The topology of GKM spaces and GKM fibrations

by

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Abstract

This thesis primarily consists of results which can be used to simplify the computation of the equivariant cohomology of a GKM space. In particular we investigate the role that equivariant maps play in the computation of these cohomology rings.

In the first part of the thesis, we describe some implications of the existence of an equivariant map π between an equivariantly formal $T-$manifold M and a GKM space $\widetilde{M}$. In particular we generalize the Chang-Skjelbred Theorem to this setting and derive some of its consequences. Then we consider the abstract setting of GKM graphs and define a category of objects which we refer to as *GKM fiber bundles.* For this class of bundles we prove a graph theoretical version of the Serre-Leray theorem. As an example, we study the projection maps from complete flag varieties to partial flag varieties from this combinatorial perspective.

In the second part of the thesis we focus on GKM manifolds M which are also $T-$Hamiltonian manifolds. For these spaces, Guillemin and Zara ([GZ]), and Goldin and Tolman ([GT]), introduced a special basis for $H_T^*(M)$, associated to a particular choice of a generic component φ of the moment map, the elements of this basis being called canonical classes. Since, for Hamiltonian T spaces, $H_T(M)$ can be viewed as a subring of the equivariant cohomology ring of the fixed point set, it is important to be able to compute the restriction of the elements of this basis to the fixed point set, and we investigate how one can use the existence of an equivariant map to simplify this computation. We also derive conditions under which the formulas we get are integral. Using the above results, we are able to prove, inter alia, positive integral formulas for the equivariant Schubert classes on a complete flag variety of type A_n, B_n, C_n and D_n. (These formulas are new, except in type A_n). More generally, we obtain positive integral formulas for the equivariant Schubert classes using fibrations of the complete flag variety over partial flag varieties, and when this fibration is a $\mathbb{C}P^1-$bundle one gets from these formulas the calculus of divided difference operators.

[GT] Goldin, R. F. and Tolman, S., Towards generalizing Schubert calculus in the symplectic category, preprint.
[GZ] Guillemin V. and Zara C., Combinatorial formulas for products of Thom classes.

In *Geometry, mechanics, and dynamics*, pages 363-405, Springer NY, 2002.

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Chapter 1

Introduction

1.1 Equivariant cohomology

Let G be a compact Lie group, and M a topological space on which G has a continuous action. Let EG be a contractible space on which G is acting freely, $BG = EG/G$ the *classifying space* for G, and $EG \to BG$ the *classifying bundle*. Since the G action on EG is free, the diagonal action on $M \times EG$ is free as well.
By the Borel construction, the G equivariant cohomology $H^*_G(M)$ of M is defined to be the ordinary cohomology of the orbit space $(M \times EG)/G$, i.e.

$$H^*_G(M) = H^*((M \times EG)/G).$$

In particular, when G acts freely on M, it is easy to check from this definition that the equivariant cohomology of M coincides with the ordinary cohomology of M/G.

Observe that the equivariant cohomology of a point is particularly rich, since it coincides with the ordinary cohomology of the classifying space BG

$$H^*_G(pt) = H^*(EG/G) = H^*(BG).$$

For example, when G is a circle S^1, then ES^1 is the unit sphere inside $\mathbb{C}^\infty$, which we denote by S^∞; as a consequence BS^1 coincides with $\mathbb{C}P^\infty$, and

$$H^*_{S^1}(pt;\mathbb{C}) = H^*(\mathbb{C}P^\infty;\mathbb{C}) = \mathbb{C}[x] \ ,$$

where x is an element of degree 2, and can be thought as the curvature of the principal S^1-bundle $S^\infty \to \mathbb{C}P^\infty$. If G is a torus $T = (S^1)^d$, then $BT = (\mathbb{C}P^\infty)^d$ and

$$H^*_T(pt;\mathbb{C}) = \mathbb{C}[x_1,\ldots,x_d] \ .$$

One can also think of $\mathbb{C}[x_1,\ldots,x_d]$ as the symmetric algebra $\mathbb{S}(\mathfrak{t}^*)$ on $\mathfrak{t}^* = (Lie(T))^*$, where $\{x_1,\ldots,x_d\}$ denotes a basis of $\mathfrak{t}^*$.

The unique map $\pi : M \to pt$ induces a map in equivariant cohomology $\pi^* : H^*_G(pt) \to H^*_G(M)$ which gives $H^*_G(M)$ the structure of a $H^*_G(pt)-$module. In this thesis we will be particularly interested in spaces for which the equivariant cohomology ring is a *free* $H^*_G(pt)-$module.

The equivariant cohomology ring also recovers information about the ordinary cohomology ring. In fact, from the inclusion $\{e\} \hookrightarrow G$, where e denotes the identity element, one has a canonical restriction map

$$r : H^*_G(M) \to H^*(M) \ .$$

In the next sections of this chapter we will exhibit categories of spaces for which the map r is surjective, and the equivariant cohomology $H^*_G(M)$ is a free $H^*_G(pt)-$module.

From now on we will restrict our attention to the case in which G is a $d-$dimensional torus T.

If M is a manifold endowed with a smooth action $\rho : T \times M \to M$, there is another interesting description of the equivariant cohomology of M due to Cartan ([7]), called the *Cartan model* for the equivariant cohomology of M.
Let $\{\xi_1,\ldots,\xi_d\}$ be a basis of the Lie algebra $\mathfrak{t} = Lie(T)$, and let $\{x_1,\ldots,x_d\}$ denote

the dual basis in $\mathfrak{t}^*$. For every element ξ in $\mathfrak{t}$, consider the vector field $\xi^{\#}$ generated by ξ, i.e.

$$\xi^{\#}(p) = \frac{d}{dt}|_{t=0}(exp(t\xi) \cdot p)$$

If $\Omega^*(M)$ denotes the ring of differential forms on M, every vector ξ in $\mathfrak{t}$ induces two derivations on $\Omega^*(M)$, the Lie derivative

$$L_\xi = L_{\xi^{\#}} : \Omega^*(M) \to \Omega^*(M) \ ,$$

and the interior product

$$\iota_\xi = \iota_{\xi^{\#}} : \Omega^*(M) \to \Omega^{*-1}(M) \ .$$

Let $(\Omega(M), d)$ denote the complex of de Rham differential forms, and $(\Omega(M)^T, d)$ the subcomplex of $(\Omega(M), d)$ composed by the differential forms which are invariant under the Lie derivative of vector fields generated by the T action. Then the Cartan complex $(\Omega_T(M), d_T)$ is defined to be

$$\Omega_T(M) = \Omega(M)^T \otimes \mathbb{S}(\mathfrak{t}^*)$$

with differential

$$d_T(\alpha \otimes f) = d\alpha \otimes f + \sum_{j=1}^{d} \iota_j \alpha \otimes x_j f$$

where ι_j denotes ι_{ξ_j}.

This complex can be regarded as a double complex

$$\Omega_T^{p,q}(M) = \Omega^{q-p}(M)^T \otimes \mathbb{S}^p(\mathfrak{t}^*)$$

with (anti)commuting differentials $d = d \otimes 1$ and $\delta = \sum_{j=1}^d \iota_j \alpha \otimes x_j$. Hence the additive structure of $H_T^*(M)$ can be computed using the spectral sequence associated to this double complex (for further details, see [16], [6]).

Another way of computing the equivariant cohomology of M is by using the Leray-

Serre spectral sequence associated to the fibration

$$\begin{array}{ccc} M & \hookrightarrow & (M \times ET)/T \\ & & \downarrow \\ & & BT \end{array} \tag{1.1}$$

which is the topological analogue of the spectral sequence associated to the Cartan double complex.

1.1.1 Equivariant formality

The E_1 term of the spectral sequence associated to the Cartan complex, thought as a double complex, is given by

$$H(M) \otimes \mathbb{S}(\mathfrak{t}^*) \; .$$

More precisely $E_1^{p,q} = H^{q-p}(M) \otimes \mathbb{S}^p(\mathfrak{t}^*)$.

When this spectral sequence collapses at the E_1 stage, we will say that M is *equivariantly formal.* So as vector spaces, as well as $\mathbb{S}(\mathfrak{t}^*)-$modules, we have

$$H_T(M) \simeq H(M) \otimes \mathbb{S}(\mathfrak{t}^*).$$

Observe that in general this is not an isomorphism of rings. But since it is an isomorphism of $\mathbb{S}(\mathfrak{t}^*)-$modules, and since the term on the right hand side is a free $\mathbb{S}(\mathfrak{t}^*)-$module, this implies that whenever M is equivariantly formal $H_T(M)$ is a *free* $\mathbb{S}(\mathfrak{t}^*)-$module.

There are a number of inequivalent conditions that imply equivariant formality. For example Goresky, Kottwitz and MacPherson analyse this property in greater generality in [12]; one of their results can be stated as follows

Theorem 1.1.1. *If the ordinary homology groups of M are generated by classes which can be represented by $T-$invariant cycles, then M is equivariantly formal.*

Another characterization of equivariant formality can be given in terms of the canonical restriction map r from the equivariant cohomology ring to the ordinary cohomology ring of M.

Theorem 1.1.2. *M is equivariantly formal if and only if the canonical restriction map*

$$r : H_T(M) \to H(M)$$

is surjective.

Observe that if K is a closed subgroup of T and M is $T-$equivariantly formal, then by Theorem 1.1.2 M is $K-$equivariantly formal as well, since the restriction map $r : H_T(M) \to H(M)$ factors through $H_K(M)$.

When M is a symplectic manifold with symplectic form ω, Kirwan ([19]) and independently Ginzburg ([10]) proved the following

Theorem 1.1.3. *M is equivariantly formal if it is compact and it admits an equivariantly closed extension of the symplectic form ω.*

1.2 Localization in equivariant cohomology

Let M be a compact manifold acted on by a torus T, and let M^T be the fixed point set of the $T-$action. In this section we will review theorems that explore how much information about the equivariant cohomology ring of M can be recovered from the fixed point set data. In particular we will first recall a theorem of Borel ([5]) and Hsiang ([17]) that studies the kernel of the restriction map $H_T(M) \to H_T(M^T)$ (in the exposition we will follow [16]). Then we will recall a theorem which is due to Atiyah and Bott ([2]), and Berline and Vergne ([3]), which gives an explicit expression for the push-forward map in equivariant cohomology $\pi_* : H_T^*(M) \to H_T^*(pt)$.

Let's recall that if A is a finitely generated $\mathbb{S}(\mathfrak{t}^*)-$module, the annihilator ideal of A is given by

$$I_A = \{f \in \mathbb{S}(\mathfrak{t}^*),\ fA = 0\}\ ,$$

and the support of A is the algebraic variety in $\mathfrak{t} \otimes \mathbb{C}$ associated to I_A, i.e.

$$\text{supp}A = \{x \in \mathfrak{t} \otimes \mathbb{C},\ f(x) = 0 \text{ for all } f \in I_A\} .$$

From the definition, it is easy to see that if A is a *free* $\mathbb{S}(\mathfrak{t}^*)-$module then supp$A = \mathfrak{t} \otimes \mathbb{C}$; moreover A is a *torsion* module if and only if supp$A$ is a proper subset of $\mathfrak{t} \otimes \mathbb{C}$.

Observe that since M is compact, there are only a finite number of subgroups K of T occurring as isotropy groups of points of M; let $\mathfrak{k}$ denote $Lie(K)$, and $H_T(\cdot)_c$ the equivariant cohomology with compact supports. Then we have the following

Theorem 1.2.1. *Let M be a compact manifold acted on by a torus T, and let X be a closed invariant $T-$submanifold. Then the $\mathbb{S}(\mathfrak{t}^*)-$modules $H_T(M \backslash X)$ and $H_T(M \backslash X)_c$ have supports contained in the set*

$$\bigcup_K \mathfrak{k} \otimes \mathbb{C} \tag{1.2}$$

where the union is over all the subgroups K which occur as isotropy groups of points of $M \setminus X$.

An important consequence of this theorem is the following

Theorem 1.2.2. *Let M be a compact manifold acted on by a torus T, X a closed invariant $T-$submanifold, and $i_X : X \hookrightarrow M$ the inclusion. Then the kernel and cokernel of the map*

$$i^* : H_T(M) \to H_T(X)$$

are supported in the set (1.2).

Let's restrict to the case in which $X = M^T$. First of all observe that

$$H_T(M^T) = H(M^T) \otimes \mathbb{S}(\mathfrak{t}^*) ,$$

hence $H_T(M^T)$ is a *free* $\mathbb{S}(\mathfrak{t}^*)-$module. Combining this fact with Theorem 1.2.2 we obtain the *abstract localization theorem*

Theorem 1.2.3. *Let $i : M^T \hookrightarrow M$ denote the inclusion of the $T-$fixed point set into M. Then the kernel of the map*

$$i^* : H_T(M) \to H_T(M^T) \tag{1.3}$$

is the module of torsion elements in $H_T(M)$.

We recall that if M is equivariantly formal then $H_T(M)$ is a *free* $\mathbb{S}(\mathfrak{t}^*)-$module. So one of the important consequences of equivariant formality is the following

Theorem 1.2.4. *Let $i : M^T \hookrightarrow M$ denote the inclusion of the $T-$fixed point set into M, and suppose that M is equivariantly formal. Then the restriction map*

$$i^* : H_T(M) \to H_T(M^T) \tag{1.4}$$

is injective.

This allows one to regard $H_T(M)$ as a subring of $H_T(M^T)$, which is a much easier object to study. In particular, when M^T is discrete, $H_T(M^T)$ is simply the ring of maps from the fixed point set to $\mathbb{S}(\mathfrak{t}^*)$, $\mathrm{Maps}(M^T, \mathbb{S}(\mathfrak{t}^*))$.

Suppose now that M is oriented and T acts on M preserving the orientation. Observe that the fibration $\pi : (M \times ET)/T \to BT$ gives rise to a push-forward map in equivariant cohomology, $\pi_* : H_T(M) \to H_T(pt) = \mathbb{S}(\mathfrak{t}^*)$, which can be thought as the integration along the fiber of π.

In the Cartan complex $(\Omega(M)^T \otimes \mathbb{S}(\mathfrak{t}^*), d_T)$, there is a natural integration operation on the equivariant forms in $\Omega(M)^T \otimes \mathbb{S}(\mathfrak{t}^*)$.
More precisely $\int_M \alpha \otimes f = f \int_M \alpha$. This produces an integration operation in equivariant cohomology

$$\int_M : H_T(M) \to \mathbb{S}(\mathfrak{t}^*)$$

which coincides with the push-forward map π_* mentioned before.

Let F denote a connected component of the fixed point set M^T, $i_F : F \hookrightarrow M$ the inclusion map and $e(\nu_F)$ the equivariant Euler class of the normal bundle ν_F of F.

The next theorem gives an explicit expression for the map π_* in terms of the fixed point set data.

Theorem 1.2.5. ***(Localization formula).***
For any $\alpha \in H_T^*(M)$

$$\pi_*(\alpha) = \int_M \alpha = \sum_{F \subseteq M^T} \int_F \frac{i_F^* \alpha}{e(\nu_F)} \tag{1.5}$$

the sum being over all the connected components F of M^T.

This theorem is due to Atiyah and Bott ([2]) and Berline and Vergne ([3]), and we will refer to it as the ABBV Localization theorem.

Observe that when F is just a point $\{p\}$, then $e(\nu_{\{p\}})$ coincides with the product of the weights $\alpha_{1,p}, \ldots, \alpha_{n,p}$ of the isotropy representation of T on $T_p(M)$. Hence if M^T is discrete the Localization formula is particularly easy, and it is given by

$$\int_M \alpha = \sum_{p \in M^T} \frac{\alpha(p)}{\prod_i \alpha_{i,p}} \ , \tag{1.6}$$

where $\alpha(p)$ denotes $i_{\{p\}}^*(\alpha)$. Observe that this identity is a formal identity in which the left hand side is an element of $\mathbb{S}(\mathfrak{t}^*)$, whereas the right hand side is a sum of elements of the ring

$$\mathbb{S}(\mathfrak{t}^*)_0 = \left\{ \frac{g}{h}, \ g \in \mathbb{S}(\mathfrak{t}^*), h \in \mathbb{S}(\mathfrak{t}^*) \setminus \{0\} \right\}$$

1.2.1 The Chang-Skjelbred Theorem

Suppose that M is a T equivariantly formal compact manifold, hence $H_T(M)$ is a free $\mathbb{S}(\mathfrak{t}^*)-$module. As we observed before, this implies that the restriction map $i^* : H_T(M) \to H_T(M^T)$ is an injection. The theorem we are going to recall in this section, due to Chang and Skjelbred ([8]), describes precisely what the image of i^* is.

Let H be a subgroup of T of codimension one which occurs as an isotropy group. Observe that the inclusion $i : M^T \hookrightarrow M$ factors through the inclusion $i_H : M^T \hookrightarrow$

M^H, i.e.

$$\begin{array}{ccc} M^T & \xrightarrow{i} & M \\ i_H \searrow & & \nearrow \\ & M^H & \end{array}$$

This induces the following maps in equivariant cohomology

$$\begin{array}{ccc} H_T(M) & \xrightarrow{i^*} & H_T(M^T) \\ \searrow & & \nearrow i_H^* \\ & H_T(M^H) & \end{array}$$

So it is clear that $i^*(H_T(M)) \subseteq \bigcap_H i_H^*(H_T(M^H))$, the intersection being over all the codimension one subtori H of T which occur as isotropy groups. The Chang-Skjelbred Theorem asserts that the converse is also true, i.e.

Theorem 1.2.6. *The image of i^* is given by*

$$\bigcap_H i_H^*(H_T(M^H)) \tag{1.7}$$

where the intersection is taken over all the codimension one subtori H of T which occur as isotropy groups.

1.3 Hamiltonian actions

Let (M, ω) be a symplectic manifold, i.e. a manifold endowed with a closed non degenerate two form ω, with a (smooth) action of a torus T.

Then the action is said to be *Hamiltonian* if there exists a $T-$invariant map

$$\psi : M \to \mathfrak{t}^*$$

which satisfies

$$\iota_{\xi^\#}\omega = -d\psi^\xi \qquad \text{for all } \xi \in \mathfrak{t} \ ,$$

where $\xi^{\#}$ is the vector field generated by ξ and ψ^{ξ} is the ($T-$invariant) function on M given by $\psi^{\xi}(p) = \langle \psi(p), \xi \rangle$. When (M, ω) is a Hamiltonian manifold with moment map ψ, we will refer to it as the triple (M, ω, ψ).

Observe that every Hamiltonian manifold (M, ω, ψ) is naturally endowed with an *equivariantly closed two form*

$$\omega + \psi \in \Omega^2_T(M),$$

since the conditions that characterize the moment map are equivalent to saying that $\omega + \psi$ is in $\Omega^2_T(M)$ and is d_T-closed.
Conversely, if (M, ω) is a symplectic manifold with a smooth $T-$action, then this action is Hamiltonian if ω can be extended to be a d_T-closed two form in $\Omega_T(M)$.

A beautiful result about Hamiltonian manifolds concerns the geometry of the image of ψ, and is due to Guillemin and Sternberg ([16]), and independently Atiyah ([1]).

Theorem 1.3.1. ***(Atiyah, Guillemin-Sternberg)*** *Let (M, ω, ψ) be a symplectic manifold with a Hamiltonian action of a torus T. Then the image of the moment map $\psi(M)$ is a convex polytope. More precisely it is the convex hull of the image of the fixed points of the $T-$action, $\psi(M^T)$.*

The existence of a Hamiltonian action on a symplectic manifold has many important consequences.
For example the components ψ^{ξ} of the moment map are ($T-$invariant) perfect Morse-Bott functions, where the critical points are precisely the fixed points of the $T-$action. Hence one can use Morse theory to understand the (equivariant) topological invariants of M.

This approach was used by Kirwan ([19]) to prove that every compact symplectic manifold with a Hamiltonian action is equivariantly formal (cfr. Theorem 1.1.3). As a consequence of equivariant formality, if M^T denotes the fixed point set of the $T-$action, and $i : M^T \to M$ the inclusion, *Kirwan's injectivity theorem* can be stated as follows

Theorem 1.3.2. ***(Kirwan)*** *Let M be a compact symplectic manifold endowed with a Hamiltonian T action. Then the restriction map*

$$i^* : H_T(M) \to H_T(M^T)$$

induced by the inclusion $i : M^T \hookrightarrow M$ is injective.

This is the analogue in the symplectic category of Theorem 1.2.4.

Another important result is the *Kirwan's surjectivity theorem*.
Suppose that zero is a regular value for the moment map $\psi : M \to \mathfrak{t}^*$. Then the preimage of this value, $Z = \psi^{-1}(0)$, is a $T-$invariant submanifold of M, with a locally free $T-$action. In particular if the action is free, then the orbit space $M_{red} = Z/T$ is a manifold. Marsden, Weinstein and Meyer (cf. [20], [22]) studied the properties of M_{red}, which is also known as the *Marsden-Weinstein-Meyer quotient*, or simply the *reduced space*.

Theorem 1.3.3. *Let (M, ω, ψ) be a $T-$Hamiltonian manifold. Suppose that 0 is a regular value for ψ and that the action of T on $Z = \psi^{-1}(0)$ is free. Let $i_Z : Z \hookrightarrow M$ denote the inclusion, and $\pi_{red} : Z \to M_{red} = Z/T$ the projection map. Then M_{red} is a manifold and the projection $\pi_{red} : Z \to M_{red}$ is a principal $T-$bundle. Moreover M_{red} has a natural symplectic form ω_{red} satisfying $i_Z^*(\omega) = \pi_{red}^*(\omega_{red})$.*

Observe that since T acts freely on Z, then the $T-$equivariant cohomology of Z can be identified with the ordinary cohomology of the reduced space M_{red}. The second crucial result of Kirwan is the *Kirwan's surjectivity theorem.*

Theorem 1.3.4. ***Kirwan*** *Let (M, ω, ψ) be a $T-$Hamiltonian manifold, zero a regular value of the moment map ψ, and $Z = \psi^{-1}(0)$, with $i_Z : Z \hookrightarrow M$. Suppose moreover that T acts freely on Z. Then the map*

$$i_Z^* : H_T^*(M) \to H_T^*(Z) \simeq H^*(M_{red})$$

is surjective.

1.4 GKM spaces

Let M be a $2n-$dimensional compact manifold endowed with an effective smooth action of a torus T.

Definition 1.4.1. *We say that M is a **GKM manifold** if the following conditions are satisfied:*

i) *M^T is discrete*

ii) *$H_T(M)$ is a free $\mathbb{S}(\mathfrak{t}^*)-$module*

iii) *For every p in M^T the weights*

$$\alpha_{1,p}, \ldots, \alpha_{n,p} \tag{1.8}$$

of the isotropy representation of T on the tangent space at p, T_pM, are pairwise linearly independent.

We want to recall a condition that is equivalent to condition iii). For the exposition we will follow [16], Chapter 11.

Theorem 1.4.2. *Let M be a $2n-$dimensional compact manifold endowed with an effective smooth action of a torus T, satisfying properties i) and ii) mentioned above. Then iii) is equivalent to the following:*

iii)' For every codimension one subtorus H of T, the connected components of M^H are at most two dimensional.

We outline the proof of this theorem.

Proof. Since, by assumption i), $H_T(M)$ is a free $\mathbb{S}(\mathfrak{t}^*)-$module, by Theorem 11.6.1 [16] every connected component of M^H contains a $T-$fixed point; let X be a connected component of M^H, and let p be a $T-$fixed point in X. If we endow M with a $T-$invariant metric g, then the exponential map

$$exp : T_pM \to M$$

intertwines the isotropy action of T on T_pM with the action of T on M. So it is clear that

$$T_pX = (T_pM)^H \ .$$

Since H is a subtorus of codimension one, its Lie algebra is given by

$$Lie(H) = \{\xi \in \mathfrak{t} \text{ s.t. } \alpha(\xi) = 0\}$$

for some $\alpha \in \mathfrak{t}^*$. Hence $\dim X > 0$ if and only if α is one of the weights of the isotropy action of T on T_pM. In particular, if this happens, then $\dim X = 2$ if and only if these weights are pairwise linearly independent. □

Now we want to give a description of these two dimensional components of M^H, where H is a codimension one subtorus of T. More precisely we want to describe their equivariant cohomology ring; then, as a consequence of the Chang-Skjelbred Theorem (cfr. section 1.2.1), we will be able to describe the equivariant cohomology of the whole manifold M. This is the famous result presented by Goresky-Kottwitz-MacPherson in [12], concerning the equivariant cohomology ring of a GKM space.

Let X be a two-dimensional connected component of M^H, where H is a subtorus of codimension one, and let $p \in X$ be a fixed point of the $T-$action. Observe that X is a compact oriented submanifold of M; moreover it is easy to see that the fixed point set X^T is discrete.

Theorem 1.4.3. *X is diffeomorphic to a two dimensional sphere S^2, and the diffeomorphism conjugates the action of T/H with the standard S^1 action on S^2, given by the rotation about an axis.*

Proof. First of all observe that X is acted effectively by the circle T/H. In order to prove that X is a two dimensional sphere S^2, it is sufficient to prove that it has positive Euler characteristic. But this follows from the fact that if ξ is a non zero vector in $Lie(T/H)$, then the corresponding vector field $\xi^\#$ on X has index one at every fixed point q in X^T.

Now endow X with a $T-$invariant metric. By the Korn-Lichtenstein Theorem (cfr.[9]) this metric is conformally equivalent to the standard metric on S^2. Hence the diffeomorphism between X and S^2 intertwines the T/H action on X with the action of a one dimensional compact connected subgroup of $SL(2,\mathbb{C})$, the group of conformal transformation on S^2. But these subgroups are all conjugate to each other. So, up to conjugation, the T/H action on X can be thought as the standard S^1 action on S^2, which is the rotation about an axis. $\square$

The standard action of S^1 on S^2 has two fixed points, the "north pole" N and "south pole" S.

Observe that if the Lie algebra of H is given by Ker(α), and $\xi \in \mathbb{Z}_T$ is a primitive element in the group lattice of T such that $\alpha(\xi) \neq 0$, then $\alpha(\xi)$ can be thought as the speed at which S^1 is rotating the sphere S^2.

Now we want to study the equivariant cohomology ring $H_T(X)$ of X.

First of all observe that X is equivariantly formal. In fact the spectral sequence associated to the Cartan complex $\Omega_T(X)$ collapses at the E_1 stage, since the cohomology of X is non zero only in dimension zero and two. Hence in particular, if N and S denote the fixed points of the $T-$action, we have that the restriction map $i^* : H^k_T(X) \to H^k_T(\{N,S\}) = \mathbb{S}^k(\mathfrak{t}^*) \oplus \mathbb{S}^k(\mathfrak{t}^*)$ is injective. Hence we can view $H_T(X)$ as a subring of $\mathbb{S}^k(\mathfrak{t}^*) \oplus \mathbb{S}^k(\mathfrak{t}^*)$.

Let H be the codimension one subtorus stabilizing X, with Lie algebra given by $\mathfrak{h}$ =Ker(α), and let $r_H : \mathbb{S}(\mathfrak{t}^*) \to \mathbb{S}(\mathfrak{h}^*)$ be the restriction map induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{t}$. The next theorem studies the image of i^*.

Theorem 1.4.4. *An element $(f,g) \in \mathbb{S}^k(\mathfrak{t}^*) \oplus \mathbb{S}^k(\mathfrak{t}^*)$ is in the image of i^* if and only if*

$$r_H(f) = r_H(g) \tag{1.9}$$

Observe that condition (1.9) is equivalent to say that

$$f - g = \alpha P, \text{ for some } P \in \mathbb{S}^{k-1}(\mathfrak{t}^*) \tag{1.10}$$

Proof. First of all notice that the following diagram

$$\begin{array}{ccc} H_T^k(X) & \longrightarrow & H_H^k(X) \\ \downarrow & & \downarrow \\ H_T^k(\{N,S\}) & \longrightarrow & H_H^k(\{N,S\}) \end{array}$$

commutes, where the horizontal arrows are obtained by restricting the action of the group T to H, and the vertical arrows are induced by the inclusion $\{N,S\} \hookrightarrow X$.

Since H acts trivially on X,

$$H_H^k(X) = H^0(X) \otimes \mathbb{S}^k(\mathfrak{h}^*) \oplus H^2(X) \otimes \mathbb{S}^{k-1}(\mathfrak{h}^*) \; .$$

Hence an element $\widetilde{\omega}$ in $H_T^k(X)$ is mapped into an element of $H_H^k(X)$ of the form

$$1 \otimes P_1 + \omega \otimes P_2,$$

where $P_1 \in \mathbb{S}^k(\mathfrak{h}^*)$, $P_2 \in \mathbb{S}^{k-1}(\mathfrak{h}^*)$ and $\omega \in H^2(X)$. So condition (1.9) follows from the commutativity of the diagram, since

$$r_H(f) = r_H(g) = P_1$$

Hence the subring of $\mathbb{S}^k(\mathfrak{t}^*) \oplus \mathbb{S}^k(\mathfrak{t}^*)$ given by the pairs (f,g) such that $r_H(f) = r_H(g)$ contains $i^*(H_T^k(X))$. In order to prove that the converse is also true, it is sufficient to consider the dimension of these rings. In fact $\dim i^*(H_T^k(X)) = \dim H_T^k(X) = \dim(H^0(X) \otimes \mathbb{S}^k(\mathfrak{t}^*) \oplus H^2(X) \otimes \mathbb{S}^{k-1}(\mathfrak{t}^*)) = \dim \mathbb{S}^k(\mathfrak{t}^*) + \dim \mathbb{S}^{k-1}(\mathfrak{t}^*)$.
If we replace condition (1.9) with condition (1.10), it is clear that the subring of $\mathbb{S}^k(\mathfrak{t}^*) \oplus \mathbb{S}^k(\mathfrak{t}^*)$ satisfying (1.10) has dimension $\dim \mathbb{S}^k(\mathfrak{t}^*) + \dim \mathbb{S}^{k-1}(\mathfrak{t}^*)$. So the conclusion follows. □

Observe that if $\widetilde{\omega}$ is an element of $H_T^k(X)$, and $i^*(\widetilde{\omega}) = (f,g)$, then condition (1.10) is also a consequence of the ABBV Theorem (cfr. Theorem 1.2.5). In fact if α is the weight of the isotropy representation at N, hence $-\alpha$ the weight of the isotropy representation at S, then the ABBV Localization theorem gives

$$\int_X \widetilde{\omega} = \frac{\widetilde{\omega}(N)}{\alpha} + \frac{\widetilde{\omega}(S)}{-\alpha} = \frac{f-g}{\alpha} \ \in \mathbb{S}^{k-1}(\mathfrak{t}^*) \ .$$

Let M be a GKM manifold. Then by definition, the equivariant cohomology ring of M is a free $\mathbb{S}(\mathfrak{t}^*)-$module. So Theorem 1.2.3 implies that the restriction map

$$i^* : H_T(M) \to H_T(M^T) = Maps(M^T, \mathbb{S}(\mathfrak{t}^*))$$

is injective.

Let $X_1, \ldots, X_N$ be the collection of embedded spheres that arise as fixed points of codimension one subtori of T. Let H_i be the stabilizer of X_i, and $\mathrm{Ker}(\alpha_i)$ its Lie algebra, for some $\alpha_i \in \mathfrak{t}^*$. Now we are ready to prove the celebrated theorem by Goresky-Kottwitz-MacPherson, presented in [12] in greater generality.

Theorem 1.4.5. *Let M be a GKM manifold, and let $i^* : H_T(M) \to H_T(M^T) = Maps(M^T, \mathbb{S}(\mathfrak{t}^*))$ be the restriction map to the fixed point set. Then an element $P \in Maps(M^T, \mathbb{S}(\mathfrak{t}^*))$ belongs to $i^*(H_T(M))$ if and only if*

$$P(p_1) - P(p_2) = \alpha_i Q, \ \textit{for some } Q \in \mathbb{S}(\mathfrak{t}^*)$$

for every pair of fixed points p_1 and p_2 such that $\{p_1, p_2\} \cap X_i = X_i^T$, where X_i is one of the embedded sphere defined before.

Proof. It is sufficient to combine the Chang-Skjelbred theorem (cfr. Section 1.2.1) and Theorem 1.4.4. ☐

1.4.1 The GKM graph of M

The information about the embedded two spheres $X_1, \ldots, X_N$, as well as the equivariant cohomology of the GKM manifold M, can be encoded in a graph, called the

GKM graph $\Gamma = (V_\Gamma, E_\Gamma)$ associated to M, which is defined as follows.

- The set of *vertices* V_Γ is given by the fixed point set M^T.
- There exists a *directed edge* e from p to q, where $(p,q) \in V_\Gamma \times V_\Gamma$, if and only if there exists an embedded sphere X_i, for some $i = 1, \ldots, N$, such that $\{p,q\} \cap X_i = X_i^T$.

Observe that if $e = (p,q)$ is an edge in E_Γ, then also $\bar{e} = (q,p)$ is an edge in E_Γ. For every edge $e = (p,q)$, we will refer to p (resp. q) as the *initial point* $i(e)$ (resp. *terminal point* $t(e)$) of e. Let $X_e = X_{\bar{e}}$ be the sphere corresponding to the edges e and $\bar{e}$.

In order to encode the information about the action of T on M, we can assign to each directed edge e in E_Γ the weight of the isotropy representation of T on $T_{t(e)}X_e$. On the graph Γ this assignment defines an *axial function*, i.e. a map

$$\alpha : E_\Gamma \to \mathfrak{t}^*$$

which clearly satisfies $\alpha(e) = -\alpha(\bar{e})$.

Now we want to define a *connection* ∇_e along an edge $e \in E_\Gamma$. Observe that the restriction of the tangent bundle TM to X_e splits equivariantly as a sum of line bundles $\mathbb{L}_i$, $i = 1, \ldots, n$

$$TM_{|X_e} = \bigoplus_{i=1}^{n} \mathbb{L}_i$$

Geometrically, a connection ∇_e along e is a bijection between the (one dimensional) complex spaces $(\mathbb{L}_i)_p$ and $(\mathbb{L}_i)_q$, where the $(\mathbb{L}_i)_p$ coincides with the tangent space at p of the sphere X_{e_i}, where $i(e_i) = p$. We can define the connection combinatorially in the following way. Let $e = (p,q)$ be a directed edge in E_Γ, and define E_p (resp. E_q) to be the subset of E_Γ composed by edges e' such that $i(e') = p$ (resp. $i(e') = q$). Then a *connection along* e is a bijection $\nabla_e : E_p \to E_q$. A *connection* ∇ on Γ is a family of connections $\nabla = \{\nabla_e\}_{e \in E_\Gamma}$ such that $\nabla_e = \nabla_{\bar{e}}^{-1}$.

Observe that, if H denotes the stabilizer of the sphere X_e, then the isotropy representation of H on the normal bundle of X_e doesn't depend on the point $x \in X_e$.

In particular this implies that the weights of the isotropy action of T at $p = i(e)$ are equal to the weights of the isotropy action of T at $q = t(e)$ modulo $\alpha(e)$.

We will say that the axial function α is *compatible* with the connection ∇ if for every edge e' in E_p we have

$$\alpha(\nabla_e(e')) - \alpha(e') = c\alpha(e)$$

for some constant c. These constants are in this case integers. In order to see this, observe that if $c(\mathbb{L}_i)$ is the Chern class of the bundle $\mathbb{L}_i$, then $c(\mathbb{L}_i)(i(e)) = \alpha(e_i)$ and $c(\mathbb{L}_i)(t(e)) = \alpha(\nabla_e(e_i))$ for all the edges $e_i \in E_p$. Hence, integrating $c(\mathbb{L}_i)$ on X_e one gets from the ABBV Localization theorem

$$c_i = \frac{\alpha(\nabla_e(e_i)) - \alpha(e_i)}{\alpha(e)}$$

So this constant coincides with the Chern number of $\mathbb{L}_i$, and hence it is an integer.

Given a GKM graph Γ together with an axial function α, we can define the cohomology ring $H^*_\alpha(\Gamma)$ of the pair (Γ, α). Let f be an element of $Maps(V_\Gamma, \mathbb{S}(\mathfrak{t}^*))$. Then f is an element of $H^*_\alpha(\Gamma)$ if and only if for every edge $e = (p, q)$, $f(p)$ and $f(q)$ have the same image in $\mathbb{S}(\mathfrak{t}^*)/\alpha(e)\mathbb{S}(\mathfrak{t}^*)$. Then Theorem 1.4.5 can be rephrased in the following way.

Theorem 1.4.6. *Given a GKM manifold M, let $H^*_T(M)$ be the equivariant cohomology ring of M. Let Γ be the GKM graph associated to M, α the associated axial function on E_Γ, and $H^*_\alpha(\Gamma)$ the cohomology ring of (Γ, α). Then as rings, as well as $\mathbb{S}(\mathfrak{t}^*)-$modules, we have*

$$H^*_T(M) \simeq H^*_\alpha(\Gamma) \tag{1.11}$$

1.4.2 Examples

Example 1.4.7 *The complex projective space $\mathbb{C}P^n$*

Let $G = SU(n+1)$, with Lie algebra $\mathfrak{g}$, and let T be the torus of diagonal matrices in

G with Lie algebra $\mathfrak{t}$. Let $\{x_i\}_{i=1}^{n+1}$ be a basis of $(\mathbb{R}^{n+1})^*$ such that $x_i(\xi_1,\ldots,\xi_{n+1}) = \xi_i$. We can identify the dual of the Lie algebra of T, $\mathfrak{t}^*$, with the subset of $(\mathbb{R}^{n+1})^*$ given by $\{\sum_{j=1}^{n+1} \mu_j x_j \quad \text{s.t.} \quad \sum_{j=1}^{n+1} \mu_j = 0\}$. We choose as a basis of $\mathfrak{t}^*$ the vectors $\alpha_j = x_j - x_{j+1}$, for all $j = 1,\ldots,n$. Define $\underline{\mu}$ to be the vector in $\mathbb{R}^{n+1}$, $\underline{\mu} = (\mu_1,\ldots,\mu_{n+1})$, such that $\mu_1 < \mu_2 = \mu_3 = \ldots = \mu_{n+1}$ and $\sum_{j=1}^{n+1} \mu_j = 0$. Let p be the point in $\mathfrak{t}^*$ given by $p = \sum_{j=1}^{n+1} \mu_j x_j$. Consider now the G coadjoint orbit through p, $\mathcal{O}_p = G \cdot p \subset \mathfrak{g}^*$. This orbit is isomorphic to G/P_p, where P_p is the stabilizer of p, which is given by $S(U(1) \times U(n))$. This space is naturally isomorphic to the complex projective space $\mathbb{C}P^n$, with symplectic form inherited by its coadjoint orbit structure. Moreover T acts on $\mathbb{C}P^n$, and the $T-$fixed points are given by

$$(\mathbb{C}P^n)^T = (G/P_p)^T = \{\sum_{j=1}^{n+1} \mu_j x_{\sigma(j)},\ \sigma \in \mathcal{S}_{n+1}\} \subset \mathfrak{t}^*$$

Observe that there are only $n+1$ distinct fixed points $p_1,\ldots,p_{n+1}$, given by $p_{\sigma(1)} = \mu_1 x_{\sigma(1)} + \mu_2(\sum_{i=2}^{n+1} x_{\sigma(i)})$, $\sigma \in \mathcal{S}_{n+1}$, since all the permutation σ in $\mathcal{S}_{n+1}$ such that $\sigma(1) = j$ are determining the same point p_j.

It is easy to check that the weights of the isotropy action of T at p_i are $\alpha_j = x_j - x_i$, for all $j \in \{1,2,\ldots,n\} \setminus \{i\}$; hence they are pairwise linearly independent.

It is well known that the action of T on $G \cdot p$ is Hamiltonian, with moment map given by $\psi : G \cdot p \hookrightarrow \mathfrak{g}^* \to \mathfrak{t}^*$, where the first arrow denotes the inclusion of $G \cdot p$ in $\mathfrak{g}^*$, and the second arrow is given by the projection of $\mathfrak{g}^*$ onto $\mathfrak{t}^*$. In particular the moment map restricted to the fixed points set $(G/P_p)^T$ is just given by the inclusion.

Since the action is Hamiltonian, by Theorem 1.1.3 this manifold is equivariantly formal; in particular $H_T(\mathbb{C}P^n)$ is a free $\mathbb{S}(\mathfrak{t}^*)-$module. Hence we can conclude that $G \cdot p \simeq \mathbb{C}P^n$, with the torus action described above, is a GKM manifold. In what follows we describe its GKM graph $\Gamma = (V_\Gamma, E_\Gamma)$ and the associated axial function.

- The set of vertices is composed by $n+1$ elements $p_1,\ldots,p_{n+1}$
- There exists a directed edge e between any two vertices p_i and p_j (with $i \neq j$); this edge corresponds to the sphere stabilized by the subtorus with Lie algebra

$\mathrm{Ker}(x_i - x_j)$.

- If (p_i, p_j) denotes the directed edge from p_i to p_j, with $i \neq j$, then $\alpha(p_i, p_j) = x_i - x_j$

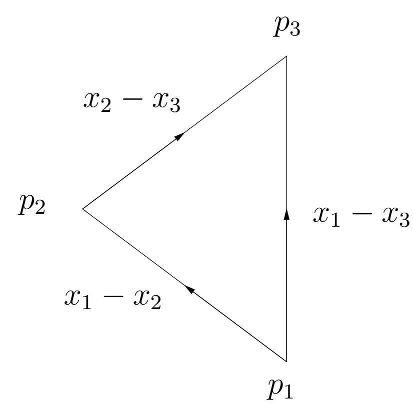


Figure 1-1: The GKM graph associated to $\mathbb{C}P^2$

In the above figure, we represent the GKM graph associated to $\mathbb{C}P^2$ and the associated axial function. Observe that for every pair of vertices p_i, p_j, with $i \neq j$, we are showing just one of the directed edges connecting p_i to p_j, the one from p_i to p_j, with $i < j$.

Example 1.4.8 *The variety of complete flags in* $\mathbb{C}^{n+1}$, $\mathcal{F}l(\mathbb{C}^{n+1})$
Let $\underline{\mu} = (\mu_1, \ldots, \mu_{n+1})$ be a generic vector in $\mathbb{R}^{n+1}$ such that $\mu_1 < \mu_2 < \ldots < \mu_{n+1}$ and $\sum_{j=1}^{n+1} \mu_j = 0$. With the same notation as in the previous example, the G coadjoint orbit through the point $p = \sum_{j=1}^{n+1} \mu_j x_j$, $G \cdot p$, is diffeomorphic to the variety of complete flags in $\mathbb{C}^{n+1}$, $\mathcal{F}l(\mathbb{C}^{n+1})$. The $T-$fixed points are in bijection with the elements of $\mathcal{S}_{n+1}$, and they are given by

$$(\mathcal{F}l(\mathbb{C}^{n+1}))^T = (G \cdot p)^T = \{\sum_{j=1}^{n+1} \mu_j x_{\sigma(j)},\ \sigma \in \mathcal{S}_{n+1}\}$$

Just like before, the symplectic structure is inherited by the structure of coadjoint orbit, and the action of T is Hamiltonian, with moment map $\psi : G \cdot p \hookrightarrow \mathfrak{g}^* \to \mathfrak{t}^*$. Moreover, if we identify the point $\sum_{j=1}^{n+1} \mu_j x_{\sigma(j)}$ with the permutation $\sigma = \sigma(1)\ldots\sigma(n+1)$ (written in the one line notation), then the weights of the isotropy action at σ are given by $\text{sign}(\sigma^{-1}(k) - \sigma^{-1}(h))(x_h - x_k)$, for all the subsets $\{h,k\}$ of $\{1,\ldots,n+1\}$. So $\mathcal{F}l(\mathbb{C}^{n+1})$ is a GKM manifold with respect to the $T-$action described above. The GKM graph $\Gamma = (V_\Gamma, E_\Gamma)$ and the axial function are described below.

- The vertices are in bijection with the elements of the permutation group $\mathcal{S}_{n+1}$. More precisely, the permutation $\sigma = \sigma(1)\ldots\sigma(n+1)$ represents the point $\mu_1 x_{\sigma(1)} + \mu_2 x_{\sigma(2)} + \ldots + \mu_{n+1} x_{\sigma(n+1)}$.

- Two vertices $\sigma, \sigma' \in \mathcal{S}_{n+1}$ are joined by an edge e if and only if σ and σ' differ by a transposition, i.e. $\sigma' = (h,k)\sigma$.

- The axial function is given by $\alpha(\sigma,\sigma') =\text{sign}(\sigma^{-1}(k) - \sigma^{-1}(h))(x_h - x_k)$.

We recall that the multiplication to the left $(h,k)*$ is swapping the values h and k in the one line notation of σ, and the right multiplication $*(i,j)$ is acting on the position i and j, i.e. $\sigma' = \sigma(i,j)$ if and only if $\sigma' = (\sigma(i),\sigma(j))\sigma$.

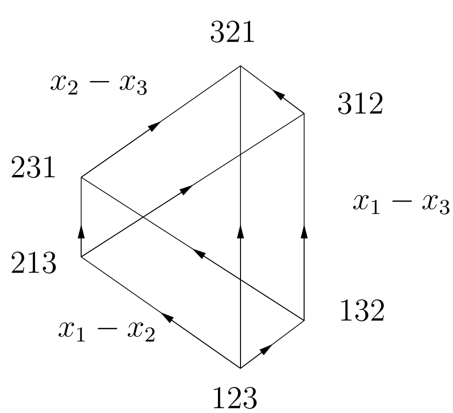


Figure 1-2: The GKM graph associated to $\mathcal{F}l(\mathbb{C}^3)$

In the above figure, parallel edges are labelled with the same axial function.

1.5 Results

In the second chapter, the first result is a generalization of Theorem 1.4.5 to spaces M which are $T-$equivariantly formal manifolds, and admit a $T-$equivariant fiber bundle $\pi : M \to \widetilde{M}$ over a GKM space $\widetilde{M}$ (cfr. Theorem 2.1.6). In particular this theorem asserts that we can compute the equivariant cohomology of M in terms of the equivariant cohomology of the base, and the equivariant cohomology of the fibers over $\widetilde{M}^T$. Then we consider the abstract setting of GKM graphs. We define the concept of "GKM fiber bundle" between two abstract GKM graphs (Γ, α) and (B, α_B), and derive the combinatorial implications of the existence of such a map at the level of their GKM graphs. In particular we prove a combinatorial analogue of Theorem 2.1.6 (cfr. Theorem 2.4.2), which can be interpreted as a discrete version of the Serre-Leray theorem. Moreover we define the concept of "holonomy group" and "invariant class" associated to a GKM fiber bundle.

In the third chapter we analyse in detail the example which was the source of inspiration for the results above: the complete flag variety $G_{\mathbb{C}}/B$ fibering over partial flag varieties $G_{\mathbb{C}}/P$. First we recall the structure of the associated GKM graphs of these spaces; then we define these projection maps combinatorially, and prove that they are GKM fiber bundles. Then we apply all the results on equivariant cohomology from the previous chapter to these examples. In particular we study the holonomy group of these bundles, and prove its connection with the subgroup of the Weyl group of G acting on the fibers. Then, by applying multiple times Theorem 2.4.2, we are able produce equivariant cohomology classes on $G_{\mathbb{C}}/B$ which are invariant under the action of the Weyl group, where $G_{\mathbb{C}}/B$ is a complete flag variety of type A_n, B_n, C_n and D_n.

In the fourth chapter we work with T Hamiltonian manifolds M which are also GKM spaces. To a generic component of the moment map φ, one can associate a collection of equivariant cohomology classes α_p, where $p \in M^T$, and show they form a basis of the equivariant cohomology of M as a module over $\mathbb{S}(\mathfrak{t}^*)$. We refer to these elements as the canonical classes associated to φ (cfr. [15], [11]). These classes do not always exist, but when they exist they are unique. Since the equivariant cohomology of M can be regarded as a subring of the equivariant cohomology of the fixed point set M^T, which is discrete in this case, it is important to be able to compute the restriction of the canonical classes to M^T. We first prove a Theorem (cfr. Theorem 4.2.3), which is a generalization of a result in [11], which allows us to compute these restrictions using different equivariantly closed two forms of M; in particular this results recover the result in [11], which only uses the equivariant symplectic form of M. Then, if $\pi : (M, \omega) \to (\widetilde{M}, \widetilde{\omega})$ is a $T-$equivariant map between $T-$Hamiltonian manifolds which are also GKM spaces, we define a category of maps, called "weight preserving maps" (as a very particular case, GKM fiber bundles are weight preserving maps). For this type of map we prove that we can use the pre-symplectic form $\pi^*(\widetilde{\omega})$ to compute these restrictions. Moreover we derive a formula that computes inductively the restriction of the canonical classes on M in terms of the canonical classes on the fibers over the fixed point set of $\widetilde{M}$. Finally, we investigate how the integrality of these formulas is related to the cohomology ring of $\widetilde{M}$.

In the fifth chapter we apply the results in the previous chapter to the case in which M is a complete flag variety. In this case, canonical classes exist and correspond to equivariant Schubert classes. Combinatorial formulas which compute the restriction of these classes to the fixed points of the T action have already been studied in the combinatorics literature (cfr. [4]). The beauty of these formulas is that they are manifestly positive and integral. In this chapter we use the structure of weight preserving maps to produce formulas that are positive and integral in the case in which M is a complete flag variety of type A_n, B_n, C_n and D_n. These formulas are not equivalent to the one found in [4], except in type A_n (cfr. [25], where the author proves the same formulas in type A_n and C_n using combinatorial tools, and the equivalence with Billey's formula in type A_n). We also prove a general positive integral formula for canonical classes on generic flag varieties, which implies as a particular case the divided difference operator identities.

The second and third chapters contain results from a joint work with V. Guillemin and C. Zara (cfr. [14]), and the fourth and fifth chapters contain results from a joint work with S. Tolman (cfr. [23]).

Chapter 2

GKM fiber bundles

Let M and $\widetilde{M}$ be $T-$equivariantly formal manifolds; suppose moreover that $\widetilde{M}$ is GKM. Then the existence of a $T-$equivariant fiber bundle $\pi : M \to \widetilde{M}$ gives information about the equivariant cohomology of M in terms of the equivariant cohomology of the base $\widetilde{M}$ and the equivariant cohomology of the fibers over the fixed point set of $\widetilde{M}$.

This is the content of the first section, in which we derive a generalization of the Chang-Skjelbred theorem, which gives as a consequence a description of the equivariant cohomology of M in terms of the GKM graph of $\widetilde{M}$ and the equivariant cohomology of the fibers over the $T-$fixed point set of $\widetilde{M}$.

In the other sections we derive the graph theoretical implications of the existence of a "GKM fiber bundle" between two abstract GKM graphs. In particular we prove a discrete version of the Serre-Leray theorem.

2.1 The Chang-Skjelbred Theorem for fiber bundles

Let $T = T^n$, M and $\widetilde{M}$ be $T-$manifolds and $\pi : M \to \widetilde{M}$ a $T-$equivariant fiber bundle. Suppose that M is $T-$equivariantly formal and $\widetilde{M}$ is GKM. Let $K_i,\ i = 1, \ldots, N$ be the codimension one isotropy groups of $\widetilde{M}$ and let $\mathfrak{k}_i$ be the Lie algebra of K_i.

Lemma 2.1.1. *If K is an isotropy group of $\widetilde{M}$, then*

$$Lie(K) = \cap \mathfrak{k}_{i_r}$$

for some multi-index $1 \leq i_1 < \ldots < i_m \leq N$.

For K a subgroup of T let $X^K = \pi^{-1}(\widetilde{M}^K)$, where $\widetilde{M}^K \subset \widetilde{M}$ denotes the set of points in $\widetilde{M}$ fixed by K. We recall ([16, Section 11.3]) that if A is a finitely generated $\mathbb{S}(\mathfrak{t}^*)-$module, then the annihilator ideal of A, I_A is defined to be

$$I_A = \{f \in \mathbb{S}(\mathfrak{t}^*),\ fA = 0\}\ ,$$

and the support of A is the algebraic variety in $\mathfrak{t} \otimes \mathbb{C}$ associated with this ideal, i.e.

$$\mathrm{supp} A = \{x \in \mathfrak{t} \otimes \mathbb{C}, f(x) = 0 \text{ for all } f \in I_A\}\ .$$

Then from the lemma and [16] Theorem 11.4.1 one gets the following.

Theorem 2.1.2. *The $\mathbb{S}(\mathfrak{t}^*)-$modules $H_T^*(M \setminus X^T)$ and $H_T^*(M \setminus X^T)_c$ are supported on the set*

$$\bigcup_{i=1}^{N} \mathfrak{k}_i \otimes \mathbb{C} \tag{2.1}$$

where $H_T^*(\ \cdot\ )_c$ denotes the equivariant cohomology with compact supports.

By [16, Section 11.3] there is an exact sequence

$$H_T^k(M \setminus X^T)_c \longrightarrow H_T^k(M) \xrightarrow{i^*} H_T^k(X^T) \longrightarrow H_T^{k+1}(M \setminus X^T)_c \tag{2.2}$$

Therefore since $H_T^*(M)$ is a free $\mathbb{S}(\mathfrak{t}^*)-$module Theorem 2.1.2 implies the following theorem.

Theorem 2.1.3. *i^* is injective and coker(i^*) is supported on $\bigcup_{i=1}^{N} \mathfrak{k}_i \otimes \mathbb{C}$*

As a consequence we get the following corollary.

Corollary 2.1.4. *If e is an element of $H_T^*(X^T)$, there exist non-zero weights $\alpha_1, \ldots, \alpha_r$ s.t. $\alpha_i = 0$ on some $\mathfrak{k}_j$ and*

$$\alpha_1 \cdots \alpha_r e \in i^*(H_T^*(M)) \tag{2.3}$$

The next theorem is a fiber bundle version of the Chang-Skjelbred theorem.

Theorem 2.1.5. *The image of i^* is the ring*

$$\bigcap_{i=1}^{N} i_{K_i}^* H_T^*(X^{K_i}) \tag{2.4}$$

where i_{K_i} denotes the inclusion of X^T into X^{K_i}.

Proof. Via the inclusion i^* we can view $H_T^*(M)$ as a submodule of $H_T^*(X^T)$. Let $e_1, \ldots, e_m$ be a basis of $H_T^*(M)$ as a free module over $\mathbb{S}(\mathfrak{t}^*)$. Then by Corollary 2.1.4 for any $e \in H_T^*(X^T)$ we have

$$\alpha_1 \cdots \alpha_r e = \sum f_i e_i, \; f_i \in \mathbb{S}(\mathfrak{t}^*) \; .$$

Then $e = \sum \frac{f_i}{p} e_i$, where $p = \alpha_1 \cdots \alpha_r$. If f_i and p have a common factor we can eliminate it and write e uniquely as

$$e = \sum \frac{g_i}{p_i} e_i \tag{2.5}$$

with $g_i \in \mathbb{S}(\mathfrak{t}^*)$, p_i a product of a subset of the weights $\alpha_1, \ldots, \alpha_r$ and p_i and g_i relatively prime.

Now suppose that K is an isotropy subgroup of $\widetilde{M}$ of codimension one and e is in the image of $H_T^*(X^K)$. By [16] Theorem 11.4.2 the cokernel of the map $H_T^*(M) \to H_T^*(X^K)$ is supported on the subset $\cup \mathfrak{k}_i \otimes \mathbb{C}$, $\mathfrak{k}_i \neq \mathfrak{k}$ of (2.1), and hence there exists weights $\beta_1, \ldots, \beta_r$, β_i vanishing on some $\mathfrak{k}_j$ but not on $\mathfrak{k}$, such that

$$\beta_i \cdots \beta_s e = \sum f_i e_i \ .$$

Thus the p_i in (2.5), which is a product of a subset of the weights $\alpha_1, \ldots, \alpha_r$, is a product of a subset of weights none of which vanish on $\mathfrak{k}$. Repeating this argument for all the codimension one isotropy groups of $\widetilde{M}$ we conclude that the weights in this subset cannot vanish on any of these $\mathfrak{k}$'s, and hence is the empty set, i.e. $p_i = 1$. Then if e is in the intersection (2.4), e is in $H_T^*(M)$. $\square$

Now suppose that $\widetilde{M} = \mathbb{C}P^1$. This action of T on $\widetilde{M}$ is effectively an action of a quotient group, T/T_1, where T_1 is the codimension one subgroup of T stabilizing $\widetilde{M}$. Moreover $\widetilde{M}^T$ consists of two points, p_i, $i = 1, 2$, and $X = X^T$ consists of the two fibers $\pi^{-1}(p_i) = F_i$. Let $T = T_1 \times S^1$. Then S^1 acts freely on $\mathbb{C}P^1 \setminus \{p_1, p_2\}$ and the quotient by S^1 of this action is the interval $(0, 1)$, so one has an isomorphism of T_1 spaces

$$(M \setminus X)/S^1 = F \times (0, 1) \ , \tag{2.6}$$

where, as a T_1-space, $F = F_1 = F_2$.

Consider now the long exact sequence (2.2). Since i^* is injective this becomes a short exact sequence

$$0 \to H_T^k(M) \to H_T^k(X) \to H_T^{k+1}(M \setminus X)_c \to 0 \ . \tag{2.7}$$

Since S^1 acts freely on $M \setminus X$ we have

$$H_T^{k+1}(M \setminus X)_c = H_{T_1}^{k+1}((M \setminus X)/S^1)_c$$

and by fiber integration one gets from (2.6)

$$H_{T_1}^{k+1}((M \setminus X)/S^1)_c = H_{T_1}^k(F) \ ,$$

so the sequence (2.7) becomes

$$0 \to H_T^k(M) \xrightarrow{i^*} H_T^k(F_1) \oplus H_T^k(F_2) \xrightarrow{r} H_{T_1}^k(F) \to 0 \tag{2.8}$$

where r is the forgetful map $H_T^k(F_i) \to H_{T_1}^k(F_i) = H_{T_1}^k(F)$.

We can summarize the previous results in the following theorem, which offers a generalization of the GKM condition for $T-$equivariantly formal manifolds M fibering equivariantly over GKM spaces $\widetilde{M}$.

Theorem 2.1.6. *Let M and $\widetilde{M}$ be $T-$manifolds and let $\pi : M \to \widetilde{M}$ be a $T-$equivariant fiber bundle. Suppose that M is $T-$equivariantly formal and $\widetilde{M}$ is GKM; let (V, E) be its GKM graph. For every $p \in V$ let $F_p = \pi^{-1}(\{p\})$ be the fiber over p. Consider the ring of maps that associate to each element $p \in V$ an equivariant cohomology class $f_p \in H_T^*(F_p)$, and let R be the subring characterized by the following condition:*

- *For every edge $e \in E$, if K denotes the stabilizer of the $\mathbb{C}P^1$ associated to e, then*

$$r_K(f_{(i(e))}) = r_K(f_{(t(e))})$$

where $r_K : H_T^*(F_p) \to H_K^*(F_{i(e)}) \simeq H_K^*(F_{t(e)})$, with $p \in \{i(e), t(e)\}$.

Then $H_T^*(M)$ is isomorphic to R.

In [14] we also offer a sheaf theoretic interpretation of the previous theorem.

2.2 GKM graphs and cohomology of GKM graphs

Let $\Gamma = (V, E)$ be a graph, where V denotes the set of vertices, and E the set of directed edges. If e is an element of E directed from p to q, we will refer to p as the *initial point* $i(e)$ of e, and q as the *terminal point* $t(e)$ of e. Hence every undirected edge will appear twice in E, once as the edge joining p to q, and once as the edge joining q to p. If e is the edge directed from p to q, we denote by $\overline{e}$ the "inverse" of e, i.e. the edge e with opposite direction, joining q to p.

Let p be an element of V, and define E_p to be the subset of E consisting of edges e such that $i(e) = p$. In this thesis we will mainly consider graphs (V, E) which are *regular*, i.e. graphs for which the cardinality of E_p doesn't depend on p, for all the vertices p in V.

Definition 2.2.1. *A* ***connection*** ∇_e *along an edge* $e \in E$ *is a bijection from* $E_{i(e)}$ *to* $E_{t(e)}$ *satisfying* $\nabla_e(e) = \overline{e}$. *A connection* ∇ *on* $\Gamma = (V, E)$ *is a family of connections* $\{\nabla_e\}_{e\in E}$ *satisfying* $\nabla_{\overline{e}} = \nabla_e^{-1}$.

Let $\mathfrak{t}$ be a finite dimensional vector space over $\mathbb{R}$, and let $\mathfrak{t}^*$ denote its dual.

Definition 2.2.2. *An* ***axial function*** α *is a map from* E *to* $\mathfrak{t}^*$ *satisfying the following properties.*

(i) For every vertex $p \in V$, *the vectors in the set* $\{\alpha(e),\ e \in E_p\}$ *are pairwise linearly independent.*

(ii) For every edge $e \in E$, $\alpha(e) = -\alpha(\overline{e})$

Observe that these properties are automatically satisfied when $\Gamma = (V, E)$ is the GKM graph associated to a GKM manifold, where the axial function is the one described in section 1.4.1.

Consider a connection ∇ on Γ. We will say that an axial function $\alpha : E \to \mathfrak{t}^*$ is *compatible* with the connection ∇ if the following condition holds.

(iii) For every edge $e = (p, q)$ in E and for every $e' \in E_p$ we have

$$\alpha(\nabla_e(e')) - \alpha(e') = c\alpha(e)$$

for some $c \in \mathbb{R}$ which depends on e and e'.

Definition 2.2.3. *Let* $\Gamma = (V, E)$ *be a regular graph,* ∇ *be a connection on* Γ *and* $\alpha : E \to \mathfrak{t}^*$ *an axial function compatible with* ∇. *Then the pair* (Γ, α) *is called an (abstract)* ***GKM graph****.*

Example 2.2.4 *The GKM graph* (K_{n+1}, α)

Let Γ be the complete graph with $n+1$ vertices $\{1, 2, \ldots, n+1\}$. Define on this graph a connection ∇ in such a way that $\nabla_{(i,j)} : E_i \to E_j$ sends (i, j) to (j, i) and (i, k) to (j, k) for $k \neq i, j$. Moreover let $\{x_1, \ldots, x_{n+1}\}$ be a basis of $\mathfrak{t}^*_{n+1}$, and consider the axial function $\alpha : E \to \mathfrak{t}^*_{n+1}$ given by $\alpha(i, j) = x_i - x_j$. Then this axial function α is compatible with ∇. Observe that the image of the axial function is given by the subspace $\{\sum_{i=1}^{n+1} \lambda_i x_i \in \mathfrak{t}^*_{n+1} \text{ s.t. } \sum_{i=1}^{n+1} \lambda_i = 0\}$. So (K_{n+1}, α) is a GKM graph. This is precisely the GKM graph associated to the complex projective space $\mathbb{C}P^n$ described in example 1.4.7.

Example 2.2.5 *The permutahedron* $(\mathcal{S}_{n+1}, \alpha)$

Let Γ be the graph such that the set of vertices is in bijection with the elements of the permutation group on $n+1$ elements $\mathcal{S}_{n+1}$, and two elements σ, σ' in $\mathcal{S}_{n+1}$ are connected by an edge if and only if they differ by a transposition, i.e. $\sigma' = \sigma(i, j)$. Such a graph is called a *permutahedron*, and we will refer to it simply as $\mathcal{S}_{n+1}$. We recall that the action of the transposition (i, j) on the right, $*(i, j)$, is swapping the elements of σ at positions i and j in the one line notation for σ, $\sigma = \sigma(1) \ldots \sigma(n+1)$; whereas the action of the transposition (h, k) on the left, $(h, k)*$ is swapping the elements h and k in the one line notation of σ. Hence $\sigma' = \sigma(i, j)$ if and only if $\sigma' = (\sigma(i), \sigma(j))\sigma$, and two permutations differ by a transposition on the right if and only if they differ by a transposition on the left. For every edge $e = (\sigma, \sigma')$, where $\sigma' = (h, k)\sigma$ define $\nabla_e : E_\sigma \to E_{\sigma'}$ to be

$$\nabla_e(u, (a, b)u) = (v, (h', k')v)$$

where $(h', k') = (h, k)(a, b)(h, k)$. An axial function α compatible with the connection ∇ defined above is the following: if $\sigma' = (h, k)\sigma$ then $\alpha(\sigma, \sigma') = \text{sign}(\sigma^{-1}(k) - \sigma^{-1}(h))(x_h - x_k)$. Hence $(\mathcal{S}_{n+1}, \alpha)$ is a GKM graph; observe that it is precisely the GKM graph associated to the variety of complete flags in $\mathbb{C}^{n+1}$, $\mathcal{F}l(\mathbb{C}^{n+1})$, described in example 1.4.8.

2.2.1 The cohomology ring of GKM graphs

Let (Γ, α) be an abstract GKM graph, where $\alpha : E \to \mathfrak{t}^*$ is an axial function compatible with some connection ∇. We want to define the cohomology ring of an abstract GKM graph. Let $\mathbb{S}(\mathfrak{t}^*)$ be the symmetric algebra on $\mathfrak{t}^*$; if $\{x_1, \ldots, x_n\}$ is a basis of $\mathfrak{t}^*$, then $\mathbb{S}(\mathfrak{t}^*)$ can be identified with the ring of polynomials $\mathbb{R}[x_1, \ldots, x_n]$ (we will use real coefficients unless otherwise stated). Then, inspired by the description of the equivariant cohomology of a GKM manifold (cfr. Theorem 1.4.6), we want to define the equivariant cohomology ring $H_\alpha^*(\Gamma)$ of a GKM graph (Γ, α). Consider the ring $\mathrm{Maps}(V, \mathbb{S}(\mathfrak{t}^*))$, where V is the set of vertices of Γ.

Definition 2.2.6. *An element f of $Maps(V, \mathbb{S}(\mathfrak{t}^*))$ is in $H_\alpha^*(\Gamma)$ if and only if for every edge e of Γ the following compatibility condition is satisfied*

$$f(t(e)) - f(i(e)) = P\alpha(e), \text{ for some } P \in \mathbb{S}(\mathfrak{t}^*) \tag{2.9}$$

This can be rephrased by saying that both $f(t(e))$ and $f(i(e))$ have the same image in $\mathbb{S}(\mathfrak{t}^*)/\alpha(e)\mathbb{S}(\mathfrak{t}^*)$, for every edge e of Γ.

The ring structure on $H_\alpha^*(\Gamma)$ is simply given by the ring structure on $\mathrm{Maps}(V, \mathbb{S}(\mathfrak{t}^*))$. Observe that $H_\alpha^*(\Gamma)$ is also an $\mathbb{S}(\mathfrak{t}^*)-$module, since for every class $f \in H_\alpha^*(\Gamma)$, the element $Pf \in \mathrm{Maps}(V, \mathbb{S}(\mathfrak{t}^*))$ such that $(Pf)(p) = Pf(p)$ still satisfies the compatibility condition (2.9), hence it is an element of $H_\alpha^*(\Gamma)$.

As for the grading, if f is an element of $H_\alpha^*(\Gamma)$ such that $f(p)$ is a homogeneous polynomial of degree k in $\mathbb{S}(\mathfrak{t}^*)$ for every $p \in V$, then f has degree $2k$ in $H_\alpha^*(\Gamma)$. If $H_\alpha^{2k}(\Gamma)$ denotes the space of cohomology classes of degree $2k$ in $H_\alpha^*(\Gamma)$, then we have

$$H_\alpha^*(\Gamma) = \bigoplus_{k \geq 0} H_\alpha^{2k}(\Gamma)$$

and the cohomology of (Γ, α) vanishes in odd dimension.

As we saw in Example 2.2.4, the image of the axial function $\alpha : E \to \mathfrak{t}^*$ might not be the entire space $\mathfrak{t}^*$. Hence for every vertex $p \in V$, define $\mathfrak{h}_p$ to be the subspace of $\mathfrak{t}^*$ generated by the image of the axial function on edges e such that $i(e) = p$, i.e.

$$\mathfrak{h}_p = \mathrm{span}\{\alpha(e),\ e \in E_p\} \subset \mathfrak{t}^*$$

Since $\Gamma = (V, E)$ is a GKM graph, it is clear that if Γ is connected then $\mathfrak{h}_p$ doesn't depend on p. Let $\mathfrak{t}_0^*$ denote this common image; then if we define α_0 to be $\alpha_0 : E \to \mathfrak{t}_0^*$, then also (Γ, α_0) is a GKM graph.

Definition 2.2.7. *An axial function $\alpha : E \to \mathfrak{t}^*$ is called effective if $\mathfrak{t}_0^* = \mathfrak{t}^*$.*

Let $\Gamma_0 = (V_0, E_0)$ be a connected subgraph of $\Gamma = (V, E)$ such that if e is an edge in E with $i(e), t(e) \in V_0$, then $e \in E_0$. Suppose moreover that the connection ∇ defined on Γ satisfies

$$\nabla_e(E_p \cap E_0) = E_q \cap E_0$$

for all the edges $e \in E_0$, with $i(e) = p$ and $t(e) = q$. Observe that in this case the axial function $\alpha : E \to \mathfrak{t}^*$ restricts to an axial function $\alpha : E_0 \to \mathfrak{t}^*$ which is compatible with the restriction of ∇ to E_0. We will refer to the pair (Γ_0, α) as a *GKM subgraph* of (Γ, α). From the definition it follows that Γ_0 is a regular graph as well, and (Γ_0, α) is a GKM graph on its own. Let (Γ_1, α_1) and (Γ_2, α_2) be two GKM graphs, where if $\Gamma_i = (V_i, E_i)$, then $\alpha_i : E_i \to \mathfrak{t}_i^*$, $i = 1, 2$.

Definition 2.2.8. *An* ***isomorphism of GKM graphs*** *from (Γ_1, α_1) to (Γ_2, α_2) is a pair (Φ, Ψ), where*

(i) $\Phi : \Gamma_1 \to \Gamma_2$ *is an isomorphism of graphs*

(ii) $\Psi : \mathfrak{t}_1^* \to \mathfrak{t}_2^*$ *is an isomorphism of linear spaces*

(iii) For every edge (p, q) of Γ_1 we have

$$\alpha_2(\Phi(p), \Phi(q)) = \Psi(\alpha_1(p, q))$$

In particular, if E_1 and E_2 denote respectively the set of edges of Γ_1 and Γ_2, then condition (i) implies that there exists a bijection Φ between these two sets, and that the isomorphism Ψ intertwines the axial function on E_1 and E_2, i.e. the following diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ {\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_2}\downarrow \\ \mathfrak{t}_1^* & \xrightarrow{\Psi} & \mathfrak{t}_2^* \end{array}$$

commutes.

2.3 Fiber bundles of GKM spaces

In what follows we will first introduce some definitions concerning particular morphisms between graphs. Then we will define the same morphisms, but in the GKM category, i.e. in the definition we will introduce compatibility conditions on the axial functions defined on the GKM graphs.

2.3.1 The complete flag variety $\mathcal{F}l(\mathbb{C}^{n+1})$ fibering over the complex projective space $\mathbb{C}P^n$

In this example we start with two GKM manifolds, the complete flag variety $\mathcal{F}l(\mathbb{C}^{n+1})$ and the complex projective space $\mathbb{C}P^n$, exhibit a $T-$equivariant fibration between these two spaces, and derive the graph theoretical implications of the existence of such a map at the level of their GKM graphs. In the next sections, motivated by this example, we will formalize some of these concepts in the abstract setting of GKM graphs.

Let $\mathcal{F}l(\mathbb{C}^{n+1})$ be the variety of complete flags in $\mathbb{C}^{n+1}$, and let $\underline{V} = (V_1, \ldots, V_{n+1})$ be a point in $\mathcal{F}l(\mathbb{C}^{n+1})$, i.e. a complete flag in $\mathbb{C}^{n+1}$, where V_i is a complex vector space of dimension i, and $V_{i+1} \supset V_i$ for all $i = 1, \ldots, n$. Let $\mathbb{C}P^n$ be the complex projective space of dimension n. Then there is a natural projection map

$$\pi : \mathcal{F}l(\mathbb{C}^{n+1}) \to \mathbb{C}P^n$$

which is given by

$$\pi(V_1, \ldots, V_{n+1}) = V_1 \ ,$$

where the typical fiber is isomorphic to a complete flag in $\mathbb{C}^n$, $\mathcal{F}l(\mathbb{C}^n)$.

Consider the description of $\mathcal{F}l(\mathbb{C}^{n+1})$ and $\mathbb{C}P^n$ as GKM spaces given in section 1.4.2.

Let $G = SU(n+1)$ and let T be the torus of diagonal matrices in G with Lie algebra $\mathfrak{t}$. Then we can identify $\mathfrak{t}^*$ with the subset of $(\mathbb{R}^{n+1})^*$ given by $\{\sum_{j=1}^{n+1} \lambda_j x_j$ s.t. $\sum_{j=1}^{n+1} \lambda_j = 0\}$, where $x_i(\xi_1, \ldots, \xi_{n+1}) = \xi_i$. Let p_0 be a point in $\mathfrak{t}^*$ given by $p_0 = \sum_{i=1}^{n+1} \mu_i x_i$, where $\underline{\mu} = (\mu_1, \ldots, \mu_{n+1})$ is a generic vector in $\mathbb{R}^{n+1}$ satisfying $\mu_1 < \mu_2 < \ldots < \mu_{n+1}$ and $\sum_{i=1}^{n+1} \mu_i = 0$. Using the Killing form we can identify the Lie algebra $Lie(G)$ with its dual $Lie(G)^*$, and hence we can consider $\mathfrak{t}^*$ as a subspace of $Lie(G)^*$. Then, the G coadjoint orbit through p_0, $G \cdot p_0$, is isomorphic to $\mathcal{F}l(\mathbb{C}^{n+1})$, with symplectic structure ω_{p_0} given by its coadjoint orbit structure. Moreover the action of T on $\mathcal{F}l(\mathbb{C}^{n+1})$ is Hamiltonian, and the moment map ψ restricted to the $T-$fixed point set is just given by the inclusion, i.e. for all $p = \sum_{i=1}^{n+1} \mu_i x_{\sigma(i)} \in (G \cdot p_0)^T$ we have

$$\psi(\sum_{i=1}^{n+1} \mu_i x_{\sigma(i)}) = \sum_{i=1}^{n+1} \mu_i x_{\sigma(i)} \ ,$$

where $\sigma \in \mathcal{S}_{n+1}$.

Let $\widetilde{p}_0$ be a point in $\mathfrak{t}^*$ given by $\widetilde{p}_0 = \widetilde{\mu}_1 x_1 + \widetilde{\mu}_2 \sum_{i=2}^{n+1} x_i$, where $\widetilde{\mu}_1 < \widetilde{\mu}_2$ and $\widetilde{\mu}_1 + n\widetilde{\mu}_2 = 0$. Then the G coadjoint orbit through $\widetilde{p}_0$ is isomorphic to $\mathbb{C}P^n$, with symplectic structure $\omega_{\widetilde{p}_0}$ inherited by its coadjoint structure. The action of T on $\mathbb{C}P^n$ is Hamiltonian, and the moment map $\widetilde{\psi}$ restricted to the $T-$fixed point set is just the inclusion, i.e. for all $p = \widetilde{\mu}_1 x_{\sigma(1)} + \widetilde{\mu}_2 \sum_{i=2}^{n+1} x_{\sigma(i)}$

$$\widetilde{\psi}(\widetilde{\mu}_1 x_{\sigma(1)} + \widetilde{\mu}_2 \sum_{i=2}^{n+1} x_{\sigma(i)}) = \widetilde{\mu}_1 x_{\sigma(1)} + \widetilde{\mu}_2 \sum_{i=2}^{n+1} x_{\sigma(i)} \ ,$$

where $\sigma \in \mathcal{S}_{n+1}$.

In this setting, the projection map π described before is given by

$$\begin{array}{rccc} \pi : & (G \cdot p_0, \omega_{p_0}, \psi) & \rightarrow & (G \cdot \widetilde{p}_0, \omega_{\widetilde{p}_0}, \widetilde{\psi}) \\ & g \cdot p_0 & \mapsto & g \cdot \widetilde{p}_0 \end{array}$$

It is well known that this projection map is a $T-$equivariant fiber bundle, with typical fiber isomorphic to a generic $SU(n)$ coadjoint orbit, which we identify with $\mathcal{F}l(\mathbb{C}^n)$.

Let $\Gamma = (V, E)$ and $B = (V_B, E_B)$ be the GKM graphs associated respectively to $\mathcal{F}l(\mathbb{C}^{n+1})$ and $\mathbb{C}P^n$, with axial functions α and α_B. Then the elements of V are in bijection with the elements of $\mathcal{S}_{n+1}$, and the bijection is given by

$$p = \sum_{i=1}^{n+1} \mu_i x_{\sigma(i)} \quad \mapsto \quad \sigma = \sigma(1) \dots \sigma(n+1)$$

The set of vertices V_B is composed by $n+1$ elements $\{1, \dots, n+1\}$, where i corresponds to the $T-$fixed point $p_i = \widetilde{\mu}_1 x_i + \widetilde{\mu}_2 \sum_{j \neq i} x_j$.

From the equivariance of π it is clear that

- Vertices of Γ are mapped to vertices of B
- If $e = (p, q)$ is an *unoriented* edge in E such that $\pi(p) \neq \pi(q)$, then $(\pi(p), \pi(q))$ is an *unoriented* edge in E_B

In the next proposition we want to give a combinatorial description of π at the level of the graphs Γ and B; in particular we want to describe how the axial function behaves with respect to π.

Proposition 2.3.1. *The fibration π mentioned above has the following properties:*

(i) If σ is a vertex of Γ, then $\pi(\sigma) = \pi(\sigma(1), \dots, \sigma(n+1)) = \sigma(1)$.

(ii) An edge $e = (\sigma, \sigma')$ in Γ, where $\sigma' = \sigma(i,j)$, $i < j$, is vertical, i.e. $\pi(\sigma) = \pi(\sigma')$, if and only if $i > 1$.

(iii) If an edge $e = (\sigma, \sigma')$ of E projects to an edge $\pi(e) = (h,k)$ of E_B, with $h \neq k$, then $\alpha(e) = \alpha_B(\pi(e)) = x_h - x_k$.

Proof. First of all observe that the projection π restricted to the $T-$fixed point set is simply given by

$$\begin{array}{cccc} \pi: & (G \cdot p_0)^T & \longrightarrow & (G \cdot \widetilde{p}_0)^T \\ & & & \\ & \sum_{i=1}^{n+1} \mu_i x_{\sigma(i)} & \longmapsto & \widetilde{\mu}_1 x_{\sigma(1)} + \widetilde{\mu}_2 \sum_{i=2}^{n+1} x_{\sigma(i)} \end{array}$$

So it is clear that, as a map from V to V_B, π sends the element $\sigma(1)\ldots\sigma(n{+}1)$ to $\sigma(1)$; hence (i) and (ii) follow immediately. Consider an edge e of Γ given by $e = (\sigma, \sigma')$ such that $(\pi(\sigma), \pi(\sigma')) = (h,k)$. This means that $\sigma(1) = h$ and $\sigma(j) = k$ for some $j > 1$. Then by definition of axial function on Γ we have that $\alpha(\sigma, \sigma') = \text{sign}(\sigma^{-1}(k) - \sigma^{-1}(h))(x_h - x_k) = x_h - x_k = \alpha_B(h,k)$, and (iii) follows. $\square$

Observe that the $T-$fixed points in the fiber $\pi^{-1}(h)$ are in bijection with the elements of $\mathcal{S}_n$; more precisely $V \cap \pi^{-1}(h) = \{\sigma \in \mathcal{S}_{n+1} \text{ s.t. } \sigma(1) = h\}$.

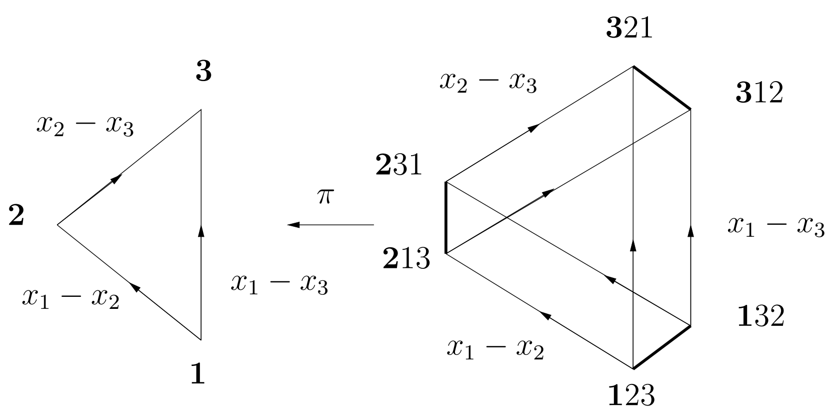


Figure 2-1: The projection $\pi : \mathcal{F}l(\mathbb{C}^3) \to \mathbb{C}P^2$ from a GKM graph theoretical point of view.

In the above figure the darker edges correspond to the fibers of π, which is $\mathbb{C}P^1$ bundle, over the fixed point set of $\mathbb{C}P^2$.

Inspired by this example, in the next sections we are going to define the discrete analogue of a fiber bundle between two GKM spaces, and derive the graph theoretical implications of the existence of such a map.

2.3.2 Morphisms of graphs

Let Γ and B be connected graphs. Then $\pi : \Gamma \to B$ is a *morphism of graphs* if

- it maps vertices of Γ to vertices of B
- if $e = (p, q)$ is an edge of Γ, then either $\pi(p) \neq \pi(q)$ and $(\pi(p), \pi(q))$ is an edge of B, or $\pi(p) = \pi(q)$.

If $e = (p, q)$ is an edge of Γ such that $\pi(p) = \pi(q)$ (resp. $\pi(p) \neq \pi(q)$), then we will call e a *vertical edge* (resp. *horizontal edge*). In what follows we will be mainly concerned with morphisms of graphs π which are surjective; so unless otherwise stated, we assume that π is surjective.

For every vertex p of Γ, one has a natural "splitting" of E_p w.r.t. π, i.e. let H_p be the set of horizontal edges of Γ with initial point p and let $E_p^{\perp}$ the set of vertical edges in E_p. Then $E_p = E_p^{\perp} \cup H_p$. Let E_B denote the set of edges of B. Then every morphism of graphs π naturally induces a map

$$\begin{array}{rccc} (d\pi)_{H_p} : & H_p & \to & (E_B)_{\pi(p)} \\ & (p, p') & \mapsto & (\pi(p), \pi(p')) \end{array}$$

Definition 2.3.2. *We will say that a morphism of graphs $\pi : \Gamma \to B$ is a* ***fibration of graphs*** *if the map $(d\pi)_{H_p}$ defined above is a bijection for every vertex p in Γ.*

Observe that every fibration of graphs has the *unique lifting property for paths*. In fact, given an edge (p_0, p_1) in E_B, let q_0 be a vertex in Γ such that $\pi(q_0) = p_0$. Then, since $(d\pi)_{q_0}$ is a bijection, there exists a unique *lift* (q_0, q_1) of the edge (p_0, p_1) with initial point q_0, i.e. a unique edge $e = (q_0, q_1)$ in E such that $(d\pi)_{q_0}(q_0, q_1) = (p_0, p_1)$. More in general, given a path γ in B, i.e. a sequence of edges $\gamma = (e_1, \ldots, e_k)$ with $i(e_1) = p_0$, let q_0 be a vertex in the fiber $\pi^{-1}(p_0)$. Then there exists a unique lift of γ starting at q_0, which is given by $(e'_1, \ldots, e'_k)$, where e'_1 is the lift of e_1 at q_0, and e'_{i+1} is the lift of the edge e_{i+1} with initial point $i(e'_{i+1}) = t(e'_i)$, for every $i = 1, \ldots, k-1$.

Consider now the fibers of the map π, $\pi^{-1}(p)$, for every vertex p of B. Let V_p be the set of vertices of the fiber over p, and let Γ_p be the subgraph of Γ with vertex set V_p. Then by definition of morphism of graphs the edge set of Γ_p is composed entirely by vertical edges. When π is a fibration of graphs it is always possible to define a map $\Phi_{p,q}$ between the set of vertices of Γ_p and Γ_q, for every edge (p, q) of B. More precisely let p' be a vertex of Γ_p, and let (p', q') be the lift of (p, q); then $\Phi_{p,q} : V_p \to V_q$ is simply defined to be $\Phi_{p,q}(p') = q'$. It is clear that $\Phi_{p,q}$ defines a bijection between V_p and V_q for all the edges (p, q) in B.

Definition 2.3.3. *We will say that a fibration of graphs $\pi : \Gamma \to B$ is a **fiber bundle** if for every edge (p, q) of B the map $\Phi_{p,q}$ is a morphism of graphs, $\Phi_{p,q} : \Gamma_p \to \Gamma_q$.*

So if (p_1, p_2) is a vertical edge over p then $(\Phi_{p,q}(p_1), \Phi_{p,q}(p_2))$ is a vertical edge over q. So for fiber bundles, the map $\Phi_{p,q}$ defines an *isomorphism* between the graphs corresponding to the fibers over p and q, i.e.

$$\Phi_{p,q} : \Gamma_p \xrightarrow{\simeq} \Gamma_q$$

2.3.3 Morphisms of GKM graphs

Now we want to define the same concepts in the GKM category. In order to do so, we first give an example of fibration between two GKM manifolds, and derive the properties that are naturally implied by the existence of such a map at the level of their GKM graphs. Then we formalize the previous properties in the category of abstract GKM graphs.

Suppose that (M, ω, J) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{J})$ are symplectic manifolds with compatible almost complex structures, acted by the same torus T, and suppose that they are GKM spaces with respect to this action. Let π be a surjective $T-$equivariant map of almost complex manifolds $\pi : (M, J) \to (\widetilde{M}, \widetilde{J})$, with surjective differential $(d\pi)_p : T_pM \to T_{\pi(p)}\widetilde{M}$. Let (Γ, α) and (B, α_B) be the GKM graphs associated to M and $\widetilde{M}$. Observe that π restricts to a morphism of graphs $\pi : \Gamma \to B$ since it is a $T-$equivariant map and, since $(d\pi)_p : T_pM \to T_{\pi(p)}\widetilde{M}$ is surjective, it is a fibration of graphs. In fact, let p be a vertex of Γ, and let $(d\pi)_{H_p} : H_p \to E_{\pi(p)}$ denote the isomorphism defined before. Then this isomorphism can be canonically defined using the axial functions α and α_B. In fact, if (p, q) is an edge in H_p, by the equivariance of π we have that $\alpha(p, q) = \pm\alpha_B(\pi(p), \pi(q))$; since π is a map of almost complex manifold, then in this case we have $\alpha(p, q) = \alpha_B(\pi(p), \pi(q))$. Since M and $\widetilde{M}$ are GKM manifolds, the map $(d\pi)_{H_p} : H_p \to E_{\pi(p)}$ can be canonically defined by saying that

$$(d\pi)_{H_p}(e') = e \text{ if and only if } \alpha(e') = \alpha_B(e)$$

Now consider a connection ∇_B on B compatible with the axial function α_B. For all the horizontal edges e' in E, we can define a connection $\nabla_{e'}$ on the set of horizontal edges H_p, with $i(e') = p$, by imposing the commutativity of the following diagram

$$\begin{array}{ccc} H_p & \xrightarrow{\nabla_{e'}} & H_q \\ \downarrow{\scriptstyle (d\pi)_{H_p}} & & \downarrow{\scriptstyle (d\pi)_{H_q}} \\ (E_B)_{\pi(p)} & \xrightarrow{(\nabla_B)_e} & (E_B)_{\pi(q)} \end{array}$$

By definition of $(d\pi)_{H_p}$, it is also clear that the axial function α is compatible with $\nabla_{e'} : H_p \to H_q$. Now we can extend the connection ∇ to be a connection defined on the whole edge set of Γ, in such a way that it sends vertical edges into vertical edges, horizontal edges into horizontal edges, and so that α is compatible with ∇.

In what follows we want to define the previous concepts in the category of abstract GKM graphs.

Definition 2.3.4. *Let (Γ, α) and (B, α_B) be GKM graphs with connections ∇, ∇_B compatible with α and α_B. Then a map $\pi : (\Gamma, \alpha) \to (B, \alpha_B)$ is a* ***GKM fibration*** *(w.r.t. ∇ and ∇_B) if the following conditions are satisfied:*

(i) π is a fibration of graphs.

(ii) For every edge e of B, if e' denotes any lift of e, then $\alpha(e') = \alpha_B(e)$.

(iii) For every edge $e = (p, q)$ of Γ, ∇_e preserves the splitting of E_p into horizontal and vertical edges, i.e. ∇_e sends horizontal edges into horizontal edges, and hence vertical edges into vertical edges.

(iv) For every edge e of B, if $e' = (p, q)$ denotes its lift, the following diagram

$$\begin{array}{ccc} H_p & \xrightarrow{\nabla_{e'}} & H_q \\ \downarrow (d\pi)_{H_p} & & \downarrow (d\pi)_{H_q} \\ (E_B)_{\pi(p)} & \xrightarrow{(\nabla_B)_e} & (E_B)_{\pi(q)} \end{array}$$

commutes.

Consider now a GKM fibration $\pi : (\Gamma, \alpha) \to (B, \alpha_B)$, which satisfies the compatibility conditions mentioned above w.r.t. ∇ and ∇_B. Let Γ_p be the graph with set of vertices $\pi^{-1}(p)$, where p is a vertex of B. Then since ∇ sends vertical edges into vertical edges, and α is compatible with ∇, it follows that the fibers (Γ_p, α) are GKM subgraphs of (Γ, α), with a connection ∇_p which is the restriction of ∇ to the vertical edges over p, and an axial function α which is compatible with ∇_p.

Given a GKM fibration $\pi : \Gamma \to B$, we would like to be able to describe how the GKM subgraphs (Γ_p, α) change as the vertex p of B changes.

Definition 2.3.5. *Let $\pi : (\Gamma, \alpha) \to (B, \alpha_B)$ be a GKM fibration. Then π is a* ***GKM fiber bundle*** *if*

(i) π is a fiber bundle

(ii) All the fibers of π are isomorphic as GKM (sub)graphs

For every edge $e = (p, q)$ of B, let $\Phi_{p,q} : \Gamma_p \to \Gamma_q$ denote the isomorphism between the graphs corresponding to the fibers $\pi^{-1}(p)$ and $\pi^{-1}(q)$.

(iii) If e' is the lift of e starting at p', then for all the vertical edges (p', p'') in $E_{p'}$ we have

$$\nabla_{e'}(p', p'') = (\Phi_{p,q}(p'), \Phi_{p,q}(p''))$$

Hence if Γ is connected all the fibers are isomorphic as GKM subgraphs. To be more precise, let $\mathfrak{t}_p^*$ be the subspace of $\mathfrak{t}^*$ generated by the values that the axial function takes on edges of Γ_p, i.e.

$$\mathfrak{t}_p^* = \text{span}\{\alpha(e), \ \ e \text{ edge in } \Gamma_p\} \subset \mathfrak{t}^*$$

Thus we can restrict the axial function on Γ_p to be $\alpha_p : \Gamma_p \to \mathfrak{t}_p^*$. Then if π is a GKM fiber bundle, for every edge (p, q) of B there exists an isomorphism of GKM graphs

$$\Upsilon_{p,q} = (\Phi_{p,q}, \Psi_{p,q}) : (\Gamma_p, \alpha_p) \to (\Gamma_q, \alpha_q)$$

where $\Phi_{p,q}$ is the isomorphism of graphs defined using the unique lifting property, and $\Psi_{p,q} : \mathfrak{t}_p^* \to \mathfrak{t}_q^*$ is a linear isomorphism intertwining the axial functions α_p and α_q. Observe that by (iii), since α is compatible with ∇, and α_p (resp. α_q) is just the restriction of α to the edges of the fiber Γ_p (resp. Γ_q), we have that if (p_1, p_2) is an edge of Γ_p and (q_1, q_2) is the corresponding edge in Γ_q, i.e. $q_i = \Phi_{p,q}(p_i), \ i = 1, 2$, then $\alpha_q(q_1, q_2) - \alpha_p(p_1, p_2) = c\alpha(p_1, q_1)$, and by definition of isomorphism of GKM graphs, $\alpha_q(q_1, q_2) = \Psi_{p,q}(\alpha_p(p_1, p_2))$. Now since $\alpha(p_1, q_1) = \alpha_B(p, q)$, we can conclude that

$$\Psi_{p,q}(\alpha_p(p_1, p_2)) - \alpha_p(p_1, p_2) = c\alpha_B(p, q) \ ,$$

where the constant c depends on $\alpha_p(p_1, p_2)$. So we can conclude that for every edge $e = (p, q)$ in B

$$\Psi_e(x) = \Psi_{p,q}(x) = x + c\,\alpha(e), \ \ \text{for all } x \in \mathfrak{t}_p^* \ . \tag{2.10}$$

If the base B is a connected graph, all the fibers of a GKM fiber bundle are isomorphic GKM spaces. So we can introduce a typical fiber (F, α_F), and for all the vertices p of B fix an isomorphism of GKM graphs from (F, α_F) to (Γ_p, α_p)

$$\rho_p = (\varphi_p, \psi_p) : (F, \alpha_F) \to (\Gamma_p, \alpha_p)$$

2.4 Cohomology of GKM fiber bundles

Let (Γ, α) and (B, α_B) be GKM graphs, and let $\pi : (\Gamma, \alpha) \to (B, \alpha_B)$ be a GKM fiber bundle. In this section we want to prove the main theorem of this chapter, which relates the cohomology ring of the total space $H_\alpha(\Gamma)$, to the cohomology ring of the base $H_{\alpha_B}(B)$ and the fiber $H_{\alpha_F}(F)$. The theorem we prove is a discrete version of the Serre-Leray theorem.

First of all, let $\rho = (\varphi, \psi) : (\Gamma_1, \alpha_1) \to (\Gamma_2, \alpha_2)$ be an isomorphism of GKM graphs. This isomorphism induces a pull back map between $\text{Maps}(V_1, \mathbb{S}(\mathfrak{t}_1^*))$ and $\text{Maps}(V_2, \mathbb{S}(\mathfrak{t}_2^*))$. More precisely let ρ^* be the map

$$\begin{array}{rccc} \rho^* : & \text{Maps}(V_2, \mathbb{S}(\mathfrak{t}_2^*)) & \to & \text{Maps}(V_1, \mathbb{S}(\mathfrak{t}_1^*)) \\ & f & \mapsto & \rho^*(f) \end{array}$$

where $\rho^*(f)(p) = \psi^{-1}(f(\varphi(p)))$, for all the vertices p in V_1, and $\psi^{-1} : \mathbb{S}(\mathfrak{t}_2^*) \to \mathbb{S}(\mathfrak{t}_1^*)$ is the isomorphism obtained by extending $\psi^{-1} : \mathfrak{t}_2^* \to \mathfrak{t}_1^*$ to be an algebra isomorphism. Since $H_{\alpha_2}(\Gamma_2)$ is a subring of $\text{Maps}(V_2, \mathbb{S}(\mathfrak{t}_2^*))$, we can restrict ρ^* to $H_{\alpha_2}(\Gamma_2)$. What we want to prove next is that the image of $H_{\alpha_2}(\Gamma_2)$ is precisely $H_{\alpha_1}(\Gamma_1)$.

Proposition 2.4.1. *The isomorphism of GKM graphs $\rho : (\Gamma_1, \alpha_1) \to (\Gamma_2, \alpha_2)$ induces an isomorphism of rings $\rho^* : H_{\alpha_2}(\Gamma_2) \to H_{\alpha_1}(\Gamma_1)$.*

Proof. We need to prove that if f is an element of $H_{\alpha_2}(\Gamma_2)$, then $\rho^*(f)$ is in $H_{\alpha_1}(\Gamma_1)$, i.e. it satisfies the compatibility condition

$$\rho^*(f)(t(e)) - \rho^*(f)(i(e)) = P\alpha_1(e)$$

for all the edges e in Γ_1, where P is an element of $\mathbb{S}(\mathfrak{t}_1^*)$.

From the definition of isomorphism of GKM graphs it follows that given an edge $e = (p, q)$ in Γ_1, then $(\varphi(p), \varphi(q))$ is an edge in Γ_2 such that $\alpha_2(\varphi(p), \varphi(q)) = \psi(\alpha_1(p, q))$. Since f is a cohomology class in $H_{\alpha_2}(\Gamma_2)$, we have $f(\varphi(q)) - f(\varphi(p)) = Q\ \alpha_2(\varphi(p), \varphi(q)) = Q\ \psi(\alpha_1(p, q))$ for some $Q \in \mathbb{S}(\mathfrak{t}_2^*)$. Then we have

$$\rho^* f(q) - \rho^* f(p) = \psi^{-1}(f(\varphi(q))) - \psi^{-1}(f(\varphi(p))) = \psi^{-1}(Q)\alpha_1(p, q)$$

So $\rho^* f$ is an element of $H_{\alpha_1}(\Gamma_1)$, and ρ^* is an isomorphism of cohomology rings. □

Now consider the ring $H_{\alpha_B}(B)$; the map π defines a pull-back map $\pi^* : H_{\alpha_B}(B) \to H_\alpha(\Gamma)$, which embeds $H_{\alpha_B}(B)$ as a subring of $H_\alpha(\Gamma)$. Observe that π^* also gives $H_\alpha(\Gamma)$ the structure of a $H_{\alpha_B}(B)-$module. We will refer to $\pi^*(H_{\alpha_B}(B))$ as the subring of *basic classes* of $H_\alpha(\Gamma)$, which we denote by $(H_\alpha(\Gamma))_{bas}$.

For every vertex p of B, consider the inclusion of the fiber Γ_p into Γ, $i_p : \Gamma_p \hookrightarrow \Gamma$, which induces a map in cohomology $i_p^* : H_\alpha(\Gamma) \to H_\alpha(\Gamma_p)$. The discrete analogue of the Serre-Leray theorem can be stated as follows.

Theorem 2.4.2. *Let $\pi : (\Gamma, \alpha) \to (B, \alpha_B)$ be a GKM fiber bundle, and let $c_1, \ldots, c_k$ be cohomology classes in $H_\alpha(\Gamma)$ such that their restrictions to the fiber Γ_p, $i_p^* c_1, \ldots, i_p^* c_k$, form a basis of $H_\alpha(\Gamma_p)$ as an $\mathbb{S}(\mathfrak{t}^*)-$module, for all the fibers Γ_p. Then $H_\alpha(\Gamma)$ is isomorphic to the free $H_{\alpha_B}(B)-$module on $c_1, \ldots, c_k$.*

Proof. First of all, it's clear that every linear combination of the classes $c_1, \ldots, c_k$ with coefficients in $(H_\alpha(\Gamma))_{bas}$ is a class in $H_\alpha(\Gamma)$. Moreover the $H_{\alpha_B}(B)-$module on $c_1, \ldots, c_k$, as a submodule of $H_\alpha(\Gamma)$, is free. In fact for every collection of basic classes $P_1, \ldots, P_k$, if the linear combination $\sum_{i=1}^k P_i c_i$ is identically zero, then also $\sum_{i=1}^k i_p^*(P_i c_i)$ is zero, for all the vertices p of B. But $i_p^*(P_i)$ is a constant polynomial on the fiber, because P_i is basic. By assumption the classes $i_p^* c_1, \ldots, i_p^* c_k$ form a basis of $H_\alpha(\Gamma_p)$ as an $\mathbb{S}(\mathfrak{t}^*)-$module, which implies that $i_p^* P_i$ is identically zero for all p, and hence P_i is zero, for all $i = 1, \ldots, k$. So the free $H_{\alpha_B}(B)-$module on $c_1, \ldots, c_k$ is embedded in $H_\alpha(\Gamma)$.

Conversely, let c be a cohomology class in $H_\alpha(\Gamma)$. Then, since the classes $i_p^*c_1, \ldots, i_p^*c_k$ are a basis for $H_\alpha(\Gamma_p)$, for every vertex p of B we have

$$i_p^*(c) = \sum_{i=1}^{k} i_p^*(P_i) i_p^* c_i$$

where the elements P_i belong to $\mathrm{Maps}(V, \mathbb{S}(\mathfrak{t}^*))$, and they are constant on each fiber Γ_p, for all $i = 1, \ldots, k$. We want to prove that in fact P_i belongs to $(H_\alpha(\Gamma))_{bas}$. Let $e = (p', q')$ be the lift of the edge $e = (p, q)$ in B at p'. Then since c and the classes c_i, $i = 1, \ldots, k$, belong to $H_\alpha(\Gamma)$ and $\alpha(e') = \alpha_B(e)$, we have $c(q') - c(p') = Q\ \alpha_B(e)$, and $c_i(q') - c_i(p') = Q_i\ \alpha_B(e)$, where Q and Q_i are elements of $\mathbb{S}(\mathfrak{t}^*)$, for all $i = 1, \ldots, k$. In particular

$$Q\ \alpha_B(e) = c(q') - c(p') = \sum_{i=1}^{k} i_q^*(P_i)(c_i(p') + Q_i\ \alpha_B(e)) - \sum_{i=1}^{k} i_p^*(P_i) c_i(p').$$

Observe that the polynomials Q and Q_i's depend on the vertex p' in Γ, and we can rewrite the above expression as

$$\sum_{i=1}^{k} (i_q^*(P_i) - i_p^*(P_i)) c_i(p') = \alpha_B(e)\ Q'(p')$$

If p'' is another vertex of Γ_p such that (p', p'') is an edge of Γ_p, we have

$$\sum_{i=1}^{k} (i_q^*(P_i) - i_p^*(P_i))(c_i(p'') - c_i(p')) = \alpha_B(e)(Q'(p'') - Q'(p'))$$

Since $\alpha(p', p'')$ divides $c_i(p'') - c_i(p')$, for all $i = 1, \ldots, k$, then $\alpha(p', p'')$ divides the right hand side of this equality. But $\alpha(p', p'')$ and $\alpha(e)$ are independent vectors, hence $\alpha(p', p'')$ must divide $Q'(p'') - Q'(p')$. So Q' is a cohomology class in $H_\alpha(\Gamma_p)$, which implies that $Q'(p') = \sum_{i=1}^{k} \beta_i c_i(p')$, for some polynomials $\beta_1, \ldots, \beta_k \in \mathbb{S}(\mathfrak{t}^*)$. Hence

$$\sum_{i=1}^{k} (i_q^*(P_i) - i_p^*(P_i) - \alpha_B(e)\beta_i) i_p^* c_i$$

is the zero class in $H_\alpha(\Gamma_p)$. But since the vectors $i_p^*c_1, \ldots, i_p^*c_k$ are a basis of $H_\alpha(\Gamma_p)$ we have that

$$i_q^*(P_i) - i_p^*(P_i) = \alpha_B(e)\beta_i$$

which implies that the elements P_i are in $(H_\alpha(\Gamma))_{bas}$. $\square$

2.4.1 The holonomy group and invariant classes

For every GKM fiber bundle it is natural to introduce the holonomy group, which is a subgroup of GKM automorphism of the typical fiber.

For every path $\gamma = (p_0, \ldots, p_m)$ in B, we can compose the isomorphisms of GKM graphs defined before, and get

$$\Upsilon_\gamma = \Upsilon_{p_{m-1},p_m} \circ \cdots \circ \Upsilon_{p_0,p_1} : (\Gamma_{p_0}, \alpha_{p_0}) \to (\Gamma_{p_m}, \alpha_{p_m})$$

which is an isomorphism of the GKM graphs corresponding to the fibers $(\Gamma_{p_0}, \alpha_{p_0})$ and $(\Gamma_{p_m}, \alpha_{p_m})$.

We can repeat the same argument for all the loops γ based at a vertex p of the base B. More precisely, let $\Omega(p)$ be the set of loops with initial and final point p. Define $\mathcal{A}_p$ to be

$$\mathcal{A}_p = \{\Upsilon_\gamma,\ \gamma \in \Omega(p)\}$$

which is a subgroup of the GKM automorphisms of the fiber Γ_p.

If we consider the typical fiber (F, α_F), for every path γ in B from p to q we can define the GKM automorphism of the fiber $\rho_\gamma = (\varphi_\gamma, \psi_\gamma) : (F, \alpha_F) \to (F, \alpha_F)$ given by

$$\rho_\gamma = \rho_q^{-1} \circ \Upsilon_\gamma \circ \rho_p$$

If we restrict our attention to the loops γ based at a vertex p of B we obtain the *holonomy group*

$$\mathrm{Hol}_p = \{\rho_\gamma,\ \gamma \in \Omega(p)\}\ ,$$

which is a subgroup of the GKM automorphisms of the typical fiber (F, α_F). Observe that for any two vertices p and q of the base, the groups Hol_p and Hol_q are conjugated through the automorphism of the fiber ρ_γ, where γ is any path in the base joining p to q.

In theorem 2.4.2, we presented a way to understand the equivariant cohomology of (Γ, α) from the cohomology of the base (B, α_B), and classes $c_1, \ldots, c_k$ in $H_\alpha(\Gamma)$ such that their restriction to each fiber (Γ_p, α_p) form a basis for $H_{\alpha_p}(\Gamma_p)$. Now we want to give a way to build these classes starting from the cohomology of a fixed fiber (Γ_p, α_p), and the subgroup of its GKM automorphisms $\mathcal{A}_p$.

Definition 2.4.3. *Let f_p be a cohomology class of the fiber (Γ_p, α_p), i.e. an element of $H_{\alpha_p}(\Gamma_p)$. Then f_p is an **invariant class** if*

$$\Upsilon_\gamma^*(f_p) = f_p, \quad \textit{for all } \Upsilon_\gamma \in \mathcal{A}_p$$

We want to prove that every invariant class in $H_{\alpha_p}(\Gamma)$ can be extended to a global class, i.e. can be extended to an element of $H_\alpha(\Gamma)$. First of all, let q be a vertex of B, and γ a path in B which starts at q and ends at p. If f_p is an invariant class of $H_{\alpha_p}(\Gamma_p)$, define f_q to be

$$f_q = \Upsilon_\gamma^*(f_p)$$

Lemma 2.4.4. *The definition of f_q doesn't depend on the path γ chosen.*

Proof. Let γ_1 and γ_2 be two paths from q to p. Then $\gamma_2 \circ \gamma_1^{-1}$ is a loop based at p. Since f_p is an invariant class, we have

$$\Upsilon^*_{\gamma_2 \circ \gamma_1^{-1}}(f_p) = \Upsilon^*_{\gamma_1^{-1}} \circ \Upsilon^*_{\gamma_2}(f_p) = f_p$$

and hence $\Upsilon^*_{\gamma_1}(f_p) = \Upsilon^*_{\gamma_2}(f_p)$. □

Let's define f to be an element of $\mathrm{Maps}(V, \mathbb{S}(\mathfrak{t}^*))$ such that for every fiber Γ_q, if $i_q : \Gamma_q \hookrightarrow \Gamma$ denotes the inclusion of the fiber Γ_q in Γ, then $i_q^*(f) = f_q$. Observe that the restriction of f to each fiber is an element of $H_{\alpha_q}(\Gamma_q)$.

Proposition 2.4.5. *The element $f : V \to \mathbb{S}(\mathfrak{t}^*))$ defined by $i_q^*(f) = f_q$ is a cohomology class in $H_\alpha(\Gamma)$.*

Proof. Since f restrict to an element of $H_{\alpha_q}(\Gamma_q)$ on each fiber Γ_q, for all the vertices q in B, we only need to prove that f satisfies the compatibility condition (2.9) on horizontal edges. Let (p_1, p_2) be an edge in B, and let (p_1', p_2') be its lift at p_1'. Then by definition

$$f(p_2') - f(p_1') = f_{p_2}(p_2') - f_{p_1}(p_1') = (\Upsilon_{\gamma_2}^* f_p)(p_2') - (\Upsilon_{\gamma_1}^* f_p)(p_1')$$

Now since $\Upsilon_{\gamma_2}^* = \Upsilon_{p_2,p_1}^* \circ \Upsilon_{\gamma_1}^*$ and $(\Upsilon_{p_2,p_1}^* f_{p_1})(p_2') = \Psi_{p_2,p_1}^{-1}(f_{p_1}(p_1'))$, then

$$f(p_2') - f(p_1') = \Psi_{p_2,p_1}^{-1}(f_{p_1}(p_1')) - f_{p_1}(p_1')$$

Hence, since by (2.10) $\Psi_{p_2,p_1}^{-1}(x) = \Psi_{p_1,p_2}(x) = x + c\alpha(p_1', p_2')$ for all $x \in \mathfrak{t}_1^*$ and $f_{p_1}(p_1') \in \mathbb{S}(\mathfrak{t}_1^*)$, we can conclude that $f(p_2') - f(p_1') = Q\alpha(p_1', p_2')$ for some $Q \in \mathbb{S}(\mathfrak{t}^*)$. ☐

Remark 2.4.6. *By the previous proposition, in order to find classes $c_1, \dots, c_k$ in $H_\alpha(\Gamma)$ such that their restrictions to the fibers Γ_p form a basis for $H_{\alpha_p}(\Gamma_p)$, it is sufficient to find classes $c_{1,p}, \dots, c_{k,p}$ which are a basis for $H_{\alpha_p}(\Gamma_p)$ and are invariant under $\mathcal{A}_p$.*

2.5 The GKM fiber bundle $\pi : \mathcal{S}_3 \to K_2$

In this section we explore in details the fibration of the complete flag variety $\mathcal{F}l(\mathbb{C}^3)$ onto the complex projective $\mathbb{C}P^2$ described in section 2.3.1 at the level of GKM graphs, prove that it is a GKM fiber bundle, describe its holonomy and give an example of an invariant class.

Let $\mathcal{S}_3$ be the permutahedron with six vertices, corresponding to the elements $123, 132, 213, 231, 312, 321$ (written in the one line notation). Then, since the axial function on the permutahedron is given by $\alpha(\sigma, \sigma') = \text{sign}(\sigma^{-1}(k) - \sigma^{-1}(h))(x_h - x_k)$, where $\sigma' = (h,k)\sigma$ (cfr. example 2.2.5), we have

$$\alpha(123,132) = \alpha(213,312) = \alpha(231,321) = x_2 - x_3$$

$$\alpha(123,213) = \alpha(132,231) = \alpha(312,321) = x_1 - x_2$$

$$\alpha(132,312) = \alpha(123,321) = \alpha(213,231) = x_1 - x_3$$

Let K_3 be the complete graph on three vertices $1,2,3$ (cfr. example 2.2.4). Here the axial function α_B is simply given by $\alpha_B(i,j) = x_i - x_j$. Consider now the projection π given by

$$\begin{array}{llll} \pi: & \mathcal{S}_3 & \longrightarrow & K_3 \\ & & & \\ & \sigma(1)\sigma(2)\sigma(3) & \mapsto & \sigma(1) \end{array}$$

In section 2.3.1, Proposition 2.3.1, we have already observed that this map is induced by the projection map of $\mathcal{F}l(\mathbb{C}^3)$ onto $\mathbb{C}P^2$, and in particular satisfies $\alpha(e) = \alpha_B(\pi(e))$ for every horizontal edge e of $\mathcal{S}_3$. Moreover it is easy to check that it is a GKM fibration with respect to the connections defined in Examples 2.2.4 and 2.2.5.

Now we want to see that it is a GKM fiber bundle. Geometrically, the projection $\pi : \mathcal{F}l(\mathbb{C}^3) \to \mathbb{C}P^2$ is a $\mathbb{C}P^1-$bundle. In the GKM picture, the three fibers over the vertices of K_3 are given by the subgraphs $\Gamma_{\{i\}}$, $i = 1,2,3$, with set of vertices $V_{\{i\}} = \{\sigma \in \mathcal{S}_3 \text{ s.t. } \sigma(1) = i\}$. These are GKM subgraphs with respect to the connection ∇ defined in example 2.2.5. Observe that, as graphs, they are isomorphic to K_2, which has vertices 1 and 2. But the axial function that the graphs $\Gamma_{\{i\}}$ inherit as GKM subgraphs of $\mathcal{S}_3$ does not coincide with the axial function on K_2; in particular it depends on i. For example $\Gamma_{\{1\}}$ has two vertices, 123 and 132, and $\alpha(123,132) = x_2 - x_3$, whereas for K_2 we have $\alpha(1,2) = x_1 - x_2$. Since the graphs $\Gamma_{\{i\}}$ correspond geometrically to the GKM graphs associated to the fibers $\pi^{-1}(p_i)$ of the $T-$equivariant fibration $\pi : \mathcal{F}l(\mathbb{C}^3) \to \mathbb{C}P^2$, where p_i is a fixed point of $\mathbb{C}P^2$ (cfr. section 2.3.1), the fact that the axial function on $\Gamma_{\{i\}}$ depends on i corresponds to the fact that the subtorus which is stabilizing the $T-$invariant fibers $\pi^{-1}(p_i)$ depends on the fixed point $p_i,\ i = 1, 2, 3$.

Now, for every pair of vertices $i, j,\ i < j$, in K_3 we want to define the maps $\Phi_{i,j} : V_{\{i\}} \to V_{\{j\}}$, where $\Phi_{i,j} = \Phi_{j,i}^{-1}$. By definition of $\Phi_{i,j}$ we have that

$$\Phi_{1,2}(123) = 213, \quad \Phi_{1,2}(132) = 231$$

$$\Phi_{2,3}(213) = 312 \quad \Phi_{2,3}(231) = 321$$

$$\Phi_{1,3}(123) = 321 \quad \Phi_{1,3}(132) = 312$$

So it is clear that the maps $\Phi_{i,j}$ are isomorphisms of graphs, and hence $\pi : \mathcal{S}_3 \to K_2$ is a GKM fiber bundle. Moreover, if $\mathfrak{t}_i^*$ denotes the subspace of $\mathfrak{t}^*$ generated by the values of the axial function α on the edges of $\Gamma_{\{i\}}$, we have $\mathfrak{t}_1^* = \mathbb{R}\langle x_2 - x_3 \rangle$, $\mathfrak{t}_2^* = \mathbb{R}\langle x_1 - x_3 \rangle$ and $\mathfrak{t}_3^* = \mathbb{R}\langle x_1 - x_2 \rangle$. The maps $\Psi_{i,j} : \mathfrak{t}_i^* \to \mathfrak{t}_j^*$ are given by

$$\Psi_{1,2}(x_2 - x_3) = x_1 - x_3$$

$$\Psi_{2,3}(x_1 - x_3) = x_1 - x_2$$

$$\Psi_{1,3}(x_1 - x_2) = -(x_2 - x_3)$$

and $\Upsilon_{i,j} = (\Phi_{i,j}, \Psi_{i,j}) : (\Gamma_i, \alpha_{\{i\}}) \to (\Gamma_j, \alpha_{\{j\}})$ defines an isomorphism of GKM graphs for every pair of vertices i, j in K_3.

Now we want to study the group of GKM automorphisms $\mathcal{A}_1$ of the fiber $\Gamma_{\{1\}}$. Observe that the loop in K_3 given by $1 \to 2 \to 3 \to 1$ lifts to the path in $\mathcal{S}_3$ with initial point 123 given by $123 \to 213 \to 312 \to 132$, and it's easy to check that $\mathcal{A}_1$ is composed by two elements, $\Upsilon_0 = Id$ and $\Upsilon_1 = (\Phi_1, \Psi_1)$ where $\Phi_1(123) = 132$ and $\Psi_1(x_2 - x_3) = -(x_2 - x_3)$.

Consider now the element $c_1 : V_{\{1\}} \to \mathfrak{t}_1^*$ given by $c_1(123) = -(x_2 - x_3)$ and $c_1(132) = x_2 - x_3$. Since $c_1(132) - c_1(123) = 2\alpha(123, 132)$, it follows that c_1 is a cohomology class in $H_{\alpha_{\{1\}}}(\Gamma_1)$. Moreover c_1 is an invariant element. So we can extend it to be a cohomology class c in $H_\alpha(\mathcal{S}_3)$ using the recipe described in section 2.4.1. If we do so we obtain

$$c(213) = -(x_1 - x_3) = -c(231)$$

$$c(312) = -(x_1 - x_2) = -c(321)$$

If 1 denotes the cohomology class in $H_\alpha(\Gamma)$ which takes value 1 at every vertex of Γ, by theorem 2.4.2 we obtain that $H_\alpha(\Gamma)$ is a free $H_{\alpha_B}(K_2)-$module on $1, c$.

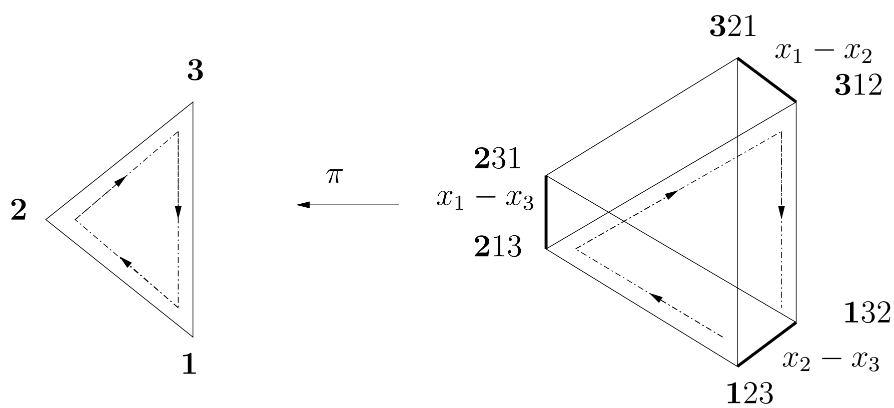


Figure 2-2: The projection $\pi : \mathcal{S}_3 \to K_2$.

In particular in the above figure, we show the lift of the path $1 \to 2 \to 3 \to 1$ at 123, and the axial function on the fibers.

In the next chapter we will see that this example is just a special case of a more general type of GKM fiber bundles, which is obtained by fibering $G_\mathbb{C}/B$ over $G_\mathbb{C}/P$, where $G_\mathbb{C}$ is a semisimple complex Lie group, P a parabolic subgroup of $G_\mathbb{C}$, and B a Borel subgroup of $G_\mathbb{C}$ contained in P.

Chapter 3

GKM fiber bundles of flag varieties

Let $G_{\mathbb{C}}$ be a connected semisimple Lie group, P a parabolic subgroup, and $G_{\mathbb{C}}/P$ the associated flag variety. In this chapter we review the GKM structure of the (generalized) flag variety $G_{\mathbb{C}}/P$, and prove that if B is a Borel subgroup of $G_{\mathbb{C}}$ contained in P, the natural projection map $G_{\mathbb{C}}/B \to G_{\mathbb{C}}/P$ is a GKM fiber bundle. Moreover, for the classical groups, we use a sequence of such bundles to produce cohomology classes on $G_{\mathbb{C}}/B$ which are invariant under the action of the Weyl group W of $G_{\mathbb{C}}$.

3.1 Flag varieties as GKM spaces

Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$, and let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t}$. Let $R \subset \mathfrak{t}^*$ denote the set of roots and W the Weyl group of G. Let $R^+ \subset R$ be a choice of positive roots in R, and R_0 the associated simple roots.

If $\langle \cdot, \cdot \rangle$ denotes a positive definite symmetric bilinear form on $\mathfrak{g}$ which is $G-$invariant, we can use it to identify $\mathfrak{g}$ and $\mathfrak{g}^*$, and also $\mathfrak{t}$ and $\mathfrak{t}^*$.

Given a point $p_0 \in \mathfrak{t}^*$, consider the $G-$coadjoint orbit $\mathcal{O}_{p_0} = G \cdot p_0$. Let $G_{p_0} \subset G$ be the stabilizer of p_0; the map which takes $g \in G$ to $g \cdot p_0 \in \mathcal{O}_{p_0}$ induces an identification $\mathcal{O}_{p_0} = G/G_{p_0}$. Moreover, there is a natural symplectic form ω on $\mathcal{O}_{p_0}$, and the action of G on $\mathcal{O}_{p_0}$ is Hamiltonian, with moment map given by the inclusion map $\mathcal{O}_{p_0} \hookrightarrow \mathfrak{g}^*$. Hence, the moment map $\psi\colon \mathcal{O}_{p_0} \to \mathfrak{t}^*$ for the T action is the composition of this inclusion with the natural projection from $\mathfrak{g}^*$ to $\mathfrak{t}^*$. Moreover, $(\mathcal{O}_{p_0}, \omega, \psi)$ is a GKM space. The GKM structure can be described as follows.

If p_0 is a generic point in $\mathfrak{t}^*$, let (Γ, α) be the GKM graph associated to $(\mathcal{O}_{p_0}, \omega, \psi)$.

- The map from the Weyl group W to $\mathfrak{t}^*$ which takes w to $w(p_0)$ induces a bijection between the elements of the Weyl group and the $T-$fixed point set of $\mathcal{O}_{p_0}$, which corresponds to the vertex set V. The moment map ψ restricted to V is the inclusion map, that is $\psi(p) = p$ for all $p \in V$.

- There exist an edge e between two vertices $p_1 = w_1(p_0)$ and $p_2 = w_2(p_0)$ if and only if $w_2 = w_1 s_\beta$, where s_β is the reflection associated to some $\beta \in R^+$. The value of the axial function on (p_1, p_2), $\alpha(p_1, p_2)$, is $w_1(\beta)$.

Remark 3.1.1. For all $\beta \in R$ and $w \in W$ the following relation

$$ws_\beta = s_{w(\beta)} w \tag{3.1}$$

holds (cfr. [18]). Since the Weyl group takes R to itself, it follows that two elements of the Weyl group W differ by a reflection over a root on the right if and only if they differ by a reflection on the left.

Consider a subset of the simple roots $\Sigma \subset R_0$, and let $W(\Sigma)$ be the subgroup of W generated by the reflections s_α, with $\alpha \in \Sigma$.

If p_0 is not a generic element of $\mathfrak{t}^*$, i.e. p_0 belongs to $\bigcap\limits_{\alpha_i \in \Sigma} \mathcal{H}_{\alpha_i}$, where $\mathcal{H}_{\alpha_i}$ is the hyperplane orthogonal to the simple root α_i and $\Sigma \neq \emptyset$, then observe that $w(p_0) = p_0$ for all the elements $w \in W(\Sigma)$.

Let $\langle \Sigma \rangle$ denote the subset of R^+ given by the roots which can be written as linear combinations of roots in Σ. Let $\mathcal{O}_{p_0}$ be the coadjoint orbit corresponding to $p_0 \in \bigcap\limits_{\alpha_i \in \Sigma} \mathcal{H}_{\alpha_i}$. Then in this case the GKM graph (Γ, α) associated to $(\mathcal{O}_{p_0}, \omega_0, \psi)$ has the following description.

- The elements of V are in bijection with the right cosets

$$W/W(\Sigma) = \{wW(\Sigma) \text{ s.t. } w \in W\} = \{[w] \text{ s.t. } w \in W\}$$

The moment map $\psi : V \to \mathbf{t}^*$ is given by $\psi([v]) = v(p_0)$. Observe that since $p_0 \in \bigcap_{\alpha_i \in \Sigma} \mathcal{H}_{\alpha_i}$, then ψ is well defined.

- Two elements $[v]$ and $[w]$ are joined by an edge if and only if $[w] = [vs_\beta] = [s_{v(\beta)}v]$ for some $\beta \in R^+ \backslash \langle\Sigma\rangle$. Observe that this condition implies that $[v] \neq [w]$. The axial function α on this edge is given by $\alpha([v], [vs_\beta]) = \alpha([v], [s_{v(\beta)}v]) = v(\beta)$.

- The connection ∇_e along the edge $e = ([v], [vs_\beta])$ is given by $\nabla_e([v], [u]) = ([s_{v(\beta)}v], [s_{v(\beta)}u])$, for all $([v], [u]) \in E_{[v]}$. It's easy to check that the axial function is compatible with this connection.

Observe that the previous case, the coadjoint orbit through a generic point $p_0 \in \mathbf{t}^*$, can be obtained from this one by setting $\Sigma = \emptyset$.

Example 3.1.2 The GKM graph associated to a generic coadjoint orbit of type B_2, $W(B_2)$

Let $G = SO(5)$, and let $\alpha_1 = x_1 - x_2$ and $\alpha_2 = x_2$ be a choice of simple roots. Let s_1 be the reflection associated to α_1, and s_2 the reflection associated to α_2. Then the GKM graph associated to a generic coadjoint orbit of type B_2, $W(B_2)$, is shown in Figure 3.1.

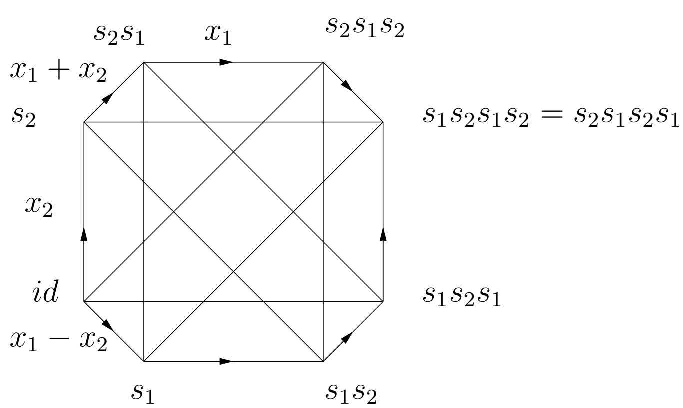


Figure 3-1: The GKM graph $W(B_2)$

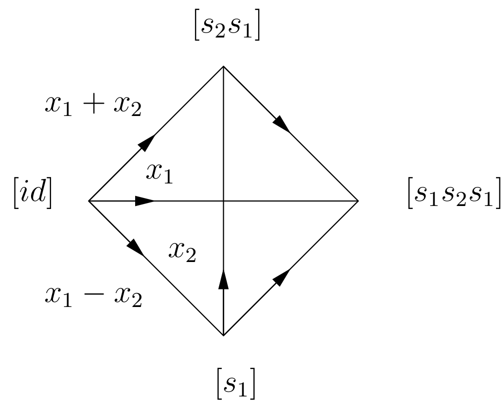


Figure 3-2: The GKM graph $W(B_2)/W(\Sigma)$.

Example 3.1.3 With the same notation of Example 3.1.2, let $\Sigma = \{\alpha_2\}$. Then $W(B_2)/W(\Sigma)$ is the GKM graph associated to $Gr_2^+(\mathbb{R}^5)$, the Grassmannian of oriented two planes in $\mathbb{R}^5$, and it is shown in Figure 3-2.

The coadjoint orbit $\mathcal{O}_{p_0}$, where $p_0 \in \bigcap\limits_{\alpha_i \in \Sigma} \mathcal{H}_{\alpha_i}$, can also be regarded as the manifold of generalized flags $G_{\mathbb{C}}/P(\Sigma)$, where $G_{\mathbb{C}}$ is the complexification of G, and $P(\Sigma)$ is the parabolic subgroup corresponding to Σ, i.e.

$$Lie(P(\Sigma)) = \mathfrak{b} \oplus \bigoplus_{\alpha \in \langle \Sigma \rangle} \mathfrak{g}_{-\alpha} \ ,$$

where $\mathfrak{b}$ is the Lie algebra of the Borel subgroup associated to R^+, and $Lie(G_{\mathbb{C}}) = \mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ is the canonical decomposition of $\mathfrak{g}_{\mathbb{C}}$, with $\mathfrak{h} \supset \mathfrak{t}$.

If $G_{\mathbb{C}}/P(\Sigma)$ is the generalized flag manifold associated to $G_{\mathbb{C}}$ and Σ, we will denote its GKM graph by $(W/W(\Sigma), \alpha)$.

Consider now two points p_0 and $\widetilde{p}_0$ in $\mathfrak{t}^*$ such that they both lie in the closure of the same Weyl chamber, and such that $G_{\widetilde{p}_0} \subset G$ (the stabilizer of $\widetilde{p}_0$) contains G_{p_0} (the stabilizer of p_0). Consider the coadjoint orbits $\mathcal{O}_{p_0}$ and $\mathcal{O}_{\widetilde{p}_0}$. Then there is a natural projection map

$$\begin{aligned} \pi : \quad \mathcal{O}_{p_0} &\rightarrow \mathcal{O}_{\widetilde{p}_0} \\ g \cdot p_0 &\mapsto g \cdot \widetilde{p}_0 \end{aligned} \tag{3.2}$$

It is well known that π is a $T-$equivariant fibration with symplectic fibers, isomorphic to $G_{\widetilde{p}_0}/G_{p_0}$.

In the language of flag varieties, this corresponds to take two subsets of simple roots Σ_1 and Σ_2 such that $\Sigma_1 \subset \Sigma_2 \subset R_0$, consider the parabolic subgroups $P(\Sigma_1) \subset P(\Sigma_2) \subset G_{\mathbb{C}}$, and the associated projection map $\pi : G_{\mathbb{C}}/P(\Sigma_1) \to G_{\mathbb{C}}/P(\Sigma_2)$, with fiber $P(\Sigma_2)/P(\Sigma_1)$.

Now consider the case in which $\Sigma_1 = \emptyset$, and hence $G_{\mathbb{C}}/P(\Sigma_1)$ is a complete flag variety. Let (W, α) be its associated GKM graph. Let Σ be a non empty subset of the simple roots R_0, and consider the partial flag $G_{\mathbb{C}}/P(\Sigma)$, with associated GKM graph $(W/W(\Sigma), \alpha)$. In the next proposition, we want to prove that the discrete analogue of the map (3.2) is a GKM fiber bundle.

Proposition 3.1.4. *The projection map* $\pi : (W, \alpha) \to (W/W(\Sigma), \alpha)$ *is a GKM fiber bundle.*

Proof. For every vertex $w \in W$ it's clear that the set of vertical edges $E_w^{\perp}$ is given by

$$E_w^{\perp} = \{(w, ws_\beta) \text{ s.t. } \beta \in \langle \Sigma \rangle\}$$

and the set of horizontal edges H_w

$$H_w = \{(w, ws_\beta) \text{ s.t. } \beta \in R^+ \setminus \langle \Sigma \rangle\}.$$

Hence the map $(d\pi)_{H_w} : H_w \to E_{[w]}$ is a bijection, and we can define it by imposing that $\alpha(e) = \alpha(\pi(e))$ for all the edges $e \in H_w$, i.e. $(d\pi)_{H_w}(w, ws_\beta) = ([w], [ws_\beta])$. Hence $\alpha(w, ws_\beta) = w(\beta) = \alpha([w], [ws_\beta])$. Moreover it is easy to check that the connections on W and $W/W(\Sigma)$ are compatible in the sense of definition 2.3.4. So π is a GKM fibration. The graph associated to each fiber is clearly isomorphic to $W(\Sigma)$. Now we need to prove that given two elements $[w]$ and $[v]$ connected by an edge e in $W/W(\Sigma)$, there exists an isomorphism of GKM graphs between the fibers $\Gamma_{[w]}$ and $\Gamma_{[v]}$. Let $[v] = [ws_\beta]$ for some $\beta \in R^+ \setminus \langle \Sigma \rangle$. Let $V_{[w]}$ be the set of vertices of $\Gamma_{[w]}$; so we have $V_{[w]} = \{wu, \quad u \in W(\Sigma)\}$. Consider the lift of $([w],[ws_\beta])$ at u. By definition we have that $\Phi_{[w],[ws_\beta]} : V_{[w]} \to V_{[ws_\beta]}$ is given by $\Phi_{[w]}(wu) = s_{w(\beta)}wu = ws_\beta u$. Hence any vertical edge (wu, wus_γ) in $\Gamma_{[w]}$, where $\gamma \in \langle\Sigma\rangle$, is sent to the edge $(s_{w(\beta)}wu, s_{w(\beta)}wus_\gamma)$. So $\Phi_{[w],[ws_\beta]} : \Gamma_{[w]} \to \Gamma_{[ws_\beta]}$ is an isomorphism of graphs. The map $\Psi_{[w],[ws_\beta]} : \mathfrak{t}^* \to \mathfrak{t}^*$ is simply given by $\Psi_{[w],[ws_\beta]}(\gamma) = s_{w(\beta)}(\gamma)$. Moreover it's straightforward to check that condition (iii) of definition 2.3.5 holds. Hence π is a GKM fiber bundle. □

Example 3.1.5 Consider $W(B_2)$ and $W(B_2)/W(\Sigma)$, as described in Example 3.1.2 and 3.1.3. Then the projection map $\pi : W(B_2) \to W(B_2)/W(\Sigma)$ is described in the following Figure.

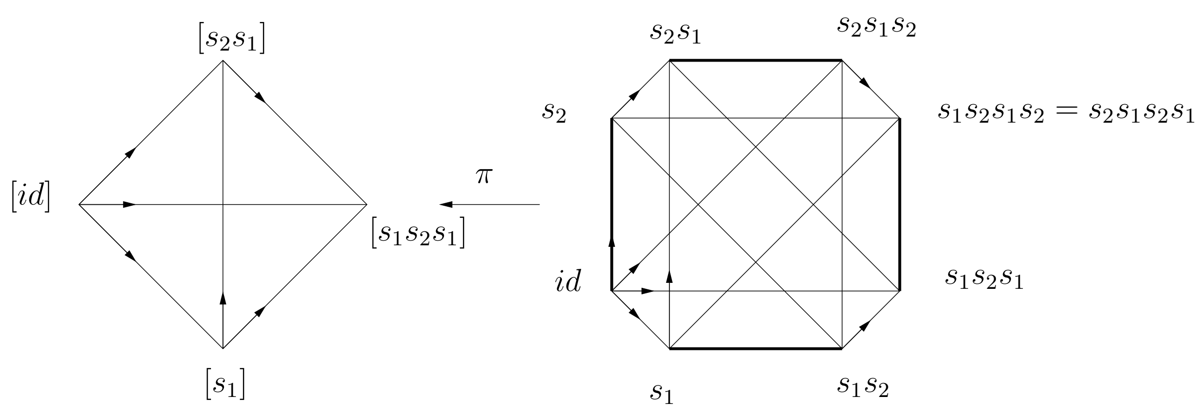


Figure 3-3: The projection $\pi : W(B_2) \to W(B_2)/W(\Sigma)$.

From the proof of the previous proposition, we can summarize the isomorphism of GKM graphs between the fibers $\Gamma_{[w]}$ and $\Gamma_{[ws_\beta]}$ in the following way.

The isomorphism of graphs $\Phi_{[w],[ws_\beta]} : \Gamma_{[w]} \to \Gamma_{[ws_\beta]}$ is given by

$$\begin{array}{rccc} \Phi_{[w],[ws_\beta]} : & \Gamma_{[w]} & \longrightarrow & \Gamma_{[ws_\beta]} \\ & v & \longmapsto & s_{w(\beta)}v \end{array}$$

whereas the isomorphism $\Psi_{[w],[ws_\beta]} : \mathfrak{t}^* \to \mathfrak{t}^*$ is given by

$$\begin{array}{rccc} \Psi_{[w],[ws_\beta]} : & \mathfrak{t}^* & \longrightarrow & \mathfrak{t}^* \\ & \delta & \longmapsto & s_{w(\beta)}(\delta) \end{array}$$

Since the typical fiber is isomorphic to $W(\Sigma)$, it's easy to see that the map $(\varphi_{[w]}, \psi_{[w]}) : (W(\Sigma), \alpha) \to (\Gamma_{[v]}, \alpha)$ defined by $\varphi_{[w]}(u) = wu$ and $\psi_{[w]}(\delta) = w(\delta)$ is an isomorphism of GKM graphs, where w is a fixed representative of the elements in $[w]$.

3.1.1 The holonomy subgroup of the GKM bundle $\pi : W \to W/W(\Sigma)$

For every element u of $W(\Sigma)$, there is a natural GKM automorphism $(\Phi_u, \Psi_u) : W(\Sigma) \to W(\Sigma)$, which is given by $\Phi_u(u') = uu'$ and $\Psi_u(\beta) = u(\beta)$. So there is an embedding of $W(\Sigma)$ into the GKM automorphisms of the GKM graph $(W(\Sigma), \alpha)$.

Now consider the group of automorphisms $\mathrm{Hol}_{[v_0]}$ of $(W(\Sigma), \alpha)$ determined by the loops in $W/W(\Sigma)$ based at some point $[v_0] \in W/W(\Sigma)$. More explicitly a loop in $W/W(\Sigma)$ based at $[v_0]$ is a sequence of vertices $[v_0], [v_1], \ldots, [v_k] = [v_0]$ such that $v_i = s_{v_{i-1}(\beta_{i-1})}(v_{i-1})$, where $\beta_{i-1} \in R^+ \setminus \langle\Sigma\rangle$, for all $i = 1, \ldots, k$. So $v_i = v_0 s_{\beta_0} \ldots s_{\beta_{i-1}}$, for all $i = 1, \ldots, k$.

If $([v_0], [v_1])$ is an edge in $W/W(\Sigma)$ such that $v_1 = s_{v_0(\beta)} v_0$ for some $\beta \in R^+ \setminus \langle\Sigma\rangle$, then for every element $v_0 u$ of the fiber $\pi^{-1}([v_0])$, with $u \in W(\Sigma)$, $\Phi_{[v_0],[v_1]}(v_0 u) = s_{v_0(\beta)}(v_0 u) = v_1 u$.

So if γ is the loop $([v_0], [v_1], \ldots, [v_k] = [v_0])$ we have that

$$\Phi_\gamma = \Phi_{[v_{k-1}],[v_k]} \circ \cdots \Phi_{[v_0],[v_1]} : \Gamma_{[v_0]} \to \Gamma_{[v_0]}$$

is given by

$$\Phi_\gamma(v_0 u) = s_{v_{k-1}(\beta_{k-1})} \cdots s_{v_1(\beta_1)} s_{v_0(\beta_0)}(v_0 u) = v_0 s_{\beta_0} s_{\beta_1} \ldots s_{\beta_{k-1}} u = v_k u \ .$$

Hence the isomorphism of the typical fiber $\varphi_\gamma : W(\Sigma) \to W(\Sigma)$ is given by

$$\varphi_\gamma(u) = (\varphi_{[v_0]}^{-1} \Phi_\gamma \varphi_{[v_0]})(u) = v_0^{-1} v_k u$$

and $\psi_\gamma : \mathfrak{t}^* \to \mathfrak{t}^*$ is given by $\psi_\gamma(\delta) = v_0^{-1}v_k(\delta)$. Since $[v_0] = [v_k]$, this implies that $w = v_0^{-1}v_k \in W(\Sigma)$.

The previous argument proves that for every loop γ in $W/W(\Sigma)$, there exists an element $w \in W(\Sigma)$ such that $(\varphi_\gamma, \psi_\gamma) : (W(\Sigma), \alpha) \to (W(\Sigma), \alpha)$ is given by

$$\begin{aligned} \varphi_\gamma(u) &= wu \\ \psi_\gamma(\beta) &= w(\beta) \end{aligned} \tag{3.3}$$

In the next proposition we want to prove that also the converse is true.

Proposition 3.1.6. *Given the GKM fiber bundle $\pi : W \to W/W(\Sigma)$, for every element w of $W(\Sigma)$ there exists a loop γ in $W/W(\Sigma)$ such that the automorphism of the fiber $(\varphi_\gamma, \psi_\gamma) : (W(\Sigma), \alpha) \to (W(\Sigma), \alpha)$ induced by γ is given by (3.3).*

Proof. Since every element in $W(\Sigma)$ can be written as a product of reflections s_{α_i}, with $\alpha_i \in \langle\Sigma\rangle$, it is sufficient to prove that for every element α_i in $\langle\Sigma\rangle$, there exists a loop γ in $W/W(\Sigma)$ such that the corresponding isomorphism of the fiber is given by (3.3).

Since the Weyl group acts transitively on R, there exists an element $w \in W$ such that $w(\alpha_i) \in R^+ \setminus \langle\Sigma\rangle$. We can write w as uv, where $u \in W(\Sigma)$ and $v = s_{\beta_1} \dots s_{\beta_k}$, with $\beta_1, \dots, \beta_k \in R^+ \setminus \langle\Sigma\rangle$. Then it's easy to check that also $u^{-1}w(\alpha_i) = s_{\beta_1} \dots s_{\beta_k}(\alpha_i)$ belongs to $R^+ \setminus \langle\Sigma\rangle$ (cfr. Lemma 4.1 in [14]). So we need to prove that there exists a loop $\gamma = ([v_0], \dots, [v_m])$ based at $[v_0]$ such that $v_m = v_0 s_{\alpha_i}$.

Observe that $v_0 v^{-1} s_{v(\alpha_i)} v = v_0 s_{\alpha_i}$, so it is sufficient to consider the loop

$$[v_0] \to [v_0 s_{\beta_k}] \to \dots \to [v_0 s_{\beta_k} \cdots s_{\beta_1}] = [v_0 v^{-1}] \to [v_0 v^{-1} s_{v(\alpha_i)}] \to$$

$$\to [v_0 s_{v(\alpha_i)} s_{\beta_1}] \to \dots \to [v_0 v^{-1} s_{v(\alpha_i)} s_{\beta_1} \cdots s_{\beta_k}] = [v_0 v^{-1} s_{v(\alpha_i)} v] = [v_0 s_{\alpha_i}]$$

and the claim follows. □

Combining the previous proposition with the definition of invariant classes we can say that

Definition 3.1.7. *A cohomology class $f \in H_\alpha(W)$ is **invariant** if and only if*

$$f(wu) = wf(u) \ \text{ for all } w, u \in W \tag{3.4}$$

In the next section we will use a sequence of such GKM fiber bundles to construct a basis for the equivariant cohomology of (W, α) composed by invariant classes.

3.2 GKM fiber bundles of classical groups

Let $G_{\mathbb{C}}/B$ be the complete flag variety in type A, B, C and D, and let (W, α) be the associated GKM graph. In this section we will use the structure of the holonomy group associated to the GKM fiber bundle $W \to W/W(\Sigma)$, for some $\Sigma \subset R_0$, to construct a basis of $H_\alpha(W)$ which is invariant under the action of the Weyl group W, in the sense of definition 3.1.7.

3.2.1 Type A_n

In type A_n the complete flag variety $G_{\mathbb{C}}/B$ is the variety of complete flags in $\mathbb{C}^{n+1}$, $\mathcal{F}l(\mathbb{C}^{n+1})$. The set of simple roots R_0 is given by $R_0 = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i = x_i - x_{i+1}$ for $i = 1, \ldots, n$, and the set of positive roots is $R^+ = \{x_i - x_j,\ 1 \leq i < j \leq n+1\}$. If we consider $\Sigma = \{\alpha_i,\ i = 2, \ldots, n\}$ then $\langle \Sigma \rangle = \{x_i - x_j,\ 2 \leq i < j \leq n+1\}$, and

$$R^+ \setminus \langle \Sigma \rangle = \{x_1 - x_j,\ j = 2, \ldots, n+1\}$$

Now we want to consider the GKM fiber bundle $\pi : W \to W/W(\Sigma)$. Observe that $W/W(\Sigma)$ is the GKM graph associated to the complex projective space $\mathbb{C}P^n$ as described in Example 1.4.7, and hence this GKM fiber bundle is the discrete analogue of the bundle described in section 2.3.1. In fact, considering $W/W(\Sigma)$ is equivalent to consider the GKM graph of the coadjoint orbit through a point $\widetilde{p}_0 \in \bigcap_{i=2}^{n+1} \mathcal{H}_{\alpha_i}$, where $\mathcal{H}_{\alpha_i}$ is the hyperplane orthogonal to α_i. If $\widetilde{p}_0 = \mu_1 x_1 + \mu_2(x_2 + \ldots + x_{n+1})$, with $\mu_1 < \mu_2$ and $\mu_1 + n\mu_2 = 0$, then the $n+1-$fixed points of $\mathcal{O}_{\widetilde{p}_0}$ are given by $\widetilde{p}_i = \mu_1 x_i + \mu_2(x_1 + \ldots + \hat{x}_i + \ldots + x_{n+1}),\ i = 1, \ldots, n+1$. Define β_i to be $x_1 - x_i,\ i = 2, \ldots, n+1$. So the set $R^+ \setminus \langle\Sigma\rangle$ is given by

$$R^+ \setminus \langle\Sigma\rangle = \{\beta_i,\ i = 2, \ldots, n+1\} .$$

Consider now the elements of W given by $Id, s_{\beta_1}, \ldots, s_{\beta_{n+1}}$. Then there is a bijection between these elements and the fixed points of $\mathcal{O}_{\widetilde{p}_0}$ given by

$$\begin{aligned} Id &\mapsto \widetilde{p}_0 \\ s_{\beta_i} &\mapsto s_{\beta_i}(\widetilde{p}_0) = \widetilde{p}_i, \end{aligned}$$

for $i = 2, \ldots, n+1$. Hence the vertices of $W/W(\Sigma)$ are given by the elements

$$\omega_1 = [Id],\ \omega_2 = [s_{\beta_2}], \ldots,\ \omega_{n+1} = [s_{\beta_{n+1}}] .$$

Observe that since $s_{\beta_j} = s_{x_i - x_j} s_{\beta_i} s_{x_i - x_j}$ for all $i \neq j$, with $i, j \in \{2, \ldots, n+1\}$, we have $[s_{\beta_j}] = [s_{x_i - x_j} s_{\beta_i}]$, and hence $\alpha(\omega_i, \omega_j) = x_i - x_j$. So, as a GKM graph, $W/W(\Sigma)$ is isomorphic to the complete graph K_{n+1} (cfr. 2.2.4).

Now we want to use Theorem 2.4.2 to build classes on W which are a basis for the cohomology of W, and are invariant under the action of the Weyl group W.

Consider the cohomology of $W/W(\Sigma)$. The element $\tau \in \text{Maps}(W/W(\Sigma), \mathbb{S}(\mathfrak{t}^*))$ given by $\tau(\omega_i) = x_i$ is a cohomology class, since

$$\tau(\omega_i) - \tau(\omega_j) = x_i - x_j = \alpha(\omega_i, \omega_j) .$$

Moreover τ is invariant under the action of the Weyl group W, i.e. $w\tau(\omega_i) = w \cdot x_i = \tau(w\omega_i)$. This also follows from the fact that the class τ, modulo rescaling, coincides with the moment map ψ on $\mathcal{O}_{\widetilde{p}_0}$ plus a constant, and the moment map on coadjoint orbits is naturally invariant under the action of the Weyl group. Then, it's easy to see that the classes $\{1, \tau, \ldots, \tau^n\}$ form a basis for the cohomology of $W/W(\Sigma)$, as a module over $\mathbb{S}(\mathfrak{t}^*)$.

Now consider the fiber of the projection $W \to W/W(\Sigma)$. In order to find cohomology classes on W such that their restriction to each fiber form a basis for the cohomology of the fiber, by Remark 2.4.6 it is sufficient to find cohomology classes on the fiber which are invariant under the action of the holonomy group of the fiber, and then extend them to the whole graph W. Since the typical fiber is isomorphic to a generic coadjoint orbit of type A_{n-1}, we can proceed inductively.

Consider the GKM fiber bundle $\pi : \mathcal{S}_3 \to K_3$ as in Example 2.5, where the projection is given by $\sigma \mapsto \sigma(1)$, for all $\sigma \in \mathcal{S}_3$. Then the typical fiber is isomorphic to $\mathcal{S}_2 \simeq K_2$, and an invariant class on $\pi^{-1}(1)$ is given by $c(123) = x_2$ and $c(132) = x_3$. If we extend c to be an element of $H_\alpha(\mathcal{S}_3)$, then by definition we obtain an element c which is invariant under the action of $\mathcal{S}_3$. By applying Theorem 2.4.2 we obtain that the cohomology of $\mathcal{S}_3$ is generated by the (invariant) classes $1, c, \tau, \tau^2, \tau c, \tau^2 c$. Let $c_{[0,0]} = 1, c_{[0,1]} = c, c_{[1,0]} = \tau, c_{[1,1]} = \tau c, c_{[2,0]} = \tau^2, c_{[2,1]} = \tau^2 c$. The values of these classes on $\mathcal{S}_3$ are summarized in the following table.

	$c_{[0,0]}$	$c_{[0,1]}$	$c_{[1,0]}$	$c_{[1,1]}$	$c_{[2,0]}$	$c_{[2,1]}$
123	1	x_2	x_1	x_1x_2	x_1^2	$x_1^2x_2$
213	1	x_1	x_2	x_2x_1	x_2^2	$x_2^2x_1$
132	1	x_3	x_1	x_1x_3	x_1^2	$x_1^2x_3$
231	1	x_3	x_2	x_2x_3	x_2^2	$x_2^2x_3$
312	1	x_1	x_3	x_3x_1	x_3^2	$x_3^2x_1$
321	1	x_2	x_3	x_3x_2	x_3^2	$x_3^2x_2$

Table 3.1: Invariant classes on $\mathcal{S}_3$

If we repeat this procedure further we obtain the following

Theorem 3.2.1. *The cohomology $H_\alpha(\mathcal{S}_n)$ has a basis $\mathcal{B}(A_{n-1})$ given by invariant classes $c_I : \mathcal{S}_n \to \mathfrak{t}^*$, where if I is the multi-index $I = [i_1, \ldots, i_{n-1}]$,*

$$c_I(w) = w \cdot (x_1^{i_1} \cdots x_{n-1}^{i_{n-1}}) = x_{w(1)}^{i_1} x_{w(2)}^{i_2} \cdots x_{w(n-1)}^{i_{n-1}}$$

and

$$\mathcal{B}(A_{n-1}) = \{c_I \ s.t.\ I = [i_1, \ldots, i_{n-1}],\ 0 \leqslant i_1 \leqslant n-1,\ 0 \leqslant i_2 \leqslant n-2, \ldots, 0 \leqslant i_{n-1} \leqslant 1\}\,.$$

3.2.2 Type B_n

The set of simple roots of B_n (for $n \geqslant 2$) is $R_0 = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i = x_i - x_{i+1}$, for $i = 1, \ldots, n-1$ and $\alpha_n = x_n$. The set of positive roots is

$$R^+ = \{x_i \mid 1 \leqslant i \leqslant n\} \cup \{x_i \pm x_j \mid 1 \leqslant i < j \leqslant n\} \; .$$

If $\Sigma = \{\alpha_2, \ldots, \alpha_n\}$, then

$$\langle \Sigma \rangle = \{x_i \mid 2 \leqslant i \leqslant n\} \cup \{x_i \pm x_j \mid 2 \leqslant i < j \leqslant n\}$$

is the set of positive roots for a root system of type B_{n-1}, and

$$R^+ \setminus \langle \Sigma \rangle = \{\beta_1 = x_1\} \cup \{\beta_j^{\pm} = x_1 \mp x_j \mid 2 \leqslant j \leqslant n\} \; .$$

Let

$$\begin{aligned} \omega_1^+ =& [id] \quad , \quad \omega_1^- = [s_{\beta_1}] \\ \omega_j^+ =& [s_{\beta_j^+}] = [s_{x_1 - x_j}] \text{ for } 2 \leqslant j \leqslant n \\ \omega_j^- =& [s_{\beta_j^-}] = [s_{x_1 + x_j}] \text{ for } 2 \leqslant j \leqslant n \; . \end{aligned}$$

Then $W/W(\Sigma) = \{\omega_1^+, \omega_1^-, \ldots, \omega_n^+, \omega_n^-\}$, and the graph structure of $W/W(\Sigma)$ is that of a complete graph with $2n$ vertices.

Geometrically, let $\widetilde{p}_0$ be a point in $\displaystyle\bigcap_{i=2}^{n} \mathcal{H}_{\alpha_i}$; for example let $\widetilde{p}_0$ be x_1. Then $W/W(\alpha)$ is the GKM graph associated to the coadjoint orbit $\mathcal{O}_{\widetilde{p}_0}$ through $\widetilde{p}_0$, which corresponds to a Grassmannian of oriented two planes in $\mathbb{R}^{2n+1}$, $\mathcal{G}r_2^+(\mathbb{R}^{2n+1})$. The $T-$fixed points of $\mathcal{O}_{\widetilde{p}_0}$ are given by $-x_1, \ldots, -x_n, x_1, \ldots, x_n$, and the bijection with the vertices of

$W/W(\Sigma)$ is given by

$$\begin{aligned} \omega_1^+ = [Id] &\mapsto & x_1 &= Id(x_1) \\ \omega_1^- = [s_{\beta_1}] &\mapsto & -x_1 &= s_{\beta_1}(x_1) \\ \omega_j^+ = [s_{\beta_j^+}] &\mapsto & x_j &= s_{\beta_j^+}(x_1),\ \ 2 \leqslant j \leqslant n \\ \omega_j^- = [s_{\beta_j^-}] &\mapsto & -x_j &= s_{\beta_j^-}(x_1),\ \ 2 \leqslant j \leqslant n \end{aligned}$$

If τ is the map $\tau\colon W/W(\Sigma) \to \mathfrak{t}^*$, $\tau(\omega_j^\epsilon) = \epsilon x_j$, with $1 \leqslant j \leqslant n$ and $\epsilon \in \{+,-\}$, then the axial function α is given by

$$\alpha(\omega_i^{\epsilon_i}, \omega_j^{\epsilon_j}) = \begin{cases} \tau(\omega_i^{\epsilon_i}) - \tau(\omega_j^{\epsilon_j}) & \text{for } 1 \leqslant i \neq j \leqslant n \\ \frac{1}{2}(\tau(\omega_i^{\epsilon_i}) - \tau(\omega_i^{-\epsilon_i})) & \text{for } 1 \leqslant i = j \leqslant n\ . \end{cases}$$

Note that although $W/W(\Sigma)$ and K_{2n} are isomorphic as graphs, they are not isomorphic as GKM graphs. One way to see that is to notice that

$$\alpha(\omega_1^+, \omega_1^-) + \alpha(\omega_1^-, \omega_2^-) + \alpha(\omega_2^-, \omega_1^+) = -x_1 \neq 0\ .$$

Nevertheless, as in the K_{2n} case, the set of classes $\{1, \tau, \ldots, \tau^{2n-1}\}$ is a basis for the free $\mathbb{S}(\mathfrak{t}^*)-$module $H_\alpha^*(W/W(\Sigma))$. Moreover observe that the class τ is precisely the restriction to the fixed point set of the moment map on $\mathcal{O}_{\widetilde{p}_0}$, and hence it is $W-$invariant. Hence all the classes $1, \tau, \ldots, \tau^{2n-1}$ are $W-$invariant.

An alternative description of the Weyl group W is that of the group of signed permutations (u, ϵ), with $u \in \mathcal{S}_n$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_j = \pm 1$. The element (u, ϵ) is represented as $(\epsilon_1 u(1), \ldots, \epsilon_n u(n))$.

Then s_{x_i} is just a change of the sign of x_i, $s_{x_i - x_j}$ corresponds to the transposition (i, j), with no sign changes, and $s_{x_i + x_j}$ corresponds to the transposition (i, j) with both signs changed. In particular, Id is the identity permutation with no sign changes, s_{β_1} is the identity permutation with the sign of 1 changed, $s_{\beta_j^+}$ is the transposition $(1, j)$ with no sign changes, and $s_{\beta_j^-}$ is the transposition $(1, j)$ with sign changes for 1 and j. In general, if $u \in \mathcal{S}_n$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{Z}_2^n$, then the element $w = (u, \epsilon) \in W$ acts by $(u, \epsilon) \cdot x_k = \epsilon_k x_{u(k)}$. Then $W/W(\Sigma)$ can be identified with $\{\pm 1, \pm 2, \ldots, \pm n\}$

by $\omega_j^\epsilon \to \epsilon j$, and the projection $\pi\colon W \to W/W(\Sigma)$ is $\pi((u,\epsilon)) = \epsilon_1 u(1)$.

For $I = [i_1, \dots, i_{n-1}]$, let $c_I\colon W \to \mathbb{S}(\mathfrak{t}^*)$ be given by

$$c_I((u,\epsilon)) = (\epsilon_1 x_{u(1)})^{i_1} \cdots (\epsilon_{n-1} x_{u(n-1)})^{i_{n-1}} .$$

Then $c_I \in (H_\alpha^*(W))^W$ is an invariant class, and we will construct a basis of the free $\mathbb{S}(\mathfrak{t}^*)-$module $H_\alpha^*(W)$ consisting of classes of type c_I, for specific multi-indices I.

When $n = 2$, the fiber over 2 is $\pi^{-1}(2) = \{(2,1),(2,-1)\}$ and is identified with $W(\Sigma) = \mathcal{S}_2 = \{1,-1\}$. A basis for $H_\alpha^*(W(\Sigma))$ is given by the invariant classes $\{c_{[0]}, c_{[1]}\}$, where $c_{[0]} \equiv 1$ and $c_{[1]}(1) = x_1$, $c_{[1]}(-1) = -x_1$. These classes are extended to the invariant classes $c_{[0,0]}$ and $c_{[0,1]}$ on W.

The classes 1, τ, τ^2, and τ^3 on the base lift to the basic classes $c_{[0,0]}$, $c_{[1,0]}$, $c_{[2,0]}$, and $c_{[3,0]}$ on W. Then a basis for the free $\mathbb{S}(\mathfrak{t}^*)-$module $H_\alpha^*(W)$ is

$$\mathcal{B}(B_2) = \{c_I \mid I = [i_1, i_2],\ , 0 \leqslant i_1 \leqslant 3, 0 \leqslant i_2 \leqslant 1\} .$$

The values of these classes on the elements of $W(B_2)$ are shown in Table 3.2. Repeating the procedure further, we get the following result.

	$c_{[0,0]}$	$c_{[0,1]}$	$c_{[1,0]}$	$c_{[1,1]}$	$c_{[2,0]}$	$c_{[2,1]}$	$c_{[3,0]}$	$c_{[3,1]}$
$(1,2)$	1	x_2	x_1	x_1x_2	x_1^2	$x_1^2x_2$	x_1^3	$x_1^3x_2$
$(1,-2)$	1	$-x_2$	x_1	$-x_1x_2$	x_1^2	$-x_1^2x_2$	x_1^3	$-x_1^3x_2$
$(-1,-2)$	1	$-x_2$	$-x_1$	x_1x_2	x_1^2	$-x_1^2x_2$	$-x_1^3$	$x_1^3x_2$
$(-1,2)$	1	x_2	$-x_1$	$-x_1x_2$	$-x_1^2$	$x_1^2x_2$	$-x_1^3$	$-x_1^3x_2$
$(2,1)$	1	x_1	x_2	x_1x_2	x_2^2	$x_2^2x_1$	x_2^3	$x_2^3x_1$
$(2,-1)$	1	$-x_1$	x_2	$-x_1x_2$	x_2^2	$-x_2^2x_1$	x_2^3	$-x_2^3x_1$
$(-2,-1)$	1	$-x_1$	$-x_2$	x_1x_2	x_2^2	$-x_2^2x_1$	$-x_2^3$	$x_2^3x_1$
$(-2,1)$	1	x_1	$-x_2$	$-x_1x_2$	x_2^2	$x_2^2x_1$	$-x_2^3$	$-x_2^3x_1$

Table 3.2: Invariant classes on $W(B_2)$

Theorem 3.2.2. *A basis of the $\mathbb{S}(\mathfrak{t}^*)-$module $H_\alpha^*(W(B_n))$ is given by the invariant classes*

$$\mathcal{B}(B_n) = \{c_I \mid I = [i_1, \dots, i_n], 0 \leqslant i_1 \leqslant 2n-1, 0 \leqslant i_2 \leqslant 2n-3, \dots, 0 \leqslant i_n \leqslant 1\} .$$

3.2.3 Type C_n

The set of simple roots of C_n (for $n \geqslant 2$) is $R_0 = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i = x_i - x_{i+1}$, for $i = 1, \ldots, n-1$ and $\alpha_n = 2x_n$. The set of positive roots is

$$R^+ = \{2x_i \mid 1 \leqslant i \leqslant n\} \cup \{x_i \pm x_j \mid 1 \leqslant i < j \leqslant n\} \,.$$

If $\Sigma = \{\alpha_2, \ldots, \alpha_n\}$, then

$$\langle \Sigma \rangle = \{2x_i \mid 2 \leqslant i \leqslant n\} \cup \{x_i \pm x_j \mid 2 \leqslant i < j \leqslant n\}$$

is the set of positive roots for a root system of type C_{n-1}, and

$$R^+ \setminus \langle \Sigma \rangle = \{\beta_1 = 2x_1\} \cup \{\beta_j^{\pm} = x_1 \mp x_j \mid 2 \leqslant j \leqslant n\} \,.$$

Let

$$\begin{aligned} \omega_1^+ =& [id] \quad , \quad \omega_1^- = [s_{\beta_1}] \\ \omega_j^+ =& [s_{\beta_j^+}] = [s_{x_1 - x_j}] \text{ for } 2 \leqslant j \leqslant n \\ \omega_j^- =& [s_{\beta_j^-}] = [s_{x_1 + x_j}] \text{ for } 2 \leqslant j \leqslant n \,. \end{aligned}$$

This is essentially the same as the B_n case, and $W(C_n) \simeq W(B_n)$ is the group of signed permutations of n letters. Then $W/W(\Sigma) = \{\omega_1^+, \omega_1^-, \ldots, \omega_n^+, \omega_n^-\}$, and the graph structure of $W/W(\Sigma)$ is that of a complete graph with $2n$ vertices. In type C_n, the graph $W/W(\Sigma)$ corresponds to the GKM graph of a complex projective space $\mathbb{C}P^{2n-1}$; in this case the axial function on $W/W(\Sigma)$ is given by

$$\alpha(\omega_i^{\epsilon_i}, \omega_j^{\epsilon_j}) = \tau(\omega_i^{\epsilon_i}) - \tau(\omega_j^{\epsilon_j}) \,,$$

It's easy to see that, although $W(B_n)$ and $W(C_n)$ are not isomorphic as GKM graphs, $\mathcal{B}(C_n) = \mathcal{B}(B_n)$ is a basis of $H_\alpha(W(C_n))$ consisting of invariant classes.

3.2.4 Type D_n

In type D_n a set of simple roots is given by $R_0 = \{\alpha_1, \ldots, \alpha_n\}$ is given by $\alpha_i = x_i - x_{i+1}$, for $i = 1, \ldots, n-1$, and $\alpha_n = x_{n-1} + x_n$. The corresponding set of positive roots is

$$R^+ = \{x_i \pm x_j,\ 1 \leqslant i < j \leqslant n\}$$

Let $\Sigma = \{\alpha_2, \ldots, \alpha_n\}$; then

$$\langle \Sigma \rangle = \{x_i \pm x_j,\ 2 \leqslant i < j \leqslant n\}$$

Observe that $\langle \Sigma \rangle$ is a set of positive roots of type D_{n-1} if $n \geqslant 4$, whereas if $n = 3$ it is of type $A_1 \times A_1$. Then we have

$$R^+ \setminus \langle \Sigma \rangle = \{\beta_i^{\pm} = x_1 \mp x_i,\ 2 \leqslant i \leqslant n\}$$

Consider a point $\widetilde{p}_0 \in \mathfrak{t}^*$ such that $\widetilde{p}_0 \in \bigcap_{i=2}^{n} \mathcal{H}_{\alpha_i}$; for example let $\widetilde{p}_0 = x_1$. Then the coadjoint orbit $\mathcal{O}_{\widetilde{p}_0}$ is isomorphic to the Grassmannian of oriented two planes in $\mathbb{R}^{2n}$, $\mathcal{G}r_2^+(\mathbb{R}^{2n})$. The $T-$fixed points are given by the elements $\pm x_i,\ i = 1, \ldots, n$.

The GKM graph associated to $\mathcal{O}_{\widetilde{p}_0}$ is $W/W(\Sigma)$, where the vertices are given by the classes

$$\begin{aligned} \omega_1^+ &= [Id] \\ \omega_1^- &= [s_{\beta_i^-} s_{\beta_i^+}] = [s_{\beta_i^+} s_{\beta_i^-}], \quad \text{for all } 2 \leqslant i \leqslant n \\ \omega_i^{\pm} &= [s_{\beta_i^{\pm}}], \quad \text{for all } 2 \leqslant i \leqslant n \end{aligned}$$

and the bijection between these vertices and the $T-$fixed points of $\mathcal{O}_{\widetilde{p}_0}$ is as in type B_n. Observe that every vertex $\omega_i^{\pm}$ is connected to every other vertex, except for $\omega_i^{\mp}$, for all $i = 1, \ldots, n$. Consider the element $\tau : W/W(\Sigma) \to \mathfrak{t}^*$ given by $\tau(\omega_i^{\epsilon}) = \epsilon x_i$, where $\epsilon \in \{+, -\}$. Observe that, as in the other types, τ is the moment map on $\mathcal{O}_{\widetilde{p}_0}$ restricted to the fixed point set.

In terms of τ, the axial function on $W/W(\Sigma)$ is given by

$$\alpha(\omega_i^{\epsilon_i}, \omega_j^{\epsilon_j}) = \tau(\omega_i^{\epsilon_i}) - \tau(\omega_j^{\epsilon_j}) = \epsilon_i x_i - \epsilon_j x_j\ .$$

The Weyl group W can be described as the group of signed permutations (u, ϵ), with an even number of sign changes. So if $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{Z}_2^n$, we have $\epsilon_1 \cdots \epsilon_n = 1$, and the element $w = (u, \epsilon)$, with $u \in \mathcal{S}_n$, acts on x_k by $w \cdot x_k = \epsilon_k x_{u(k)}$. Observe that, as before, the elements $1, \tau, \ldots, \tau^{2n-2}$ are $W-$invariant elements of $H_\alpha(W/W(\Sigma))$; but we need another element of degree $n-1$ (hence an element of $H_\alpha^{2n-2}(W/W(\Sigma))$) to obtain a basis for $H_\alpha(W/W(\Sigma))$ (as a module over $\mathbb{S}(\mathfrak{t}^*)$). Observe that if we take $\eta = x_1 \cdots x_n \tau^{-1}$, this is in $H_\alpha^{2n-2}(W/W(\Sigma))$, and it is invariant. Using a Vandermonde type argument, it can be proved that the classes $1, \tau, \ldots, \tau^{2n-2}, \eta$ form a basis of $W-$invariant classes of $H_\alpha(W/W(\Sigma))$.

In terms of the description of the Weyl group given above, the vertices of $W/W(\Sigma)$ can be identified with the set $\{\pm 1, \ldots, \pm n\}$, where $\omega_i^\pm$ corresponds to $\pm i$, and the projection $\pi : W \to W/W(\Sigma)$ is given by $\pi(u, \epsilon) = \epsilon_1 u(1)$.

For every multi-index $I = [i_1, \ldots, i_n]$ consider the class $c_I : W \to \mathbb{S}(\mathfrak{t}^*)$ given by

$$c_I(u, \epsilon) = (\epsilon_1 x_{u(1)})^{i_1} \cdots (\epsilon_{n-1} x_{u(n-1)})^{i_{n-1}} \ .$$

Then c_I is a $W-$invariant class in $H_\alpha(W)$.

Repeating an argument similar to the other types, we have

Theorem 3.2.3. *A basis of $W-$invariant elements of $H_\alpha(W)$ is given by the elements c_I, for $I \in \mathcal{P}_n$, where $\mathcal{P}_2 = \{[0,0], [1,0], [2,0], [0,1]\}$ and $\mathcal{P}_n = [i_1, \ldots, i_n]$ is defined inductively by the following rule*

- $0 \leqslant i_1 \leqslant 2n-2$ *and* $[i_2, \ldots, i_n] \in \mathcal{P}_{n-1}$ *or*
- $i_1 = 0$ *and* $[i_2 - 1, \ldots, i_n - 1] \in \mathcal{P}_{n-1}$.

(For more details about this construction, see [14], section 5.4).

Chapter 4

Canonical classes

Let M be a $T-$Hamiltonian manifold with discrete fixed point set M^T. As we observed in the introduction, in this case the equivariant cohomology ring of M can be viewed as a subring of the equivariant cohomology ring of M^T.

In this chapter we introduce a special basis for the equivariant cohomology of M associated to a generic component of the moment map, and give formulas that compute the restriction of this basis to the fixed point set of the T action.

In particular let $\pi : M \to \widetilde{M}$ be a $T-$equivariant fibration between $T-$Hamiltonian GKM spaces. We explore how to use the symplectic structure of $\widetilde{M}$, together with the graph theoretical implications of the existence of such a map, to derive formulas that compute the restriction of the elements of this basis to the fixed point set. Moreover we explore how the integrality of such formulas is related to the cohomology of $\widetilde{M}$.

4.1 Definition of canonical classes

Let (M, ω) be a compact symplectic manifold with a Hamiltonian action of a torus T. Let $\psi : M \to \mathfrak{t}^*$ be the moment map. We recall that ψ is a $T-$invariant map satisfying

$$\iota_{\xi^\#}\omega = -d\psi^\xi \text{ for all } \xi \in \mathfrak{t}$$

where $\xi^{\#}$ is the vector field generated by ξ, and ψ^{ξ} is the $T-$invariant function defined by $\psi^{\xi}(p) = \langle \psi(p), \xi \rangle$. Moreover, if d_T denotes the Cartan differential on $\Omega_T(M)$ defined in the introduction, then $\omega + \psi$ is a d_T-closed equivariant two form. So every Hamiltonian manifold (M, ω, ψ) is naturally endowed with an equivariant cohomology class $[\omega + \psi] \in H_T^2(M)$.

Suppose that the fixed point set M^T is discrete. Fix a generic $\xi \in \mathfrak{t}$, i.e. a vector ξ in $\mathfrak{t}$ such that $\eta(\xi) \neq 0$, for each weight $\eta \in \mathfrak{t}^*$ of the isotropy action of T on T_pM, for every fixed point $p \in M^T$. Then the ξ component of the moment map ψ, $\varphi = \psi^{\xi}$, is a perfect Morse function with critical set corresponding to the fixed point set M^T. Let $\nu^-(p)$ be the negative normal bundle of φ at p. Then T has an isotropy action on $\nu^-(p)$ with one fixed point, p. So the Morse index of φ at p is even for all $p \in M^T$. We define $\lambda(p)$ to be half of the Morse index of φ at p.

Consider the weights of the isotropy action of T on $\nu^-(p)$. By our choice of the moment map, they are positive weights, i.e. every such weight satisfies $\eta(\xi) > 0$. Let Λ_p^- denote the product of the weights on the negative normal bundle $\nu^-(p)$.

Definition 4.1.1. *Let T be a torus which acts on a symplectic manifold (M, ω) in a Hamiltonian fashion, with moment map $\psi : M \to \mathfrak{t}^*$. Let $\varphi = \psi^{\xi}$ be a generic component of the moment map. Then $\alpha_p \in H_T^{2\lambda(p)}(M, \mathbb{R})$ is the* ***canonical class*** *at p (w.r.t. φ) if*

1. *$\alpha_p(p) = \Lambda_p^-$*

2. *$\alpha_p(q) = 0$ for all $q \in M^T \setminus \{p\}$ s.t. $\lambda(q) \leq \lambda(p)$*

We will say that a canonical class is ***integral*** *if $\alpha_p \in H_T^{2\lambda(p)}(M, \mathbb{Z})$.*

These classes were introduced by Goldin and Tolman ([11]) in the symplectic category, and they coincide in a particular case with the "Thom classes" introduced by Guillemin and Zara ([15]) in the GKM category. They can be thought as an equivariant Poincaré dual to the closure of the unstable manifolds of φ.

Canonical classes do not always exist (cfr. Example 2, [11]); but if they do for all the fixed points $p \in M^T$, then the set $\{\alpha_p\}_{p \in M^T}$ is a basis for $H_T(M)$ as a module over $H_T(\{pt\})$ (cfr. Proposition 2.2, [11]). Moreover the set of canonical classes is uniquely determined by φ (cfr. Lemma 2.6, [11]). Since the restriction map $i^* : H_T(M) \to H_T(M^T) = \mathrm{Maps}(M^T, \mathbb{S}(\mathfrak{t}^*))$ is injective (cfr. Theorem 1.3.2), in order to understand the equivariant cohomology ring of M it is sufficient to find formulas that compute the image of $\{\alpha_p\}_{p \in M^T}$ under i^*. As a consequence, one can also derive formulas for the multiplicative structure constants ([11], [15]). In the next sections we will provide formulas that compute $i^*(\{\alpha_p\})$, for all $p \in M^T$, which are a generalization of the one provided in [11].

4.2 A generalized path formula for canonical classes

Let M be a Hamiltonian T manifold with moment map $\psi : M \to \mathfrak{t}^*$ and discrete fixed point set M^T. Let $\varphi = \psi^\xi : M \to \mathbb{R}$ be a generic component of the moment map, and suppose that there exists a canonical class $\alpha_p \in H_T^{2\lambda(p)}(M, \mathbb{R})$ for each fixed point $p \in M^T$. In this section we give a formula that computes $i^*(\alpha_p)$ for all $p \in M^T$.

First we define a graph $\Gamma = (V, E)$ determined by the canonical classes associated to φ (which are unique, as we mentioned before); we will refer to it as the **canonical graph**.

- The vertex set V is the fixed point set M^T; we label each vertex with the moment map image $\psi(p)$.
- The directed edge set E is given by

$$E = \{(r, r') \in V \times V \mid \lambda(r') - \lambda(r) = 1 \text{ and } \alpha_r(r') \neq 0\} ;$$

  we label each edge $(r, r') \in E$ by

$$\xi(r, r') = \frac{\alpha_r(r')}{\Lambda^-_{r'}} ,$$

  which is an element of $\mathbb{S}(\mathfrak{t}^*)_0$, the field of fractions of $\mathbb{S}(\mathfrak{t}^*)$.

A **path** γ in Γ from p to q is a sequence of $k+1$ vertices $\gamma = (\gamma_1, \ldots, \gamma_{k+1})$, where

$\gamma_1 = p$, $\gamma_{k+1} = q$ and $(\gamma_i, \gamma_{i+1}) \in E$ for all $1 \leq i \leq k$. We define the **length** of this path to be k, the number of edges, and we will refer to it as $|\gamma|$. We denote by $\Sigma(p, q)$ the set of all paths from p to q in Γ.

Before proving the first theorem, we recall two Lemmas from [11].

Lemma 4.2.1. *Let $\varphi = \psi^\xi$ be a generic component of the moment map and $\alpha_p \in H_T^{2\lambda(p)}(M;\mathbb{R})$ the canonical class w.r.t. φ at a fixed point p. Then*

- *$\alpha_p(q) = 0$ for all $q \in M^T$ so that $q \neq p$ and $\varphi(q) \leq \varphi(p)$.*

Lemma 4.2.2. *Let $\varphi = \psi^\xi$ be a generic component of the moment map. Fix $p \in M^T$. Let γ be a (integral) class in $H_T^{2i}(M;\mathbb{R})$. If $\gamma(q) = 0$ for all $q \in M^T$ so that $\lambda(q) < \lambda(p)$, then*

- *$\gamma(p)$ is a (integral) multiple of Λ_p^-; in particular,*
- *if $\lambda(p) > i$ then $\gamma(p) = 0$.*

The next theorem gives an inductive formula for computing the restriction of a canonical class α_p to another fixed point $q \in M^T$, $\alpha_p(q)$, for all $p, q \in M^T$. In particular this formula depends on the value of $\alpha_r(r')$ for all the fixed points $r, r' \in M^T$ such that $\lambda(r') - \lambda(r) = 1$. In the next section we will give an explicit expression for $\alpha_r(r')$ when M is a GKM space.

Theorem 4.2.3. *Let a torus T act on a compact symplectic manifold (M, ω) with discrete fixed point set and moment map $\psi\colon M \to \mathfrak{t}^*$. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that for all $p \in M^T$ there exists a canonical class $\alpha_p \in H_T^{2\lambda(p)}(M;\mathbb{R})$. Let (V, E) be the associated canonical graph.*

Given fixed points p and q, let $\Sigma(p, q)$ denote the set of paths from p to q in (V, E). For each $r \in M^T$ pick a closed equivariant two form $\omega_r + \psi_r \in \Omega_T^2(M)$. Assume that $\psi_{\gamma_i}(q) \neq \psi_{\gamma_i}(\gamma_i)$ for all $\gamma = (\gamma_1, \ldots, \gamma_{|\gamma|+1}) \in \Sigma(p, q)$ and $1 \leq i \leq |\gamma|$. Then

$$\alpha_p(q) = \Lambda_q^- \sum_{\gamma \in \Sigma(p,q)} \prod_{i=1}^{|\gamma|} \frac{\psi_{\gamma_i}(\gamma_{i+1}) - \psi_{\gamma_i}(\gamma_i)}{\psi_{\gamma_i}(q) - \psi_{\gamma_i}(\gamma_i)} \xi(\gamma_i, \gamma_{i+1}) \tag{4.1}$$

Proof. Our convention is that if $p \neq q$ and $\Sigma(p,q)$ is empty, then $\alpha_p(q) = 0$. Observe that if $p \neq q$ and $\lambda(q) \leq \lambda(p)$ there are no terms in the sum, hence $\alpha_p(q) = 0$, which is true from the definition of canonical class. If $\lambda(q) - \lambda(p) = 1$, the formula above is true because of the definition of $\xi(p,q)$. So the claim is true if $\lambda(q) - \lambda(p) \leq 1$.

We can always assume that $\psi_r(r) = 0$, by replacing ψ_r with $\psi_r' = \psi_r - \psi_r(r)$ for all $r \in M^T$; observe that for all $\gamma = (\gamma_1, \ldots, \gamma_{|\gamma|+1})$ in $\Sigma(p,q)$, ψ_{γ_i}' satisfies $\psi_{\gamma_i}'(q) \neq \psi_{\gamma_i}'(\gamma_i) = 0$ for all $1 \leq i \leq |\gamma|$. We will refer to ψ_r' simply as ψ_r.

Formula (4.1) can be written as

$$\alpha_p(q) = \sum_{(p,r)\in E} \frac{\psi_p(r)}{\psi_p(q)} \xi(p,r)\alpha_r(q) \tag{4.2}$$

and we prove (4.2) by induction.

Let $\alpha_p \in H_T^{2\lambda(p)}(M;\mathbb{R})$ be the canonical class at p. By definition, $\alpha_p(p) = \Lambda_p^-$ and $\alpha_p(s) = 0$ for all $s \in M^T$ such that $s \neq p$ and $\lambda(s) \leq \lambda(p)$. Consider $\alpha_p(\omega_p + \psi_p) \in H_T^{2\lambda(p)+2}(M;\mathbb{R})$. This form vanishes at all fixed points s such that $\lambda(s) \leq \lambda(p)$. By Lemma 4.2.2 this implies that for all $r \in M^T$ such that $\lambda(r) - \lambda(p) = 1$, its restriction $\alpha_p(r)\psi_p(r)$ to r is a multiple of Λ_r^-. We can conclude that $\frac{\alpha_p(r)\psi_p(r)}{\Lambda_r^-} = \xi(p,r)\psi_p(r)$ is a real number.

Consider the class

$$\alpha_p(\omega_p + \psi_p) - \sum_{(p,r)\in E} \xi(p,r)\psi_p(r)\alpha_r \in H_T^{2\lambda(p)+2}(M;\mathbb{R}).$$

Since $\alpha_r(r) = \Lambda_r^-$ for all $r \in M^T$, this class vanishes on all fixed points s such that $\lambda(s) \leq \lambda(p) + 1$. By the second part of Lemma 4.2.2 this implies that it is the zero class, therefore

$$\alpha_p(\omega_p + \psi_p) = \sum_{(p,r)\in E} \psi_p(r)\xi(p,r)\alpha_r$$

Restricting to q and dividing by $\psi_p(q)$ (which is not zero by assumption) we obtain (4.2). ∎

Remark 4.2.4. *Observe that by the definition of canonical graph and by Lemma*

4.2.1, for every edge (r,r') in E we have $\varphi(r) < \varphi(r')$. So if $\gamma = (\gamma_1, \ldots, \gamma_{|\gamma|+1})$ is a path in the canonical graph, then $\varphi(\gamma_i) < \varphi(q)$ for all $1 \leq i \leq |\gamma|$. This implies that $\psi(\gamma_i) \neq \psi(q)$ for all $1 \leq i \leq |\gamma|$. Hence the equivariant symplectic form $\omega + \psi$ always satisfies the condition required by Theorem 4.2.3. If we choose $\omega_r + \psi_r = \omega + \psi$ for all $r \in M^T$, equation (4.1) recovers precisely the formula found by Goldin and Tolman in [11] (cfr. Theorem 1.2).

Our next goal is to simplify formula (4.1) as much as possible.

For example, consider a family of closed equivariant two forms $\overline{\omega}_1 + \overline{\psi}_1, \ldots, \overline{\omega}_k + \overline{\psi}_k$ in $\Omega^2_T(M)$ such that for all $(r,r') \in E$ and $j \in \{1, \ldots, k\}$

$$\text{either } \overline{\psi}_j^{\xi}(r) < \overline{\psi}_j^{\xi}(r') \text{ or } \overline{\psi}_j(r) = \overline{\psi}_j(r') \; .$$

Moreover suppose that for each $(r,r') \in E$, there exists $j \in \{1, \ldots, k\}$ such that $\overline{\psi}_j(r) \neq \overline{\psi}_j(r')$.

We can define the *height* of an edge $(r,r') \in E$ to be

$$h(r,r') = \min \left\{ j \in \{1, \ldots, k\} \mid \overline{\psi}_j(r) \neq \overline{\psi}_j(r') \right\} .$$

Fix $q \in M^T$; consider a point $r \in M^T \setminus \{q\}$ and a path $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{|\gamma|+1})$ from r to q. By assumption, there exists $j \in \{1, \ldots, k\}$ such that $\overline{\psi}_j^{\xi}(\gamma_1) < \overline{\psi}_j^{\xi}(\gamma_2)$. Moreover, $\overline{\psi}_j^{\xi}(\gamma_i) \leq \overline{\psi}_j^{\xi}(\gamma_{i+1})$ for all $i \in \{1, \ldots, |\gamma|\}$. Therefore, $\overline{\psi}_j^{\xi}(r) < \overline{\psi}_j^{\xi}(q)$, which implies that $\overline{\psi}_j(r) \neq \overline{\psi}_j(q)$. So for all $r \in M^T \setminus \{q\}$ such that $\Sigma(r,q) \neq \emptyset$ we can define

$$h(r,q) = \min \left\{ j \in \{1, \ldots, k\} \mid \overline{\psi}_j(r) \neq \overline{\psi}_j(q) \right\} .$$

Since for all $(r,r') \in E$ either $\overline{\psi}_j^{\xi}(r) < \overline{\psi}_j^{\xi}(r')$ or $\overline{\psi}_j(r) = \overline{\psi}_j(r')$, it follows that if $\overline{\psi}_j(\gamma_i) = \overline{\psi}_j(q)$ for some $i \in \{1, \ldots, |\gamma|\}$ then $\overline{\psi}_j(\gamma_i) = \overline{\psi}_j(\gamma_{i+1}) = \overline{\psi}_j(q)$ as well. Therefore,

$$h(\gamma_i, q) \leq h(\gamma_{i+1}, q) \quad \text{and} \quad h(\gamma_i, q) \leq h(\gamma_i, \gamma_{i+1}). \tag{4.3}$$

The first refinement of Theorem 4.2.3 is the following Corollary.

Corollary 4.2.5. *Let a torus T act on a compact symplectic manifold (M,ω) with discrete fixed point set and moment map $\psi\colon M \to \mathfrak{t}^*$. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that there exists a canonical class $\alpha_p \in H_T^{2\lambda(p)}(M;\mathbb{R})$ for all $p \in M^T$; let (V,E) be the associated canonical graph. Pick closed equivariant two forms $\overline{\omega}_1+\overline{\psi}_1,\dots,\overline{\omega}_k+\overline{\psi}_k$ in $\Omega^2_T(M)$ such that for all $(r,r')\in E$ and $j\in\{1,\dots,k\}$*

$$\text{either } \overline{\psi}_j^\xi(r) < \overline{\psi}_j^\xi(r') \text{ or } \overline{\psi}_j(r) = \overline{\psi}_j(r')$$

Assume that for each $(r,r')\in E$ there exists $j\in\{1,\dots,k\}$ such that $\overline{\psi}_j(r)\neq\overline{\psi}_j(r')$ and for all $(r,r')\in E$ define

$$h(r,r') = \min\left\{ j\in\{1,\dots,k\} \mid \overline{\psi}_j(r)\neq\overline{\psi}_j(r')\right\}.$$

Given p and q in M^T, let $\Sigma(p,q)$ denote the set of paths in (V,E) from p to q. Then

$$\alpha_p(q) = \Lambda_q^- \sum_{\gamma\in C(p,q)} \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(\gamma_{i+1}) - \overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(\gamma_i)}{\overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(q) - \overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(\gamma_i)} \xi(\gamma_i,\gamma_{i+1}), \quad \textit{where}$$

$$C(p,q) = \left\{ \gamma\in\Sigma(p,q) \mid \ h(\gamma_1,\gamma_2)\le h(\gamma_2,\gamma_3)\le\cdots\le h(\gamma_{|\gamma|},\gamma_{|\gamma|+1})\right\}.$$

Proof. As we observed before, for all $r\in M^T\setminus\{q\}$ s.t. $\Sigma(r,q)\neq\emptyset$ we can define $h(r,q) = \min\{j\in\{1,\dots,k\} \mid \ \overline{\psi}_j(r)\neq\overline{\psi}_j(q)\}$. We apply Theorem 4.2.3 choosing the equivariantly closed two form $\omega_r+\psi_r$ associated to $r\in M^T$ to be $\overline{\omega}_{h(r,q)} + \overline{\psi}_{h(r,q)}$ if $r\neq q$ and $\Sigma(r,q)\neq\emptyset$. Then Theorem 4.2.3 implies that

$$\alpha_p(q) = \Lambda_q^- \sum_{\gamma\in\Sigma(p,q)} \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}_{h(\gamma_i,q)}(\gamma_{i+1}) - \overline{\psi}_{h(\gamma_i,q)}(\gamma_i)}{\overline{\psi}_{h(\gamma_i,q)}(q) - \overline{\psi}_{h(\gamma_i,q)}(\gamma_i)} \xi(\gamma_i,\gamma_{i+1}).$$

Let γ be one of the paths in $\Sigma(p,q)$. Observe that if $\overline{\psi}_{h(\gamma_i,q)}(\gamma_i) = \overline{\psi}_{h(\gamma_i,q)}(\gamma_{i+1})$, then γ contributes 0 to the formula above. Therefore only the paths γ in $\Sigma(p,q)$ which satisfy

$$h(\gamma_i,\gamma_{i+1}) \le h(\gamma_i,q) \text{ for all } i=1,\dots,|\gamma| \tag{4.4}$$

have a non zero contribution. Using (4.3) together with (4.4), the paths in $\Sigma(p,q)$ which give non zero contribution satisfy

$$h(\gamma_i,\gamma_{i+1}) = h(\gamma_i,q) \leq h(\gamma_{i+1},q) = h(\gamma_{i+1},\gamma_{i+2})$$

for all $i = 1,\ldots,|\gamma|-1$. □

4.3 Canonical classes on GKM spaces

Let M be a compact symplectic manifold with a Hamiltonian action of a torus T and moment map $\psi : M \to \mathfrak{t}^*$. Suppose that the fixed point set M^T is discrete. Then M is equivariantly formal (cfr. Theorem 1.1.3) and therefore $H_T(M)$ is a free $\mathbb{S}(\mathfrak{t}^*)-$module. Then M is a GKM manifold if for every codimension one subtorus K of T, the connected components of the fixed submanifold M^K have dimension at most two. This is equivalent to requiring the weights of the isotropy action of T on T_pM be pairwise linearly independent.

Every GKM manifold is naturally endowed with a graph, called the GKM graph, where

- the set of vertices coincide with the set of fixed points M^T; we label each vertex $p \in M^T$ with its moment map image $\psi(p) \in \mathfrak{t}^*$
- given $p \neq q$, there exist a directed edge (p,q) from p to q exactly if there exists a codimension one subtorus $K \subset T$ so that p and q are contained in the same connected component N of M^K. We label each directed edge (p,q) by the weight $\eta(p,q)$ associated to the isotropy representation of T on $T_qN \simeq \mathbb{C}$.

We denote the set of vertices of the GKM graph by V, and the set of edges by E_{GKM} (to avoid confusion with the set of edges E of the canonical graph).

Observe that if $(p,q) \in E_{\text{GKM}}$ then $(q,p) \in E_{\text{GKM}}$. Moreover for all $(p,q) \in E_{\text{GKM}}$ $\eta(p,q) = -\eta(q,p)$ and $\psi(q)-\psi(p)$ is a positive multiple of $\eta(p,q)$. The set of weights in the tangent space at $p \in V$ is

$$\Pi_p = \{\eta(r,p) \mid (r,p) \in E_{\mathrm{GKM}}\}.$$

Consider a generic component of the moment map $\varphi = \psi^\xi$. Then the set of weights Π_p^- in the negative normal bundle of φ at p is the set of positive weights in the tangent bundle, that is,

$$\Pi_p^- = \{\eta(r,p) \mid (r,p) \in E_{\mathrm{GKM}} \text{ and } \eta(r,p)(\xi) > 0\}.$$

Observe that Π_p^- also coincides with the set $\{\eta(r,p) \mid (r,p) \in E_{\mathrm{GKM}} \text{ and } \varphi(r) < \varphi(p)\}$. Hence, $\Lambda_p^- = \prod_{\eta \in \Pi_p^-} \eta$ and $\lambda(p) = |\Pi_p^-|$ is the number of edges $(r,p) \in E_{\mathrm{GKM}}$ such that $\varphi(r) < \varphi(q)$.

On GKM manifolds it is possible to find conditions that ensure the existence of canonical classes and when canonical classes α_p exist, it is possible to compute explicitly $\alpha_p(q)$ for all $q \in M^T$ such that $\lambda(q) - \lambda(p) = 1$ (cfr. [11], [15]).

Definition 4.3.1. *Let (p,q) be an edge in E_{GKM}. We say that (p,q) is* ***increasing*** *if $\varphi(p) < \varphi(q)$.*
A path $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ in (V, E_{GKM}) is an ***increasing path*** *if the edges (γ_i, γ_{i+1}) are increasing for all $i = 1, \dots, |\gamma|$.*
A generic component of the moment map φ is called ***index increasing*** *if $\lambda(p) < \lambda(q)$ for every increasing edge $(p,q) \in E_{\mathrm{GKM}}$.*

If φ is index increasing, integral canonical classes exist and it is possible to compute the restriction of a canonical class α_p to $q \in M^T$, for all p and q in M^T such that $\lambda(q) - \lambda(p) = 1$. More specifically, given $\eta \in \mathfrak{t}^*$ s.t. $\langle \eta, \xi \rangle \neq 0$, let $\rho_\eta : \mathbb{S}(\mathfrak{t}^*) \to \mathbb{S}(\mathfrak{t}^*)$ be the homomorphism of symmetric algebras induced by the projection map which sends $X \in \mathfrak{t}^*$ to $X - \frac{\langle X, \xi \rangle}{\langle \eta, \xi \rangle}\eta \in \xi^\circ \subset \mathfrak{t}^*$, where $\xi^\circ = \{\beta \in \mathfrak{t}^* \mid \beta(\xi) = 0\}$.

For any $(p,q) \in E_{\mathrm{GKM}}$, following [15], we define

$$\Theta(p,q) = \frac{\rho_{\eta(p,q)}(\Lambda_p^-)}{\rho_{\eta(p,q)}(\frac{\Lambda_q^-}{\eta(p,q)})} \in \mathbb{S}(\mathfrak{t}^*)_0$$

where $\mathbb{S}(\mathfrak{t}^*)_0$ denotes the ring of fractions of $\mathbb{S}(\mathfrak{t}^*)$. Observe that neither $\rho_{\eta(p,q)}(\Lambda_p^-)$ nor $\rho_{\eta(p,q)}(\frac{\Lambda_q^-}{\eta(p,q)})$ are zero, since by the GKM assumption the weights at each fixed point are pairwise linearly independent. The theorem below was proved in [15] over the rationals and then extended to the integers in [11].

Theorem 4.3.2. *Let (M,ω,ψ) be a GKM space and (V, E_{GKM}) the associated GKM graph. Let $\varphi = \psi^\xi$ be a generic component of the moment map; assume that φ is index increasing. Then*

1. *For each $p \in M^T$ there exists a canonical class $\alpha_p \in H_T^{2\lambda(p)}(M;\mathbb{Z})$.*

2. *If p and q are vertices such that $\lambda(q) - \lambda(p) = 1$, then*

$$\alpha_p(q) = \begin{cases} \Lambda_q^- \dfrac{\Theta(p,q)}{\eta(p,q)} & \text{if } (p,q) \in E_{\mathrm{GKM}}, \text{ and} \\ \\ 0 & \text{if } (p,q) \notin E_{\mathrm{GKM}} \end{cases} \tag{4.5}$$

3. *$\Theta(p,q)$ is a non-zero integer.*

Let (M,ω,ψ) and $(\widetilde{M},\widetilde{\omega},\widetilde{\psi})$ be GKM spaces; let (V, E_{GKM}) and $(\widetilde{V}, \widetilde{E}_{\mathrm{GKM}})$ be the associated GKM graphs.

Let $\pi\colon M \to \widetilde{M}$ be an equivariant map. An edge $e = (p,q) \in E_{\mathrm{GKM}}$ is said to be *vertical* (with respect to π) if $\pi(p) = \pi(q)$ and *horizontal* (with respect to π) if $\pi(p) \neq \pi(q)$.

From the equivariance of π it is easy to see that the following conditions are satisfied:

- $\pi(V) \subset \widetilde{V}$.

- If $e = (p,q) \in E_{\mathrm{GKM}}$ is a horizontal edge, then $\pi(e) = (\pi(p), \pi(q))$ is an edge in $\widetilde{E}_{\mathrm{GKM}}$.

- If $e \in E_{\mathrm{GKM}}$ is a horizontal edge then $\eta(e) = \pm\eta(\pi(e))$.

Definition 4.3.3. *Let π be an equivariant map between two GKM spaces. Then π is* **weight preserving** *if $\eta(e) = \eta(\pi(e))$ for all horizontal edges $e \in E_{\mathrm{GKM}}$.*

Observe that if π is weight preserving and has surjective differential, then one can find connections on Γ and $\widetilde{\Gamma}$ in such a way that π is a GKM fibration (cfr. section 2.3.3).

Remark 4.3.4. If J and $\widetilde{J}$ are compatible almost complex structure on (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ respectively, then any equivariant map $\pi\colon M \to \widetilde{M}$ which intertwines J and $\widetilde{J}$ is weight preserving.

Lemma 4.3.5. *Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be GKM spaces, and $\pi\colon M \to \widetilde{M}$ a weight preserving equivariant map. Let $\varphi = \psi^\xi$ be a generic component of the moment map; assume that φ is index increasing. Define an oriented graph with vertex set $V = M^T$ and edge set*

$$E = \{(r, r') \in E_{\mathrm{GKM}} \mid \lambda(r') - \lambda(r) = 1\},$$

where (V, E_{GKM}) is the GKM graph associated to M. Then for all the horizontal edges $(r, r') \in E$

$$\widetilde{\psi}^\xi(\pi(r)) < \widetilde{\psi}^\xi(\pi(r'))$$

Proof. Let (r, r') be an edge in E. Since $\lambda(r') > \lambda(r)$ and $\varphi = \psi^\xi$ is index increasing we have $\varphi(r) < \varphi(r')$. Since π is an equivariant weight preserving map, if $(r, r') \in E_{\mathrm{GKM}}$ is horizontal with respect to π, then $\psi(r') - \psi(r)$ and $\widetilde{\psi}(r') - \widetilde{\psi}(r)$ are both positive multiples of $\eta(r, r')$, and the conclusion follows. □

Definition 4.3.6. *Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be GKM spaces, and let $\pi\colon M \to \widetilde{M}$ be an equivariant fibration. We will say that π has* **symplectic fibers** *if the restriction of ω to $\widehat{M}_p = \pi^{-1}(\pi(p))$ is symplectic for all $p \in M^T$.*

Given a generic $\xi \in \mathfrak{t}$, consider the Morse functions $\varphi = \psi^\xi$ on M, and $\widetilde{\varphi} = \widetilde{\psi}^\xi$ on $\widetilde{M}$. Let $\widehat{\Lambda}_p^-$ and $\widetilde{\Lambda}_{\pi(p)}^-$ denote respectively the equivariant Euler class of the negative normal bundle of $\varphi|_{\widehat{M}_p}$ at $p \in \widehat{M}_p$ and the equivariant Euler class of the negative normal bundle of $\widetilde{\varphi} = \widetilde{\psi}^{\xi}$ at $\pi(p) \in \widetilde{M}$.

Observe that for every fixed point $p \in M^T$ the $T-$invariant fiber is a Hamiltonian space $(\widehat{M}_p, \omega_{\widehat{M}_p}, \psi_{|_{\widehat{M}_p}})$ and is a GKM space with GKM graph $(\widehat{V}_p, (\widehat{E}_{\text{GKM}})_p)$, where $\widehat{V}_p = V \cap \widehat{M}_p^T$ and $(\widehat{E}_{\text{GKM}})_p = \{(q, r) \in E_{\text{GKM}} \mid \pi(q) = \pi(r) = \pi(p)\}$. Moreover the set of weights in the tangent space at any point $p \in \widehat{M}_p$ is

$$\widehat{\Pi}_p = \{\eta(r, p) \mid (r, p) \in E_{\text{GKM}} \text{ and } \pi(r) = \pi(p)\} \; .$$

Let $(\widetilde{V}, \widetilde{E}_{\text{GKM}})$ be the GKM graph associated to $\widetilde{M}$, and suppose that π is weight preserving. If $\widetilde{\Pi}_s = \{\eta(r, s) \mid (r, s) \in \widetilde{E}_{\text{GKM}}\}$, then

$$\widetilde{\Pi}_{\pi(p)} = \{\eta(r, p) \mid (r, p) \in E_{\text{GKM}} \text{ and } \pi(r) \neq \pi(p)\}$$

and

$$\Pi_p = \widetilde{\Pi}_{\pi(p)} \coprod \widehat{\Pi}_p \tag{4.6}$$

In particular, if $\widehat{\Pi}_p^-$ and $\widetilde{\Pi}_{\pi(p)}^-$ denote respectively the set of weights in the negative normal bundle of $\varphi|_{\widehat{M}_p}$ at p and the set of weights in the negative normal bundle of $\widetilde{\varphi} = \widetilde{\psi}^{\xi}$ at $\pi(p) \in \widetilde{M}^T$, then

$$\Pi_p^- = \widetilde{\Pi}_{\pi(p)}^- \coprod \widehat{\Pi}_p^- . \tag{4.7}$$

and

$$\Lambda_p^- = \widetilde{\Lambda}_{\pi(p)}^- \widehat{\Lambda}_p^- \tag{4.8}$$

Therefore if $\widetilde{\lambda}(s)$ is half of the Morse index of $\widetilde{\varphi}$ at $s \in \widetilde{M}^T$ and $\widehat{\lambda}(p)$ half of the Morse index of $\varphi_{|\widehat{M}_p}$ at $p \in \widehat{M}_p^T$, then

$$\lambda(p) = \widetilde{\lambda}(\pi(p)) + \widehat{\lambda}(p) \quad \text{for all } p \in M^T \tag{4.9}$$

since by (4.7) $\widetilde{\lambda}(s) = |\widetilde{\Pi}_s^-|$ and $\widehat{\lambda}(p) = |\widehat{\Pi}_p^-|$.

Proposition 4.3.7. *Let $\{(M_i, \omega_i, \psi_i)\}_{i=0}^k$ be a sequence of GKM spaces so that M_0 is* *a point and* $p_i\colon M_{i+1} \to M_i$ *a weight preserving equivariant map for each* $i \in \{0, \ldots, k-1\}$. *Let* $\varphi_k = \psi_k^\xi\colon M_k \to \mathbb{R}$ *be a generic component of the moment map; assume that* φ_k *is index increasing. Define an oriented graph with vertex set*

$$V = (M_k)^T$$

and edge set

$$E = \{(r, r') \in E_{\mathrm{GKM}} \mid \lambda(r') - \lambda(r) = 1\},$$

where (V, E_{GKM}) *is the GKM graph associated to* M_k. *Also, for all distinct* r *and* r' *in* V, *define*

$$h(r, r') = \min\{j \in \{1, \ldots, k\} \mid \pi_j(r) \neq \pi_j(r')\}$$

where $\pi_k = Id : M_k \to M_k$ *and otherwise* $\pi_j = p_j \circ p_{j+1} \circ \cdots \circ p_{k-1} : M_k \to M_j$ *for all* $j = 0, \ldots, k-1$. *Let* $\overline{\psi}_j = \pi_j^*(\psi_j) : M_k \to \mathfrak{t}^*$ *for all* $j = 0, \ldots, k$. *Then*

1. *For all* $p \in M_k^T$ *there exists a unique canonical class* $\alpha_p \in H_T^{2\lambda(p)}(M_k; \mathbb{Z})$.

2. *Given* p *and* q *in* M^T, *let* $\Sigma(p, q)$ *denote the set of paths from* p *to* q *in* (V, E); *then*

$$\alpha_p(q) = \Lambda_q^- \sum_{\gamma \in C(p,q)} \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(\gamma_{i+1}) - \overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(\gamma_i)}{\overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(q) - \overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(\gamma_i)} \cdot \frac{\Theta(\gamma_i, \gamma_{i+1})}{\eta(\gamma_i, \gamma_{i+1})},$$

   where

$$C(p, q) = \{\gamma \in \Sigma(p, q) \mid h(\gamma_1, \gamma_2) \leq h(\gamma_2, \gamma_3) \leq \cdots \leq h(\gamma_k, \gamma_{k+1})\}.$$

Proof. By Theorem 4.3.2, integral canonical classes exist, and (V, E) is precisely the canonical graph associated to $(M_k, \omega_k, \psi_k, \varphi_k)$. Moreover by definition for all $(r, r') \in E$

$$\xi(r, r') = \frac{\alpha_r(r')}{\Lambda_{r'}^-} = \frac{\Theta(r, r')}{\eta(r, r')}.$$

For each $j \in \{1, \ldots, k\}$, consider the closed equivariant 2-forms

$\overline{\omega}_j + \overline{\psi}_j = \pi_j^*(\omega_j + \psi_j) \in \Omega_T^2(M_k)$. Since each p_i is an equivariant weight preserving map, π_j is also an equivariant weight preserving map for all j. Hence, Lemma 4.3.5 implies that for all $(r, r') \in E$ and $j \in \{1, \ldots, k\}$

$$\text{either } \overline{\psi}_j^\xi(r) < \overline{\psi}_j^\xi(r') \text{ or } \pi_j(r) = \pi_j(r') \tag{4.10}$$

Hence for all $(r, r') \in E$ either $\overline{\psi}_j^\xi(r) < \overline{\psi}_j^\xi(r')$ or $\overline{\psi}_j(r) = \overline{\psi}_j(r')$. Moreover, since $\pi_k = Id$, it is obvious that $\overline{\psi}_k(r) \neq \overline{\psi}_k(r')$ for all $(r, r') \in E$, and the second claim is an immediate consequence of Corollary 4.2.5. $\square$

Lemma 4.3.8. *Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be GKM spaces. Let $\pi\colon M \to \widetilde{M}$ be a weight preserving equivariant fibration with symplectic fibers. Let $\varphi = \psi^\xi$ be a generic component of the moment map; assume that φ is index increasing.*

Define an oriented graph with vertex set $V = M^T$ and edge set

$$E = \{(r, r') \in E_{\text{GKM}} \mid \lambda(r') - \lambda(r) = 1\},$$

where (V, E_{GKM}) is the GKM graph associated to M.

1. *There exists a canonical class $\widehat{\alpha}_s$ on the fiber $\widehat{M}_s = \pi^{-1}(\pi(s))$ with respect to the restriction $\varphi_{|_{\widehat{M}_s}}$ for all $s \in M^T$.*

2. *For all r and q in M^T such that $\pi(r) = \pi(q)$, let $\widehat{\Sigma}(r, q)$ be the set of paths from r to q in $\Sigma(r, q)$ such that every edge is vertical. Then*

$$\widehat{\alpha}_r(q) = \frac{\Lambda_q^-}{\widetilde{\Lambda}_{\pi(q)}^-} \sum_{\gamma \in \widehat{\Sigma}(r,q)} \prod_{i=1}^{|\gamma|} \frac{\psi(\gamma_{i+1}) - \psi(\gamma_i)}{\psi(q) - \psi(\gamma_i)} \frac{\Theta(\gamma_i, \gamma_{i+1})}{\eta(\gamma_i \gamma_{i+1})}$$

Proof. We have already observed that for all $s \in M^T$ the fiber $\widehat{M}_s$ is a GKM space and the GKM graph of $\widehat{M}_s$ is just the restriction of the GKM graph for M to $\widehat{M}_s$.

By (4.9), for all $p \in M^T$, $\lambda(p) = \widetilde{\lambda}(\pi(p)) + \widehat{\lambda}(p)$. Therefore $\varphi_{|_{\widehat{M}_s}}$ is index increasing on $\widehat{M}_s$. Moreover $\lambda(p) - \lambda(q) = \widehat{\lambda}(p) - \widehat{\lambda}(q)$ for all p and q such that $\pi(p) = \pi(q)$. Hence, canonical classes exist by Theorem 4.3.2. The canonical graph associated to $\widehat{M}_s$ is just the restriction of the canonical graph of M restricted to $\widehat{M}_s$. For all $r, s \in \widehat{M}_s^T$ such that $\lambda(s) - \lambda(r) = 1$ (hence $\widehat{\lambda}(s) - \widehat{\lambda}(r) = 1$) define $\widehat{\Theta}(s, r)$ to be

$$\widehat{\Theta}(s,r) = \frac{\rho_{\eta(s,r)}(\widehat{\Lambda}_r^-)}{\rho_{\eta(s,r)}(\frac{\widehat{\Lambda}_s^-}{\eta(s,r)})} .$$

Since $\pi(s) = \pi(r)$, by (4.8) we have

$$\widehat{\Theta}(s,r) = \frac{\rho_{\eta(s,r)}(\widehat{\Lambda}_r^-)}{\rho_{\eta(s,r)}(\frac{\widehat{\Lambda}_s^-}{\eta(s,r)})} = \frac{\rho_{\eta(s,r)}(\Lambda_r^-)}{\rho_{\eta(s,r)}(\frac{\Lambda_s^-}{\eta(s,r)})} = \Theta(s,r) . \tag{4.11}$$

Equivalently, if we define $\widehat{\xi}(r, s)$ to be $\dfrac{\widehat{\alpha}_r(s)}{\widehat{\Lambda}_s^-}$, then

$$\xi(r,s) = \widehat{\xi}(r,s)$$

and the conclusion follows from Theorem 4.2.3 and Remark 4.2.4. ☐

Corollary 4.3.9. *Assume that the hypotheses of Lemma 4.3.8 hold. Then given p and s in M^T, let $\overline{\Sigma}(p, s)$ denote the set of paths γ from p to s in (V, E) such that $\pi(\gamma_i) \neq \pi(\gamma_{i+1})$ for all i. Let $\overline{\psi} : M \to \mathfrak{t}^*$ be $\pi^*(\widetilde{\psi})$.*

Then for every p and q in M^T,

$$\alpha_p(q) = \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in \overline{\Sigma}(p,s)} \widetilde{\Lambda}_{\pi(q)}^- \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}(\gamma_{i+1}) - \overline{\psi}(\gamma_i)}{\overline{\psi}(q) - \overline{\psi}(\gamma_i)} \cdot \frac{\Theta(\gamma_i, \gamma_{i+1})}{\eta(\gamma_i, \gamma_{i+1})} \right) \widehat{\alpha}_s(q).$$

Proof. By lemma 4.3.8 for all $s \in M^T$ there exists a canonical class $\widehat{\alpha}_s$ on $\widehat{M}_s$ with respect to $\varphi|_{\widehat{M}_s}$. Moreover for all s and q in M^T such that $\pi(s) = \pi(q)$, the restriction $\widehat{\alpha}_s(q)$ is given by

$$\widehat{\alpha}_s(q) = \frac{\Lambda_q^-}{\widetilde{\Lambda}_{\pi(q)}^-} \sum_{\gamma \in \widehat{\Sigma}(s,q)} \prod_{i=1}^{|\gamma|} \frac{\psi(\gamma_{i+1}) - \psi(\gamma_i)}{\psi(q) - \psi(\gamma_i)} \frac{\Theta(\gamma_i, \gamma_{i+1})}{\eta(\gamma_i \gamma_{i+1})} , \tag{4.12}$$

where $\widehat{\Sigma}(s, q)$ is the set of paths from s to q in the sub-graph of (V, E) with vertices

$V \cap \widehat{M}_q^T$.

We can apply Proposition 4.3.7, where $(M, \omega, \psi) = (M_2, \omega_2, \psi_2)$ and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi}) = (M_1, \omega_1, \psi_1)$, and $\pi = \pi_1 : M \to \widetilde{M}$. Observe that for all p, q in M^T and every path $\gamma = (\gamma_1, \ldots, \gamma_{|\gamma|+1})$ in $C(p, q)$, there exists $j \in \{1, \ldots, |\gamma|+1\}$ such that $h(\gamma_i, \gamma_{i+1}) = 1$ for all $i = 1, \ldots, j-1$, and $h(\gamma_i, \gamma_{i+1}) = 2$ if $i = j, \ldots, |\gamma|$. Since $h(\gamma_i, \gamma_{i+1}) = h(\gamma_i, \gamma_k) = 1$ for all $i \leq k-1$, the subpath $(\gamma_1, \ldots, \gamma_j)$ belongs to $\overline{\Sigma}(p, \gamma_j)$ (and $\gamma_i \notin \widehat{M}_q^T$ for all $i < j$), and $(\gamma_j, \ldots, \gamma_{|\gamma|+1})$ belongs to $\widehat{\Sigma}(\gamma_j, \gamma_{|\gamma|+1})$ (hence $\gamma_i \in \widehat{M}_q^T$ for all $i \geq j$). Conversely every path $\gamma = (\gamma_1, \ldots, \gamma_{|\gamma|+1})$ that can be broken into two paths $(\gamma_1, \ldots, \gamma_j) \in \overline{\Sigma}(\gamma_1, \gamma_j)$ and $(\gamma_j, \ldots, \gamma_{|\gamma|+1}) \in \widehat{\Sigma}(\gamma_j, \gamma_{|\gamma|+1})$ belongs to $C(\gamma_1, \gamma_{|\gamma|+1})$.

Hence by Proposition 4.3.7 we have that for all $p, q \in M^T$

$$\alpha_p(q) = \Lambda_q^- \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in \overline{\Sigma}(p,s)} \overline{\Upsilon}(\gamma) \right) \left(\sum_{\gamma \in \widehat{\Sigma}(s,q)} \widehat{\Upsilon}(\gamma) \right) \tag{4.13}$$

where

$$\overline{\Upsilon}(\gamma) = \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}(\gamma_{i+1}) - \overline{\psi}(\gamma_i)}{\overline{\psi}(q) - \overline{\psi}(\gamma_i)} \cdot \frac{\Theta(\gamma_i, \gamma_{i+1})}{\eta(\gamma_i, \gamma_{i+1})}$$

and

$$\widehat{\Upsilon}(\gamma) = \prod_{i=1}^{|\gamma|} \frac{\psi(\gamma_{i+1}) - \psi(\gamma_i)}{\psi(q) - \psi(\gamma_i)} \cdot \frac{\Theta(\gamma_i, \gamma_{i+1})}{\eta(\gamma_i, \gamma_{i+1})}$$

Combining equations (4.12) and (4.13), the conclusion follows immediately. □

4.4 Integrality and positivity

In this section we investigate conditions under which the path formula for the computation of the restriction of the canonical classes to the fixed point set is integral and positive, in the sense specified below.

Let (M, ω, ψ) be a GKM space and $\varphi = \psi^\xi$ a generic component of the moment map. We define the **set of weights of** M to be

$$\Pi(M) = \{\eta(r, p) \mid (r, p) \in E_{\mathrm{GKM}}\}$$

and the **set of positive weights of** M to be $\{\alpha \in \Pi(M) \mid \alpha(\xi) > 0\}$.

Given any subring (with unit) $A \subset \mathbb{Q}$, let $\mathbb{S}(A \otimes \ell^*)$ denote the symmetric algebra associated to the A module $A \otimes \ell^*$, where $\ell^* \subset \mathfrak{t}^*$ is the weight lattice. If $\{x_1, \ldots, x_n\}$ is a basis for ℓ^*, then $\mathbb{S}(A \otimes \ell^*) = A[x_1, \ldots, x_n]$.

Proposition 4.4.1. *Let $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be a $T-$Hamiltonian space which is a GKM space with respect to this action. Let $\widetilde{\varphi} = \widetilde{\psi}^{\xi}$ be a generic component of the moment map. Assume that $H^*(\widetilde{M}, A) \simeq H^*(\mathbb{C}P^{\frac{1}{2}\dim \widetilde{M}}, A)$ as rings, and that $[\widetilde{\omega}]$ is an integral class, i.e. it lies in the image of $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$. Given $x \in \widetilde{M}^T$ and a subset $S \subset \{y \in \widetilde{M}^T \mid \widetilde{\varphi}(y) < \widetilde{\varphi(x)}\}$, then*

$$\widetilde{\Lambda}_x^- \prod_{y \in S} \frac{1}{\widetilde{\psi}(x) - \widetilde{\psi}(y)}$$

is an element of $A_+\{\alpha \in \Pi(M) \mid \alpha(\xi) > 0\}$, the semiring of $\mathbb{S}(A \otimes \ell^*)$ generated by $A_+ = \{t \in A \mid t \geq 0\}$ and the set of positive weights of $\widetilde{M}$.

Proof. First observe that since the fixed points are isolated and $\widetilde{\varphi}$ is a perfect Morse function, there exists one fixed point of index $2i$, for all $i \in \{0, \ldots, \frac{\dim(\widetilde{M})}{2}\}$. Moreover the GKM graph associated to $\widetilde{M}$ is a complete graph, and

$$\widetilde{\varphi}(y) < \widetilde{\varphi}(x) \quad \text{if and only if} \quad \lambda(y) < \lambda(x) \tag{4.14}$$

(cfr. also Proposition 3.4 in [24]). Then for all $y \in S$, $\widetilde{\psi}(x) - \widetilde{\psi}(y) = n\eta(y, x)$ for some $n \in \mathbb{N}$, which implies that

$$\widetilde{\Lambda}_x^- \prod_{\widetilde{\varphi}(y) < \widetilde{\varphi}(x)} \frac{1}{(\widetilde{\psi}(x) - \widetilde{\psi}(y))} = q \quad \in \mathbb{Q}_+$$

and hence

$$\widetilde{\Lambda}_x^- \prod_{y \in S} \frac{1}{\widetilde{\psi}(x) - \widetilde{\psi}(y)} \in \mathbb{Q}_+\{\alpha \in \Pi(M) \mid \alpha(\xi) > 0\}.$$

We need to prove that $q \in A_+$. Consider the equivariant cohomology class

$$\beta = \prod_{\widetilde{\varphi}(y) < \widetilde{\varphi}(x),\ y \in M^T} [\widetilde{\omega} + \widetilde{\psi} - \widetilde{\psi}(y)] \quad \in H_T^{2\lambda(x)}(M; \mathbb{Z})$$

Since $\widetilde{M}$ is an index increasing GKM space (cfr. (4.14)), integral canonical classes exist by Theorem 4.3.2. Hence, by Lemma 4.2.2, $\beta = m\alpha_x$ for some $m \in \mathbb{N}$, where α_x is the canonical class at x w.r.t. $\widetilde{\varphi}$, and $\frac{1}{m}[\widetilde{\omega}]^{\lambda(x)}$ is an integral class. Since by assumption $H^*(\widetilde{M}, A) \simeq H^*(\mathbb{C}P^{\frac{1}{2}\dim \widetilde{M}}, A)$, we have $\frac{1}{m} \in A_+$. But

$$\frac{\alpha_x(x)}{\beta(x)} = \widetilde{\Lambda}_x^- \prod_{\widetilde{\varphi}(y) < \widetilde{\varphi}(x)} \frac{1}{(\widetilde{\psi}(x) - \widetilde{\psi}(y))} = \frac{1}{m}$$

and the conclusion follows. □

Corollary 4.4.2. *Assume that the hypotheses of Corollary 4.3.9 hold. Moreover suppose that* $H^*(\widetilde{M}, A) \simeq H^*(\mathbb{C}P^{\frac{1}{2}\dim \widetilde{M}}, A)$ *as rings.*

Given p *and* s *in* M^T, *let* $\overline{\Sigma}(p, s)$ *denote the set of paths* γ *from* p *to* s *in* (V, E) *such that* $\pi(\gamma_i) \neq \pi(\gamma_{i+1})$ *for all* i. *Define* $\overline{\psi} : M \to \mathfrak{t}^*$ *to be* $\pi^*(\widetilde{\psi})$.

Then for every p *and* q *in* M^T,

$$\alpha_p(q) = \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in \overline{\Sigma}(p,s)} P(\gamma) \right) \widehat{\alpha}_s(q)$$

where

$$P(\gamma) = \widetilde{\Lambda}_{\pi(q)}^- \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}(\gamma_{i+1}) - \overline{\psi}(\gamma_i)}{\overline{\psi}(q) - \overline{\psi}(\gamma_i)} \cdot \frac{\Theta(\gamma_i, \gamma_{i+1})}{\eta(\gamma_i, \gamma_{i+1})}$$

lies in the symmetric algebra $\mathbb{S}(A \otimes \ell^*)$. *Moreover if* $\Theta(r, r') > 0$ *for all* $(r, r') \in E$, *then* $P(\gamma)$ *lies in* $A_+\{\alpha \in \Pi(M) \mid \alpha(\xi) > 0\}$, *the semiring of* $\mathbb{S}(A \otimes \ell^*)$ *generated by* $A_+ = \{t \in A \mid t \geq 0\}$ *and the set of positive weights of* M.

Proof. Since $H^2(\widetilde{M}, A) = H^2(\mathbb{C}P^{\frac{1}{2}\dim \widetilde{M}}, A)$, the value of $P(\gamma)$ will be the same for any closed equivariant two form $\widetilde{\omega} + \widetilde{\psi} \in \Omega_T^2(\widetilde{M})$ which is not exact. So we can assume that $[\widetilde{\omega} + \widetilde{\psi}]$ is a non zero integral class, i.e. it lies in $H_T^2(\widetilde{M}, \mathbb{Z})$. Then

$\dfrac{\overline{\psi}(r') - \overline{\psi}(r)}{\eta(r,r')} \in \mathbb{Z}$ for all $(r,r') \in E$, and since π is weight preserving, it is a positive integer. Moreover observe that for all the paths $\gamma \in \overline{\Sigma}(p,q)$ $\overline{\psi}^{\xi}(\gamma_i) < \overline{\psi}^{\xi}(q)$, for all $i = 1, \ldots, |\gamma|$. The conclusion follows by combining part three of Theorem 4.3.2, Corollary 4.3.9 and Proposition 4.4.1. □

Suppose that $\{(M_i, \omega_i, \psi_i)\}_{i=0}^{k}$ is a sequence of GKM spaces such that M_0 is a point, and let $p_i : M_{i+1} \to M_i$ be a weight preserving equivariant map for each $i = 0, \ldots, k-1$. Moreover assume that each p_i is a fibration with symplectic fiber F_i and $H^*(F_i, A) \simeq H^*(\mathbb{C}P^{\frac{1}{2}\dim \widetilde{M}}, A)$ as rings. Then we can apply Proposition 4.3.7 and by an argument similar to the one used in the proof of Corollary 4.4.2, we get the following result (see [23]).

Proposition 4.4.3. *Assume that the hypotheses of Proposition 4.3.7 hold. If each p_i is a fibration with symplectic fiber F_i, with $H^*(F_i, A) \simeq H^*(\mathbb{C}P^{\frac{1}{2}\dim \widetilde{M}}, A)$ as rings, then given $p, q \in M^T$ we have*

$$\alpha_p(q) = \sum_{\gamma \in C(p,q)} \Xi(\gamma)$$

where

$$\Xi(\gamma) = \Lambda_q^- \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(\gamma_{i+1}) - \overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(\gamma_i)}{\overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(q) - \overline{\psi}_{h(\gamma_i,\gamma_{i+1})}(\gamma_i)} \cdot \frac{\Theta(\gamma_i,\gamma_{i+1})}{\eta(\gamma_i,\gamma_{i+1})}$$

lies in the symmetric algebra $\mathbb{S}(A \otimes \ell^*)$. Moreover, if $\Theta(r,r') > 0$ for all $(r,r') \in E$ then $\Xi(\gamma)$ lies in $A_+\{\alpha \in \Pi(M) \mid \alpha(\xi) > 0\}$, the semiring of $\mathbb{S}(A \otimes \ell^*)$ generated by $A_+ = \{t \in A \mid t \geq 0\}$ and the set of positive weights of M.

Chapter 5

Canonical classes on flag varieties

In this chapter we apply the results proved in Chapter 4 to the case of complete flag varieties. In particular we prove integral formulas for canonical classes on complete flag varieties of type A_n, B_n, C_n and D_n. Canonical classes correspond to equivariant Schubert classes on flag varieties, and the formulas we find are new, except in type A_n. Moreover we prove a general integral formula that implies the divided difference operator identities.

5.1 Existence of canonical classes

Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$, and $T \subset G$ a maximal torus with Lie algebra $\mathfrak{t}$. Let $R \subset \mathfrak{t}^*$ denote the set of roots, R^+ a choice of positive roots in R, and R_0 the associated simple roots. Let $\langle \cdot, \cdot \rangle$ be a positive definite symmetric bilinear form on $\mathfrak{g}$ which is $G-$invariant (e.g. negative the Cartan-Killing form), which we use to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\mathfrak{t}$ with $\mathfrak{t}^*$. Let W be the Weyl group of G.

Consider a generic point p_0 in $\mathfrak{t}^*$; we choose p_0 in such a way that $\langle p_0, \alpha \rangle < 0$ for all $\alpha \in R^+$. If $\mathcal{O}_{p_0}$ denotes the $G-$coadjoint orbit through p_0, we have already observed (cfr. section 3.1) that $\mathcal{O}_{p_0}$ is a GKM space with respect to the residual $T-$action. We recall the structure of the GKM graph (V, E_{GKM}).

- The vertices are in bijection with the elements of the Weyl group; more precisely the bijection is given by sending $w \in W$ to $w(p_0) \in \mathfrak{t}^*$. The restriction of the moment map ψ to the set of vertices is the inclusion.

- There exist an edge e between two vertices $p_1 = w_1(p_0)$ and $p_2 = w_2(p_0)$ if and only if $w_2 = w_1 s_\beta$ for some $\beta \in R^+$. The weight associated to the directed edge (p_1, p_2) is $\eta(p_1, p_2) = w_1(\beta)$. Since $w_1 s_\beta = s_{w_1(\beta)} w_1$, we can also say that p_1 and p_2 are joined by an edge if and only if $w_2 = s_\alpha w_1$ for some $\alpha \in R$. In this case we can define the weight associated to (p_1, p_2) to be the unique $\alpha \in R$ such that $w_2 = s_\alpha w_1$ and $\langle p_2, \alpha \rangle > 0$. It's easy to check that the two definitions of weight given above are equivalent.

Let $w_2 = s_\alpha w_1$, $p_1 = w_1(p_0)$ and $p_2 = w_2(p_0)$. Then

$$\psi(p_2) - \psi(p_1) = p_2 - s_\alpha(p_2) = 2\frac{\langle p_2, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha = 2\frac{\langle p_2, \alpha \rangle}{\langle \alpha, \alpha \rangle}\eta(p_1, p_2). \tag{5.1}$$

Hence, as required, $\psi(p_2) - \psi(p_1)$ is a positive multiple of $\eta(p_1, p_2)$. Since $\langle \cdot, \cdot \rangle$ is $\mathfrak{g}$ invariant

$$\langle p_2, \alpha \rangle = \langle w_2(p_0), \alpha \rangle = \langle p_0, w_2^{-1}(\alpha) \rangle = \langle p_0, w_1^{-1} s_\alpha(\alpha) \rangle = -\langle p_0, w_1^{-1}(\alpha) \rangle. \tag{5.2}$$

In particular, the set of weights Π_{p_2} in the tangent bundle at p_2 is

$$\Pi_{p_2} = \{\alpha \in R \mid \langle p_0, w_2^{-1}(\alpha) \rangle > 0\}. \tag{5.3}$$

Fix a generic $\xi \in \mathfrak{t}$ such that $\alpha(\xi) > 0$ for all α in R^+ and let $\varphi = \psi^\xi \colon \mathcal{O}_{p_0} \to \mathbb{R}$ be a generic component of the moment map; it's easy to see that φ achieves its minimum value at p_0 and therefore $\alpha(\xi) < 0$ for every weight $\alpha \in \Pi_{p_0}$. Since $\langle p_0, \alpha \rangle < 0$ for all $\alpha \in R^+$, for any point $p = w(p_0)$ we can rewrite (5.3) as

$$\Pi_p = \{\alpha \in R \mid w^{-1}(\alpha) \in -R^+\} = w(-R^+)$$

The set of weights Π_p^- in the negative normal bundle of p is the set of positive weights in the tangent bundle at p, i.e. if η belongs to Π_p^- then $\eta(\xi) > 0$ and hence $\eta \in R^+$.

This implies that

$$\Pi_p^- = R^+ \cap w(-R^+) = w\left(-R^+ \cap w^{-1}(R^+)\right). \tag{5.4}$$

In particular, $\lambda(p) = |R^+ \cap w(-R^+)|$.

We will need the following standard facts about root systems (cfr. [18]). Every element w of the Weyl group W can be written as a product of simple reflections, i.e. $w = s_{i_1} \cdots s_{i_r}$, where $s_{i_j} = s_{\alpha_{i_j}}$ and $\alpha_{i_j} \in R_0$ for all $j = 1, \dots, r$. The **length** of w, denoted by $l(w)$, is the smallest r for which such an expression exists. We refer to any such expression with $r = l(w)$ as a **reduced expression** for w (by convention $l(1) = 0$).

Given $w \in W$ and $\gamma \in R^+$, $l(ws_\gamma) > l(w)$ if and only if $w(\gamma) \in R^+$.

Let $w = s_{i_1} s_{i_2} \dots s_{i_r}$ be a reduced expression for $w \in W$ and $\beta_j = s_{i_r} \cdots s_{i_{j+1}}(\alpha_{i_j})$, with $\beta_r = \alpha_{i_r}$, where $s_{i_j} = s_{\alpha_{i_j}}$ for some $\alpha_{i_j} \in R_0$ for all $j = 1, \dots, r$. Then

$$R^+ \cap w^{-1}(-R^+) = \{\beta_1, \beta_2, \dots, \beta_r\} \tag{5.5}$$

Moreover, the β_j are all distinct.

Combining (5.4) and (5.5), we see that for any $p = w(p_0) \in V$,

$$\lambda(p) = l(p), \quad \text{and,} \tag{5.6}$$

$$\Pi_p^- = w\left(-R^+ \cap w^{-1}(R^+)\right) = \{\eta_1, \eta_2, \dots, \eta_r\} \tag{5.7}$$

where $\eta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$, with $\eta_1 = \alpha_{i_1}$.

We are ready to prove the following.

Lemma 5.1.1. *Let the maximal torus T of a compact simple Lie group G act on a generic coadjoint orbit $\mathcal{O}_{p_0} \subset \mathfrak{g}^*$. Let (V, E_{GKM}) be the associated GKM graph. Let $\psi\colon M \to \mathfrak{t}^*$ be the moment map and $\varphi = \psi^\xi\colon \mathcal{O}_{p_0} \to \mathbb{R}$ a generic component of the moment map. Then φ is index increasing.*

Proof. Consider an edge $(p_1, p_2) \in E_{\mathrm{GKM}}$ so that $\varphi(p_2) > \varphi(p_1)$; let $\alpha = \eta(p_1, p_2)$.

By the description of (V, E_{GKM}), there exists w_1 and w_2 in W so that $p_1 = w_1(p_0)$, $p_2 = w_2(p_0)$, and $w_2 = s_\alpha w_1$. Since $\psi(p_2)-\psi(p_1)$ is a positive multiple of $\eta(p_1,p_2)$, the fact that $\varphi(p_2) > \varphi(p_1)$ implies that $\alpha(\xi) > 0$, that is $\alpha \in \Pi_{p_2}^-$; by (5.4) this implies that $\alpha \in R^+ \cap w_2(-R^+)$. Hence $\delta = -w_2^{-1}(\alpha) = w_1^{-1}(\alpha)$ is a positive root, and $\alpha = w_1(\delta)$ is a positive root. This implies that $l(w_1 s_\delta) > l(w_1)$, which is equivalent to $l(w_2) = l(s_\alpha w_1) > l(w_1)$. Finally, by (5.6), $\lambda(w_2) > \lambda(w_1)$ as required. □

Since a generic component of the moment map is index increasing, Theorem 4.3.2 implies that integral canonical classes exist.

We recall that to any edge $(p,q) \in E_{\mathrm{GKM}}$ one associates an element $\Theta(p,q)$ in the field of fractions $\mathbb{S}(\mathfrak{t}^*)_0$ (cfr. section 4.3). When $\lambda(q) - \lambda(p) = 1$, this $\Theta(p,q)$ is a non zero integer (cfr. Theorem 4.3.2).

Proposition 5.1.2. *Let T be a maximal torus in a compact simple Lie group G, acting on a generic coadjoint orbit $\mathcal{O}_{p_0} \subset \mathfrak{g}^*$. Let (V, E_{GKM}) be the associated GKM graph. Let $\psi\colon M \to \mathfrak{t}^*$ be the moment map and $\varphi = \psi^\xi\colon \mathcal{O}_{p_0} \to \mathbb{R}$ a generic component of the moment map. Then*

$$\Theta(p,q) = 1$$

for all $(p,q) \in E_{\mathrm{GKM}}$ with $\lambda(q) - \lambda(p) = 1$.

Proof. Let Π_p^- (resp. Π_q^-) denote the set of weights in the negative normal bundle of φ at p (resp. q) and α the weight associated to (p,q). Observe that in order to prove that $\Theta(p,q) = 1$, it is sufficient to find a bijection $f : \Pi_p^- \to \Pi_q^- \setminus \{\alpha\}$ such that for all $\eta \in \Pi_p^-$, $f(\eta) - \eta = c\alpha$ for some constant c (depending on η).

Let $q = w(p_0)$ and $w = s_1 s_2 \cdots s_l$ be a reduced expression for w (where $s_i = s_{\alpha_{j_i}}$ for some $\alpha_{j_i} \in R_0,\ i = 1, \ldots, l$). By the Strong Exchange Condition (cfr. [18]) $p = w'(p_0)$, where $w' = s_1 \cdots \widehat{s_j} \cdots s_l$ for some (unique) $j = 1, \ldots, l$.
Let $\widetilde{w} = s_1 s_2 \cdots s_{j-1}$. We have

$$q = s_1 s_2 \cdots s_l(p_0) = \widetilde{w} s_j s_{j+1} \cdots s_l(p_0) = s_{\widetilde{w}(\alpha_j)} \widetilde{w} s_{j+1} \cdots s_l(p_0) = s_{\widetilde{w}(\alpha_j)}(p)$$

So, for the particular reduced expression of w above, $\alpha = \widetilde{w}(\alpha_j)$. Moreover

$$\Pi_p^- = \{\alpha_1, s_1(\alpha_2), \dots, s_1 s_2 \cdots s_{j-2}(\alpha_{j-1}), s_1 \cdots s_{j-1}(\alpha_{j+1}), \dots, s_1 \cdots \widehat{s_j} \cdots s_{l-1}(\alpha_l)\}$$

$$\Pi_q^- \setminus \{\alpha\} = \{\alpha_1, s_1(\alpha_2), \dots, s_1 s_2 \cdots s_{j-2}(\alpha_{j-1}), s_1 \cdots s_j(\alpha_{j+1}), \dots, s_1 \cdots s_{l-1}(\alpha_l)\}.$$

Define $f : \Pi_p^- \to \Pi_q^- \setminus \{\alpha\}$ to be

$$f(s_1 \cdots s_k(\alpha_{k+1})) = s_1 \cdots s_k(\alpha_{k+1}) \quad \text{if} \quad 1 \leq k < j-1 \ ,$$

and

$$f(s_1 \cdots \widehat{s_j} \cdots s_k(\alpha_{k+1})) = s_1 \cdots s_k(\alpha_{k+1}) \quad \text{if} \quad j \leq k < l$$

For every k s.t. $j \leq k < l$ we have $s_1 \cdots s_k(\alpha_{k+1}) = s_{\widetilde{w}(\alpha_j)}(s_1 \cdots \widehat{s_j} \cdots s_k(\alpha_{k+1}))$, hence

$$f(s_1 \cdots \widehat{s_j} \cdots s_k(\alpha_{k+1})) - s_1 \cdots \widehat{s_j} \cdots s_k(\alpha_{k+1}) \equiv 0 \bmod \alpha (= \widetilde{w}(\alpha_j)) \ .$$

Since on the other weights f is the identity, the claim follows immediately. □

Consider two points p_1 and p_2 in $\mathfrak{t}^*$ which lie in the closure of the same Weyl chamber, and such that $G_{p_2} \subset G$ (the stabilizer of p_2) contains G_{p_1} (the stabilizer of p_1). Consider the $G-$coadjoint orbits $\mathcal{O}_{p_1}$ and $\mathcal{O}_{p_2}$. Then there is a natural projection map

$$\begin{array}{rccc} \pi : & \mathcal{O}_{p_1} & \to & \mathcal{O}_{p_2} \\ & g \cdot p_1 & \mapsto & g \cdot p_2 \end{array} \tag{5.8}$$

We have already observed (cfr. section 3.1) that π is a $T-$equivariant fibration with symplectic fibers isomorphic to G_{p_2}/G_{p_1}. Moreover π is a weight preserving map, since it is a map of almost complex manifold. In [14] we prove a much stronger result: if Γ_1 and Γ_2 denote the GKM graphs associated respectively to $\mathcal{O}_1$ and $\mathcal{O}_2$, then $\pi : \Gamma_1 \to \Gamma_2$ is a GKM fiber bundle (cfr. Theorem 4.2).

Now we apply the results of Chapter 3 to derive integral formulas for canonical classes on generic coadjoint orbits of type A, B, C and D.

5.2 Generic coadjoint orbit of type A_n

Let $G = SU(n+1)$, and let T be the torus of diagonal matrices in G with Lie algebra $\mathfrak{t}$. Then $\mathcal{S}_{n+1}$, the group of permutations on $n+1$ elements, is the Weyl group W of G. Let $\{x_i\}_{i=1}^{n+1}$ be the basis of $(\mathbb{R}^{n+1})^*$ given by $x_i(\xi_1, \dots, \xi_{n+1}) = \xi_i$. We can identify the dual of the Lie algebra of T, $\mathfrak{t}^*$, with the subset of $(\mathbb{R}^{n+1})^*$ given by

$$\left\{ \sum_{j=1}^{n+1} \mu_j x_j \mid \sum_{j=1}^{n+1} \mu_j = 0 \right\} .$$

A choice of simple roots is given by $\alpha_j = x_j - x_{j+1}$, for all $j = 1, \dots, n$. For all $i = 0, \dots, n$ define $\mu^i = (\mu^i_1, \dots, \mu^i_{n+1})$ to be a vector in $\mathbb{R}^{n+1}$ such that $\mu^i_1 < \mu^i_2 < \ldots < \mu^i_i < \mu^i_{i+1} = \ldots = \mu^i_{n+1}$ and $\sum_{j=1}^{n+1} \mu^i_j = 0$. Let p^i be the point $p^i = \sum_{j=1}^{n+1} \mu^i_j x_j$ in $\mathfrak{t}^*$. The coadjoint orbit $\mathcal{O}_{p^i} = G \cdot p^i$ is isomorphic to G/G_{p^i}, where

$$G_{p^i} = S(U(1) \times \ldots \times U(1) \times U(n-i+1)) .$$

The $T-$fixed points are given by

$$(G/G_{p^i})^T = \{ \sum_{j=1}^{i} \mu^i_j x_{\sigma(j)}, \;\; \sigma \in \mathcal{S}_{n+1} \} \subset \mathfrak{t}^* .$$

The moment map ψ_i restricted to the fixed point set is given by the inclusion $\psi_i : (G/G_{p^i})^T \hookrightarrow \mathfrak{t}^*$. Observe that G/G_{p^0} is just the point p^0 and G/G_{p^n} is isomorphic to $\mathcal{F}l(\mathbb{C}^{n+1})$, the manifold of complete flags in $\mathbb{C}^{n+1}$.

For all $i = 0, \dots, n$ define $M_i = G/G_{p^i}$, and consider the projection map $p_i : M_{i+1} \to M_i$ as defined in (5.8); then the fiber of p_i, $i = 0, \dots, n-1$ is isomorphic to $U(n-i+1)/U(n-i)$, which is the complex projective space $\mathbb{C}P^{n-i}$. So the sequence of spaces $\{(M_i, \omega_i, \psi_i)\}_{i=0}^n$ is a sequence of GKM spaces, and the map $p_i : M_{i+1} \to M_i$ is a weight preserving $T-$equivariant fibration with symplectic fiber isomorphic to $\mathbb{C}P^{n-i}$, for all $i = 0, \dots, n-1$.

Let $\pi_i : M_n \to M_i$ be the composition $\pi_i = p_i \circ p_{i+1} \circ \cdots \circ p_{n-1}$. The restrictions of the maps p_i's and π_i's to the fixed points, $p_i : (M_{i+1})^T \to (M_i)^T$ and

$\pi_i : (M_n)^T \to (M_i)^T$, are given by

$$p_i(\sum_{j=1}^{n+1} \mu_j^{i+1} x_{\sigma(j)}) = \sum_{j=1}^{n+1} \mu_j^i x_{\sigma(j)}, \quad \text{hence} \quad \pi_i(\sum_{j=1}^{n+1} \mu_j^n x_{\sigma(j)}) = \sum_{j=1}^{n+1} \mu_j^i x_{\sigma(j)} \ .$$

Let r and r' be points in $(M_n)^T$, where $r = \sum_{j=1}^{n+1} \mu_j^n x_{\sigma(j)}$ and $r' = \sum_{j=1}^{n+1} \mu_j^n x_{\sigma'(j)}$. Then

$$\pi_i(r) = \pi_i(r') \Longleftrightarrow \pi_j(r) = \pi_j(r') \ \ \forall\, 0 \le j \le i \Longleftrightarrow \sigma(j) = \sigma'(j) \ \ \forall\, 0 \le j \le i$$

For any pair of elements r, r' in $(M_n)^T$ define $h(r, r')$ to be

$$h(r, r') = \min\{j \in \{0, \dots, n\} \mid \quad \pi_j(r) \neq \pi_j(r')\}$$

(cfr. Proposition 4.3.7). It's clear that $h(r, r') = a$ if and only if $\sigma(j) = \sigma'(j)$, for all $0 \le j < a$ and $\sigma(a) \neq \sigma'(a)$. If (r, r') is an edge in E, hence in E_{GKM}, then $r' = s_\beta r$ for some reflection $s_\beta \in W$, $\beta \in R$, where $\beta = x_{\sigma(h)} - x_{\sigma(k)}$ for some $h < k$; in this case $h(r, r') = h$.

Let's recall the following notation: $\sigma' = \sigma(h, k)$ means that σ' is obtained from σ by swapping the elements at positions h and k in the one line notation of $\sigma = \sigma(1) \dots \sigma(n+1)$; $\sigma' = (i, j)\sigma$ means that we are swapping the elements i and j in the one line notation of σ. Hence $\sigma' = \sigma(h, k)$ if and only if $\sigma' = (\sigma(h), \sigma(k))\sigma$. Observe that if $\sigma' = \sigma(h, k)$, with $h < k$, then the height $h(\sigma(p^n), \sigma'(p^n))$ of the edge $(\sigma(p^n), \sigma'(p^n))$ is given by h.

Consider now the canonical classes $\{\alpha_p\}_{p \in M_n^T}$ associated to $\varphi_n = \psi_n^\xi$. These classes exist by Lemma 5.1.1. For any $p, q \in (M_n)^T$ define

$$C(p, q) = \{\gamma \in \Sigma(p, q) \mid h(\gamma_1, \gamma_2) \le h(\gamma_2, \gamma_3) \le \ldots \le h(\gamma_{|\gamma|}, \gamma_{|\gamma|+1})\} \ .$$

Proposition 5.2.1. *Let G/G_{p^n} be a generic coadjoint orbit of type A_n. Fix p and q in $(G/G_{p^n})^T$. Let $\alpha_p \in H_T^{2\lambda(p)}(M_n; \mathbb{Z})$ be the canonical class associated to $\varphi_n = \psi_n^\xi$.*

(1) A path $\gamma = (\sigma_1(p^n), \dots, \sigma_{l+1}(p^n))$ is an element of $C(\sigma_1(p^n), \sigma_{l+1}(p^n))$, if and

only if

(a) $l(\sigma_{j+1}) = l(\sigma_j) + 1$ *and* $\sigma_{j+1} = \sigma_j(h_j, k_j)$ *with* $h_j < k_j$, *for all* $j = 1, \dots, l$

(b) $h_1 \leq h_2 \leq \dots \leq h_l$

(2) (c) *For all* $p, q \in (G/P_{p^n})^T$

$$\alpha_p(q) = \Lambda_q^- \sum_{\substack{\gamma \in C(p,q) \\ \gamma=(\sigma_1(p^n),\dots,\sigma_{l+1}(p^n))}} \prod_{j=1}^{l} \frac{1}{x_{\sigma_j(h_j)} - x_{\sigma_{l+1}(h_j)}}$$

(d) *For all* $\gamma = (\sigma_1(p^n), \dots, \sigma_{l+1}(p^n))$ *in* $C(p,q)$

$$\Xi(\gamma) = \Lambda_q^- \prod_{j=1}^{l} \frac{1}{x_{\sigma_j(h_j)} - x_{\sigma_{l+1}(h_j)}}$$

is a polynomial with positive integer coefficients in the simple roots, i.e. it belongs to $\mathbb{Z}_{\geq 0}[\alpha_1, \dots, \alpha_n]$.

Proof. Condition (a) is equivalent to saying that γ belongs to $\Sigma(\sigma_1(p^n), \sigma_{l+1}(p^n))$. Then, from what we observed before, if $\sigma_{j+1} = \sigma_j(h_j, k_j)$ with $h_j < k_j$, then $h(\sigma_j(p^n)\sigma_{j+1}(p^n)) = h_j$, for all $j = 1, \dots, l$. So part (1) follows from the definition of $C(p,q)$.

Now observe that the sequence of GKM spaces $\{(M_i, \omega_i, \psi_i)\}_{i=0}^n$ satisfies the hypotheses of Proposition 4.3.7. Define $\overline{\psi}_j = \pi_j^*(\psi_j) : M_n \to \mathfrak{t}^*$ for all j. Then $\overline{\psi}_i(\sigma(p^n)) = \sum_{m=1}^{n+1} \mu_m^i x_{\sigma(m)}$, where $\mu_{i+1}^i = \dots = \mu_{n+1}^i$. Hence, if $h(\sigma(p^n)\sigma'(p^n)) = h$, we have $\overline{\psi}_h(\sigma'(p^n)) - \overline{\psi}_h(\sigma(p^n)) = \mu_h^h(x_{\sigma'(h)} - x_{\sigma(h)}) + \mu_{h+1}^h \sum_{m=h+1}^{n+1}(x_{\sigma'(m)} - x_{\sigma(m)})$. But $\sum_{m=h}^{n+1}(x_{\sigma'(m)} - x_{\sigma(m)}) = 0$ because $\{\sigma(h), \dots, \sigma(n+1)\} = \{\sigma'(h), \dots, \sigma'(n+1)\}$, hence

$$\begin{aligned}
&\overline{\psi}_h(\sigma'(p^n)) - \overline{\psi}_h(\sigma(p^n)) = \\
&\mu_h^h(x_{\sigma'(h)} - x_{\sigma(h)}) + \mu_{h+1}^h \sum_{m=h+1}^{n+1}(x_{\sigma'(m)} - x_{\sigma(m)}) - \mu_{h+1}^h \sum_{m=h}^{n+1}(x_{\sigma'(m)} - x_{\sigma(m)}) = \\
&(\mu_{h+1}^h - \mu_h^h)(x_{\sigma(h)} - x_{\sigma'(h)})
\end{aligned}$$

In particular if $\sigma' = \sigma(h,k)$ with $h < k$ then

$$\overline{\psi}_h(\sigma'(p^n)) - \overline{\psi}_h(\sigma(p^n)) = (\mu^h_{h+1} - \mu^h_h)(x_{\sigma(h)} - x_{\sigma(k)}) = (\mu^h_{h+1} - \mu^h_h)\eta(\sigma(p^n), \sigma'(p^n))$$

Recall that for every path $\gamma = (\sigma_1(p^n), \ldots, \sigma_{l+1}(p^n))$ in $C(\sigma_1(p^n), \sigma_{l+1}(p^n))$, $h(\sigma_i(p^n), \sigma_{i+1}(p^n)) = h(\sigma_i(p^n), \sigma_{l+1}(p^n))$ for all $i = 1, \ldots, l$ (cfr. Corollary 4.2.5).

Combining the above computations with Proposition 4.3.7 and 5.1.2, we obtain claim (c). Part (d) follows from Proposition 4.4.3. □

5.3 Generic coadjoint orbit of type B_n

Let $G = SO(2n+1)$ and T a maximal compact subtorus in G with Lie algebra $\mathfrak{t}$. Let $\{x_i\}_{i=1}^n$ be the basis of $(\mathbb{R}^n)^* \simeq \mathfrak{t}^*$ given by $x_i(\mu_1, \ldots, \mu_n) = \mu_i,\ i = 1, \ldots, n$. Let R be the set of roots of G, $\alpha_1 = x_1 - x_2, \ldots, \alpha_{n-1} = x_{n-1} - x_n, \alpha_n = x_n$ a choice of simple roots R_0, and R^+ the associated set of positive roots. The Weyl group W of G is the group on signed permutations of n elements.

Consider a vector in $\mathbb{R}^n$ $(\mu_1, \ldots, \mu_n)$ such that $\mu_1 < \mu_2 < \ldots < \mu_n < 0$ and let p_0 be the point in $\mathfrak{t}^*$ given by $p_0 = \sum_{j=1}^n \mu_j x_j$.

The $T-$fixed points of the $G-$coadjoint orbit $\mathcal{O}_{p_0} = G \cdot p_0 \simeq G/T$ are given by

$$\mathcal{O}^T_{p_0} = \left\{ \sum_{j=1}^n (-1)^{\epsilon_j} \mu_j x_{\sigma(j)},\ \epsilon_j \in \{0,1\}\ j = 1, \ldots, n,\ \sigma \in \mathcal{S}_n \right\} \subset \mathfrak{t}^*$$

Let's denote $\mathcal{O}_{p_0}$ by M, and let ω be the canonical symplectic form associated to it. The moment map ψ restricted to the fixed point set is given by the inclusion $\psi : \mathcal{O}^T_{p_0} \hookrightarrow \mathfrak{t}^*$. Let ξ be a generic vector in $\mathfrak{t}$ such that $\alpha(\xi) > 0$ for all $\alpha \in R^+$. Then $\varphi = \psi^\xi : M \to \mathbb{R}$ is a Morse function, and the point p_0 is the minimum of φ.

Now let's consider the set of canonical classes $\{\alpha_p\}_{p \in M^T}$ associated to φ, which exists by Lemma 5.1.1. The main result of this section is an explicit inductive *positive integral formula* for $\alpha_p(q)$ for all $p, q \in M^T$, i.e. a formula in which each term is a polynomial in the simple roots with positive integer coefficients.

Let $\widetilde{p_0}$ be the point $-x_1 \in \mathfrak{t}^*$. Then the coadjoint orbit $G \cdot \widetilde{p_0} = \mathcal{O}_{\widetilde{p_0}}$ is isomorphic to $Gr_2^+(\mathbb{R}^{2n+1})$, the Grassmannian of oriented two planes in $\mathbb{R}^{2n+1}$. Let's denote it by $\widetilde{M}$; let $\widetilde{\omega}$ be the canonical symplectic structure on $\widetilde{M}$, and $\widetilde{\psi} : \widetilde{M} \to \mathfrak{t}^*$ the moment map.

The GKM graph $(\widetilde{V}, \widetilde{E}_{\text{GKM}})$ associated to $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ is a complete graph with $2n$ vertices,

$$\widetilde{V} = \{\pm x_i,\ i = 1, \dots, n\}$$

and as before the moment map restricted to the fixed point set is given by the inclusion $\widetilde{\psi} : \mathcal{O}_{\widetilde{p_0}}^T = \widetilde{V} \hookrightarrow \mathfrak{t}^*$. The same choice of the polarizing vector ξ gives a Morse function $\widetilde{\varphi} = \widetilde{\psi}^\xi : \widetilde{M} \to \mathbb{R}$, with critical points $\{-x_1, -x_2, \dots, -x_n, x_n, \dots, x_2, x_1\}$. We have $\widetilde{\lambda}(-x_i) = i - 1$, $\widetilde{\lambda}(x_i) = n - i$, for all $i = 1, \dots, n$, where $\widetilde{\lambda}(p)$ denotes half of the Morse index of $\widetilde{\varphi}$ at p. So the critical points are listed by increasing Morse index; hence the points $-x_1$ and x_1 are respectively the minimum and the maximum of $\widetilde{\varphi}$.

Consider the projection map

$$\pi : (M, \omega, \psi) \to (\widetilde{M}, \widetilde{\omega}, \widetilde{\psi}) \tag{5.9}$$

given by $\pi(g \cdot p_0) = g \cdot \widetilde{p_0}$, for all $g \in G$. Recall that the fixed point set M^T of M can be described as $\{w(p_0),\ w \in W\}$, and the map π restricted to the fixed point set is simply given by

$$\begin{array}{rccc} \pi_{|_{M^T}} : & M^T & \to & \widetilde{M}^T \\ & w(p_0) & \mapsto & w(\widetilde{p_0}) \ . \end{array}$$

Since p_0 and $\widetilde{p_0}$ lie in the closure of the same Weyl chamber, the map π is a weight preserving $T-$equivariant fibration with symplectic fibers. Moreover observe that the fiber over a point $\pm x_h \in \widetilde{M}^T$ is a generic coadjoint orbit of type B_{n-1}. The subgroup W_h of W generated by the elements s_{x_i} and $s_{x_i - x_j}$ with $i, j \in \{1, \dots, n\} \setminus \{h\}$ acts transitively on the fixed points of the fiber.

For every p and $q \in M^T$, let $\overline{\Sigma}(p,q)$ be the set of paths in $\Sigma(p,q)$ which are horizontal with respect to π (cfr. Corollary 4.3.9). Given $\gamma \in \overline{\Sigma}(p,q)$, let $\widetilde{\gamma} = \pi(\gamma)$

be its projection to $\widetilde{E}_{\mathrm{GKM}}$ and $V(\widetilde{\gamma})$ the set of vertices of $\widetilde{\gamma}$,

$$V(\widetilde{\gamma}) = \{\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_{|\gamma|+1}\} \subseteq \widetilde{V} ,$$

with $\widetilde{\gamma}_i = \pi(\gamma_i)$, for all $i = 1, \ldots, |\gamma|+1$. Observe that $\widetilde{\gamma}_i = \pm x_j$, for some $j = 1, \ldots, n$; so we can define $-\widetilde{\gamma}_i$ to be $\mp x_j$

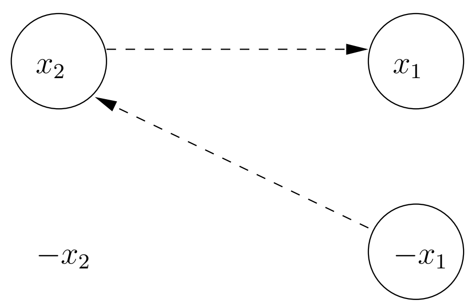
Definition 5.3.1. *Given a path $\gamma \in \overline{\Sigma}(p, q)$, let $\widetilde{\gamma} = \pi(\gamma)$. The path γ is said to be **uneven** if the following conditions are satisfied.*

(i) $\widetilde{\gamma}_{|\gamma|+1} \in R^+$,

(ii) $-\widetilde{\gamma}_{|\gamma|+1} \in V(\widetilde{\gamma})$, *and*

(iii) $\max\{h \mid x_h \in V(\widetilde{\gamma})\} \neq \max\{h \mid -x_h \in V(\widetilde{\gamma})\}$.

*Otherwise γ is said to be **even**.*



In the above figure we show the projection $\pi(\gamma) = (-x_1, x_2, x_1)$ of an uneven path γ.

Observe that from the definition of uneven path, condition (ii) implies that the set

$$\{j \mid x_j \text{ and } -x_j \in V(\widetilde{\gamma})\}$$

is non empty. For every uneven path define

$$k(\gamma) = \max\{j \mid x_j \text{ and } -x_j \in V(\widetilde{\gamma})\} .$$

Observe that (iii) implies that $k(\gamma) < n$.

Definition 5.3.2. *Given a path $\gamma \in \overline{\Sigma}(p,q)$, let $\widetilde{\gamma} = \pi(\gamma)$. The path γ is said to be **relevant** if it is either even or if it is uneven and $x_{k(\gamma)+1} \in V(\widetilde{\gamma})$. Let $R(p,q) \subset \overline{\Sigma}(p,q)$ denote the set of **relevant paths**.*

Observe that by Lemma 4.3.8, canonical classes on the fiber of π exist. The main theorem of this section is the following.

Theorem 5.3.3. *Let $\pi : (M,\omega,\psi) \to (\widetilde{M},\widetilde{\omega},\widetilde{\psi})$ be the projection map (5.9). Let $\varphi = \psi^{\xi} : M \to \mathbb{R}$ be a generic component of the moment map, and consider the canonical classes $\{\alpha_p\}_{p\in M^T}$ associated to φ. For every $s \in M^T$ consider the canonical class $\widehat{\alpha}_s$ on the fiber $\widehat{M}_s = \pi^{-1}(\pi(s))$, and let $R(p,s)$ be the set of relevant paths from p to s.*

For every $\gamma \in R(p,s)$ define $Q(\gamma)$ to be

(i)

$$Q(\gamma) = \widetilde{\Lambda}^-_{\pi(\gamma_{|\gamma|+1})} \prod_{i=1}^{|\gamma|} \frac{\pi(\gamma_{i+1}) - \pi(\gamma_i)}{\pi(\gamma_{|\gamma|+1}) - \pi(\gamma_i)} \frac{1}{\eta(\gamma_i,\gamma_{i+1})}$$

if γ is even;

(ii)

$$Q(\gamma) = 2\widetilde{\Lambda}^-_{\pi(\gamma_{|\gamma|+1})} \left(\prod_{i=1}^{|\gamma|} \frac{\pi(\gamma_{i+1}) - \pi(\gamma_i)}{\pi(\gamma_{|\gamma|+1}) - \pi(\gamma_i)} \frac{1}{\eta(\gamma_i,\gamma_{i+1})} \right) \frac{\pi(\gamma_{|\gamma|+1})}{\eta(-x_{k(\gamma)+1}, \pi(\gamma_{|\gamma|+1}))}$$

if γ is uneven.

Then

(1) For all p and q in M^T

$$\alpha_p(q) = \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in R(p,s)} Q(\gamma) \right) \widehat{\alpha}_s(q),$$

(2) $Q(\gamma)$ is a polynomial in the simple roots with positive integer coefficients, i.e. it belongs to $\mathbb{Z}_{\geq 0}[\alpha_1,\ldots,\alpha_n]$.

Before proving Theorem 5.3.3 we need to give another characterization of even and uneven paths, and make the expression of the terms $Q(\gamma)$ more explicit.

If (V, E_{GKM}) is the GKM graph associated to a GKM space (M, ω, ψ), we define the **magnitude** $m^{\psi}(e)$ of an edge $e = (r, s) \in E_{\mathrm{GKM}}$ to be

$$m^{\psi}(e) = \frac{\psi(s) - \psi(r)}{\eta(e)} .$$

Observe that for the GKM space $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ it is possible to define the magnitude of (r, s) for any pair of vertices r, s in $\widetilde{V}$, since $(\widetilde{V}, \widetilde{E}_{\mathrm{GKM}})$ is a complete graph. Since from now on we will only consider the magnitude of edges in $(\widetilde{V}, \widetilde{E}_{\mathrm{GKM}})$, we will denote $m^{\widetilde{\psi}}(e)$ simply by $m(e)$, for all $e \in \widetilde{E}_{\mathrm{GKM}}$.

Definition 5.3.4. *Given a path* $\gamma \in \overline{\Sigma}(p, q)$, *let* $\widetilde{\gamma} = \pi(\gamma)$. *The set of* **skipped vertices** *of* γ *is*

$$SV(\widetilde{\gamma}) = \left\{ s \in \widetilde{V} \;\middle|\; \widetilde{\psi}^{\xi}(s) < \widetilde{\psi}^{\xi}(\widetilde{\gamma}_{|\gamma|+1}) \right\} \setminus V(\widetilde{\gamma}).$$

Proposition 5.3.5. *Let* $\gamma = (\gamma_1, \ldots, \gamma_{|\gamma|+1})$ *be a path in* (V, E_{GKM}) *which is horizontal and increasing. Define*

$$P(\gamma) = \widetilde{\Lambda}^{-}_{\pi(\gamma_{|\gamma|+1})} \prod_{i=1}^{|\gamma|} \frac{\pi(\gamma_{i+1}) - \pi(\gamma_i)}{\pi(\gamma_{|\gamma|+1}) - \pi(\gamma_i)} \frac{1}{\eta(\gamma_i, \gamma_{i+1})}$$

If $\widetilde{\gamma}$ *denotes* $\pi(\gamma)$ *then*

(i) $P(\gamma) = \prod_{s \in SV(\widetilde{\gamma})} \eta(s, \widetilde{\gamma}_{|\widetilde{\gamma}|+1})$ *if and only if one of the following happens*

(a) $\widetilde{\gamma}_1 \in R^+$ *or* $\widetilde{\gamma}_{|\widetilde{\gamma}|+1} \notin R^+$

(b) $-\widetilde{\gamma}_{|\widetilde{\gamma}|+1} \notin V(\widetilde{\gamma})$ *and* $\max\{h \mid x_h \in V(\widetilde{\gamma})\} \neq \max\{h \mid -x_h \in V(\widetilde{\gamma})\}$

(c) $-\widetilde{\gamma}_{|\widetilde{\gamma}|+1} \in V(\widetilde{\gamma})$ *and* $\max\{h \mid x_h \in V(\widetilde{\gamma})\} = \max\{h \mid -x_h \in V(\widetilde{\gamma})\}$

(ii) $P(\gamma) = 2 \prod_{s \in SV(\widetilde{\gamma})} \eta(s, \widetilde{\gamma}_{|\widetilde{\gamma}|+1})$ *if and only if* $\widetilde{\gamma}_{|\widetilde{\gamma}|+1} \in R^+$, $\widetilde{\gamma}_1 \notin R^+$, $-\widetilde{\gamma}_{|\widetilde{\gamma}|+1} \notin V(\widetilde{\gamma})$

and $\max\{h \mid x_h \in V(\widetilde{\gamma})\} = \max\{h \mid -x_h \in V(\widetilde{\gamma})\}$

(iii) $P(\gamma) = \dfrac{1}{2} \prod_{s\in SV(\widetilde{\gamma})} \eta(s, \widetilde{\gamma}_{|\widetilde{\gamma}|+1})$ *if and only if* $\widetilde{\gamma}_{|\widetilde{\gamma}|+1} \in R^+$, $-\widetilde{\gamma}_{|\widetilde{\gamma}|+1} \in V(\widetilde{\gamma})$ *and* $\max\{h \mid x_h \in V(\widetilde{\gamma})\} \neq \max\{h \mid -x_h \in V(\widetilde{\gamma})\}$.

Hence $P(\gamma)$ *is a polynomial with positive integer coefficients in the positive roots if and only if* γ *is even.*

As usual we set the empty product to be equal to one.

Proof. First of all it's easy to see that if $\widetilde{\gamma} = (\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_{|\widetilde{\gamma}|+1}) = \pi(\gamma)$, then

$$P(\gamma) = \prod_{i=1}^{|\widetilde{\gamma}|} \frac{m(\widetilde{\gamma}_i, \widetilde{\gamma}_{i+1})}{m(\widetilde{\gamma}_i, \widetilde{\gamma}_{|\widetilde{\gamma}|+1})} \prod_{s\in SV(\widetilde{\gamma})} \eta(s, \widetilde{\gamma}_{|\widetilde{\gamma}|+1}) \tag{5.10}$$

In fact observe that $\pi(\gamma_{|\gamma|+1}) \neq \pi(\gamma_i)$ for all $i = 1, \ldots, |\gamma|$, since π is a weight preserving map and γ is horizontal and increasing. Then observe that

$$\frac{\pi(\gamma_{i+1}) - \pi(\gamma_i)}{\eta(\gamma_i, \gamma_{i+1})} = \frac{\widetilde{\psi}(\pi(\gamma_{i+1})) - \widetilde{\psi}(\pi(\gamma_i))}{\eta(\pi(\gamma_i), \pi(\gamma_{i+1}))} = m(\pi(\gamma_i), \pi(\gamma_{i+1}))$$

and

$$\frac{\widetilde{\Lambda}^-_{\pi(\gamma_{|\gamma|+1})}}{\pi(\gamma_{|\gamma|+1}) - \pi(\gamma_i)} = \frac{\widetilde{\Lambda}^-_{\pi(\gamma_{|\gamma|+1})}}{m(\pi(\gamma_i), \pi(\gamma_{|\gamma|+1}))\eta(\pi(\gamma_i), \pi(\gamma_{|\gamma|+1}))} .$$

Moreover since γ is an increasing horizontal path, and since π is a weight preserving map, the path $\widetilde{\gamma} = \pi(\gamma)$ is increasing as well. Hence by definition of $SV(\widetilde{\gamma})$ we have that

$$\prod_{i=1}^{|\gamma|} \frac{\widetilde{\Lambda}^-_{\pi(\gamma_{|\gamma|+1})}}{\eta(\pi(\gamma_i), \pi(\gamma_{|\gamma|+1}))} = \prod_{s\in SV(\widetilde{\gamma})} \eta(s, \pi(\gamma_{|\gamma|+1})) ,$$

and (5.10) follows. Then observe that since $\widetilde{\gamma}$ is increasing, its sequence of vertices is an ordered subsequence of

$$(-x_1, -x_2, \ldots, -x_n, x_n, \ldots, x_2, x_1)$$

The only edges in $\widetilde{E}_{\text{GKM}}$ of magnitude 2 are the ones connecting opposite vertices $-x_j$

and x_j, for some $j = 1, \dots, n$; all the others have magnitude 1. Then, since $\widetilde{\gamma}$ is an increasing path, it can have at most one edge of type $(-x_j, x_j)$ for some $j = 1, \dots, n$, and if such an edge exists then $j = \max\{h \mid x_h \in V(\widetilde{\gamma})\} = \max\{h \mid -x_h \in V(\widetilde{\gamma})\}$. Hence we have that $\prod_{i=1}^{|\widetilde{\gamma}|} m(\widetilde{\gamma}_i, \widetilde{\gamma}_{i+1}) = 2$ precisely if $\widetilde{\gamma}$ contains an edge of type $(-x_j, x_j)$, where $j = \max\{h \mid x_h \in V(\widetilde{\gamma})\} = \max\{h \mid -x_h \in V(\widetilde{\gamma})\}$, otherwise the product equals 1.

Similarly $\prod_{i=1}^{|\widetilde{\gamma}|} m(\widetilde{\gamma}_i, \widetilde{\gamma}_{|\widetilde{\gamma}|+1}) = 2$ precisely if $\widetilde{\gamma}$ contains the edge $(-\widetilde{\gamma}_{|\widetilde{\gamma}|+1}, \widetilde{\gamma}_{|\widetilde{\gamma}|+1})$.
Combining these facts and (5.10) the claim follows. □

We recall that for every uneven path γ, if $\widetilde{\gamma} = \pi(\gamma)$, $k(\gamma)$ is defined to be $\max\{j \mid x_j \text{ and } -x_j \in V(\widetilde{\gamma})\}$, which from the definition of uneven path is a non empty set, and $k(\gamma) < n$. The proof of Theorem 5.3.3 is based on the following two lemmas (the proofs of which are given at the end of this section).

Lemma 5.3.6. *Let γ be a horizontal path in $\overline{\Sigma}(p,q)$ and let $\widetilde{\gamma} = \pi(\gamma)$. If γ is uneven then either $x_{k(\gamma)+1}$ or $-x_{k(\gamma)+1}$ belongs to $V(\widetilde{\gamma})$.*

For every path $\widetilde{\gamma} = \pi(\gamma)$ satisfying this property define $\widetilde{\gamma}'$ to be the path obtained from $\widetilde{\gamma}$ by replacing the vertex $x_{k(\gamma)+1} \in V(\widetilde{\gamma})$ (or $-x_{k(\gamma)+1} \in V(\widetilde{\gamma})$) with $-x_{k(\gamma)+1}$ (or $x_{k(\gamma)+1}$). For example if $\widetilde{\gamma} = (\dots, -x_{k(\gamma)}, -\widehat{x_{k(\gamma)+1}}, \dots, x_{k(\gamma)+1}, x_{k(\gamma)}, \dots)$ then $\widetilde{\gamma}' = (\dots, -x_{k(\gamma)}, -x_{k(\gamma)+1}, \dots, \widehat{x_{k(\gamma)+1}}, x_{k(\gamma)}, \dots)$.

Lemma 5.3.7. *Let γ be a horizontal path in $\overline{\Sigma}(p,q)$ and let $\widetilde{\gamma} = \pi(\gamma)$. If γ is an uneven path then there exists $\gamma' \in \overline{\Sigma}(p,q)$ such that $\pi(\gamma') = \widetilde{\gamma}'$.*

Hence the uneven paths which are in the image of $\overline{\Sigma}(p,q)$ always come in pairs. We are now ready to prove Theorem 5.3.3

Proof of Theorem 5.3.3. Observe that if $\pi : (M, \omega, \psi) \to (\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ is the projection (5.9), then $\pi^*(\widetilde{\psi})(r) = \pi(r)$ for all $r \in M^T$. By Corollary 4.3.9 it follows that for all $p, q \in M^T$

$$\alpha_p(q) = \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in \overline{\Sigma}(p,s)} \widetilde{\Lambda}^-_{\pi(q)} \prod_{i=1}^{|\gamma|} \frac{\pi(\gamma_{i+1}) - \pi(\gamma_i)}{\pi(q) - \pi(\gamma_i)} \cdot \frac{\Theta(\gamma_i, \gamma_{i+1})}{\eta(\gamma_i, \gamma_{i+1})} \right) \widehat{\alpha}_s(q).$$

Hence by definition of $P(\gamma)$ and Proposition 5.1.2 we have

$$\alpha_p(q) = \sum_{s\in\widehat{M}_q^T} \Big(\sum_{\gamma\in\overline{\Sigma}(p,s)} P(\gamma)\Big)\widehat{\alpha}_s(q)\ .$$

In order to prove part (1) of Theorem 5.3.3 it is enough to prove that for all $p, s \in M^T$

$$\sum_{\gamma\in\overline{\Sigma}(p,s)} P(\gamma) = \sum_{\gamma\in R(p,s)} Q(\gamma)$$

Observe that for every relevant path $\gamma \in R(p,s)$, we have

$$Q(\gamma) = \begin{cases} P(\gamma) & \text{if } \gamma \text{ is even} \\ \\ 2P(\gamma)\dfrac{\pi(s)}{\eta(-x_{k(\gamma)+1}, \pi(s))} & \text{if } \gamma \text{ is uneven} \end{cases}$$

By Lemma 5.3.6 and 5.3.7, the set of uneven paths in $\overline{\Sigma}(p,s)$ contains pairs of paths γ and γ', where the set of vertices of $\widetilde{\gamma}' = \pi(\gamma')$ is obtained from the set of vertices of $\widetilde{\gamma} = \pi(\gamma)$ by replacing $x_{k(\gamma)+1}$ (or $-x_{k(\gamma)+1}$) with $-x_{k(\gamma)+1}$ (or $x_{k(\gamma)+1}$). Hence either γ or γ' is relevant. Suppose that γ is relevant, i.e. $x_{k(\gamma)+1} \in V(\widetilde{\gamma})$ (or equivalently $-x_{k(\gamma)+1} \in SV(\widetilde{\gamma})$). Observe that by definition of γ and γ' we have
$SV(\widetilde{\gamma}) \setminus \{-x_{k(\gamma)+1}\} = SV(\widetilde{\gamma}') \setminus \{x_{k(\gamma)+1}\}$. Hence by Proposition 5.3.5 (*iii*)

$$\frac{P(\gamma)}{\eta(-x_{k(\gamma)+1}, \pi(s))} = \frac{1}{2}\frac{\prod_{r\in SV(\widetilde{\gamma})} \eta(r, \pi(s))}{\eta(-x_{k(\gamma)+1}, \pi(s))} = \frac{1}{2}\frac{\prod_{r\in SV(\widetilde{\gamma}')} \eta(r, \pi(s))}{\eta(x_{k(\gamma)+1}, \pi(s))} = \frac{P(\gamma')}{\eta(x_{k(\gamma)+1}, \pi(s))}$$

So

$$
\begin{aligned}
P(\gamma)+P(\gamma^{'}) &= \frac{P(\gamma)}{\eta(-x_{k(\gamma)+1},\pi(s))}\eta(-x_{k(\gamma)+1},\pi(s)) + \frac{P(\gamma')}{\eta(x_{k(\gamma)+1},\pi(s))}\eta(x_{k(\gamma)+1},\pi(s)) \\
&= \frac{P(\gamma)}{\eta(-x_{k(\gamma)+1},\pi(s))}(\pi(s)+x_{k(\gamma)+1}+\pi(s)-x_{k(\gamma)+1}) \\
&= 2P(\gamma)\frac{\pi(s)}{\eta(-x_{k(\gamma)+1},\pi(s))} = Q(\gamma).
\end{aligned}
$$

This proves part (1) of Theorem 5.3.3.

Now observe that if γ is an even relevant path then $P(\gamma)=Q(\gamma)$. By Proposition 5.3.5 (i) and (ii), $P(\gamma)$ is a polynomial with positive integer coefficients in the positive roots, hence with positive integer coefficients in the simple roots.

If γ is an uneven relevant path then it follows from Proposition 5.3.5 (iii) that

$$
Q(\gamma) = 2P(\gamma)\frac{\pi(s)}{\eta(-x_{k(\gamma)+1},\pi(s))} = \prod_{r\in SV(\widetilde{\gamma})\setminus\{-x_{k(\gamma)+1}\}} \eta(r,\pi(s))\pi(s)
$$

which is clearly a polynomial with positive integer coefficients in the positive roots, hence with positive integer coefficients in the simple roots. □

Example 5.3.8 Let M be a generic coadjoint orbit of type B_2. Then the associated GKM graph (V, E_{GKM}) has eight vertices, $p_0=-2x_1-x_2$, $p_1=-2x_1+x_2$, $q_0=-x_1-2x_2$, $q_1=x_1-2x_2$, $r_0=-x_1+2x_2$, $r_1=x_1+2x_2$, $s_0=2x_1-x_2$, $s_1=2x_1+x_2$. Moreover, with the convention for $\varphi=\psi^{\xi}: M\to\mathbb{R}$ chosen before, the minimum and maximum of φ are respectively p_0 and s_1. The canonical graph associated to the canonical classes w.r.t. φ is shown in Figure 5-1. Observe that it is a subgraph of (V, E_{GKM}).

The graph $(\widetilde{V}, \widetilde{E}_{\text{GKM}})$ associated to the degenerate coadjoint orbit $\widetilde{M}$ has four vertices, $p=-x_1$, $q=-x_2$, $r=x_2$ and $s=x_1$. It's easy to see that $\pi(p_0)=\pi(p_1)=p$, $\pi(q_0)=\pi(q_1)=q$, $\pi(r_0)=\pi(r_1)=r$ and $\pi(s_0)=\pi(s_1)=s$. We want to compute $\alpha_{p_1}(s_1)$ and $\alpha_{q_1}(s_1)$ using $\pi: M\to\widetilde{M}$.

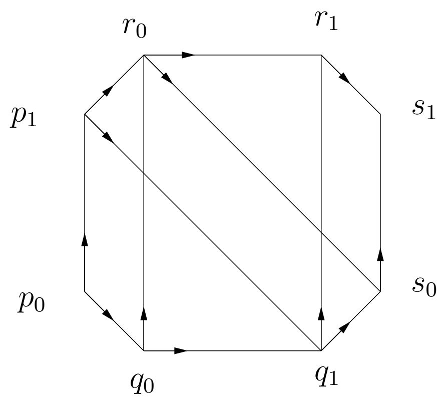


Figure 5-1: An example of canonical graph.

Let's first compute $\alpha_{p_1}(s_1)$. Since $\pi^{-1}(\pi(s_1)) = \{s_0, s_1\}$, we need to find the sets of paths $\overline{\Sigma}(p_1, s_0)$ and $\overline{\Sigma}(p_1, s_1)$, and the corresponding subsets of relevant paths $R(p_1, s_0)$ and $R(p_1, s_1)$. It's easy to see that

- $\overline{\Sigma}(p_1, s_0) = \{\gamma, \gamma^{'}\}$, where $\gamma = (p_1, r_0, s_0)$ and $\gamma^{'} = (p_1, q_1, s_0)$. Then, since $\widetilde{\gamma} = \pi(\gamma) = (p, r, s) = (-x_1, x_2, x_1)$ and $\widetilde{\gamma}^{'} = (p, q, s) = (-x_1, -x_2, x_1)$, the paths γ and $\gamma^{'}$ are uneven and

$$P(\gamma) = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad P(\gamma^{'}) = \frac{1}{2}(x_1 - x_2) \ .$$

  By definition γ is relevant, hence $R(p_1, s_0) = \{\gamma\}$ and $Q(\gamma) = P(\gamma) + P(\gamma^{'}) = x_1$.

- $\overline{\Sigma}(p_1, s_1) = \{\gamma^{''}\}$, where $\gamma^{''} = (p_1, q_1, r_1, s_1)$ (and $\widetilde{\gamma}^{''} = \pi(\gamma^{''}) = (-x_1, -x_2, x_2, x_1)$). Hence, since $\overline{\Sigma}(p_1, s_1)$ is composed by one path only, $\gamma^{''}$ must be even, and hence relevant. Moreover $Q(\gamma^{''}) = 1$.

By Theorem 5.3.3, we have that

$$\alpha_{p_1}(s_1) = Q(\gamma)\widehat{\alpha}_{s_0}(s_1) + Q(\gamma^{''})\widehat{\alpha}_{s_1}(s_1)$$

where the fiber over $\pi(s_1)$ has two fixed points, s_0 and s_1. Since s_0 is the minimum in the fiber, and s_1 the maximum, we have $\widehat{\alpha}_{s_0}(s_1) = 1$ and $\widehat{\alpha}_{s_1}(s_1) = x_2$, and we can conclude that

$$\alpha_{p_1}(s_1) = x_1 + x_2 \ .$$

Now we compute $\alpha_{q_1}(s_1)$. It's easy to check that $\overline{\Sigma}(q_1, s_0) = \{(q_1, s_0)\}$ and $\overline{\Sigma}(q_1, s_1) = \{(q_1, r_1, s_1)\}$; hence both sets are composed by even paths. For $\gamma_1 = (q_1, r_1, s_1)$ we have $Q(\gamma_1) = 2x_1$, whereas for $\gamma_2 = (q_1, s_0)$, $Q(\gamma_2) = x_1(x_1 - x_2)$. By Theorem 5.3.3 we have

$$\alpha_{q_1}(s_1) = 2x_1\widehat{\alpha}_{s_1}(s_1) + x_1(x_1 - x_2)\widehat{\alpha}_{s_0}(s_1) = 2x_1x_2 + x_1(x_1 - x_2) = x_1(x_1 + x_2)$$

It remains to prove Lemmas 5.3.6 and 5.3.7.

Let's introduce some notation.

If $\alpha_1 = x_1 - x_2, \alpha_2 = x_2 - x_3, \ldots, \alpha_{n-1} = x_{n-1} - x_n, \alpha_n = x_n$ are the simple roots, let's denote by $s_1, s_2, \ldots, s_n$ the associated reflections. We recall that they satisfy the following relations

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } i = 1, \ldots, n-2 \\ s_{n-1} s_n s_{n-1} s_n &= s_n s_{n-1} s_n s_{n-1} && \\ s_i s_j &= s_j s_i && \text{for all } i, j \in \{1, \ldots, n\} \text{ s.t. } |i - j| \geq 2 \end{aligned}$$

Moreover we have that for all $l \in \{1, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, n\}$ with $j \notin \{l, l+1\}$

$$\begin{aligned} s_{x_l} &= s_l s_{x_{l+1}} s_l \\ s_{x_l \pm x_j} &= s_l s_{x_{l+1} \pm x_j} s_l \\ s_{x_l + x_{l+1}} &= s_l s_{x_l + x_{l+1}} s_l \end{aligned} \tag{5.11}$$

The proofs of Lemmas 5.3.6 and 5.3.7 will be a consequence of the next result.

Proposition 5.3.9. *Let $\widetilde{\gamma}$ be an increasing path in $(\widetilde{V}, \widetilde{E}_{\mathrm{GKM}})$ starting at $-x_l$ and ending at x_l, for some $l = 1, \ldots, n-1$. Suppose that $\widetilde{\gamma}$ satisfies one of the following two conditions* (*a*) $\{-x_{l+1}, x_{l+1}\} \cap V(\widetilde{\gamma}) = \emptyset$

(*b*) $-x_{l+1} \notin V(\widetilde{\gamma})$ *and* $x_{l+1} \in V(\widetilde{\gamma})$.

If $\widetilde{\gamma}$ *is of type* (*a*), *let* $\widetilde{\gamma}'$ *be the path in* $(\widetilde{V}, \widetilde{E}_{\mathrm{GKM}})$ *obtained from* $\widetilde{\gamma}$ *by adding the vertices* $-x_{l+1}$ *and* x_{l+1}. *If* $\widetilde{\gamma}$ *is of type* (*b*), *let* $\widetilde{\gamma}'$ *be the path obtained from* $\widetilde{\gamma}$ *by replacing the vertex* x_{l+1} *with* $-x_{l+1}$, *i.e. if* $\widetilde{\gamma} = (-x_l, -\widehat{x}_{l+1}, \ldots, x_{l+1}, x_l)$ *then* $\widetilde{\gamma}' = (-x_l, -x_{l+1}, \ldots, \widehat{x}_{l+1}, x_l)$. *Then in both cases the path* $\widetilde{\gamma}'$ *is increasing. Moreover, for every such pair of paths, consider the lifts* γ *and* γ' *(resp. of* $\widetilde{\gamma}$ *and* $\widetilde{\gamma}'$*) starting at the same point* $p \in V$. *Then* γ *and* γ' *end at the same point.*

Proof. It's easy to see that $\widetilde{\gamma}'$ is an increasing path in both cases. Then, since π is a GKM fibration (cfr. section 3.1), it is sufficient to prove that if $\widetilde{\gamma} = (\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_{|\widetilde{\gamma}|+1})$ with $\widetilde{\gamma}_{i+1} = s_{\beta_i}\widetilde{\gamma}_i$ for $\beta_i \in R$ and $i = 1, \ldots, |\widetilde{\gamma}|$, and $\widetilde{\gamma}' = (\widetilde{\gamma}'_1, \ldots, \widetilde{\gamma}'_{|\widetilde{\gamma}'|+1})$ with $\widetilde{\gamma}'_{j+1} = s_{\delta_j}\widetilde{\gamma}'_j$ for $\delta_j \in R$ and $j = 1, \ldots, |\widetilde{\gamma}'|$, then $s_{\beta_{|\widetilde{\gamma}|}} s_{\beta_{|\widetilde{\gamma}|-1}} \cdots s_{\beta_1} = s_{\delta_{|\widetilde{\gamma}'|}} s_{\delta_{|\widetilde{\gamma}'|-1}} \cdots s_{\delta_1}$. Denote $s_{\beta_{|\widetilde{\gamma}|}} s_{\beta_{|\widetilde{\gamma}|-1}} \cdots s_{\beta_1}$ by w and $s_{\delta_{|\widetilde{\gamma}'|}} s_{\delta_{|\widetilde{\gamma}'|-1}} \cdots s_{\delta_1}$ by w'. Using the identities (5.11) it follows that

(*a*1) If $\widetilde{\gamma} = (-x_l, x_l)$ (hence $\widetilde{\gamma}' = (-x_l, -x_{l+1}, x_{l+1}, x_l)$) then $w = s_{x_l} = s_l s_{x_{l+1}} s_l = w'$

(*a*2) If the length of $\widetilde{\gamma}$ is greater than one then we have $w = s_{x_l \pm x_h} w_0 s_{x_l \pm x_i}$ and $w' = s_l s_{x_{l+1} \pm x_h} w_0 s_{x_{l+1} \pm x_i} s_l$ for some $h, i > l+1$, where w_0 is a product of reflections (or possibly an empty product, which we set to be the identity) which commutes with s_l, i.e. $s_l w_0 s_l = w_0$. Hence $w' = s_l s_{x_{l+1} \pm x_h} s_l w_0 s_l s_{x_{l+1} \pm x_i} s_l = s_{x_l \pm x_h} w_0 s_{x_l \pm x_i} = w$.

(*b*1) If $\widetilde{\gamma} = (-x_l, -\widehat{x}_{l+1}, x_{l+1}, x_l)$ and $\widetilde{\gamma}' = (-x_l, -x_{l+1}, \widehat{x}_{l+1}, x_l)$ then $w = s_l s_{x_l + x_{l+1}}$ and $w' = s_{x_l + x_{l+1}} s_l$, hence $w = w'$.

(*b*2) If the length of $\widetilde{\gamma}$ is greater than two then $w = s_l s_{x_{l+1} \pm x_h} w_0 s_{x_l \pm x_i}$ and $w' = s_{x_l \pm x_h} w_0 s_{x_{l+1} \pm x_i} s_l$ for some $h, i > l+1$, where w_0 is a product of reflections (possibly the empty product) which commutes with s_l.
Hence $w = s_l s_{x_{l+1} \pm x_h} s_l w_0 s_l s_{x_l \pm x_i} = s_{x_l \pm x_h} w_0 s_{x_{l+1} \pm x_i} s_l = w'$.

□

Proof of Lemma 5.3.6. Let $\gamma \in \overline{\Sigma}(p,q)$ be an uneven path, and let $k = \max\{j \text{ s.t. } x_j \text{ and } -x_j \in V(\widetilde{\gamma})\}$. Suppose that neither x_{k+1} nor $-x_{k+1}$ belongs to $V(\widetilde{\gamma})$. Let $\widetilde{\gamma}'$ be the path obtained from $\widetilde{\gamma}$ by adding the vertices $-x_{k+1}$ and x_{k+1}. Then by Proposition 5.3.9 $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ are increasing paths which lift to increasing paths $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ and $\gamma' = (\gamma_1', \dots, \gamma_{|\gamma|+3}')$ both starting at p and ending at q. But this is impossible since $\gamma \in \overline{\Sigma}(p,q)$ implies that $\lambda(q) - \lambda(p) = |\gamma|$. On the other hand since γ' is increasing too, and M is an index increasing GKM space we would have $\lambda(\gamma_{i+1}') - \lambda(\gamma_i') \geq 1$ for all $i = 1, \dots, |\gamma|+2$, which would imply $\lambda(q) - \lambda(p) \geq |\gamma|+2$. □

Proof of Lemma 5.3.7. By Lemma 5.3.6 either $-x_{k+1}$ or x_{k+1} belongs to $V(\widetilde{\gamma})$, with $\widetilde{\gamma} = \pi(\gamma)$ and $\gamma \in \overline{\Sigma}(p,q)$. Suppose that $x_{k+1} \in V(\widetilde{\gamma})$, and let $\widetilde{\gamma}'$ be the path defined as in Proposition 5.3.9 (b). Then the lift $\gamma' = (\gamma_1', \dots, \gamma_{|\gamma'|+1}')$, with $\gamma_1' = p$, is an increasing path which ends at q, with the same length as γ. Hence it must be $\lambda(\gamma_{i+1}') - \lambda(\gamma_i') = 1$, for all $i = 1, \dots, |\gamma|$, which implies that $\gamma' \in \overline{\Sigma}(p,q)$. □

5.4 Generic coadjoint orbit of type C_n

Let $G = Sp(n)$, the quaternionic unitary group $U(n, \mathbb{H})$ and T a maximal torus in G with Lie algebra $\mathfrak{t}$. Consider the basis $\{x_i\}_{i=1}^n$ of $(\mathbb{R}^n)^* \simeq \mathfrak{t}^*$ such that $x_i(\mu_1, \dots, \mu_n) = \mu_i$. Let R be the roots of G, and $\alpha_i = x_i - x_{i+1}$, $i = 1, \dots, n-1$, $\alpha_n = 2n$ a choice of simple roots. In this case the Weyl group W of G is the group of signed permutations on n elements.

For all $i = 0, \dots, n$ let $\mu^i = (\mu_1^i, \dots, \mu_i^i, \mu_{i+1}^i, \dots, \mu_n^i)$ be a vector in $\mathbb{R}^n$ such that $\mu_1^i < \ldots < \mu_i^i < \mu_{i+1}^i = \ldots = \mu_n^i = 0$. Then every such vector determines a point p^i in $\mathfrak{t}^*$, $p^i = \sum_{j=1}^i \mu_j^i x_j$. The coadjoint orbit $\mathcal{O}_{p^i} = G \cdot p^i$ of the point p^i is isomorphic to G/G_{p^i}, where in this case $G_{p^i} = S^1 \times \ldots \times S^1 \times U(n-i; \mathbb{H})$. The $T-$fixed set $(G/G_{p^i})^T$ is given by

$$(G/G_{p^i})^T = \{\sum_{j=1}^{i} (-1)^{\epsilon_j} \mu_j^i x_{\sigma(j)},\ \epsilon_j \in \{0,1\}\ \forall\, j = 1, \dots, i,\ \sigma \in \mathcal{S}_n\} \subset \mathfrak{t}^*$$

and the moment map ψ_i restricted to the fixed point set is given by the inclusion,

$\psi_i : (G/G_{p^i})^T \hookrightarrow \mathfrak{t}^*$.

Let M_i be G/G_{p^i}, $i = 0, \ldots, n$, and observe that the fiber of the natural projection $p_i : M_{i+1} \to M_i$ is $U(n-i, \mathbb{H})/(S^1 \times U(n-i-1, \mathbb{H}))$, which is isomorphic to a projective space $\mathbb{C}P^{2(n-i)-1}$. Hence $\{(M_i, \omega_i, \psi_i)\}_{i=0}^n$ is a sequence of GKM spaces, and for all $i = 0, \ldots, n-1$ the map $p_i : M_{i+1} \to M_i$ is a weight preserving $T-$equivariant fibration with symplectic fibers isomorphic to $\mathbb{C}P^{2(n-i)-1}$.

Let $\pi_i : M_n \to M_i$ be the composition $\pi_i = p_i \circ \cdots \circ p_{n-1}$. The maps p_i's and π_i's restricted to the fixed point sets, $p_i : (M_{i+1})^T \to (M_i)^T$, $\pi_i : (M_n)^T \to (M_i)^T$, are given by

$$p_i(\sum_{j=1}^{i+1}(-1)^{\epsilon_j}\mu_j^{i+1}x_{\sigma(j)}) = \sum_{j=1}^{i}(-1)^{\epsilon_j}\mu_j^{i}x_{\sigma(j)} \ ,$$

$$\pi_i(\sum_{j=1}^{n}(-1)^{\epsilon_j}\mu_j^{n}x_{\sigma(j)}) = \sum_{j=1}^{i}(-1)^{\epsilon_j}\mu_j^{i}x_{\sigma(j)}$$

From the definitions, for any $r, r' \in M_n^T$ such that $r = \sum_{j=1}^n (-1)^{\epsilon_j}\mu_j^n x_{\sigma(j)}$ and $r' = \sum_{j=1}^n (-1)^{\epsilon'_j}\mu_j^n x_{\sigma'(j)}$ we have

$$\pi_i(r) = \pi_i(r') \iff \pi_j(r) = \pi_j(r') \ \forall\ 0 \leq j \leq i \iff \epsilon_j = \epsilon'_j \text{ and } \sigma(j) = \sigma'(j) \ \forall\ 0 \leq j \leq i$$

For any pair of points r, r' in $(M_n)^T$ define

$$h(r, r') = \min\{j \in \{0, \ldots, n\} \mid \pi_j(r) \neq \pi(r')\}$$

(cfr. Proposition 4.3.7). So $h(r, r') = h$ if and only if $\sigma(j) = \sigma'(j)$, and $\epsilon_j = \epsilon'_j$ for all $0 \leq j < h$ and $(-1)^{\epsilon_h}x_{\sigma(h)} - (-1)^{\epsilon'_h}x_{\sigma'(h)} \neq 0$. In particular if (r, r') in an edge in $E \subset E_{\text{GKM}}$, then $r' = s_\beta r$ for some $\beta \in R$. Then β can be either $x_{\sigma(h)} \pm x_{\sigma(k)}$ for some h, k s.t. $1 \leq h < k \leq n$ or $2x_{\sigma(h)}$ for some $h = 1, \ldots, n$; in both cases we have $h(r, r') = h$.

Consider now the canonical classes $\{\alpha_p\}_{p \in M_n^T}$ associated to $\varphi_n = \psi_n^\xi$, which exist and are integral by Lemma 5.1.1 and Theorem 4.3.2. For any $p, q \in (M_n)^T$ let $C(p, q)$ be the set of paths as defined in Proposition 4.3.7.

Proposition 5.4.1. *Let G/G_{p^n} be a generic coadjoint orbit of type C_n. Fix p and q in $(G/G_{p^n})^T$. Let $\alpha_p \in H_T^{2\lambda(p)}(M_n;\mathbb{Z})$ be the canonical class associated to $\varphi_n = \psi_n^\xi$.*

(1) A path $\gamma = (w_1(p^n), \ldots, w_{l+1}(p^n))$, where $w_j \in W$ and $w_j(p^n) = \sum_{m=1}^{n} \mu_m^n (-1)^{\epsilon_m^j} x_{\sigma_j(m)}$ for all $j = 1, \ldots, l+1$, is an element of $C(w_1(p^n), w_{l+1}(p^n))$ if and only if

(a) $l(w_{j+1}) = l(w_j)+1$ and $w_{j+1} = s_{\beta_j} w_j$ where β_j is either $x_{\sigma_j(h_j)} \pm x_{\sigma_j(k_j)}$, for some h_j, k_j such that $1 \leq h_j < k_j \leq n$, or $2x_{\sigma_j(h_j)}$ for some $h_j = 1, \ldots, n$, and $\sigma_j \in \mathcal{S}_n$, for all $j = 1, \ldots, l$.

(b) $h_1 \leq h_2 \leq \ldots \leq h_l$

(2) (c) For all $p, q \in (G/G_{p^n})^T$

$$\alpha_p(q) = \Lambda_q^- \sum_{\substack{\gamma \in C(p,q) \\ \gamma = (w_1(p^n), \ldots, w_{l+1}(p^n))}} \prod_{j=1}^{l} \frac{1}{\left((-1)^{\epsilon_{h_j}^{h_j}} x_{\sigma_j(h_j)} - (-1)^{\epsilon_{h_j}^{l+1}} x_{\sigma_{l+1}(h_j)} \right)}$$

(d) For every path $\gamma = (w_1(p^n), \ldots, w_{l+1}(p^n))$ in $C(p,q)$

$$\Xi(\gamma) = \Lambda_q^- \prod_{j=1}^{l} \frac{1}{\left((-1)^{\epsilon_{h_j}^{h_j}} x_{\sigma_j(h_j)} - (-1)^{\epsilon_{h_j}^{l+1}} x_{\sigma_{l+1}(h_j)} \right)}$$

is a polynomial with positive integer coefficients in the simple roots, i.e. it belongs to $\mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_n]$.

Proof. Part (a) is equivalent to saying that γ belongs to $\Sigma(w_1(p^n), w_{l+1}(p^n))$. Then, for what we observed before, $h(w_j(p^n), w_{j+1}(p^n)) = h_j$ for all $j = 1, \ldots, l$; so claim (1) follows immediately.

Let $\overline{\psi}_j = \pi_j^*(\psi_j) : M_n \to \mathfrak{t}^*$. Observe that if r, s are points in M_n^T, with $r = \sum_{j=1}^n (-1)^{\epsilon_j} \mu_j^n x_{\sigma(j)}$, $s = \sum_{j=1}^n (-1)^{\epsilon'_j} \mu_j^n x_{\sigma'(j)}$ and $h(r,s) = h$ then

$$\overline{\psi}_h(s) - \overline{\psi}_h(r) = \mu_h^n \left((-1)^{\epsilon'_h} x_{\sigma'(h)} - (-1)^{\epsilon_h} x_{\sigma(h)} \right)$$

The conclusion in part (c) follows by applying Proposition 4.3.7 together with Proposition 5.1.2. Part (d) follows from Proposition 4.4.3. ∎

Example 5.4.2 Let M be a generic coadjoint orbit of type C_2. The GKM graph $\Gamma = (V, E_{\mathrm{GKM}})$ associated to it is the same as the one associated to a coadjoint orbit of type B_2, but the axial functions are different. Let V be the set of vertices of Γ as described in Example 5.3.8. Suppose that we want to compute $\alpha_{p_1}(s_1)$. $C(p_1, s_1)$ is composed by three paths, $\gamma_1 = (p_1, r_1, s_0, s_1)$, $\gamma_2 = (p_1, q_0, s_0, s_1)$ and $\gamma_3 = (p_1, r_1, q_1, s_1)$, and their contributions are $\Xi(\gamma_1) = x_1 - x_2$, $\Xi(\gamma_2) = x_1 + x_2$ and $\Xi(\gamma_3) = 2x_2$. Hence Proposition 5.4.1 gives

$$\alpha_{p_1}(s_1) = (x_1 - x_2) + (x_1 + x_2) + (2x_2) = 2(x_1 + x_2)$$

Remark 5.4.3. *In type B_n one could still apply the same argument shown in type C_n, and the formula for the canonical classes on a generic coadjoint orbit of type B_n is the same as in type C_n (cfr. Proposition 5.4.1 part (c)). Notice however that if $\alpha_1, \dots, \alpha_n$ denote the simple roots in type B_n and $A = \mathbb{Z}[\frac{1}{2}]$, the single contributions $\Xi(\gamma)$ (cfr. Proposition 5.4.1 part (d)) belong to $A_+[\alpha_1, \dots, \alpha_n]$ (cfr. section 4.4). This comes from Proposition 4.4.3, since the fibers of the maps p_i are isomorphic to Grassmannians of oriented two planes in $\mathbb{R}^{2k+1}$, and $H^*(Gr_2^+(\mathbb{R}^{2k+1}), A) \simeq H^*(\mathbb{C}P^{2k-1}, A)$.*

5.5 Generic coadjoint orbit of type D_n

Let M be a generic coadjoint orbit of type D_n. In this section we prove an inductive integral formula for the canonical classes on M. Since the exposition and the proofs are very similar to the ones given in section 5.3, we will either omit or just outline them.

Let $G = SO(2n)$, and $\alpha_1 = x_1 - x_2$, $\alpha_2 = x_2 - x_3, \dots, \alpha_{n-1} = x_{n-1} - x_n$, $\alpha_n = x_{n-1} + x_n$ a choice of simple roots. Let $(\mu_1, \dots, \mu_n)$ be a vector in $\mathbb{R}^n$ such that $\mu_1 < \mu_2 < \dots < \mu_n < 0$, and consider $p_0 = \sum_{j=1}^n \mu_j x_j \in \mathfrak{t}^*$. Let M be the $G-$coadjoint orbit $G \cdot p_0$. Let $\widetilde{p}_0 = -x_1$; then the $G-$coadjoint orbit through $\widetilde{p}_0$, $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$, is isomorphic to $Gr_2^+(\mathbb{R}^{2n})$, the Grassmannian of oriented two planes in $\mathbb{R}^{2n}$. Let $(\widetilde{V}, \widetilde{E}_{\mathrm{GKM}})$ be the GKM graph associated to it. Then $\widetilde{V} = \{\pm x_i,\ i = 1, \dots, n\}$.

As for the edges of $\widetilde{E}_{\text{GKM}}$, there exists an edge between any two vertices, except for the pairs of vertices $-x_i, x_i,\ i = 1, \ldots, n$ (so it is not a complete graph).
If we choose a generic $\xi \in \mathfrak{t}$ such that $\alpha(\xi) > 0$ for all the positive roots $\alpha \in R^+$, then $\widetilde{\psi}^\xi$ has the following critical points, listed in non decreasing Morse index:

$$-x_1, -x_2, \ldots, -x_n, x_n, \ldots, x_2, x_1$$

Observe that now $-x_n$ and x_n have the same Morse index.
Let

$$\pi : M \to \widetilde{M} \tag{5.12}$$

be the projection of $G \cdot p_0$ onto $G \cdot \widetilde{p}_0$. Consider the set of horizontal paths $\overline{\Sigma}(p, q)$ in the canonical graph associated to the canonical classes $\{\alpha_p\}_{p \in M^T}$ w.r.t. $\varphi = \psi^\xi$. Since π is a weight preserving equivariant fibration, we can apply Corollary 4.3.9 to compute the restriction of the canonical classes to the fixed point set. In particular if $\overline{\psi} = \pi^*(\widetilde{\psi})$, for every $p, q \in M^T$, then

$$\alpha_p(q) = \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in \overline{\Sigma}(p,s)} \Xi(\gamma) \right) \widehat{\alpha}_s(q) \ ,$$

where

$$\Xi(\gamma) = \widetilde{\Lambda}^-_{\pi(q)} \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}(\gamma_{i+1}) - \overline{\psi}(\gamma_i)}{\overline{\psi}(q) - \overline{\psi}(\gamma_i)} \cdot \frac{1}{\eta(\gamma_i, \gamma_{i+1})} \ .$$

Before starting, we want to exhibit an explicit computation that shows that in this case the single "horizontal contribution" $\Xi(\gamma)$ is not integral in the weights.

Example 5.5.1 Let $n = 4$, so that $G \cdot \widetilde{p}_0 \simeq Gr_2^+(\mathbb{R}^8)$. Suppose that we want to compute $\alpha_{p_0}(q)$, where q is the maximum of φ. The only fixed point $s \in \widehat{M}_q^T$ for which $\overline{\Sigma}(p_0, s)$ is not empty is the point $s = -\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 - \mu_4 x_4$, which is also the minimum of $\varphi_{|_{\widehat{M}_q^T}}$ on $\widehat{M}_q^T$. The set of paths $\overline{\Sigma}(p_0, s)$ is composed by two paths, γ and γ': their projections $\widetilde{\gamma}$ and $\widetilde{\gamma'}$ are given by $\widetilde{\gamma} = (-x_1, -x_2, -x_3, x_4, x_3, x_2, x_1)$ and

$\widetilde{\gamma}' = (-x_1, -x_2, -x_3, -x_4, x_3, x_2, x_1)$, and their contributions are given by

$$\Xi(\gamma) = \frac{x_1 + x_4}{2x_1} \quad \text{and} \quad \Xi(\gamma') = \frac{x_1 - x_4}{2x_1} \,.$$

Let $\widetilde{\gamma}$ be a path in $(\widetilde{V}, \widetilde{E}_{\mathrm{GKM}})$ and $V(\widetilde{\gamma})$ the set of its vertices.

Definition 5.5.2. *Let γ be a path in $\overline{\Sigma}(p,q)$, and let $\widetilde{\gamma} = (\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_{|\gamma|+1})$ be $\pi(\gamma)$. We say that γ is* ***uneven*** *if $-\widetilde{\gamma}_{|\widetilde{\gamma}|+1} \in V(\widetilde{\gamma})$. It is* ***even*** *otherwise.*

If γ is an uneven path, define $k(\gamma)$ to be $\max\{i \mid -x_i \text{ and } x_i \text{ belong to } V(\widetilde{\gamma})\}$, where $\widetilde{\gamma} = \pi(\gamma)$. By definition this set is non empty, and $k(\gamma) < n$, since $(-x_n, x_n)$ cannot be an edge of $\widetilde{\gamma}$.

Definition 5.5.3. *A path $\gamma \in \overline{\Sigma}(p,q)$ is said to be* ***relevant*** *if either γ is even or γ is uneven and $x_{k(\gamma)+1} \in V(\widetilde{\gamma})$, where $\widetilde{\gamma} = \pi(\gamma)$. Let's denote this subset of $\overline{\Sigma}(p,q)$ by $R(p,q)$.*

Since π is a $T-$equivariant fibration with symplectic fibers, by Lemma 4.3.8 canonical classes on the fiber exist. The next theorem gives an inductive integral formula for computing the restriction of the canonical classes to the $T-$fixed point set.

Theorem 5.5.4. *Let $\pi : (M, \omega, \psi) \to (\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be the projection map (5.12). Let $\varphi = \psi^{\xi} : M \to \mathbb{R}$ be a generic component of the moment map, and consider the canonical classes $\{\alpha_p\}_{p \in M^T}$ associated to φ. For every $s \in M^T$ consider the canonical class $\widehat{\alpha}_s$ on the fiber $\widehat{M}_s = \pi^{-1}(\pi(s))$, and let $R(p,s)$ be the set of relevant paths from p to s.*

For every path $\gamma = (\gamma_1, \ldots, \gamma_{|\gamma|+1})$ in $R(\gamma_1, \gamma_{|\gamma|+1})$, define $Q(\gamma)$ to be

$$Q(\gamma) = \widetilde{\Lambda}^-_{\pi(\gamma_{|\gamma|+1})} \prod_{i=1}^{|\gamma|} \frac{\pi(\gamma_{i+1}) - \pi(\gamma_i)}{\pi(\gamma_{|\gamma|+1}) - \pi(\gamma_i)} \frac{1}{\eta(\gamma_i, \gamma_{i+1})}$$

if γ is even, and

$$Q(\gamma) = 2\widetilde{\Lambda}^-_{\pi(\gamma_{|\gamma|+1})}\left(\prod_{i=1}^{|\gamma|}\frac{\pi(\gamma_{i+1})-\pi(\gamma_i)}{\pi(\gamma_{|\gamma|+1})-\pi(\gamma_i)}\frac{1}{\eta(\gamma_i,\gamma_{i+1})}\right)\frac{\pi(\gamma_{|\gamma|+1})}{\eta(-x_{k(\gamma)+1},\pi(\gamma_{|\gamma|+1}))}$$

if γ is uneven.

Then

(1) For all p, q in M^T

$$\alpha_p(q) = \sum_{s\in\widehat{M}^T_q}(\sum_{\gamma\in R(p,s)} Q(\gamma))\widehat{\alpha}_s(q)$$

(2) $Q(\gamma)$ is a polynomial in the simple roots with positive integer coefficients, i.e. it belongs to $\mathbb{Z}_{\geq 0}[\alpha_1,\ldots,\alpha_n]$.

The next Proposition gives another characterization of even and uneven paths, and shows that the non integral contributions come from uneven paths.

Proposition 5.5.5. *Let $\gamma = (\gamma_1,\ldots,\gamma_{|\gamma|+1})$ be a path in (V, E_{GKM}) which is horizontal and increasing. Define $P(\gamma)$ to be*

$$P(\gamma) = \widetilde{\Lambda}^-_{\pi(\gamma_{|\gamma|+1})}\prod_{i=1}^{|\gamma|}\frac{\pi(\gamma_{i+1})-\pi(\gamma_i)}{\pi(\gamma_{|\gamma|+1})-\pi(\gamma_i)}\frac{1}{\eta(\gamma_i,\gamma_{i+1})}$$

Then if $\widetilde{\gamma} = \pi(\gamma)$

$$P(\gamma) = \begin{cases} \displaystyle\prod_{r\in SV(\widetilde{\gamma})\setminus\{-\widetilde{\gamma}_{|\widetilde{\gamma}|+1}\}} \eta(r,\widetilde{\gamma}_{|\widetilde{\gamma}|+1}) & \text{if } \gamma \text{ is even} \\ \\ \dfrac{\displaystyle\prod_{r\in SV(\widetilde{\gamma})}\eta(r,\widetilde{\gamma}_{|\widetilde{\gamma}|+1})}{2\widetilde{\gamma}_{|\widetilde{\gamma}|+1}} & \text{if } \gamma \text{ is uneven} \end{cases}$$

We recall that for any uneven path γ, $k(\gamma)$ is defined to be
$\max\{j \mid x_j \text{ and } -x_j \in V(\widetilde{\gamma})\}$, where $\widetilde{\gamma} = \pi(\gamma)$, which from the definition of uneven path is a non empty set, and $k(\gamma) < n$.

Lemma 5.5.6. *Let γ be a horizontal path in $\overline{\Sigma}(p,q)$. If γ is uneven and $\widetilde{\gamma} = \pi(\gamma)$ then either $x_{k(\gamma)+1}$ or $-x_{k(\gamma)+1}$ belongs to $V(\widetilde{\gamma})$.*

For every path $\widetilde{\gamma} = \pi(\gamma)$ satisfying this property define $\widetilde{\gamma}'$ to be the path obtained from $\widetilde{\gamma}$ by replacing the vertex $x_{k(\gamma)+1} \in V(\widetilde{\gamma})$ (or $-x_{k(\gamma)+1} \in V(\widetilde{\gamma})$) with $-x_{k(\gamma)+1}$ (or $x_{k(\gamma)+1}$).

Lemma 5.5.7. *Let γ be a horizontal path in $\overline{\Sigma}(p,q)$. If γ is an uneven path then there exists $\gamma^{'} \in \overline{\Sigma}(p,q)$ such that $\pi(\gamma^{'}) = \widetilde{\gamma}^{'}$.*

Hence the uneven paths which are in the image of $\overline{\Sigma}(p,q)$ always come in pairs.

Proof of Theorem 5.5.4. The proof of this theorem can be carried on as in the B_n case, observing that by Corollary 4.3.9 and Proposition 5.1.2

$$\alpha_p(q) = \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in \overline{\Sigma}(p,s)} P(\gamma) \right) \widehat{\alpha}_s(q)$$

Then, if γ is even, $P(\gamma) = Q(\gamma)$; if γ is uneven and relevant then $P(\gamma) + P(\gamma') = Q(\gamma)$, where γ' is defined in Lemma 5.5.7.

As for the integrality and positivity of $Q(\gamma)$, observe that if γ is even, then Proposition 5.5.5 implies that $Q(\gamma) = P(\gamma)$ belongs to $\mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_n]$. If γ is uneven and relevant, let γ' be the path defined before and let $\widetilde{\gamma} = \pi(\gamma)$. It's easy to check that

$$Q(\gamma) = P(\gamma) + P(\gamma') = \prod_{r \in SV(\widetilde{\gamma}) \setminus \{-x_{k(\gamma)+1}\}} \eta(r, \widetilde{\gamma}_{|\widetilde{\gamma}|+1})$$

which is clearly a polynomial with positive integer coefficients in the positive roots, hence in the simple roots. □

Example 5.5.8 Let M be a generic coadjoint orbit of type D_4 through the point $p_0 = -4x_1 - 3x_2 - 2x_3 - x_4$. Suppose that we want to compute $\alpha_p(q)$, where $p = -4x_3 - 3x_1 - 2x_2 - x_4$ and $q = 4x_2 - 3x_1 - 2x_3 + x_4$. There are precisely two paths in $R(p,q)$, γ_1 and γ_2, and their projection onto $\widetilde{M}$ is given by $\widetilde{\gamma}_1 = (-x_3, -x_4, x_3, x_2)$ and $\widetilde{\gamma}_2 = (-x_3, x_4, x_3, x_2)$. Their contribution is given by $Q(\gamma_1) = (x_1 + x_2)(x_2 - x_4)$

and $Q(\gamma_2) = (x_1 + x_2)(x_2 + x_4)$. Moreover it's easy to check that q is the minimum on the fiber over $\pi(q)$. So $\widehat{\alpha}_s(q) \neq 0$ if and only if $s = q$, where $s \in (\pi^{-1}(\pi(q)))^T$, and $\widehat{\alpha}_q(q) = 1$. Hence Theorem 5.5.4 gives

$$\alpha_p(q) = Q(\gamma_1) + Q(\gamma_2) = (x_1 + x_2)(x_2 - x_4) + (x_1 + x_2)(x_2 + x_4) = 2x_2(x_1 + x_2)$$

5.6 A general integral formula

Let G be a compact simple Lie group, $T \subset G$ a maximal compact torus in G with Lie algebra $\mathfrak{t}$, p_0 a generic element of $\mathfrak{t}^*$, and $\widetilde{p}_0$ an element in the closure of the Weyl chamber containing p_0. We have already observed that if $G \cdot p_0$ and $G \cdot \widetilde{p}_0$ are the $G-$ coadjoint orbit through p_0 and $\widetilde{p}_0$, the natural projection $\pi : G \cdot p_0 \to G \cdot \widetilde{p}_0$ is a $T-$equivariant weight preserving fibration. So we can use Corollary 4.3.9 to compute the restriction of the canonical classes to the fixed point set. More precisely, for every $p, q \in (G \cdot p_0)^T$, let $q_0, \ldots, q_N$ be the elements of $(\pi^{-1}(\pi(q)))^T$. Then, combining Corollary 4.3.9 with Proposition 5.1.2, we have

$$\alpha_p(q) = \sum_{j=0}^{N} \left(\sum_{\gamma \in \overline{\Sigma}(p, q_j)} \Xi(\gamma) \right) \widehat{\alpha}_{q_j}(q) \ ,$$

where

$$\Xi(\gamma) = \widetilde{\Lambda}^-_{\pi(q)} \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}(\gamma_{i+1}) - \overline{\psi}(\gamma_i)}{\overline{\psi}(q) - \overline{\psi}(\gamma_i)} \cdot \frac{1}{\eta(\gamma_i, \gamma_{i+1})} \ . \tag{5.13}$$

However the single "horizontal contribution" $\Xi(\gamma)$ is not in general in $\mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_n]$, where $\alpha_1, \ldots, \alpha_n$ denote the simple roots (see Example 5.5.1). This strongly depends on the cohomology ring of $G \cdot \widetilde{p}_0$ (see also Corollary 4.4.2).

In this section we show how to combine the contributions $\Xi(\gamma)$ to get an integral formula, for any $T-$equivariant weight preserving map $\pi : G \cdot p_0 \to G \cdot \widetilde{p}_0$.

In particular we prove that when π is a $\mathbb{C}P^1$ bundle, as a consequence of this formula one gets the divided difference operator identities.

We recall the following combinatorial description of π. Let R be the set of roots of G, R^+ a choice of positive roots, and R_0 the associated simple roots. Let W be the Weyl group of G. Given a subset of simple roots $\Sigma \subset R_0$, let $\langle \Sigma \rangle$ denote the subset of R^+ given by the roots which can be written as linear combinations of roots in Σ. Moreover, let $W(\Sigma)$ be the subgroup of W generated by the reflections s_α, with $\alpha \in \Sigma$. If

$$\widetilde{p}_0 \in \bigcap_{\alpha_i \in \Sigma} \mathcal{H}_{\alpha_i}$$

lies in the closure of the Weyl chamber containing p_0, then the projection $\pi : G \cdot p_0 \to G \cdot \widetilde{p}_0$ induces a map at the level of the GKM graphs W and $W/W(\Sigma)$ associated to these spaces. This map $\pi : W \to W/W(\Sigma)$ is a GKM fiber bundle (see section 3.1). Let E_{GKM} be the edge set of the GKM graph associated to $G \cdot p_0$, and $V \simeq W$ the set of vertices. Let $\varphi = \psi^\xi : G \cdot p_0 \to \mathbb{R}$ be a generic component of the moment map. We recall that an edge $e \in E_{\mathrm{GKM}}$ is said to be increasing if $\varphi(i(e)) < \varphi(t(e))$.

For every simple root $\alpha \in R_0$, define $\Phi_\alpha : V \to V$ to be

$$\Phi_\alpha(w(p_0)) = ws_\alpha(p_0) \ .$$

More in general, for every element u of the Weyl group, define $\Phi_u : V \to V$ to be

$$\Phi_u(w(p_0)) = wu^{-1}(p_0) \ .$$

Observe that if $u = s_{i_1} \cdots s_{i_m}$, with $s_{i_j} = s_{\alpha_{i_j}}$ and $\alpha_{i_j} \in R_0$ for all $j = 1, \dots, m$, then

$$\Phi_u = \Phi_{s_{i_1} \cdots s_{i_m}} = \Phi_{\alpha_{i_1}} \circ \cdots \circ \Phi_{\alpha_{i_m}} \ .$$

The main result of this section will be a consequence of the following lemmas.

Lemma 5.6.1. *Let $p_1 = w_1(p_0)$ and $p_2 = w_2(p_0)$, for some $w_1, w_2 \in W$, and α a simple root. Consider the edge $(p_1, p_2) \in E_{\mathrm{GKM}}$, where $w_2 = w_1 s_\delta$ for some $\delta \in R^+ \setminus \{\alpha\}$. Then $(\Phi_\alpha(p_1), \Phi_\alpha(p_2)) \in E_{\mathrm{GKM}}$ and $\eta(p_1, p_2) = \eta(\Phi_\alpha(p_1), \Phi_\alpha(p_2))$.*

Moreover if (p_1, p_2) *is an increasing edge in* E_{GKM}, *then* $(\Phi_\alpha(p_1), \Phi_\alpha(p_2))$ *is increasing as well.*

Proof. Since by assumption $w_2 = w_1 s_\delta$ and $\delta \in R^+$, $\eta(p_1, p_2) = w_1(\delta)$. Then $\Phi_\alpha(p_2) = w_2 s_\alpha(p_0) = w_1 s_\delta s_\alpha(p_0) = s_{w_1(\delta)}\Phi_\alpha(p_1)$; hence $(\Phi_\alpha(p_1), \Phi_\alpha(p_2)) \in E_{\mathrm{GKM}}$. Moreover it's easy to check that the condition $\delta \in R^+ \setminus \{\alpha\}$ implies $\langle \Phi_\alpha(p_2), w_1(\delta)\rangle > 0$, hence $\eta(\Phi_\alpha(p_1), \Phi_\alpha(p_2)) = w_1(\delta)$. We recall that for every edge (p, q) in E_{GKM}, $\psi(q) - \psi(p)$ is a positive multiple of $\eta(p, q)$. So, since $\eta(p_1, p_2) = \eta(\Phi_\alpha(p_1), \Phi_\alpha(p_2))$, if (p_1, p_2) is an increasing edge, then $(\Phi_\alpha(p_1), \Phi_\alpha(p_2))$ is increasing as well. $\square$

Lemma 5.6.2. *Under the same hypotheses of Lemma 5.6.1, suppose that* $\lambda(p_2) - \lambda(p_1) = 1$. *If* $(p_1, \Phi_\alpha(p_1))$ *is increasing, then* $(p_2, \Phi_\alpha(p_2))$ *is increasing as well. Equivalently if* $(\Phi_\alpha(p_2), p_2)$ *is increasing, then* $(\Phi_\alpha(p_1), p_1)$ *is increasing as well. Moreover* $\lambda(\Phi_\alpha(p_2)) - \lambda(\Phi_\alpha(p_1)) = 1$

Proof. Suppose that $(p_1, \Phi_\alpha(p_1))$ is increasing but $(p_2, \Phi_\alpha(p_2))$ is not increasing. Then by Lemma 5.1.1, $\lambda(\Phi_\alpha(p_2)) < \lambda(p_2)$, and by Lemma 5.6.1, $\lambda(\Phi_\alpha(p_1)) < \lambda(\Phi_\alpha(p_2))$. But this implies that $\lambda(p_1) < \lambda(\Phi_\alpha(p_1)) < \lambda(\Phi_\alpha(p_2)) < \lambda(p_2)$, which contradicts the fact that $\lambda(p_2) - \lambda(p_1) = 1$.

For the last claim, it's easy to see that since α is a simple root, then by (5.6) $\lambda(\Phi_\alpha(p_2)) - \lambda(p_2) = \lambda(\Phi_\alpha(p_1)) - \lambda(p_1)$ is either 1 or -1, and the conclusion follows. $\square$

Consider the canonical classes on $G \cdot p_0$ associated to φ, and let $(V, E) = (W, E)$ be the associated canonical graph. Fix a subset of simple roots $\Sigma \subset R_0$ and consider the projection map $\pi : W \to W/W(\Sigma)$. Let $\overline{\Sigma}(p, s)$ be the set of horizontal paths in $\Sigma(p, s)$, i.e. paths $\gamma = (\gamma_1, \ldots, \gamma_{|\gamma|+1})$ such that $\gamma_1 = p, \gamma_{|\gamma|+1} = s$, $(\gamma_i, \gamma_{i+1}) \in E$ and $\pi(\gamma_i) \neq \pi(\gamma_{i+1})$ for all $i = 1, \ldots, |\gamma|$. Observe that for every horizontal edge (γ_i, γ_{i+1}) of γ, if $\gamma_i = w_i(p_0)$ and $\gamma_{i+1} = w_{i+1}(p_0)$, $w_i, w_{i+1} \in W$, then $w_{i+1} = w_i s_\delta$ for some $\delta \in R^+ \setminus \langle \Sigma \rangle$.

Let $q_0 = u_0(p_0)$ be the minimum of $\varphi_{|_{\pi^{-1}(\pi(s))}}$ on $\pi^{-1}(\pi(s))$, the fiber of π over $\pi(s)$. Recall that by our choices p_0 is the minimum of φ on $G \cdot p_0$ (cfr. section 5.1).

Lemma 5.6.3. *The set of horizontal paths $\overline{\Sigma}(p_0, q_0)$ is non empty.*

Moreover let v be an element of $W(\Sigma)$. Then

$$\lambda(\Phi_{v^{-1}}(p_0)) = \lambda(\Phi_{v^{-1}}(q_0)) - \lambda(q_0) ,$$

and $\Phi_{v^{-1}}$ defines a bijection between $\overline{\Sigma}(p_0, q_0)$ and $\overline{\Sigma}(\Phi_{v^{-1}}(p_0), \Phi_{v^{-1}}(q_0))$.

Proof. Consider the canonical class α_{p_0}. Since p_0 is the minimum of φ, $\alpha_{p_0}(q) = 1$ for every $q \in V$. As we observed at the beginning of this section,

$$\alpha_{p_0}(q) = \sum_{j=0}^{N} \left(\sum_{\gamma \in \overline{\Sigma}(p_0, q_j)} \Xi(\gamma) \right) \widehat{\alpha}_{q_j}(q) ,$$

where $\Xi(\gamma)$ is given by (5.13). Observe that since q_0 is the minimum of φ on the fiber containing q, by Lemma 4.2.1 $\widehat{\alpha}_{q_j}(q_0) \neq 0$ if and only if $j = 0$, and $\widehat{\alpha}_{q_0}(q_0) = 1$. Hence if $q = q_0$ the above formula gives

$$\alpha_{p_0}(q_0) = \sum_{\gamma \in \overline{\Sigma}(p_0, q_0)} \Xi(\gamma) .$$

Since $\alpha_{p_0}(q_0) = 1$ this implies that $\overline{\Sigma}(p_0, q_0)$ is non empty.

Let $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_m}}$ be a reduced word for $v \in W(\Sigma)$, where $\alpha_{i_j} \in \Sigma$ for all $j = 1, \ldots, m$. Consider the path $\gamma = (\gamma_1, \ldots, \gamma_{|\gamma|+1})$ in $\overline{\Sigma}(p_0, q_0)$.
Since $\lambda(p_0) = 0 < \lambda(\Phi_{\alpha_{i_1}}(p_0)) = 1$, by Lemma 5.1.1 $(p_0, \Phi_{\alpha_{i_1}}(p_0)) = (\gamma_1, \Phi_{\alpha_{i_1}}(\gamma_1))$ is an increasing edge. Since (γ_1, γ_2) is an horizontal edge s.t. $\lambda(\gamma_2) - \lambda(\gamma_1) = 1$, combining Lemma 5.6.1 and 5.6.2 we have that $(\gamma_2, \Phi_{\alpha_{i_1}}(\gamma_2))$ is an increasing edge, hence $\lambda(\Phi_{\alpha_{i_1}}(\gamma_2)) - \lambda(\gamma_2) = 1$.
Moreover $\lambda(\Phi_{\alpha_{i_1}}(\gamma_2)) - \lambda(\Phi_{\alpha_{i_1}}(\gamma_1)) = 1$. By repeating the same argument for all the edges (γ_i, γ_{i+1}) of γ we can conclude that $(q_0, \Phi_{\alpha_{i_1}}(q_0))$ is an increasing edge and $\lambda(\Phi_{\alpha_{i_1}}(q_0)) - \lambda(q_0) = 1$. Moreover $\Phi_{\alpha_{i_1}}(\gamma) = (\Phi_{\alpha_{i_1}}(\gamma_1), \ldots, \Phi_{\alpha_{i_1}}(\gamma_{|\gamma|+1})$ is an element of $\overline{\Sigma}(\Phi_{\alpha_{i_1}}(p_0), \Phi_{\alpha_{i_1}}(q_0))$. Vice versa, for every element $\gamma' \in \overline{\Sigma}(\Phi_{\alpha_{i_1}}(p_0), \Phi_{\alpha_{i_1}}(q_0))$, $\Phi_{\alpha_{i_1}}(\gamma')$ is an element of $\overline{\Sigma}(p_0, q_0)$.

Now consider the points $p_j = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_j}}(p_0) = \Phi_{\alpha_{i_j}}(p_{j-1})$ for all $j = 1, \ldots, m$.

Then $p_m = \Phi_{v^{-1}}(p_0) = v(p_0)$. Observe that since $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_m}}$ is a reduced expression, then $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_j}}$ is a reduced expression for all $j = 1, \dots, m$. This implies that $\lambda(p_j) = j$, hence $\lambda(p_{j+1}) - \lambda(p_j) = 1$ for all $j = 1, \dots, m-1$.

We can repeat the argument shown above for the edge $(p_0, \Phi_{\alpha_{i_1}}(p_0))$ multiple times, for all the increasing edges $(p_j, \Phi_{\alpha_{i_j}}(p_j))$, $j = 1, \dots, m$, and the conclusion follows.

□

Lemma 5.6.4. *Let p, q be elements of V and v an element of $W(\Sigma)$ such that $q = \Phi_{v^{-1}}(q_0)$. Then if $\overline{\Sigma}(p,q) \neq \emptyset$*

$$\lambda(p) - \lambda(\Phi_v(p)) = \lambda(q) - \lambda(q_0) \ .$$

Moreover Φ_v defines a bijection between $\overline{\Sigma}(p,q)$ and $\overline{\Sigma}(\Phi_v(p), q_0)$.

Proof. Let $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_m}}$ be a reduced word for v. Observe that $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_j}}$ is a reduced expression for all $j = 1, \dots, m$. Let $v_j = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_j}}$ and define $q_j = \Phi_{v_j^{-1}}(q_0) = \Phi_{\alpha_{i_j}}(q_{j-1})$ for all $j = 1, \dots, m$; observe that $q_m = q$.

Since $\Phi_{v_j^{-1}}(p_0) = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_j}}(p_0)$ and $\lambda(\Phi_{v_j^{-1}}(p_0)) = j$, by Lemma 5.6.3 $\lambda(q_j) = \lambda(q_0) + j$. So $(q_{j-1}, \Phi_{\alpha_{i_j}}(q_{j-1}))$ is an increasing edge in E_{GKM} s.t. $\lambda(\Phi_{\alpha_{i_j}}(q_{j-1})) - \lambda(q_{j-1}) = 1$ for all $j = 1, \dots, m$.

At this point the conclusion follows by applying an argument similar to the one used in the proof of Lemma 5.6.3. □

Remark 5.6.5. *The previous Lemma also proves that if α is a simple root in Σ, $\lambda(q) - \lambda(\Phi_\alpha(q)) = 1$ and $\lambda(p) - \lambda(\Phi_\alpha(p)) = -1$ then $\overline{\Sigma}(p,q) = \emptyset$.*

Now we are ready to prove the main result of this section.

Theorem 5.6.6. *Let $\Sigma \subset R_0$ be a subset of simple roots and π the projection map $\pi : G \cdot p_0 \to G \cdot \widetilde{p}_0$, where $\widetilde{p}_0 \in \cap_{\alpha_i \in \Sigma} \mathcal{H}_{\alpha_i}$ lies in the closure of the Weyl chamber containing p_0. Consider the set of canonical classes $\{\alpha_p\}_{p \in (G \cdot p_0)^T}$ associated to a generic component of the moment map $\varphi : G \cdot p_0 \to \mathbb{R}$.*

Fix $p, q \in (G \cdot p_0)^T$. Let $q_0, \ldots, q_N$ be the elements of $(\pi^{-1}(\pi(q)))^T$, where q_0 is the minimum of φ on $\pi^{-1}(\pi(q))$. Consider $v_0, \ldots, v_N$ in $W(\Sigma)$ such that $q_j = \Phi_{v_j^{-1}}(q_0)$ and define $p_j = \Phi_{v_j}(p)$ for all $j = 0, \ldots, N$.
Let $J = \{j \in \{0, \ldots, N\} \mid \lambda(p) - \lambda(p_j) = \lambda(q_j) - \lambda(q_0)\}$. Then

$$\alpha_p(q) = \sum_{j \in J} \alpha_{p_j}(q_0) \widehat{\alpha}_{q_j}(q) \tag{5.14}$$

Proof. Since π is a weight preserving $T-$equivariant fibration between GKM spaces, we can apply Corollary 4.3.9. Hence, if $\overline{\psi} = \pi^*(\widetilde{\psi}) : G \cdot p_0 \to \mathfrak{t}^*$, where $\widetilde{\psi}$ is the moment map on $G \cdot \widetilde{p}_0$, we have that for every p, q in $(G \cdot p_0)^T$

$$\alpha_p(q) = \sum_{j=0}^{N} \left(\sum_{\gamma \in \overline{\Sigma}(p, q_j)} \widetilde{\Lambda}^-_{\pi(q)} \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}(\gamma_{i+1}) - \overline{\psi}(\gamma_i)}{\overline{\psi}(q) - \overline{\psi}(\gamma_i)} \cdot \frac{1}{\eta(\gamma_i, \gamma_{i+1})} \right) \widehat{\alpha}_{q_j}(q) , \tag{5.15}$$

where we also use the fact that $\Theta(r, r') = 1$ for all the edges (r, r') of the canonical graph (cfr. Proposition 5.1.2).

Observe that for all the simple roots α in Σ and all the points $r \in (G \cdot p_0)^T$, $\overline{\psi}(r) = \overline{\psi}(\Phi_\alpha(r))$ since $\pi(r) = \pi(\Phi_\alpha(r))$. By Lemma 5.6.1 $\eta(r, r') = \eta(\Phi_\alpha(r), \Phi_\alpha(r'))$ for all the horizontal edges $(r, r') \in E_{\mathrm{GKM}}$. Moreover by definition $\widetilde{\Lambda}^-_{\pi(q)} = \widetilde{\Lambda}^-_{\pi(\Phi_\alpha(q))}$.

In (5.15) we can restrict the sum to the fixed points q_j such that $\overline{\Sigma}(p, q_j) \neq \emptyset$. Let v_j be the element in $W(\Sigma)$ such that $q_j = \Phi_{v_j^{-1}}(q_0)$ for all $j = 0, \ldots, N$. If $\overline{\Sigma}(p, q_j) \neq \emptyset$, then by Lemma 5.6.4 there exists a bijection between $\overline{\Sigma}(p, q_j)$ and $\overline{\Sigma}(p_j, q_0)$, where $p_j = \Phi_{v_j}(p)$ and $\lambda(p) - \lambda(p_j) = \lambda(q_j) - \lambda(q_0)$. So (5.15) can be written as

$$\alpha_p(q) = \sum_{j \in J} \left(\sum_{\gamma \in \overline{\Sigma}(p_j, q_0)} \widetilde{\Lambda}^-_{\pi(q_0)} \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}(\gamma_{i+1}) - \overline{\psi}(\gamma_i)}{\overline{\psi}(q_0) - \overline{\psi}(\gamma_i)} \frac{1}{\eta(\gamma_i, \gamma_{i+1})} \right) \widehat{\alpha}_{q_j}(q) . \tag{5.16}$$

For any $r \in (G \cdot p_0)^T$ consider $\alpha_r(q_0)$. Since q_0 is the minimum of $\varphi_{|_{\pi^{-1}(\pi(q_0))}}$, by Lemma 4.2.1 $\widehat{\alpha}_s(q_0) = 0$ for all $s \in (\pi^{-1}(\pi(q_0)))^T \setminus \{q_0\}$. Moreover $\widehat{\alpha}_{q_0}(q_0) = 1$. So

Corollary 4.3.9 gives

$$\alpha_r(q_0) = \sum_{\gamma \in \overline{\Sigma}(r,q_0)} \widetilde{\Lambda}^-_{\pi(q_0)} \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}(\gamma_{i+1}) - \overline{\psi}(\gamma_i)}{\overline{\psi}(q_0) - \overline{\psi}(\gamma_i)} \frac{1}{\eta(\gamma_i, \gamma_{i+1})} \ . \tag{5.17}$$

The conclusion follows combining (5.16) and (5.17). □

We can restate Theorem 5.6.6 in a more combinatorial way.

Theorem 5.6.7. *Let $\Sigma \subset R_0$ and consider the projection $\pi : W \to W/W(\Sigma)$. Fix $u, w \in W$. Consider the elements $u_0, \ldots, u_N$ of the set $\pi^{-1}(\pi(u)) \subset W$ and let u_0 be the unique element satisfying $l(u_0) = \min\{l(u_j),\ j = 0, \ldots, N\}$.*

Let $v_0, \ldots, v_N$ be elements in $W(\Sigma)$ such that $u_j = u_0 v_j$, for all $j = 0, \ldots, N$. Define $J = \{j \in \{0, \ldots, N\} \mid l(w) - l(wv_j^{-1}) = l(u_j) - l(u_0)\}$. Then

$$\alpha_w(u) = \sum_{j \in J} \alpha_{wv_j^{-1}}(u_0) \widehat{\alpha}_{u_j}(u) \ . \tag{5.18}$$

5.6.1 The divided difference operator identities

Canonical classes on generic coadjoint orbits coincide with equivariant Schubert classes. The divided difference operator has a natural action on equivariant Schubert classes, which is described as follows.

First of all, let's identify the fixed points of the $T-$action on $G \cdot p_0$ with the elements of the Weyl group, $w(p_0) \mapsto w$. Consider the canonical classes associated to φ, $\{\alpha_w\}_{w \in W}$. Let α be a simple root, and s_α the associated reflection. Then the divided difference operator ∂_α acts on α_w in the following way

$$\partial_\alpha \alpha_w(u) = \frac{\alpha_w(us_\alpha) - \alpha_w(u)}{u(\alpha)} \tag{5.19}$$

Equivariant Schubert classes satisfy the following identities, which we will refer to as the *divided difference operator identities* (cfr. [4])

$$\partial_\alpha \alpha_w(u) = \begin{cases} \alpha_{ws_\alpha}(u) & \text{if } l(w) > l(ws_\alpha) \\ \\ 0 & \text{if } l(w) < l(ws_\alpha) \end{cases} \tag{5.20}$$

In this section we prove how the identities (5.20) are an easy consequence of Theorem 5.6.7.

Fix u and w in W, and consider the projection map $\pi : W \to W/W(\alpha)$, where $\Sigma = \{\alpha\}$. Suppose that $l(w) > l(ws_\alpha)$ and $l(us_\alpha) > l(u)$ (the case in which $l(us_\alpha) < l(u)$ is similar); recall that by (5.6), this implies that u is the minimum of fiber $\pi^{-1}(\pi(u))$, since $\pi^{-1}(\pi(u))^T = \{u, us_\alpha\}$. Observe that since α is a simple root $l(us_\alpha) - l(u) = 1$ and $l(w) - l(ws_\alpha) = 1$; hence by Remark 5.6.5 $\overline{\Sigma}(ws_\alpha, us_\alpha) = \emptyset$.

Then Theorem 5.6.7 implies that

- $\alpha_{ws_\alpha}(us_\alpha) = \alpha_{ws_\alpha}(u)\widehat{\alpha}_u(us_\alpha)$. Since u is the minimum of the fiber, $\widehat{\alpha}_u(us_\alpha) = 1$. Hence $\partial_\alpha \alpha_{ws_\alpha}(u) = 0$

- $\alpha_w(us_\alpha) = \alpha_{ws_\alpha}(u)\widehat{\alpha}_{us_\alpha}(us_\alpha) + \alpha_w(u)\widehat{\alpha}_u(u)$. Now observe that $\widehat{\alpha}_{us_\alpha}(us_\alpha) = \widehat{\Lambda}^-_{us_\alpha} = u(\alpha)$ and $\widehat{\alpha}_u(u) = 1$. Hence the previous equation gives

$$\alpha_{ws_\alpha}(u) = \frac{\alpha_w(us_\alpha) - \alpha_w(u)}{u(\alpha)} = \partial_\alpha \alpha_w(u)$$

5.7 Connections with Billey's formula

In [4], Billey proves a manifestly positive integral formula for the restriction of equivariant Schubert classes on flag varieties $G_{\mathbb{C}}/B$ to the fixed point set of the T action, where $G_{\mathbb{C}}$ is a semisimple Lie group and B a Borel subgroup. More precisely, the formula can be stated in the following way.

Let $\alpha_1, \ldots, \alpha_n$ be a choice of simple roots, and $s_1, \ldots, s_n$ the associated reflections. For every element u of the Weyl group W, let $s_{a_1} \cdots s_{a_N}$ be a reduced expression for u, where $a_i \in \{1, \ldots, n\}$ for all $i = 1, \ldots, N$. Then (cfr. [4] sec. 4)

Theorem 5.7.1. ***(Billey's formula)*** *For every fixed reduced word $s_{a_1} \cdots s_{a_N}$ of u*

$$\alpha_w(u) = \sum_{(a_{j_1},\ldots,a_{j_m}) \in J(w,u)} \prod_{k=1}^{m} s_{a_1} s_{a_2} \cdots s_{(a_{j_k}-1)}(\alpha_{a_{j_k}}) \tag{5.21}$$

where $J(w,u)$ is the set of ordered subsequences $(a_{j_1}, \ldots, a_{j_m})$ of $(a_1, \ldots, a_N)$ such that $s_{a_{j_1}} \cdots s_{a_{j_m}}$ is a reduced word for w.

The key point in the proof of this Theorem is the use of the divided difference operator identities, which are also a consequence of Theorem 5.6.7.

In [25], Zara proves that in type A_n there is a bijection between the path contributions given in Proposition 5.2.1 and the Billey's contributions, for a special choice of a reduced word for u. In type B_n, C_n and D_n the positive integral formula we exhibit in sections 5.3, 5.4 and 5.5 are not equivalent to Billey's formula. In what follows we give counter examples in each type.

- Consider a generic coadjoint orbit of type B_2, and consider $\alpha_{q_1}(s_1)$ as in Example 5.3.8. It's easy to see that if $q_1 = w(p_0)$, then the only reduced word for w is given by $s_{\alpha_1} s_{\alpha_2}$. If $s_1 = u(p_0)$, then u has two reduced words: $s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}$ and $s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$. In the first case there are precisely three positive integral contributions in Billey's formula, whereas in the second case there is precisely one. Since in Example 5.3.8 we only had two positive integral contributions, the two formulas cannot be equivalent.

- Consider a generic coadjoint orbit of type C_2, as in Example 5.4.2. Suppose that we want to compute $\alpha_{p_1}(s_1)$ using Billey's formula. Let $p_1 = w(p_0)$ and $s_1 = u(p_0)$. Then the only reduced word for w is given by s_{α_2}, and the two reduced words for u are given by $s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}$ and $s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$. Independently on the reduced word chosen for u, there are only two positive integral contributions in Billey's formula, given by $2x_1$ and $2x_2$; whereas in Example 5.4.2 we had three positive integral contributions.

- Consider a generic coadjoint orbit of type D_4 as in Example 5.5.1, and suppose we want to compute $\alpha_p(q)$ using Billey's formula. Let $p = w(p_0)$ and $q = u(p_0)$. The only reduced word for w is given by $s_{\alpha_2}s_{\alpha_1}$, whereas there are precisely two reduced words for u, given by $s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_1}$ and $s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}$. In both cases the contributions in Billey's formula are given by $(x_2-x_3)(x_1+x_2)$ and $(x_2+x_3)(x_1+x_2)$, which are different from the contributions we obtained in Example 5.5.1.

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