1 + e_{41} \frac{1}{1 + G}
\]

\[
E_1 = \frac{e_{41}}{1 + G}
\]

\[
-\frac{1 + G}{T} (t - t_1) + e_{41} \frac{1}{1 + G}
\]

The final value of $e_1$ is $e_{11} = \frac{e_{41}}{1 + G}$ and of $e_3$ is $e_{31} = e_{11} - e_{41} = \frac{e_{41}}{1 + \frac{1}{G}}$

$\sim e_{41}$ if $G \gg 1$, which is commonly the case in practical applications.

Figure 8 shows $e_1$ as a function of time in this case. Thus, after a time $t = t_1$

$\approx (t_0) \ (\text{Eq. (19)})$, $e_1$ has reached the value $e_1 = E_1$. From there on it decreases approxi-
mately exponentially towards its final value $e_{11}$ with a time-constant $\frac{T}{1 + G}$. If $e_{10}$

$\sim e_{12}$ there is an inflexion point between $t = 0$ and $t = t_1$. The other inflexion point

falls exactly at $t = t_1$.
An important application is the case when \( e_{40} + e_{41} = 0 \), i.e., asymmetrical square-wave voltage. In this case the initial value \( e_{10} \) of \( e_1 \) is

\[
e_{10} = -\frac{e_{40}}{1 + \frac{1}{G}} + e_{41} = \frac{2 + \frac{1}{G}}{1 + \frac{1}{G}} e_{41} \approx 2e_{41}, \text{ when } G \gg 1 \quad (23)
\]

or \( x_0 = 4a \) in (19). Thus,

\[
\frac{t(o)}{T} = \frac{1}{2} \ln \left[ 1 + \frac{8 a^2}{b^2} \right] + \frac{a}{\sqrt{b^2 - a^2}} \left[ \text{ctn}^{-1} \left( -\frac{a}{\sqrt{b^2 - a^2}} \right) - \text{ctn}^{-1} \left( \frac{3a}{\sqrt{b^2 - a^2}} \right) \right] \quad (24)
\]

Putting \( \frac{a}{b} = \alpha \)

\[
\frac{t(o)}{T} = \frac{1}{2} \ln \left[ 1 + 8\alpha^2 \right] + \frac{1}{\sqrt{\frac{1}{\alpha^2} - 1}} \left[ \text{ctn}^{-1} \left( -\frac{1}{\sqrt{\frac{1}{\alpha^2} - 1}} \right) - \text{ctn}^{-1} \left( \frac{3}{\sqrt{\frac{1}{\alpha^2} - 1}} \right) \right]
\]

(24a)

If \( \alpha \ll 1 \),

\[
\frac{t(o)}{T} \approx 4\alpha^2 + \alpha \left[ \text{ctn}^{-1} (-\alpha) - \text{ctn}^{-1} 3\alpha \right] \approx 4\alpha^2 + 4\alpha^2 = 8\alpha^2
\]

Thus

\[
t_1 \approx t(o) = 8\alpha^2 T.
\]

(25)

This approximation, however, is only valid when \( e_{10} \gg e_1 \), because, by definition, \( t_1 = 0 \)
for $e_{10} \leq E_1$. In this case

$$e_1 \psi (e_{10} - \frac{e_{41}}{1 + G}) \psi \left( \frac{1 + G}{t} + \frac{e_{41}}{1 + G} \right). \quad (26)$$

The value of $\alpha$ corresponding to $e_{10} = E_1$ in this case of symmetrical $e_4$ voltage is:

$$\alpha = \frac{e_{41}}{2} \cdot \frac{1}{\sqrt{E_1 E_2}} = \frac{1}{4G} \cdot \quad (27)$$

It is interesting to know which values correspond to $\alpha = 1$, i.e., the limiting value in this case:

$$e_{41} = 2\sqrt{E_1 E_2}$$

$$e_{11} = \frac{e_{41}}{1 + G} = \frac{2\sqrt{E_1 E_2}}{1 + \frac{E_2}{E_1}} \approx \frac{2E_1}{\sqrt{G}}$$

$$e_{10} \approx 2e_{41} \approx 4E_1 \sqrt{G}$$

$$\frac{e_{10}}{e_{11}} \approx 2G.$$

Let us now treat the on-off case when $e_{41}$ is such that $a < b$, but where the negative part, $e_{40}$, of the impressed square-wave voltage $e_4$ completely stops the oscillation of the klystron. This means that $e_{20} = e_{30} = 0$ at the end of the negative half period. Thus the initial value of the repeller voltage $e_{10} = e_{41} = 2a$ and from Eq. (19),

$$\int(a) = \frac{1}{2} \ln \left( 1 + \frac{a}{\sqrt{b^2 - a^2}} [\text{ctn}^{-1}(-\frac{a}{\sqrt{b^2 - a^2}}) - \text{ctn}^{-1}\frac{a}{\sqrt{b^2 - a^2}}] \right) = \frac{2}{a^2 - 1} \left( \frac{1}{\sqrt{\frac{1}{a^2} - 1}} \right)$$

where $\alpha = \frac{a}{b}$ as above. If $\alpha < 1$, but greater than $\frac{1}{2\sqrt{6}}$, Eq. (29) can be approximated as,
\[ \frac{t(0)}{T} \approx \frac{2}{\frac{1}{\alpha^2} - 1} \sim \frac{1}{\alpha^2} \tag{30} \]

and

\[ t_1 = t(\alpha) = 2\alpha^2 t \]

The case when \( \alpha < \frac{1}{2\sqrt{0}} \) (i.e., \( e_{10}(e_1) \)) leaves the shortest build-up time. Then \( t_1 = 0 \) and only the exponential portion of the \( e_1 \) curve is left. It is clearly seen that if \( e_{41} \) is exactly zero there will be no voltage across the output condenser \( C \) and therefore \( e_4 \) will follow \( e_4 \) instantaneously (with the assumption of instant response \( e_2 = D(e_1) \)).

When \( a > b \), we can have either the case shown in Fig. 4c where there is only one equilibrium point, or, the case shown in Fig. 4b where there are two stable and one unstable equilibrium points. If in the last case \( e_{10} \) has a value to the left of the unstable point, the left-hand stable point is attained, and if it has a value to the right then the right-hand stable point is attained. The right-hand stable equilibrium point (and the equilibrium point in Fig. 4c) gives no stabilization, because then \( \frac{da_{41}}{de_{41}} > 1 \). The mathematical expression for \( e_1 \) as a function of time using the approximation for the discriminator curve in Eq. (11) is:

\[ \frac{1}{T} = \frac{x + x}{2x_b} \ln \frac{x - x - x}{x - x - x} - \frac{x - x}{2x_b} \ln \frac{x - x + x}{x - x + x}, \tag{31} \]

where

\[ x = e_1, \quad x_a = \frac{e_{41}}{2}, \quad x_o = e_{10}, \quad x_b = \sqrt{\left(\frac{e_{41}}{2}\right)^2 - 1} \]

3. Experimental Results

Most experiments were carried out with the type of equipment described by Lawrance. The only change was the introduction of a square-wave voltage as shown in Fig. 9. The wavelength of the radio frequency was about 3.2 cm.
Figure 9. Schematic diagram of experimental circuit.

\[ R_1 = R_2 = 51 \text{k}\]
\[ C_2 = 0.5 \mu\text{f}\]
\[ C_1 = \text{Adjustable capacitor}\]
\[ V_4 = \text{Square-wave generator, G. M.}\]
\[ O = \text{Oscilloscope, Dumont Type 208}\]

Signals at the 60-cps line frequency were present in the first equipment used, and these signals caused considerable difficulty when photographs were taken of the circuit response voltages. The type of stabilizer developed by Zaffarano showed much less of this type of interference and was therefore used in place of the above. The square-wave voltage was introduced in exactly the same way and the only change made was to bypass the output cathode follower of the stabilizer. The shielded lead in Fig. 9 was connected to the plate of the phase detector tube (6AS6). The description of the results of the experimental part of the work can best be made with reference to the accompanying photographs. Information about voltages, etc., is given in the following table:

**TABLE I.**

<table>
<thead>
<tr>
<th>Figure No.</th>
<th>Voltage ( V_4 ) Peak-to-peak (volts)</th>
<th>Frequency of ( V_4 ) (cps)</th>
<th>( C_1 ) (( \mu\text{f} ))</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>16.0</td>
<td>500</td>
<td>0</td>
<td>medium</td>
</tr>
<tr>
<td>11</td>
<td>16.2</td>
<td>500</td>
<td>0</td>
<td>medium</td>
</tr>
<tr>
<td>12</td>
<td>1.6</td>
<td>500</td>
<td>0</td>
<td>medium</td>
</tr>
<tr>
<td>13</td>
<td>1.6</td>
<td>500</td>
<td>0.01</td>
<td>medium</td>
</tr>
<tr>
<td>14</td>
<td>0.2</td>
<td>500</td>
<td>0.01</td>
<td>medium</td>
</tr>
<tr>
<td>16</td>
<td>1.6</td>
<td>5000</td>
<td>0</td>
<td>low</td>
</tr>
<tr>
<td>17</td>
<td>1.6</td>
<td>5000</td>
<td>0</td>
<td>high</td>
</tr>
</tbody>
</table>
The time constant, $T$, is obtained by reducing the circuit in Fig. 9 to the corresponding RC-network in Fig. 1. With an internal plate resistance of the 6AS6 tube equal to one megohm and an external plate resistance of 120 k, $R = 52$ k and $C = \text{the sum of } C_1 \text{ and the internal capacitance of 1000 } \mu\text{f}$.

Thus,

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu\text{f}$</td>
<td>$\text{Hsec}$</td>
</tr>
<tr>
<td>0</td>
<td>52</td>
</tr>
<tr>
<td>0.01</td>
<td>5700</td>
</tr>
</tbody>
</table>

Figure 10. The symmetrical case where $V_4$ has been adjusted to give a rather long time $t_1$, i.e., less than, but nearly equal to $R$. The steep portion just before $e_1 = E_1$ is clearly seen, as well as the exponential part that follows the very steep portion.
Figure 11. The voltage $V_4$ has been slightly increased compared with the previous figure, in order to make $a > b$.

Figure 12. The voltage $V_4$ has been reduced to a tenth of the value in Fig. 10, and the time $t_1$ has been appreciably shortened.
Figure 13. In order to widen the peak, the time constant, $T$, has been made eleven times greater than in Fig. 12.

Figure 14. The voltage $V_4$ has been further reduced and the same time constant has been used as in Fig. 13. The double trace is a result of 60-cps interference.
Figure 15. The square-wave voltage $V_4$ of 500 cps for reference.

Figure 16. The frequency of $V_4$ has been increased to 5000 cps. The loop gain is rather low, and the curve shows very much the same properties as exhibited in Fig. 10.
Figure 17. Same as Fig. 16 but with higher loop gain. Here damped oscillations appear, an occurrence which cannot be explained by the previous theory. These oscillations seem to be due partly to overloading the amplifier, a condition which introduces limiting of the discriminator curve. Using the cathode-follower after the phase-detector tube gave still more oscillations.

Figure 18. The square-wave voltage \( V_4 \) of 5000 cps for reference.
APPENDIX

Derivation of the integral:

\[ I = \int \frac{x}{x^2 - 2ax + b^2} \, dx \]

The denominator has the roots

\[ x_{1,2} = a \pm \sqrt{a^2 - b^2} \]

\[ \frac{x}{x^2 - 2ax + b^2} = \frac{x}{(x-x_1)(x-x_2)} = \frac{x_1}{x_1-x_2} \cdot \frac{x_2}{x_2-x_1} = \frac{1}{x-x_1} - \frac{1}{x-x_2} \]

\[ I = \frac{1}{x_1-x_2} [x_1 \ln(x-x_1) - x_2 \ln(x-x_2)] \]

\[ = \frac{1}{2} \ln(x-x_1)(x-x_2) + \frac{1}{x_1-x_2} \left( \frac{x_1+x_2}{2} \right) \ln \frac{x-x_1}{x-x_2} \]

\[ a^2 < b^2 \]

\[ x_{1,2} = x_r \pm jx_1 = a \pm j\sqrt{b^2 - a^2} \]

Figure 19.
Referring to Fig. 19, we note that

\[ \frac{x - x_1}{x - x_2} = e^{j\varphi} \quad \frac{\ln x - x_1}{x - x_2} = j\varphi \]

\[ \varphi = -2 \cot^{-1} \frac{x - x_r}{x_1} \]

\[ I = \frac{1}{2} \ln[(x - x_r)^2 + x_1^2] - \frac{x_r}{x_1} \cot^{-1} \frac{x - x_r}{x_1} + \frac{\ln(x - x_0) + \frac{x_0}{x - x_0}}{a^2 = b^2} \]

\[ x_1 = x_2 = x_0 = a \]

\[ I = \ln(x - x_0) + \frac{x_0}{x - x_0} \]

\[ a > b^2 \]

\[ \begin{align*}
  x_1 &= x_a + x_b = a + \sqrt{a^2 - b^2} \\
  x_2 &= x_a - x_b = a - \sqrt{a^2 - b^2}
\end{align*} \]

\[ I = \frac{x_a + x_b}{2x_b} \ln \left( x - x_a - x_b \right) - \frac{x_a - x_b}{2x_b} \ln \left( x - x_a + x_b \right). \]

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References

