Modeling and Measurement of Microwave Effects
in High-$T_c$ Long Josephson Junctions

by

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B. S., Physics (1988)
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Submitted to the Department of Physics
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ABSTRACT

A circuit model is presented for Josephson junctions (JJ) that solves the nonlinear long junction equation subject to a nonuniform current distribution. This extended resistively shunted junction (ERSJ) model consists of a parallel array of ideal resistively shunted JJs connected by inductors. The junction array is connected to an array of current sources that simulate the time and space-dependent current distribution. This model can describe the creation, annihilation and motion of Josephson vortices. The results explain the experimentally measured step structure in the power dependence of the effective resistance in YBCO Josephson junctions. The calculated reactance also fits the experimental data much better than previous models. This model contributes to a better understanding of the power-handling characteristics of high-$T_c$ microwave devices, in which the power losses are believed to result from Josephson-junction effects associated with imperfections in the films.

The model also predicts second-harmonic generation with a highly nonlinear and non-monotonic power dependence. I have measured the second-harmonic generated by a 1.8-GHz YBCO stripline resonator with an engineered step-edge JJ. I have found a second-harmonic signal with a power dependence consistent with my calculation, in the temperature range where the step structure in the resistance is clearest.

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CHAPTER 1

INTRODUCTION

Superconducting thin films have a tremendous potential for improving the state of the art in electronic devices. Shorter time constants lead to faster devices and higher frequency operation. Low losses can potentially reduce energy consumption and circuit heating. Low resistance also results in filters with significantly higher quality factors $Q$ or smaller bandwidth. However, one of the major difficulties in producing superconducting devices is that they have limited current-carrying capacity above which power losses become nonlinear.

High-$T_c$ superconductors such as YBCO superconduct at higher temperatures and magnetic fields, relative to conventional superconductors. They are also more difficult to prepare with good yield and quality. These thin films have been plagued by nonlinear losses which occur for rf current levels well below those predicted for the bulk properties. The losses are believed to
be caused by grain boundaries and defect structures in the films\textsuperscript{1,2}. These defects represent weak links for superconductivity, which are associated with Josephson junctions (JJ). Significant efforts are being made today to understand these nonlinear losses in order to improve the quality of high-$T_c$ thin-film devices.

The coupled-grain model has been relatively successful at explaining the temperature and magnetic-field dependence of the resistance data of some films\textsuperscript{3}. However, the reactance data cannot easily be described by this model. The need for greater understanding of the Josephson effect associated with grain boundaries prompted an in-depth study of individual JJs in high-$T_c$ thin films. This study is being conducted on YBCO thin-film microwave (rf) resonators with engineered long Josephson junctions. The stripline pattern and the Josephson junction are depicted in Fig. 1.1. The two ground planes, above and below the stripline, are not pictured. The resonator provides a driving-current density with a precise microwave frequency and a nonuniform spatial distribution that can be determined experimentally. The device used in my experiment has a fundamental resonant-mode frequency of 1.8 GHz.

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\textsuperscript{1} J. Halbritter, J. Supercond. \textbf{10}, 91 (1997)


Measurement of the resistance due to engineered JJs reveals a power-dependent step structure as shown in Fig. 1.2. The onset of nonlinear resistance occurs well below the junction critical current, as determined from the dc measurements and the depicted fit. The resistively shunted junction (RSJ) model for short junctions produces a step structure in the resistance. However, this step structure begins near the critical current, and once again the reactance data is not explained. Averaging the RSJ resistance over the nonuniform current distribution of the film has been demonstrated to provide an overall good fit to resistance data for some JJs\(^4\), but not for the measured reactance. The current-density-averaged-RSJ model fit is depicted by the solid curve in Fig. 1.2. The fitting parameters are \(I_c = 29\) mA and \(R_n = 9\) m\(
\Omega\). Furthermore, current-density averaging removes the steps, that are seen experimentally. In addition, second-harmonic generation that cannot be explained by any of the above models, has been measured in high-\(T_c\) thin films.

---

The inadequacy of the above models has produced the necessity to model the long Josephson junction more accurately. In this thesis, I will present a model for vortex dynamics in long Josephson junctions. This model takes into account the nonuniform current-density distribution of the film without resorting to a critical-state model. This model supports Josephson vortices which I will demonstrate are responsible for the resistance step structure, increased reactance and second-harmonic generation of the device.

In Chapter 2, I will present the basic model for the long junction and the nonlinear differential equations that govern its electrodynamic behavior. Then I will develop in detail an extended resistively shunted junction (ERSJ) model. In Chapter 3, I will explain the Josephson vortex dynamics of the solutions to the model and the consequences of these vortices on the power handling and even-harmonic generation. In Chapter 4, I will present the experimental techniques, my experimental data and an analysis of that data based on the ERSJ predictions.
CHAPTER 2

EXTENDED RESISTIVELY SHUNTED JUNCTION MODEL

In these microwave measurements we use stripline resonators with an engineered Josephson junction to study rf Josephson effects in thin films. In this chapter, I will describe these junctions, their parameters and the differential equations governing their behavior. Then I will present the extended resistively shunted junction (ERSJ) model and show how we calculate rf properties from this model.

2.1 The Modeled System

A Josephson junction with a dimension much longer than the Josephson penetration depth $\lambda_j$ is a long junction. The junctions we are modeling have a $\lambda_j$ of about 2

\[ \text{Figure 2.1: Uniform Josephson Junction in a Stripline with the Coordinate System.} \]
\( \mu \text{m, and are 150-\mu m in width} \ W \) (see Fig. 2.1). Thus, these are long junctions, which are described by the sine-Gordon equation\(^5\,^6\),

\[ \lambda_j^2 \nabla^2 \phi = \sin \phi + \tau_j \frac{\partial \phi}{\partial t} + \tau_{RC} \tau_j \frac{\partial^2 \phi}{\partial t^2} , \]  

(2.1)

where \( \phi(x, z, t) \) is the gauge-invariant phase difference across the junction, \( \tau_j \) is the Josephson time constant and \( \tau_{RC} \) is the capacitive time constant. The three terms on the right-hand side of Eqn. 2.1 represent the Josephson, normal, and displacement currents, respectively.

The film thickness \( h \) is 0.14 \( \mu \text{m} \), for the modeled junctions. Since \( h < \lambda_j \), the junction properties can be assumed to be uniform in the \( z \)-direction. When \( \tau_{RC} \) is much less than the other time constants, we can neglect the capacitive term. We estimate \( \tau_{RC} \) to be approximately 0.2 ps from Zhang's\(^7\) data on a YBCO JJ. The smallest \( \tau_j \) we have measured is about 3 ps, corresponding to \( I_c R_n = 1 \text{ mV} \). The rf period \( \tau_{rf} \) is 0.56 ns. Thus, we will drop the capacitive term in our analysis. Our model is capable of including this term, if needed. Thus we can use the one-dimensional, over-damped form of the long-junction equation,

\[ \frac{J(x, t)}{J_c} = \lambda_j^2 \frac{\partial^2 \phi}{\partial x^2} = \sin \phi + \tau_j \frac{\partial \phi}{\partial t} , \]  

(2.2)

---


\(^6\) J. McDonald and J. R. Clem, unpublished

where $J(x, t)$ is the total current density crossing the junction and $J_c$ is the junction critical current density.

The superconducting current density $J_s$ through the junction is in the $y$-direction and is given by the current-phase relationship

$$J_s = J_c \sin \phi .$$ (2.3)

The electric field $E$ is also in the $y$-direction and is given by

$$E = \frac{\Phi_0}{2\pi d_e} \frac{\partial \phi}{\partial t} = \frac{V}{d_e} ,$$ (2.4)

where the magnetic flux quantum is given by $\Phi_0 = \frac{\hbar \pi}{e} = 2.068 \times 10^{-15} \text{ T m}^2$, $V$ is the voltage, and $d_e$ is the effective electrical thickness that is approximately equal to the junction interlayer distance $d$ (see Fig. 2.1). The magnetic field $B$ is in the $z$-direction and has a magnitude given by

$$B = \frac{\Phi_0}{2\pi d_m} \frac{\partial \phi}{\partial \chi} ,$$ (2.5)

where $d_m$ is the effective magnetic thickness,

---

8 T. P. Orlando, op. cit.

\[ d_m = 2 \lambda_i + d_e , \]  

(2.6)

where \( \lambda_i \) is the London penetration depth of the film.

We can express the time and distance constants from Eqn. 2.1 as follows. The Josephson penetration depth \( \lambda_j \) is related to the parameters \( J_c \) and \( d_m \) by

\[ \lambda_j = \sqrt{\frac{\Phi_0}{2\pi \mu_0 J_c d_m}} . \]  

(2.7)

The Josephson frequency \( f_j \) and time constant \( \tau_j \) are given by

\[ f_j = \frac{1}{\tau_j} = \frac{2\pi}{\Phi_0} J_c R_n = \frac{2\pi}{\Phi_0} J_c \rho_n d_e , \]  

(2.8)

where the junction critical current is \( I_c = J_c hW \), the normal resistance is \( R_n = \rho_n \frac{d_e}{hW} \) and \( \rho_n \) is the normal resistivity. The RC time constant \( \tau_{RC} \) is

\[ \tau_{RC} = R_n C_j = \rho_n \epsilon . \]  

(2.9)

where \( C_j = \frac{\epsilon hW}{d_e} \) is the junction capacitance.

The junction is driven by the current of the stripline resonator. This current varies in time according to the frequency of the resonator \( \omega_{rf} \) as

\[ I(t) = I_a \sin(\omega_{rf} t) , \]  

(2.10)
where \( I_o \) is the current amplitude and \( t \) is time. The spatial current density distribution \( J_f(x) \) that reflects the effects of the London penetration depth \( \lambda_f \), and the geometry of the device has been calculated by Sheen\(^\text{10}\). This distribution assumes that the current can be approximated as uniform in the \( z \)-direction when the film thickness is less than or approximately equal to \( \lambda_f \). Our film thickness \( h \) is 0.14 \( \mu \text{m} \) and \( \lambda_f \) is approximately 0.2 \( \mu \text{m} \). The current density distribution \( J_f(x) \) is shown in Fig. 2.2 and is found to be sharply peaked at the edges. We define the dimensionless spatial current density distribution \( j_f(x) \) as

\[ j(x) = \frac{W J_j(x)}{\int_{-W/2}^{W/2} J_f(x) dx}. \]  

The complete driving current density distribution \( J(x, t) \) seen by the junction can then be written as

\[ J(x, t) = \frac{J_s}{hW} j(x) \sin(\omega_{nt} t). \]  

This functional form of \( J(x, t) \) is valid if the peak current density \( J_s(\pm W/2, t) \) is much less than the pair-breaking critical current density of the stripline.

We must solve Eqn. 2.2 subject to the current density distribution given by Eqn. 2.12. The driving function is then fixed by the current density distribution \( j(x) \), the frequency \( \omega_{nt} \), and the current amplitude \( J_s \). The overall Josephson junction is characterized by the parameters \( \rho_n \), \( J_s \), and \( \lambda_r \). The spatial boundary condition must reflect the fact that current cannot pass through the edge of the junction or the stripline. While this condition is difficult to express mathematically, it is easily accounted for in the model that I will present in the next section. Finally we impose the steady-state boundary condition that has the following periodicity with respect to the driving frequency

\[ \Phi(x, t) = \Phi(x, t + \tau_{nt}) \].
While there are an infinite number of solutions that can satisfy the periodic boundary conditions, the steady-state solution is unique, because it is the one solution that minimizes the losses.

The solutions to the long-junction equation are Josephson vortices. The particular solution in Fig. 2.3 for a stationary isolated vortex centered at the origin is

\[ \phi(x) = -2 \arcsin \frac{1}{\cosh \frac{x}{\lambda_j}}. \]  

(2.14)

The circulating current density associated with this vortex has a spatial dependence given by

\[ B(x)/\mu_0 J_0 \lambda_j \]

\[ J(x)/J_c \]

\[ \phi(x)/2\pi \]

\[ x/\lambda_j \]

**Figure 2.3:** Analytical Solution for an Isolated Stationary Josephson Vortex: Josephson Phase \( \phi(x) \), Magnetic Field \( B(x) \), Current Density \( J(x) \)

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11 T. P. Orlando, op. Cit.
and the magnetic field has a spatial dependence given by

\[
B(x) = \frac{\mu_0 J \lambda_j}{\cosh \frac{x}{\lambda_j}}.
\]  

(2.16)

The general solution includes combinations of any number of moving vortices and antivortices. Antivortices are identical to vortices except that they have the opposite magnetic polarity and the circulating current flows in the opposite direction. The moving vortex is not just a translation of the stationary one. The time dependence in Eqn. 2.2 introduces a normal current that changes the shape of the vortex. Nor is constructing solutions with multiple vortices a simple matter of making linear combinations of isolated vortices. The additional vortices must be added in a way that is consistent with the nonlinearity of Eqn. 2.2. The shape of each vortex depends on its velocity, and its position relative to all other vortices and boundaries. The flux quantization built into the long-junction equation guarantees that all vortices contain one fluxon, a magnetic flux quantum \( \Phi_0 \). These particular nonlinear dynamic solutions are normally accomplished through a numerical method such as our model, which I will present in the next section.
2.2 The Model

2.2.1 Model Description

We use the circuit model in Fig. 2.4 to simulate the long junction. This long-junction model consists of a parallel array of \( N \) ideal resistively shunted junctions (RSJ)\(^{12}\), coupled by lateral inductors. Each JJ element obeys the short-junction limit of Eqn. 2.2, which is equivalent to the RSJ equations:

\[
I_f(n, t) = I_c \sin \phi(n, t) \quad ; \quad V_f(n, t) = \frac{\Phi_0}{2\pi} \frac{d\phi(n, t)}{dt}.
\] (2.17)

The RSJ equations are valid for junction widths small compared with \( \lambda_f \). Thus, each junction in the array represents part of the long-junction consistent with this limit. If the capacitive term that we dropped from Eqn. 2.1 were needed, we would simply add a parallel capacitor to each RSJ unit. This junction array is excited by an array of current sources that simulate the time and space-dependent current distribution. The model constitutes a periodic lattice of \( N \) unit cells indexed by \( n \), each unit cell containing a sub-circuit. Each sub-circuit contains a JJ indicated in Fig. 2.4 by an \( \mathbf{X} \), a resistor, two inductors and two current sources. The circuit is then solved by JSIM\(^{13}\) a computer program that is like SPICE\(^{14}\), but includes JJ circuit elements. From the time-

\(\text{---}\)


Figure 2.4: Circuit Diagram of the Extended Resistively Shunted Junction Model. The circuit elements represented by an X are the ideal JJs.

Varying phases and voltages of the model junctions, we can then calculate the rf impedance, fields and currents of the long junction.

Since the modeled JJ is uniform, the only junction-model parameters used in the circuit are the normal resistance $R_n$, the critical current $I_c$, and the lateral inductance $L_l$. The $R_n$ and $I_c$ are easily calculated from the resistivity in the normal state $\rho_n$ and the critical current density $J_c$, of the modeled junction (Fig. 2.1) as follows:

$$\frac{1}{R_n} = \frac{hW}{\rho_n d_e} = \sum_{n=1}^{N} \frac{1}{R_n(n)} \quad , \tag{2.18}$$
\[ I_c = J_c \ hW = \sum_{n=1}^{N} I_c(n) \ . \] (2.19)

The \( L_l \) is given by

\[ L_l = \frac{\Phi_0}{4\pi I_c} \left( \frac{W}{\lambda} \right)^2 \approx \sum_{n=1}^{N-1} L_l(n) \ . \] (2.20)

which is derived from Ampere’s law by relating the magnetic flux to the lateral current. The derivation is included as Appendix A.

Each of the \( N \) unit cells in the model, indexed by \( n \), simulates a fraction of the long junction equal to \( \frac{\Delta W(n)}{W} \). If all \( \Delta W(n) \) are equal, then all of the unit cells are identical except for the amplitudes of their rf current sources. The current from each source is given by

\[ I(n, t) = \sin(\omega_{rf} t) \ \frac{I_0}{W} \int_{\Delta W(n)} j(x) dx \ . \] (2.21)

The unit-cell resistances \( R_n(n) \) are determined from \( R_n \) by the usual parallel combination relationship,

\[ R_n(n) = R_n \ \frac{W}{\Delta W(n)} = \frac{\rho_n d_e}{h\Delta W(n)} \ . \] (2.22)

The unit-cell critical currents \( I_c(n) \) are found by
\[ I_c(n) = I_c \frac{\Delta W(n)}{W} = J_c h \Delta W(n) \quad . \tag{2.23} \]

The unit-cell lateral inductances \( L_k(n) \) for each inductor are given by

\[ L_k(n) = L_1 \frac{\Delta L(n)}{W} = \frac{\Phi_0 \Delta L(n)}{4\pi J_c h \lambda_j^2} \quad , \tag{2.24} \]

where \( \Delta L(n) \) is the distance between the centers of the regions \( \Delta W(n) \) and \( \Delta W(n+1) \), corresponding to the junctions connected by the \( n \)-th inductor.

\[ 2.2.2 \quad \text{Output} \]

A wealth of information can be extracted from the time and space varying phase \( \phi(n, t) \).

In this section, I will describe how and what information I calculate from \( \phi \). All of the values and functions that I calculate can be mathematically derived from a phase solution to the ERSJ model, as a function of time and space; however, in practice, I use both \( \phi(n, t) \) and the circuit element voltages \( V(n, t) \) calculated by JSIM. The junction voltages are in fact related to the time derivative of the phase, according to

\[ V_j(n, t) = \frac{\Phi_0}{2\pi} \frac{\partial \phi(n, t)}{\partial t} \quad . \tag{2.25} \]
The accuracy of the numerical calculation of a time derivative is related to the size of the time step. JSIM uses smaller internal time steps than those that it returns as output; thus the voltage is more accurate than calculating a time derivative from $\phi(n, t)$.

$\phi(n, t)$ and $V_j(n, t)$ are converted to $\phi(x, t)$ and $E(x, t)$, by using Eqn. 2.4 and $x(n)$ which is the center of each region $\Delta W(n)$. The $J_s(x, t)$ and $B(x, t)$ are calculated from $\phi(x, t)$ according to Eqns. 2.3 and 2.5, respectively. The magnetic field can be calculated equivalently from the LI product for the lateral inductors. The normal current $J_s(x, t)$ is related to $E(x, t)$ by

$$E(x, t) = \rho_n J_n(x, t). \tag{2.26}$$

The total current density $J(x, t)$ is the sum of the supercurrent density $J_s(x, t)$ and the normal current density $J_n(x, t)$. We can then integrate $J(x, t)$ and $B(x, t)$ over the width $W$ of the junction to get the total current $I(t)$ and total magnetic flux $\Phi(x(t))$, respectively. The power density $p(x, t)$ is

$$p(x, t) = E(x, t) J(x, t). \tag{2.27}$$

The vortices produced by the Josephson junctions are centered at positions where $\phi(x, t)$ is an odd integer multiple of $\pi$.

The simplest way to calculate resistance is to sum the power $P(t) = I(t)V(t)$ dissipated per rf period $\tau_{rf}$ in all of the resistors. The effective resistance $R_{\text{eff}}$ is then given by

$$R_{\text{eff}} = \frac{1}{I_{\text{rms}}^2 \tau_{rf}} \sum_{n=1}^{N} \int_{-W/2}^{W/2} \int_{-\tau_{rf}}^{\tau_{rf}} J_n(x, t)^2 dt dx, \tag{2.28}$$
where $I_{rms}$ is the root mean squared current ($I_{rms} = \frac{I_a}{\sqrt{2}}$), $I_a$ is the rf current amplitude, $\Delta t$ is the time step used in the computation, and the sum over $n$ includes all $N$ of the unit cells in the model.

While this method is the easiest way to calculate resistance, it contains no other useful information. However, the Fourier analysis in time of the voltage can be used to calculate both resistance and reactance as well as harmonic generation\(^\text{15}\). Since the electric field has significant spatial dependence, we must define the effective voltage $V(t)$. $V(t)$ is determined from the total power $P(t)$ and total current $I(t)$ to be

$$V(t) = \frac{P(t)}{I(t)} = \frac{\sum_{n=1}^{N} \sum_{I} V(n, t) I(n, t)}{I_a \sin(\omega_{rf} t)}, \quad (2.29)$$

where the sum over the set $\{I\}$ includes the junction, resistor, and two inductors in each unit cell $n$. The lateral inductors are needed, because they represent the magnetic-field energy stored in the long junction. Now we can calculate the Fourier components of the time varying effective voltage. The in-phase components $R_m$ are given by

$$R_m(I_a) = \frac{\int V(t) I_a \sin(m \omega_{rf} t) \, dt}{\tau_{rf} I_{rms}^2}, \quad (2.30)$$

and the out-of-phase components $X_m$ are given by

\(^{15}\) J. McDonald, op. cit.
\[
X_m(I_a) = \frac{\tau_{\text{rf}} \cdot \int V(t) I_a \cos(m \omega_{\text{rf}} t) \, dt}{\tau_{\text{rf}} I_{\text{rms}}^2}, \tag{2.31}
\]

where \( m = \{0, 1, 2, \ldots\} \) indicates the Fourier component or harmonic index. The \( R_0 \) component is always zero and \( X_0 \) is the dc component. It can be verified that Eqn. 2.28 is equivalent to Eqn. 2.30 when \( m = 1 \), thus \( R_1 = R_{\text{eff}} \). The \( X_1 \) component is the effective reactance. The \( R_2 \) and \( X_2 \) components are the in and out of phase components of the second harmonic generated by the circuit and so forth. Harmonics are generated only at frequencies given by

\[
f_m = mf_1, \tag{2.32}
\]

where \( f_1 = f_{\text{rf}} \) is the fundamental or driving frequency and \( f_m \) is the frequency of the \( m \)-th harmonic.

### 2.2.3 Range of Validity

The most critical restriction on this model is the junction spacing \( \Delta W \). The \( \Delta W \) must be less than \( \lambda_j \) to insure proper coupling between neighboring discrete junctions in the model. Eqn. 2.1 shows that \( \lambda_j \) is the applicable length scale for this junction and the Josephson vortices. Thus, we cannot model the junction with RSJ unit cells or points spaced greater than \( \lambda_j \). In §2.1 we saw that the size of a vortex is approximately \( 2\lambda_j \). The smallest number of points necessary to describe a vortex is two. Furthermore, the model becomes exact in position space as \( \Delta W^2 \ll \lambda_j^2 \). Experience with running this simulation shows that setting \( \Delta W = \lambda_j \) results in a reasonable
approximate solution. Doubling the junction spacing $\Delta W$ results in significant errors. These errors occur largely because the flux quantization built into the differential equation results in flux pinning between discrete junctions. Simulations have also verified that the accuracy becomes excellent when the junction density $\frac{\lambda_j}{\Delta W}$ is increased to five junctions per $\lambda_j$. This is because $\lambda_j$ enters into the equations as a squared quantity. Thus we expect the errors to decrease like $\left(\frac{\Delta W}{\lambda_j}\right)^2$.

The steady-state periodic boundary condition $\phi(x, t) = \phi(x, t + \tau_R)$ is achieved by running the simulation long enough for the transients to die off. Five rf periods in time are normally sufficient to resolve spatial differences less than 1 $\mu$m, as measured by vortex positions.

As with any numerical calculation, the density of points in space and time is critical to the accuracy of the calculation. However, the advantages of using more points must be balanced against the cost in computer power and time. Calculating one junction solution for phase and voltage as a function of time and space at a single current amplitude can take hours on a PC. Calculating the power dependence takes more than a week.

The number of junctions must meet the above condition that $N = \frac{W}{(\Delta W)_{avg}} > \frac{W}{\lambda_j}$. One or two junctions per Josephson penetration depth are sufficient to achieve errors in the vortex positions of less than half of $\lambda_j$. This error estimate was made by running the simulation with identical parameters, but gradually decreasing $\Delta W$ until the solution was clearly no longer changing to any reasonable degree of accuracy. The error associated with $\Delta W$ is then the difference between the convergent vortex penetration distance and that calculated using a given $\Delta W$. Increasing the number of junctions per $\lambda_j$ to five or ten results in errors much less than $\lambda_j$. 
JSIM internally adjusts the time steps used to solve the circuit, but I set the minimum to 100 steps per shortest possible time constant. The Fourier analysis is then done over some lesser number of points in time, but using a sufficient number to achieve an accurate result within a few percent.

2.2.4 Verifying the Results by Comparing the Limits

To verify that the model is working properly, I calculated various limiting cases, using the same computer program. Making $L_i$ very small, we simulated a Josephson penetration depth greater than the junction width. This limit yielded the small junction or RSJ model result as expected. Then we increased $L_i$, which is equivalent to decreasing $\lambda_p$, but we did not decrease the junction spacing. The result was the same as averaging the RSJ-model solutions over the current distribution\(^{16}\). This is the expected result, since in this limit, the lateral current is negligible and the junctions are uncoupled. Decreasing the junction spacing sufficiently to keep the proper coupling while increasing $L_i$ is computationally prohibitive. Furthermore, the actual $\lambda_p$ is normally much greater than $\lambda_i$.

In this chapter, I described the ERSJ model in sufficient detail so that it can be understood, reproduced or improved upon. For the remainder of this document I will only discuss details of the model if they are pertinent. Unless otherwise stated, the model is being used so that it conforms to all of the specifications described in this chapter. The subsequent chapters discuss the results obtained by running ERSJ simulations and I then compare these results to experimental data.

\(^{16}\) D. E. Oates, op. cit.
CHAPTER 3

MODELING RESULTS

In this chapter, I present my analysis of the modeling results obtained from many ERSJ simulations. I will do this by describing the physics responsible for the rf long-JJ behavior and support this analysis with appropriate examples of ERSJ solutions. This analysis includes power handling and harmonic generation, both of which can be explained by Josephson-vortex dynamics.

3.1 Power Regimes

The nonlinear behavior of a long JJ depends significantly on the amount of current crossing the junction. I will begin this analysis by considering the following regimes: quasilinear (low-power), vortex (intermediate-power) and saturation (high-power). In considering the power-handling characteristics of a superconducting device, we want to look specifically at the amount of current in the device. The measure of improved power handling in superconductors is not the ability to absorb more heat and energy, but rather the ability to carry more current with
fewer losses and greater linearity. The goal then is to keep the resistance small while increasing the total current. Zero resistance only exists for direct current (dc) superconductivity. While the supercurrent never directly dissipates energy, a changing supercurrent induces an electric field that drives lossy normal currents. When we refer to the power dependence we will express it as a function of the ratio of the rf current amplitude $I_a$ to the junction critical current $I_c$. Fig. 3.1 plots an example of the calculated resistance and reactance (both normalized to $R_n$) as a function of $I_a/I_c$. The three regimes, which will be explained below, are labeled in Fig. 3.1.
3.1.1 Quasilinear (Low-Power) Regime

In order for Eqn. 2.2 to approach a linear limit, the phase $\phi$ everywhere must remain small relative to $\pi$. Then the approximation $\phi = \sin \phi$ is valid. It is clear from Eqn. 2.3 that this requires the supercurrent density $J$ to be small compared with the junction critical current density $J_c$. We define $I_c$ as the rf current amplitude which makes the peak current density in the stripline equal to the critical current density of the junction. The $I_c$ is given by

$$I_c = I_c J_f \left( \pm \frac{W}{2} \right). \quad (3.1)$$

The strict requirement for complete linearity is that $I_a \ll I_c$. In this limit, the ideal JJs in the ERSJ model can each be accurately replaced by inductors $L_j$ such that

$$L_j(n) = \frac{\Phi_0}{2\pi I_c(n)}. \quad (3.2)$$

Thus our array of RSJ junctions will behave like an array of inductors and resistors. The resulting impedance depends on the rf frequency but not on the amplitude of the current. This result for junctions is physically the same as the low-power limit for superconducting thin films where the low-power impedance depends on the square of the rf frequency.

Even the linear equivalent circuit described above diverts current away from the edges. When the current through the inductive (junction) channel changes in time, a voltage results across the inductor-resistor pair. This has two effects: First, current flows through the resistor.
The normal or resistive channel is where all of the power losses occur in this linear circuit as well as in the fully nonlinear long junction. Second, current flows through the lateral inductors. This occurs in order to minimize the energy losses in the resistive channel. Redirected current then flows across the junction at another location, through a combination of the resistive and inductive channels. The resistive-channel power density is given by

\[ p_n(x, t) = E(x, t) J_n(x, t) = \rho_n I^2_n(x, t) = E^2(x, t)/\rho_n. \]  \hspace{1cm} (3.3)

Losses are thus minimized by both reducing the total current through the resistive channel as well as by making the normal current more uniform in time and space. Current redistribution through the lateral inductors reduces losses in both ways.

Current continuity requires that the lateral current crosses the junction at some location farther from the edge, and returns through the opposing lateral inductors. Thus the lateral current on each side of the junction is equal and opposite. Furthermore, the same current redistribution occurs inward from the opposite edge. In the physical junction, current flowing along such a path results in an increased magnetic field within the junction. The product of the lateral current and the lateral inductivity per unit width in the circuit model is equivalent to the magnetic field in the physical JJ. Additionally Eqn. 2.5 results in the following flux-phase relationship for any region \( \Delta x \) of the junction with a phase difference \( \Delta \phi \),

\[ \Phi(\Delta x, t) = \int_{\Delta x} d_m B_z(x, t) \, dx = \frac{\Phi_0}{2\pi} \Delta \phi, \]  \hspace{1cm} (3.4)
where $d_m$ is the effective magnetic thickness of the junction. I used this relationship to insure that JSIM correctly accounted for flux quantization. I verified that the LI product (normalized to $\Phi_0$) of any inductor connecting two junctions is always equal to the difference between the Josephson phases $\phi$ (normalized to $2\pi$) of the two junctions.

Another consequence of current redistribution is that the peak current density across the junction is less than the peak current density in the stripline. In fact as long as current redistribution results in the junction current density remaining less than $J_c$ everywhere, then at worst only a small portion of the junction can be strongly nonlinear for only part of the rf cycle. The size of this nonlinear region is of the length scale $\lambda_n$, and thus it is small compared to the whole long junction. This means that the junction behavior can in some cases be effectively linear even when $I_a \geq I_c$. We do not know of any analytical formulation that can be used to predict the current amplitude that separates the effectively linear regime from the nonlinear regime in the power domain. I only refer to the linear regime as the range where $I_a < I_c$. I will refer to the quasilinear regime as including the linear regime as well as the effectively linear region described above.

3.1.2 Vortex (Intermediate-Power) Regime

The quasilinear regime comes to an abrupt end when there is sufficient current redistribution to nucleate a Josephson vortex. Increasing the driving current (or power) results in more current redistribution, greater flux penetration and a higher peak current density. The sine-Gordon equation guarantees that when the phase at the edge of the junction is equal to $\pm \pi/2$, then the supercurrent density $J_s(\pm W/2)$ must be equal to $\pm J_c$ and the flux penetration in the junction
must be $\pm \Phi_0/4$. Similarly when $\phi(\pm W/2) = \pi$, then $J_x(\pm W/2) = 0$ and the flux is equal to $\pm \Phi_0/2$.

This is a highly unstable situation which is best described as half of a vortex, whose core is located exactly at the edge of the junction. Vortex nucleation occurs when the vortex core crosses the boundary. If nucleation occurs, the vortex quickly moves completely into the junction. We define this current level as the fluxon-nucleation current $I_f$. There is no analytical formulation defining $I_f$ which appears to depend on $\lambda_j$ and the spatial distribution of the driving current. The calculations I have run to date indicate that any dependence of $I_f$ on the time constants is small. $I_f$ can be calculated numerically using our model or it can be measured experimentally. $I_f$ separates the low-power regime from the vortex regime (see Fig. 3.1).

Since a vortex has no transport current, a nucleated vortex may remain in the junction at an rf node, when the net current through the junction is zero. An effect of the steady-state boundary condition is that the solution minimizes losses consistent with the boundary conditions and driving function. Thus vortices should be present at the rf nodes if they result in reduced losses per rf cycle. I will describe the vortex dynamics in detail in §3.3. For now I will simply note that the steady-state solutions can be classified by the number of vortices present at the rf nodes. Vortices are nucleated at the edge of the junction. They penetrate some distance to a turning point which occurs at the rf node. Then they reverse their direction of motion and move out of the junction through the same edge where they entered. The creation and destruction of vortices involves motion across the boundary at the edge of the junction. Fig. 3.2 illustrates the vortex trajectories for a complete solution involving four vortices: one per edge per rf node.

We know that losses occur only in the resistive channel and that the normal-current density is proportional to the electric field. Eqn. 2.4 tells us that $E(x, t)$ is proportional to the time
derivative of $\phi$. Therefore, vortex motion results in losses. Furthermore, we can plot the power dissipation density $p_n(x, t)$ that is given by

$$p_n(x, t) = J_n(x, t) E(x, t)$$

(3.5)

where $J_n$ is the normal current density. Fig 3.3, the surface plot of the calculated power dissipation density associated with the four vortex trajectories in Fig. 3.2, depicts the losses as a function of time and space. The surface is a complete steady-state solution, including one full rf period on the time axis and the entire junction width on the position axis. The volume under this

Figure 3.2: Four Vortex Trajectory Solution, $N = 80$ JJs.
Power Dissipation Density

\[ \frac{t}{\tau_{rf}} \text{ Time} \]

\[ \frac{W}{2} \text{ Position} \]

\[ \frac{J_n E}{J_c^2 \rho_n} \]

\[ I_a = 0.38 I_c \]
\[ I_c R_n = 0.225 \text{ mV} \]
\[ \lambda_j = 2.434 \mu\text{m} \]
\[ f_{rf} = 3 \text{ GHz} \]

Figure 3.3: Surface Plot of the Power Dissipation Density for a Four Vortex Solution. The surface depicts a complete steady-state solution, including one full rf period on the time axis and the entire junction width on the position axis. The volume under this surface is proportional to the power dissipated per rf cycle. The spikes are vortex creation and destruction events. The parabola-like ridges connecting the spikes are the losses associated with vortex motion. These ridges correspond to the trajectories in Fig. 3.2.
surface is proportional to the power dissipated per rf cycle and to the resistance. The spikes are vortex creation and destruction events. The spikes are connected by faintly-visible parabola-like ridges. The ridges are the losses associated with vortex motion, and thus have the same parabola-like shape as the vortex trajectories in Fig. 3.2. This demonstrates the fact that vortex motion dissipates energy and that the creation and destruction of vortices are high-speed high-energy events. Since losses are related to the vortex velocity, we expect the effective resistance to increase with the number and speed of the vortices. Furthermore, the vast majority of the losses occur during nucleation and annihilation events, of which there must be an integer number. This explains the step structure in the effective resistance. Each step then indicates the addition of another pair of vortices, one vortex on each side of the junction.

The reactance is a measure of the energy stored per rf cycle. The junction and the vortices store energy in three forms. The first is the superelectron momentum, or kinetic inductance. In the circuit model this energy is included in the JJ circuit elements whose potential energy\(^{17}\) \(U(n, t)\) is given by

\[
U(n, t) = I_c(n) \frac{\Phi_0}{2\pi} (1 - \cos \phi(n, t)).
\]  

\(3.6\)

The time-dependent inductance \(L_f(t)\) associated with this energy is\(^{18}\)

\[
L_f(t) = \frac{\Phi_0}{2\pi I_c \cos \phi(t)}.
\]

\(3.7\)

\(^{17}\) T. P. Orlando, op. cit.

\(^{18}\) T. P. Orlando, ibid.
Eqn. 3.2 is the linear limit of Eqn. 3.7. The second form of energy storage is the magnetic flux. The magnetic energy stored is related to the number of vortices, since each vortex contains one fluxon. The third form is electric-field energy, which is stored in moving vortices and is small because the junction capacitance is small. We ignore the electric-field energy by not including the capacitance in our model. Since the long junction stores energy in vortices, we expect the effective reactance to increase with the vortex density.

3.1.3 Saturation (High-Power) Regime

As \( I_a \) increases well beyond \( I_c \), the supercurrent channel becomes saturated and the majority of the current must flow through the normal channel. This requires that the electric field must increase and thus so does the vortex velocity. In this limit, the resistance approaches a value proportional to \( R_n \). If the limiting junction-current-density distribution were uniform, then the resistance would approaches \( R_n \). If there is any remaining nonuniform current distribution then the resulting resistance is increased beyond \( R_n \) by a factor related to the \( P \) dependence of the power,

\[
R_{\text{eff}}(I_a) = \frac{2\hbar}{I_a^2 \tau_{\text{rf}}} \int \int E(x, t) J_n(x, t) dx dt = \frac{2\rho_n h}{I_a^2 \tau_{\text{rf}}} \int \int J_n(x, t)^2 dx dt .
\]  

(3.8)

We would expect the supercurrent channel to become saturated when \( I_a \) is approximately equal to \( I_c \).

There is also a limit to the number of vortices that can fit in the junction at one instant in time. The size of a vortex is approximately \( 2\lambda_p \). Thus the junction becomes full as the number of
vortices approaches \( \frac{W}{2\lambda_j} \). The approach to this limit is marked by the onset of vortex/antivortex annihilations at the center of the junction. When a vortex nucleated on one edge collides with an antivortex from the other edge, the two vortices annihilate as their current densities and magnetic fields cancel. The stored energy of both vortices is dissipated through the resistive channel during the collision. Two of these events are displayed in Fig. 3.4 where the trajectory lines cross the center \( (x = 0) \) of the junction. The first such annihilation occurs at a lower vortex density than predicted above. Specifically, the maximum number of vortices present in this solution is twenty rather than the 30 vortices predicted. This occurs because vortices are not tightly packed in the

\[ I_{r\xi} = 0.225 \text{ mV}, f_{r\xi} = 3 \text{ GHz}, \lambda_j = 2.434 \text{ \mu m}, N = 80 \text{ JJs} \]

**Figure 3.4**: Forty Vortex Trajectory Solution with Two Center Annihilations. The center annihilations occur three hundredths of an rf period before each rf node. \( I_{r\xi} = 0.225 \text{ mV}, f_{r\xi} = 3 \text{ GHz}, \lambda_j = 2.434 \text{ \mu m}, N = 80 \text{ JJs} \)
junction. The vortex dynamics, that will be explained in more detail in §3.3, result in vortices reaching the center sooner than predicted above.

In the example given, the first center annihilation occurs at approximately $I_a = I_c$, as we might expect. Therefore, we consider the region above $I_c$ to be the saturation regime. However, the exact $I_a$ where the first such annihilation occurs may vary with the junction parameters and can only be determined from the simulations.

In the vortex regime, the reactance increases with increasing $I_a$ and vortex density. Annihilations at the center of the junction limit the vortex density. Once the limit is reached, no additional magnetic flux can be stored. In this regime, the reactance stops increasing.

Thus we see that the resistance approaches low and high-power limits that are independent of the current amplitude. The region between the limits has a highly nonlinear power dependence. The inductance also has a low-power limit, and increases in the vortex regime. We have concluded that the magnetic flux becomes saturated in the high-power limit. At the present time we have not studied the high-power inductance behavior sufficiently to completely understand the saturation regime. We know that as the power is increased, so is the electric field and vortex velocity. Thus far, we have neglected the electric-field energy. In order to study the high-power limit, we will need also to consider the junction capacitance, because as the vortex velocity increases, so does the energy stored in the electric field.
3.2 Circuit Analogy

In the above section, I explained the calculated nonlinear power dependence of the impedance based on the vortex dynamics of the solutions to the ERSJ model. I also discussed the energy storage and dissipation in terms of the analogous channels in the circuit model and the long junction. To complete the analogy, I will review the components of the circuit model, the physical long junction, and the sine-Gordon equation (Eqn. 2.2). The normal or resistive channel dissipates energy but does not store it. This channel is represented by normal electrons in the junction, the resistors in the ERSJ model and the voltage \( \frac{d\phi}{dt} \) term in Eqn. 2.2. The superelectron channel does not dissipate any energy, but rather it stores kinetic energy in the form of superelectron momentum. The superelectrons are represented by the JJ elements in the ERSJ model and the nonlinear \( \sin \phi \) term of the sine-Gordon equation. The lateral inductors in the ERSJ model represent the supercurrent in the YBCO that can move laterally (x-direction) before crossing the junction. An equal and opposite lateral current on each side of the junction creates a magnetic field in the junction. The magnetic flux in the junction is analogous to the LI product of the lateral inductors in the circuit, and the Laplacian term in the sine-Gordon equation. These correspond to stored energy but not to energy dissipation. The flux-phase and phase-supercurrent relationships inherent in Eqn. 2.2 are preserved by the circuit, therefore the circuit model must simulate flux-quantized Josephson vortices. This concludes our comparison. Henceforth, I will refer to the channels and fields without distinguishing between the ERSJ model, the sine-Gordon equation (Eqn. 2.2) or the physical long JJ since these three descriptions are all equivalent.
3.3 Vortex Dynamics

I have already discussed the steady-state vortex dynamics which are found in the calculated solutions. In this section, I will specifically address the reasons for the resulting dynamics so that the particular solutions can be understood rather than just accepted as the calculated result. In the process of discussing the vortex dynamics I will also address some of the more subtle characteristics of the solutions to Eqn. 2.2.

3.3.1 Wave-Particle Duality

Solutions to the sine-Gordon equation correspond to electromagnetic waves, while the Josephson vortices correspond to coherent wave packets. Flux quantization gives these wave packets a particle nature, since the vortices can be considered as magnetic dipoles confined to move in one dimension. Since they cannot rotate, each has a fixed polarity, defined by the orientation of the magnetic flux or equivalently the direction of the circulating current.

3.3.2 Forces on Vortices

The attractive and repulsive interactions between vortices are consistent with magnetic dipoles, as explained below. The actual interaction mechanism is the Lorentz force\(^{19}\). Consider the isolated stationary vortex pictured in Fig. 2.3. The Lorentz force on the vortex current density due to the self field is outward from the vortex core. The symmetry of the magnetic field and current density results in a zero net force on this vortex. A net force results from the addition of an externally applied current. A vortex \((B_z > 0)\) subject to a positive external current \((\gamma-\)

direction) will experience a net force in the x-direction. Changing either the polarity of the vortex or the direction of the external current will reverse the direction of the net Lorentz force. The current density of a neighboring vortex is experienced as an external current, with the appropriate resulting force. Thus like polarity vortices repel and opposite polarity vortices attract.

Vortex interactions can also be looked at in terms of minimizing the total energy of the vortices. The isolated stationary vortex in Fig. 2.3 is the lowest energy state for a single vortex. Vortices of opposite polarity decrease their net stored energy by moving closer together and they ultimately cancel each other out. Vortices of like polarity add magnetic fields. This results in more net stored energy due to the $B^2$ dependence of the field energy. Also the net current density distribution has an additional node, indicating a higher energy state. Thus, vortices of like polarity minimize their energy by repelling each other. This phenomenon also describes the effects of the boundary condition at the edge of the junction ($x = \pm \frac{W}{2}$). As with all constraints, the boundary condition can only increase the energy of the solution. For a vortex without an externally applied current, the boundary condition results in the total current density being zero at the boundary. This distorts the shape of the vortex, resulting in a higher energy state. Thus the boundary repels vortices of either polarity, within the length scale $\lambda_j$.

The last force operating on vortices is a drag force. A moving vortex has more stored energy than a stationary one. The additional energy has two forms. One is the electric field which is not present in a stationary vortex. The second is due to the distortion of the vortex shape from that of the stationary state. It is clear from the homogeneous equation (Eqn. 2.1), that a nonzero time derivative will change the shape of the vortex solution. The total additional energy is synonymous with the vortex kinetic energy. In the over-damped case, the momentum is
negligible. As the vortex moves, the resistive channel immediately dissipates the energy of the driving current density, except during nucleation when some of the incident energy is used to create the new vortex. As the vortex slows, its shape and stored energy will approach that of the stationary vortex.

The dominant force on the vortices in our solutions is the Lorentz force due to the driving current density. The directions of motion for all of the vortices in Figs. 3.2 and 3.4 are consistent with the direction of the rf current. Note that each vortex has a turning point at the moment when the rf current changes direction (the rf node). The effects of the other conservative forces are perturbations to the trajectories that would result from only the driving current and the damping force. In the vortex trajectory plots, such as Figs. 3.2 and 3.4, the slope of the trajectory is inversely proportional to the velocity of the vortex. In Fig. 3.4, the annihilating vortices accelerate as the vortex separation reaches the length scale of $\lambda_r$, demonstrating the attractive interaction. I will address the boundary interactions in §3.3.4.

3.3.3 Vortex Velocity

It is well know that vortices tend to approach a fixed velocity. More precisely, in the absence of any net forces, vortices travel at the Swihart velocity. The Swihart velocity represents the relativistic limit of Eqn. 2.1. Our vortices obey the over-damped limit, where vortex momentum is negligible. The over-damped vortices will stop moving, as soon as the net external force is zero. This is apparent in Fig. 3.2 because the vortices do not overshoot the rf nodes. The small overshoot in Fig. 3.3 is due to neighboring vortex interactions.

\[ \text{(20) J.C. Swihart, J. Appl. Phys. 32, 461 (1961)} \]
In order to relate this to our situation we must include the capacitive term in the analysis. In the limit that $\tau_{RC} >> \tau_p$, the displacement current is much larger than the normal current, and therefore we would ignore the drag term. Such a vortex stores both electric and magnetic field energy, but does not dissipate energy. This vortex should have a constant velocity $v$, equal to the ratio of the space and time constants,

$$
v = \frac{\lambda_j}{\sqrt{\tau_j^t RC}} = \sqrt{\frac{d_e}{d_m \mu_0 \varepsilon}} = C \sqrt{\frac{d_e}{d_m}},
$$

where $C$ is the speed of light in the junction, $d_m$ is the effective magnetic thickness of the junction and $d_e$ is the effective electrical thickness of the junction.

### 3.3.4 Image Vortices

The use of image vortices is extremely helpful in understanding the complete solutions to the sine-Gordon equation (Eqn. 2.2). While the magnetic flux associated with each vortex is quantized, the total flux in the junction is not quantized, when there is an applied current. The real vortices certainly dominate those solutions which include real vortices. However, in the low-power limit, we have Meissner-like solutions without vortices, and within each class of vortex solutions there is a contribution from Meissner-like flux penetration. These are similar to the Abrikosov-vortex state in type II superconductors, where the flux is equal to the flux quantum $\Phi_0$ times the number of vortices plus or minus some Meissner-like flux. Similarly, the total time-varying flux in our long junction solutions is also equal to the number of vortices times the flux.
quantum plus or minus some additional flux that is less than a flux quantum. In order to understand these variations we can use image vortices.

Orlando and Delin\textsuperscript{21} describe how the short-junction solutions can be understood as a vortex in time passing the junction. This is equivalent to viewing the long-junction solutions from a fixed point in space, where you would see vortices passing your position and the local Josephson phase developing accordingly. A complete sine-Gordon solution can be described by real vortices inside the junction and image vortices outside the junction. Even the Meissner-like solutions can be viewed this way. Such solutions are expected to include current density and magnetic field decreasing exponentially like \( \lambda \). This is consistent with image vortices in the limit where the images are many penetration depths away from the edge.

Now we have a method for understanding the solutions in terms of vortex dynamics that is valid for all solutions, even those without vortices. We can now use this method to understand the boundary interaction and explain why the destructive losses in Fig 3.3 are greater than the nucleation losses. For the purpose of clarifying this analysis I define the following three events. An annihilation event is a collision between a vortex and an antivortex, in which the pair permanently cancels out. A destruction event is the process that results in the elimination of a vortex at the edge of the junction. Nucleation is the process that creates a new vortex. Nucleation occurs at the edge of the junction. In theory, nucleation (of a vortex/antivortex pair) can occur inside the junction, but such an event requires an external energy source.

Image vortices can be used to match the boundary conditions for the sine-Gordon equation, just as image charges are used in electrostatic problems, with the caveat that the

\textsuperscript{21} T. P. Orlando, op. cit.
superposition of vortices is nonlinear. The image vortex must have the same polarity as the real vortex, since the interaction is always repulsive. However, the sine-Gordon equation inhibits $\phi$ from changing by more than $2\pi$ in the distance $\lambda_r$. Thus the cores of two vortices of the same polarity cannot be separated by a distance less than $\lambda_r$. This analysis implies that a vortex near the edge of the junction will experience a tremendous short-range repulsive force. For a nucleation event, this immediately accelerates the vortex away from the edge and into the junction. For a destruction event the boundary presents a steep barrier. The Josephson phase at the edge $\phi(\pm \frac{W}{2})$ must develop and the flux must flow in the direction required by the Lorentz force, due to the driving current. The motion of the vortex out of the junction would satisfy the time development requirement on the flux and phase; however, the vortex cannot cross the boundary because of the barrier inherent to the boundary condition. The conflict is resolved by the nucleation of a new antivortex that moves in and annihilates the first vortex. These annihilation events are clearly visible in Fig. 3.4. Thus vortex destruction consists of a nucleation and an annihilation event, possibly in very close proximity. Clearly, this will dissipate more energy than a simple nucleation event, as shown in Fig 3.3. This example illustrates the fact that there are always two ways to satisfy the flux-penetration requirements, but that the steady-state solution selects out the lower-energy state.

We can now say that of the three events defined above, only nucleation and annihilation are elementary. Destruction consists of a nucleation and an annihilation. Fig. 3.4 shows that these annihilations occur deeper within the junction, for each additional vortex. A characteristic of these solutions is that the two elements of each destruction event become more distinct as the number of vortices increases.
We can summarize Josephson vortex behavior in terms of over-damped magnetic dipoles confined to one-dimensional motion. The dominant force on them is the Lorentz force due to the driving current density. They have the usual magnetic dipole interactions between each other and are repelled by the junction boundaries.

3.4 Even-Harmonic Generation

In general, even-harmonics are not generated by symmetric circuits. Harmonic generation is given by the Fourier expansion of the effective voltage as in Eqns. 2.30 and 2.31. A symmetric device, for this purpose, is one whose response to a positive current is the same as it is to a negative current. In other words, the circuit equations are symmetric under a sign change in the driving current. Based on this generalization, we would not expect the ERSJ model to generate even harmonics. Actually, the ERSJ model does produce second harmonics, which are explained below as another consequence of the vortex dynamics.

The symmetry of the solutions in Figs. 3.2 and 3.4 is readily apparent. The current density distribution is symmetric in position at all times, just as is the driving current density distribution. The magnetic field and thus vortex polarities are antisymmetric under the same spatial symmetry operation. All fields and current densities, including the driving current, are antisymmetric under a simple time translation of one-half rf period. This temporal symmetry is the requirement for a solution with no even-harmonic generation. Thus these two solutions produce no even harmonics.

From our understanding of the vortex dynamics and in order to preserve both of the above symmetry operations, we might expect that solutions should include vortices in sets of four.
Figure 3.5: Six Vortex Trajectory Solution demonstrating the temporal symmetry breaking that generates even harmonics, $N = 80$ JJs.

Any solution containing a number of vortices that is not an integer multiple of four, must break at least one of the above symmetries. Breaking the spatial symmetry does not result in even harmonics, but breaking the temporal symmetry does. Furthermore, for those solutions where vortices remain at locations many penetration depths from the center, the opposing edges can be assumed to be independent. Since the steady-state solution is unique, and the conditions at each edge are symmetric, then the two independent solutions will have the same symmetry as the conditions at the edges. Thus, I will focus the remainder of this discussion solely on the temporal symmetry.
Our calculated solutions contain vortices in sets of two without breaking the spatial symmetry. The spatially symmetric addition of two vortices requires that they be added in one half of the rf cycle but not in the other. This destroys the temporal symmetry and generates even harmonics. The trajectories of a six-vortex even-harmonic state are plotted in Fig. 3.5. Since the additional pair of vortices can be added to either half of the rf cycle, we see that this solution is doubly degenerate as required for spontaneous symmetry breaking.

Based on the simulations discussed above, we now expect to observe even-harmonic solutions. One way to explain why they occur can be understood by considering some homogeneous solutions to our equations. The short-junction equations are equivalent to a pendulum. The pendulum has two equilibrium states. A stable equilibrium when $\phi = 0$, and an unstable equilibrium when $\phi = \pi$. Similarly, a vortex in a long junction has a stable equilibrium position at the center of the junction. A vortex whose core is positioned on an isolated edge or boundary, such as $x = \pm \frac{W}{2}$, is in unstable equilibrium. Since our solutions are steady state and not stationary, these unstable equilibrium solutions do not occur. However, the fact that they exist is sufficient to break the symmetry of the steady-state solutions.

Now that we have established the potential for symmetry breaking, the next question must be which symmetry should be broken. An asymmetric solution will occur only if it results in fewer losses than the symmetric solution that would otherwise exist. The fully symmetric solutions can be divided into four regions in space and time, such that the solution in each region has a symmetry relationship to each of the other regions. Thus the entire solution is defined by the solution in any one region. If each region is independent of the others, then the lowest energy solution always obeys the same symmetry which is contained in the differential equations and the
boundary conditions. However, if there is an interaction mechanism between the regions, then that interaction can potentially reduce the total losses. So far I have not encountered any calculated solution without spatial symmetry. This is consistent with the previous assertion that the two edges, can be assumed to be independent except in the saturation regime. I have not studied this regime in sufficient depth to find a spatially asymmetric solution, but based on this analysis I believe such solutions should exist. The solution depicted in Fig. 3.5 demonstrates that temporally asymmetric solutions exist and thus must have fewer losses than the symmetric solution that would otherwise exist. The fact that asymmetric solutions exist, implies an interaction between the two halves of the rf cycle. The presence of vortices at the rf nodes provides an interaction mechanism between the two halves of the rf cycle. The symmetric solutions have the same number of vortices at each of the rf nodes, while the number of vortices at each of the nodes differs by two in the asymmetric solutions. The symmetric solution that would exist without temporal symmetry breaking approaches the unstable stationary state previously described as a vortex centered at the edge of the junction.

We now understand that vortices come in pairs in the steady-state solutions. An odd number of vortex pairs will generate even harmonics, while an even number of vortex pairs will not. The calculated alternating second-harmonic strength is plotted in Fig. 3.6 as $R_2$. The $X_2$ component is negligible. Each step in the power-dependent resistance $R_1$ is due to the addition of a pair of spatially symmetric vortices. There is an even-harmonic step at $I_f$ in Fig. 3.6 that is too narrow to be resolved. This is the asymmetric two-vortex state.

The alternating nature of the solution symmetry is responsible for the alternating length of the resistance steps. Since the asymmetry of any asymmetric solution can always be attributed
to just two vortices, the solution approaches a symmetric solution as the total number of vortices increases with increasing power level. This explains why, as the power increases, the strength of the calculated second-harmonic signal decreases and the length of the asymmetric step approaches that of the symmetric step.

3.5 Summary

In this chapter, I have demonstrated that the ERSJ model is capable of explaining the existence of a step structure in the power dependent resistance as a consequence of flux-quantized Josephson vortices. Furthermore, the model gives us a list of power-handling characteristics, that we can expect to find in long Josephson junctions. Specifically, we expect a quasilinear, low-
power regime, where no vortices are nucleated and only Meissner-like flux penetration occurs. This regime generates no second harmonics, has small losses and is expected to exhibit a power-independent impedance. The onset of Josephson-vortex nucleation is abruptly marked by significantly increased and highly nonlinear losses as a function of power level. The reactance also is expected to increase significantly with the onset of Josephson-vortex nucleation. Even harmonics are generated with a nonlinear non-monotonic power dependence. Furthermore, this vortex regime can begin at rf current amplitudes far below $I_c$. The junction behavior at each of the two edges can be approximated as independent of each other, since there is no significant interaction between the vortices near each edge. Finally, a saturation regime is reached when $I_a \approx I_c$. This marks the onset of the vortex/antivortex annihilation process at the center of the junction. In this high-power limit, the magnetic flux and reactance both saturate with increasing vortex density, and the resistance approaches a limit that is independent of power level.
CHAPTER 4

EXPERIMENTAL

The ERSJ model was inspired by the measured step structure in the power-dependent long-JJ resistance and an inability to explain the measured reactance. Second-harmonic generation was not anticipated. When the calculations clearly produced even-harmonic solutions, we set out to measure the second-harmonic generation of physical long junctions. In this chapter, I will describe the type of device measured, the experimental technique, and the results.

We conducted these measurements on a YBCO microwave stripline resonator with an engineered step-edge JJ. The device selected is the same one that was used to present the current-density-averaged RSJ model\textsuperscript{22}. This resonator produced the clearest resistance-step structure of the available step-edge devices, so we expected it would represent the best prospect for generating a clear second-harmonic signature. Since then, new devices with sapphire-bicrystal

\textsuperscript{22} D.E. Oates, op. cit.
grain-boundary junctions have been produced and measured. Those results are being published elsewhere\textsuperscript{23}.

4.1 Overview

In this section I present the measured data and an overview of some general topics that impact on the experimental results.

The results of my experimental measurement are shown in Fig. 4.1. Each data set is plotted as the measured second-harmonic power vs. the incident driving power. The significance of the two modes will be explained in §4.2.1. The power levels portrayed in these measurements are extremely low. Thus noise and spurious second harmonics are of great concern and are thoroughly addressed in the following sections. The noise floor is evident in each data set. Each data set also contains a background second-harmonic signal with a slope approximately equal to two. In order to distinguish this spurious second harmonic from the desired signal, I have plotted a fit to the spurious signal. The fitting parameters are given in each legend. The desired signal is that which remains above the noise floor and the fit to the background second harmonic. Clearly, the only significant JJ second-harmonic signal measured is found in the mode-one 60-K data.

4.1.1 General Second-harmonic Generation

One difficulty in measuring second harmonics was separating all of the other second-harmonic generation from the desired signal. All of the active components in the system generate

Second-Harmonic Data

Figure 4.1: Mode-One and Mode-Two Second-Harmonic Power measured at 40 K and 60 K. The line in each plot is a fit to the background second-harmonic signal. The JJ second-harmonic signal is that which remains above the noise and the background fit.

second-harmonic signals. Furthermore, resonators without junctions also generate second-harmonic signals. Second-harmonic distortion usually has a quadratic power dependence, just as first-harmonic response is usually linear. Thus second harmonics are expected to have a response with a slope of two, and first harmonics a slope of one on a log-log scale. The second-harmonic signal we are looking for is distinctly not quadratic in its power dependence. Since the second-harmonic signal due to the junction is extremely small, I used various techniques, which I will describe below, to reduce the spurious second-harmonic signal.
A system of filters and amplifiers was used to isolate the desired signal. Because of the generally quadratic nature of second-harmonic generation, it was essential to filter any noise, at the first possible point in the circuit. My general strategy was to filter the source of any second-harmonic signal before the resonator, and then to filter out the fundamental frequency after the resonator, and finally to amplify the remaining second-harmonic signal before measuring it.

4.1.2 Magnetic Field Effects

The spurious second-harmonic signal, which is present in all of the above data, was generated in the resonator since I measured all other second-harmonic signals to be much smaller without the resonator present. The strength of this signal was related to the magnetic field in which the resonator was cooled. We believe these second harmonics are due to magnetic flux that is trapped in asymmetric pinning sites. When the device is cooled in a magnetic field, Josephson vortices may remain in JJs and Abrikosov vortices may remain in the stripline. These vortices can become trapped in potential wells that result from imperfections in the superconductor. Based on the analysis in Chapter 3, we expect a trapped vortex (Josephson or Abrikosov) to oscillate under the influence of an rf current. If the pinning potential is asymmetric, then even harmonics will be generated. We expect the dominant term in the second-harmonic response to be quadratic in power. This accounts for the fact that the fits of power out vs. power in have a slope of approximately two. The total pinned-flux second harmonic should be directly related to the number of trapped fluxons which is roughly proportional to the strength of the magnetic field when the resonator was cooled. The exact relationship, depends on the shape of
each occupied pinning potential. Since different pinning sites may be occupied each time the device is cooled, the second harmonic may be different each time the device is cooled.

I experimented with using various magnetic-shielding techniques to exclude magnetic fields from the resonator. These shielding techniques are addressed in §4.2.3. I did not conduct a comprehensive study of the field dependence of this second-harmonic signal, since it was not the signal I wanted to measure. However, I verified that the strength of this signal depended on the cooling field. The ambient field was of order 1 G and varied from day to day, depending on the other experiments being conducted at the laboratory. As I reduced the field from 1 G to 1 μG, I found that the strength of the spurious second harmonic was also reduced.

4.2 Device, Equipment and Procedure

4.2.1 The Stripline Resonator

The resonator consists of a 150-μm wide YBa$_2$Cu$_3$O$_7$ (YBCO) film patterned into a stripline on a LaAlO$_3$ substrate, with YBCO ground planes. As depicted in Fig. 4.2, a step-edge junction is positioned at the center of the approximately 2-cm length $L$, and cuts across the entire width $W$ (See Fig. 2.1). The normal metal barrier material is YBa$_2$CoCu$_2$O$_7$. The frequency of the first or fundamental resonant mode $f_1$ is approximately 1.8 GHz. This first mode is a half-wavelength standing wave of the resonator.
The second mode is a full-wavelength standing wave. The overtone frequencies are approximately given by

$$f_i \approx l f_1 ,$$  \hspace{1cm} (4.1)

where $l$ is the mode number. The first mode has a current-density maximum at the junction while the second mode has a node, as shown in Fig. 4.3. Thus we expect the first mode to be maximally affected and the second mode to be minimally affected by the junction properties. Measurements at the two mode frequencies provide a method for distinguishing the junction effects from the film effects in the same device. Fig 4.4 demonstrates the difference between resistance measurements.
for modes one and two. The mode-two result is similar to that which is found in resonators without a fabricated junction. The low-power limit is proportional to the frequency squared. The sharp rise at high power is due to the film critical current $I_c$. Mode two includes both of these features, but the intermediate response differs due to the long-JJ effects.

Harmonic distortion occurs only at frequencies that are precisely given by

$$f_{1,m} = mf_1,$$  \hspace{1cm} (4.2)

where $m$ is the harmonic index, just as in Eqn. 2.32. The resonant mode frequencies are only approximated by Eqn. 4.1, while the harmonic frequencies are exact. The resonant bandwidth is much smaller that the resonant mode frequency deviations from Eqn. 4.1. Thus harmonic generation can be measured without interference from the higher resonance modes.

4.2.2 Experimental Equipment Configuration

Figure 4.5 is a schematic diagram of the experimental equipment used to measure the second-harmonic. In this section I will describe each component and its purpose. The computer controls the connected components and collects the data through the General Purpose Interface Bus (GPIB). The GPIB is a cable which allows two-way communications between the computer and the electronic components. The computer can direct the equipment to perform almost any function that a human operator could do from the keypads. The computer automatically steps the components through the experiment and records the measured data.

All microwave signals were ultimately measured on the HP8563H Spectrum Analyzer. The second-harmonic signals were measured with a 1-Hz resolution bandwidth in order to make
Figure 4.5: Schematic Diagram of the Experimental Equipment used for the measurement of the second-harmonic signal.

the noise floor as low as possible. I then averaged as many as 30 measurements per data point.

Signal averaging increased the accuracy of the measurement and lowered the effective noise floor. The signal frequency and amplitude of each measurement were automatically recorded by computer.

The output circuit (see Fig. 4.5) is used to filter out the fundamental frequency and then amplify the second-harmonic signal. The output circuits provided 27-dB gain for mode one and 19-dB gain for mode two. The JJ second-harmonic signal was more than 100 dB below the
fundamental. Without high-pass filtering, the fundamental frequency would have generated more second-harmonic distortion in the amplifier than the second-harmonic signal from the JJ. The amplifier was necessary, because the second-harmonic signal at low input powers was below the noise floor of the spectrum analyzer. The output circuit was removed to measure signals that did not require amplification.

The rf input signal (1.8 and 3.6 GHz) was generated by an HP8673C Synthesized Signal Generator. The input power was stepped at the synthesizer for each new data point. The source signal was then filtered with a low-pass filter to remove the second harmonic generated by the synthesizer. The signal-generator power setting was recorded by the computer.

The HP85645A Tracking Source was used in place of the previously mentioned input circuit, for measuring the resistance and reactance. The raw data for this measurement is the resonant frequency, frequency shift, insertion loss, and 3-dB bandwidth.

The reference source is an additional signal generator that is used only to produce the 10-MHZ reference signal, that is necessary to phase synchronize the signal generator and the spectrum analyzer. Normally one of the system components is designated as a reference, but this resulted in spurious second harmonics. An independent reference source was required to isolate the signal source from the spectrum analyzer.

The resonator was cooled in a Janus Varitemp cryostat. Rough temperature control is achieved by setting the helium vapor flow rate. The temperature at the inlet valve was regulated by a temperature controller with a temperature sensor and a resistive heater. This first temperature controller was set a few degrees below the desired temperature. The second temperature controller was connected to a temperature sensor and resistive heater attached
directly to the resonator package on the probe. This system resulted in temperature deviations of less than 0.01 K at 60 K. Temperature control failed only when the helium flow rate changed dramatically. Precise temperature control was crucial, because small deviations in temperature can shift the resonant frequency farther than its bandwidth. Since the frequency of the signal generator was fixed, frequency shifts could not be tolerated. The temperature of the resonator package was recorded by the computer.

I used five circuit configurations in the experiment. The configuration used to measure the impedance is the one described above, which substituted the tracking signal generator for the synthesizer. The impedance configuration bypasses the output circuit. There were four configurations used for measuring harmonics. Two input and two output circuits were tailored, one each for the mode-one and mode-two measurements. The four configurations are designated by mode one or two and by the presence or absence of the output circuit. Each of these four circuits was used to measure both first- and second-harmonic signals, by setting the spectrum analyzer to the appropriate frequency.

4.2.3 Magnetic Shielding

The following is a description of the magnetic shielding that I used, and the resulting field. I wrapped the outside of the Dewar in two double sheets of μ-metal and added two bottom sheets. I measured the magnetic field inside this shielding by placing a directional field detector in the Dewar without the probe. The shields always exhibited two moments. One was fixed relative to the shield and the other one acted to cancel the ambient field. By arranging the orientation of the shields, to oppose and cancel the ambient field I minimized the field inside the Dewar to 530
mG, roughly perpendicular to the probe axis. The remaining field, is largely due to the fixed moment of the shield, not the ambient field. The shielding moment is still available to cancel any stray or changing fields in the laboratory. Next, I used three μ-metal cans attached to the probe inside the Dewar. The outer two cans reduced the field to approximately 200 μG. I measured the shielding of the inner can separately. The inner can reduced a field of 10 G to less than 10 mG, which was the lower limit of the only available field sensor that fit inside the one-inch diameter of the shield. Finally, I combined all of these shields and placed the device in the Dewar so that the direction of any remaining field inside the shields would be in the plane of the device. Thus I estimate the cooling field to be on the order of 1 μG. All of the above data (Fig. 4.1) were measured under this minimized magnetic-field condition.

4.2.4 Noise and Spurious Second Harmonic

In this section I describe the noise level and the spurious second-harmonic signals and how I dealt with them. First, I cover the random thermal noise. Second, I address the spurious second harmonic generated in the measurement equipment.

The overall thermal noise floor was a constant and readily apparent in all data sets. This floor was approximately -133 dBm for all harmonic measurements using a 1-Hz resolution bandwidth. The noise floor was primarily due to the Spectrum Analyzer. Additional amplification would not have improved the signal-to-noise ratio, because it would have raised the noise floor.

The next noise source was the second harmonic generated by the circuit components. In order to insure that the noise and spurious second-harmonic levels were properly quantified and
controlled, I ran the experiment many times without the resonator. Each of the four circuit configurations was tested for its worst-case second-harmonic generation. I say this is the worst case, because the resonator has an insertion loss (approximately 20 dB) at the fundamental frequency and a larger insertion loss at the second-harmonic frequency. The results of these noise runs always had two features as seen in Fig. 4.6. First, they contained the power-independent thermal noise described above. Second, they contained a slope of two second harmonic that was generated by the circuit components. The slope of the circuit noise from the mode-one circuit was 2.01±0.04. The slope of the mode-two circuit noise was 1.89±0.05. The resulting second-harmonic circuit noise was at least 20 dB below any second-harmonic in the final measurements in Fig. 4.1. Some of these noise data also contained the high-power saturation limit of the amplifiers.

The overall slope-two background shown in all of the second-harmonic measurments originates in the resonator itself. This signal is believed to be due to trapped magnetic fluxons, as explained in §4.1.2 above.

Since the thermal noise is random, it can be averaged out by many measurements, thereby reducing the effective thermal-noise floor. The circuit and flux noise, however, each have
a specific phase relationship to the driving source. Our calculations predict that the junction second-harmonic signal is either in phase with the driving current or π out of phase. If we knew the amplitude and phase of the spurious second harmonic, we could subtract it out of our measurements. We can estimate the amplitude from the slope of the measured second harmonic at higher power where we expect the JJ second harmonic to be negligible. However, the phase relationship is not known and may be power dependent. Thus we cannot simply subtract out the noise source from the measured data. Furthermore, these coherent noise sources can add constructively or destructively with the JJ signal.

4.2.5 Calibration

I measured the insertion loss or gain of each component at all pertinent frequencies. I further verified that each response was linear over the power range from the lowest accessible power to the highest power level of the measurement. The high-power flattening evident in the data (see Fig. 4.1) is due to the critical current of the stripline resonator. No test signal was available to verify linearity down to -160 dBm. The signal generator produced a signal down to -90 dBm. Attenuators were used to produce a signal down to the thermal noise level of -133 dBm. It is assumed that the measurement circuits remain linear down to the thermal noise level of the resonator. The results of these measurements were used to calibrate the final data.

The noise runs described in §4.2.4, were part of a larger set of noise-calibration runs. These were performed over the available power range with all circuit configurations, and once each with the probe (but no resonator) cooled to 40 and 60 K. Then eight different data runs were conducted, with the resonator. First and second harmonics were measured with each of the
four circuit configurations. All of these data were then compared to check the second-harmonic noise levels, to verify the range of linearity for the entire system, and to double check the calibration for each data run.

4.3 Analysis

4.3.1 Film Linearity

The first issue to be addressed in analyzing this device is that our model assumes the stripline is completely in its linear power regime. This assumption allowed us to use Eqn. 2.12 in the model, which assumes that the driving-current-density distribution is separable into time and space-dependent parts. In this section, I will show that the linear stripline assumption necessary for Eqn. 2.12 requires that \( I_e \) be two orders of magnitude below the critical current \( I_c \) of the stripline. This requirement exists despite the fact that the resistance of the stripline is effectively linear up to almost \( I_c \).

Determining the linear regime of the stripline is identical to the analysis of the JJ in §3.1.1. The critical current of the stripline \( I_c \) is of order 1 A, as is readily apparent from Fig. 4.7. The \( I_c \) is related to the depairing critical current density \( J_{pair} \) of the stripline by \( I_c = J_{pair} hW \). The local requirement for linearity is that \( J \) must be much less than \( J_{pair} \). The film current density distribution, pictured in Fig. 2.2, has a density at the edge that is 10.5 times the average current density. Just as I defined \( I_e \) for the JJ, I now define \( I_c \) as the rf current amplitude at which the stripline current density distribution obtained from a linear assumption (Fig. 2.2) would result in a peak current density equal to \( J_{pair} \). Thus
\[ I_a \ll I_{ef} = \frac{I_{ef}}{10.5} \approx 95 \text{ mA} \]  

is the strict condition for complete linearity everywhere in the stripline. If we take the much less than symbol to mean an order of magnitude, then Eqn. 4.3 indicates that the linear limit should only be valid for rf current amplitudes that are two orders of magnitude below the critical current of the stripline.

In §3.1.1, I made the case for effective linearity in the JJ existing up to a higher current amplitude, \( I_e \). Similarly, stripline resonators demonstrate effective linearity up to almost the critical current \( I_{ef} \) as demonstrated by the mode-two data in Fig. 4.4. The argument in §3.1.1 is based on the assertion that the nonlinear effects are confined to a small portion of the space and time domain in the steady-state solution. However, the local nonlinearity in the stripline occurs at the time and place where Josephson-vortex nucleation occurs. Thus the above linearity condition must be interpreted strictly with respect to the current-density distribution seen by the junction and employed in the ERSJ model. The data in Fig. 4.7 demonstrate the power-level and temperature dependence of the effective resistance for mode one. The resistance step structure becomes rounded and disappears as the strict linear limit described above becomes invalid (above 10 mA).

At this time we do not have a model that can simulate a driving current density distribution whose time and space-dependent parts are not separable. From our analysis of vortex nucleation in Chapter 3, we know that Josephson vortices are created when current redistribution at the junction results in the penetration of a single fluxon \( \Phi_0 \). As the linear power dependence
assumption above becomes invalid, we expect that current redistribution will occur in the film away from the junction. Thus we have flux penetration into the film occurring at the rf peaks, when nucleation is expected in the junction. The result is that the resistance step structure becomes rounded, but the overall behavior is still the same. Specifically, significant power losses still occur well below the stripline and junction critical currents, but without the step structure.

My second-harmonic data are taken at 40 and 60 K. We can expect that the ERSJ model will be valid for the temperatures and powers where clear steps appear in the measured resistance. Outside this region we cannot be sure if we should see JJ second harmonics, and if we do, we expect the structure to be smeared. Based on Fig. 4.7, we expect that we may be able to measure the JJ second harmonics for the first few resistance steps at 60 K, but that we cannot
expect to measure a clear JJ second-harmonic structure at 40 K. The data in Fig 4.1 show a JJ second-harmonic signal at 60 K but not at 40 K.

4.3.2 Nonuniform Junction

Physical junctions are certainly not as uniform as our model junction. However, we cannot measure the spatial dependence of the junction properties, which is especially important because vortex nucleation and losses occur over a region of length scale \( \lambda_v \), at the lower end of the vortex regime (see Figs. 3.1 and 3.2).

We concluded in Chapter 3 that for solutions at the low end of the vortex regime, the two edges of the JJ behave independently. If the local properties of the junction are different at each of the edges, then we must expect that the height and onset of the steps due to each edge may be different. If the properties are close, this may just cause a smearing of the step structure. If the differences are large then we would expect to see two distinct sets of steps. Since we only see the first few steps with this device, we cannot expect to be able to discern the height and width patterns found in the calculations (see Fig. 3.6).

In Chapter 3, I described the alternating nature of the steps in the power dependence of the resistance. Since the periodicity of the calculated steps is repeated every two steps, we need to experimentally measure at least four steps in order to verify the periodicity. If the properties at each edge of the junction are sufficiently different to resolve two distinct step structures, then we must measure eight steps in order to see the periodic behavior repeat itself. Even the most optimistic interpretation of the data in Fig. 4.7 includes only three or four steps, thus this is insufficient to verify the periodicity predicted by the ERSJ model.
4.3.3 Josephson-Junction Second-harmonic Signal

Now that I have presented all of the difficulties involved in resolving the JJ second-harmonic signal, I will describe the observed characteristics and show how they are consistent with our model. The only part of the data presented in Fig. 4.1 that can be clearly attributed to the JJ is in mode one, at 60 K from -43 to -10 dBm power in. In this section, I will first explain the other features of the data and then the JJ second-harmonic features.

The flattening that occurs in the high-power limit is the result of the stripline approaching the normal state. In other words, the critical current $I_c$ of the stripline is being exceeded. This feature is not seen in the mode-one 40-K data because the measurement was not taken above 0 dBm.

The noise floor of each data set is evident in the low-power limit. The noise floor varies from -158 to -153 dBm. These variations are related to the different measuring circuits used, the number of averages taken and the temperature (see §4.2.4).

The spurious second-harmonic signal that has a slope of approximately two is fitted with a straight line in each data set. The fitting parameters are given in each legend. The fit has a slope of approximately 1.9 in both of the mode-two data sets where junction effects are expected to be negligible. The fitted slope is less for the mode-one data where the junction contributes. This spurious second-harmonic signal that we believe is due to trapped magnetic flux is described in §4.1.2.

The JJ second-harmonic features are distinguished from the other features as follows. First, the JJ signal must be above the thermal-noise floor and distinguishable from the slope-two
spurious signal. Second, the second-harmonic effect must occur in the mode-one measurements but not in the mode-two measurements. The signal that remains above the noise floor and above the fitted line is attributable to the JJ. The following analysis describes how this JJ signal is consistent with the ERSJ prediction.

The distinguishing characteristics of the calculated JJ second harmonic are that the signal is highly nonlinear and non-monotonic. Furthermore, the nonlinearity is distinctly not quadratic in power level. These properties are highly unusual and not expected to occur by any other known mechanism. The portion of the measured second-harmonic data that we attribute to the JJ has both of these properties. Non-monotonic behavior is clearly evident at -38, -32 and possibly -25 dBm on the power-in axis. Furthermore, this JJ data cannot be approximated by a quadratic fit.

Analysis of the four data plots in Fig. 4.1 shows that the highly nonlinear and non-monotonic behavior in Mode 1 at 60 K is not present at the other temperature or mode. Section 4.3.1 tells us that the distinguishing structure of the JJ second harmonic should become washed out or should disappear as the power is increased as in the 60-K data (Fig. 4.1). Since the resistance step structure is not observed at 40 K (Fig. 4.7), we do not expect to see the JJ second-harmonic structure at 40 K (Fig. 4.1).

While these preliminary data do not provide a conclusive proof of a long-JJ second-harmonic signature, the data are consistent with our current understanding of the long JJ and Josephson-vortex dynamics. Measurement of long-JJ second harmonics is continuing and the results are expected to be published at some future date.
In this chapter, I will discuss some ERSJ fits to experimental impedance data. Then I cover the calculations necessary to explore the consequences of varying the model parameters. Finally, I present a list of adaptations that can be made to expand this model for the general application to any Josephson junction problem.

5.1 Measured Impedance

While I developed this model, Youssef M. Habib continued the experimental study of long JJs in stripline resonators such as the one that I described in my experiments (see Chapter 4). He has used my ERSJ model to fit his experimental impedance data in Fig. 5.1. These data and the experimental procedure are presented in detail in the article submitted to Physical Review B\textsuperscript{24}. The resonator devices used in these measurements are prepared on a sapphire bicrystal substrate.

\textsuperscript{24} Y.M. Habib, op. cit.
with a 24-degree-misorientation angle. A grain-boundary Josephson junction is formed at the grain boundary where the two sapphire crystals are joined. Otherwise, these devices are essentially the same as the one that I described above in Chapter 4.

These fits were achieved by starting with the $I_c$ and $R_s$ values that were measured under dc conditions. The final parameters that were used as input to the ERSJ model were varied by no more than a factor of two from the dc measured values. The resistance and reactance fits represented by the solid lines are calculated from the same ERSJ solution.

The dashed lines are the fit to the same data using the current-density-averaged RSJ model described in Chapter 1. The ERSJ fit constitutes an improvement in the resistance fit, primarily because both the fit and the experimental data contain a step structure and because the fitting parameters are restricted as described above. We expect that continued work with the ERSJ model will improve the matching of the measured steps with the calculated steps. The

Figure 5.1: Experimentally Measured Impedance of a Grain-Boundary Josephson Junction in a 3-GHz Stripline Resonator. The ERSJ model fits are shown as solid lines. The current averaged RSJ model fits are shown as dashed lines. $I_f$ indicates the fluxon-nucleation current.
current averaged RSJ model has no steps to match. Furthermore, the current averaged RSJ model is clearly not capable of describing the experimentally measured reactance while the ERSJ model provides an excellent fit to the reactance up to the end of this calculation. These data are presented to demonstrate the fact that the ERSJ model represents a significant improvement in describing the impedance of long Josephson junctions.

5.2 Varying the Junction/Equation Parameters

This model can now be used to conduct a comprehensive study of the long-junction and sine-Gordon parameters, and their effects on rf impedance. The results of this study may provide a set of guidelines for reducing nonlinear losses that are due to the Josephson effect. The analysis below includes some of the preliminary conclusions drawn from the simulations we have conducted so far. To obtain comprehensive results for the various Josephson junctions and parameters will take a great deal more time, due to the number of independent parameters and the amount of time required for each calculation.

5.2.1 Time Domain

The over-damped short-junction (RSJ) limit of the sine-Gordon equation can be expressed in reduced units. The reduced form of the equation is

\[ \tau_{JR} \frac{\partial \phi}{\partial t_R} + \sin \phi = I_R \sin(2\pi t_R) , \]  

(5.1)

where the reduced-units parameters are:
\[ \tau_{JR} = \frac{\tau_I}{\tau_{rf}} = \frac{\Phi_0}{2\pi I_c R_n} \frac{f_{rf}}{t_R} ; \quad t_R = f_{rf} t ; \quad I_R = \frac{I_R}{I_c} . \] (5.2)

The reduced impedance is

\[ Z(I_R) = \frac{R(I_R) + iX(I_R)}{R_n} = R_n(I_R) + iX_n(I_R) . \] (5.3)

The periodic boundary condition is \( \phi(t_R) = \phi(t_R + 1) \). This formulation is convenient because it gives the complete set of solutions with only two independent parameters \( (I_R, \tau_{JR}) \). The general effect of varying \( \tau_{JR} \) is to change the height and width of the step structure in \( R_n(I_R) \). The high-power limit is one, and the first step is at \( I_R = 1 \approx I_{cR} = I_c/1_c \), independent of \( \tau_{JR} \).

This formulation can be expanded to include the capacitive term

\[ \tau_{CR}^2 \frac{\partial^2 \phi}{\partial t_R^2} + \frac{\kappa_{JR}}{\kappa_{RF}} \frac{\partial \phi}{\partial t_R} + \sin \phi = I_R \sin(2\pi t_R) , \] (5.4)

where the reduced-units capacitive time constant \( \tau_{CR} \) is

\[ \tau_{CR} = \tau_{RF} \frac{\kappa_I}{\kappa_{RF}^2} = \frac{\Phi_0}{2\pi I_c} \frac{f_{RF}^2 C_J}{2\pi I_c} . \] (5.5)

In this case the complete set of solutions is given by a three parameter \( (I_R, \tau_{JR}, \tau_{CR}) \) family of functions.
5.2.2 Space Domain

The reduced formulation for the long-junction equation is more complicated. The equation is given by

\[-\lambda_R^2 \frac{\partial^2 \phi}{\partial x_R^2} + \tau_{CR}^2 \frac{\partial^2 \phi}{\partial t_R^2} + \tau_{JR} \frac{\partial \phi}{\partial t_R} + \sin \phi = I_R j(x_R) \sin(2\pi t_R). \tag{5.6}\]

where $x_R = x/W$ and $\lambda_R = \lambda/W$. Recall from §2.1 that $j(x)$ is already defined in reduced units. For a fixed $j(x_R)$ this formulation results in a four-parameter $(I_R, \tau_{JR}, \tau_{CR}, \lambda_R)$ family of solutions. However, the spatial dependence of the driving current density depends on two parameters, $\frac{\lambda_I}{W}$ and $\frac{\lambda_I}{h}$. Note that even though we can assume that $j(x_R)$ is uniform in the z-direction, the distribution function still depends on the height $h$. Since $\lambda_I = 2\lambda_I + d$, the spatial distribution introduces two additional parameters in general or one more in the limit where $\lambda_I \gg d$. This last limit is appropriate for grain-boundary junctions.

The complete set of solutions to the full long-junction equation is then a six-parameter $(I_R, \tau_{JR}, \tau_{CR}, \lambda_R, \frac{h}{W}, \frac{d}{W})$ family of functions. Thus far, we have only looked at over-damped solutions where we have kept $j(x_R)$ fixed. The results from varying $\lambda_R$ are that $I_I$ changes from $I_c$ in the short-junction limit to $I_c$ in the small $\lambda_R$ limit.

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25 T. Van Duzer, op. cit.
5.3 Model Modifications and Improvements

There are numerous adaptations of this model that can greatly expand its application. In this section, I explain how these adaptations might be accomplished and why they might be interesting. Unless otherwise specified, these adaptations have not been implemented yet, and therefore there are no results to analyze at present.

I stated previously that the junction capacitance can be modeled by adding a parallel capacitor to each unit cell. This would be necessary to model a system where the assumption that $\tau_{RC} \ll \tau_j$ is not valid, or where vortex velocities approach the Swihart velocity.

I have developed the circuit element equations to allow for each $\Delta W(n)$ to be different while still modeling a uniform junction. This freedom is maintained for various purposes. If a simulation is being run where the solution is known to be linear over some spatial region, then the restriction that $\Delta W < \lambda_j$ may be relaxed in that region. Another reason for maintaining this freedom is to model more precisely the film current distribution, which changes at the edges on the length scale of the London penetration depth $\lambda_J$. Furthermore, physical junctions are not uniform over long lengths. Modeling nonuniform junctions would necessitate that variations in $\Delta W$ accompany variations in $\lambda_J$. Eqns. 2.18, 2.19 and 2.20 are valid for spatially dependent $\rho_s(x)$, $J_s(x)$ and $\lambda_J(x)$. This flexibility can be used to study pinning sites, edge defects, or other defects introduced in processing, deliberately or otherwise.

The array of current sources can be configured to simulate other types of current distributions. They can simulate a dc current without modification, by setting the rf frequency to zero. The spatial current density distribution $j_s(x)$ can be changed to represent the driving current density distribution due to an externally applied magnetic field rather than an applied current. The
current density distribution due to a driving current is symmetric in $x$ while that which is due to a
driving magnetic field (z-direction) is antisymmetric. Lastly, any combination of these driving
sources can be simultaneously simulated by the parallel addition of multiple arrays of current
sources in the circuit model.

Finally, if the assumption that $h \ll \lambda_J$ is not valid, then we cannot assume that the
current density and other properties are uniform in the z-direction. The model, as described so
far, is a one-dimensional array. The z-direction problem can be dealt with by constructing a two-
dimensional array of RSJ unit cells. This would result in the magnetic field having $x$ and $z$
components, and vortex motion in the same directions. However, currents and the electric field
would still have only a y-component in the junction.

The above adaptations have all been for the purpose of modeling different types of
Josephson junctions. The ERSJ circuit can also be used as a subcircuit in a larger network. In
this way it could be used to represent a network of grain boundaries in a film.

5.4 Conclusion

The overall objective of my research has been to improve our understanding of long-
Josephson-junction effects in high-$T_c$ thin films, so that this knowledge may be applied to grain-
boundary defects and improve the linearity of future high-$T_c$ thin-film devices. I have used the
extended resistively shunted junction model to provide an understanding of rf Josephson vortex
dynamics and a prediction for the power dependence of the impedance and harmonic generation
associated with the Josephson junction. The effects of temperature and magnetic field can be
predicted from their effects on the junction parameters.
To the best of my knowledge this work represents the first model that explains both the microwave resistance and reactance of long Josephson junctions. The ERSJ model predicted second-harmonic generation with an unusual power dependence. I measured the second-harmonic data presented here in order to experimentally verify the modeling results. While these measurements are not conclusive, they are consistent with the model. Second-harmonic generation represents a potential Josephson-junction signature. The fact that previously unexplained second-harmonics were measured in superconducting thin-films may provide evidence supporting the theory that grain-boundary Josephson effects are responsible for the onset of nonlinear losses in high-$T_c$ thin films.

The ERSJ model represents tremendous potential for future research. Since this model is equivalent to solving the sine-Gordon equation, it is generalizable for application to any Josephson junction. The sine-Gordon equation (Eqn. 2.1) describes Josephson junctions in type I and low-$T_c$ superconductors as well as high-$T_c$ superconductors. Therefore, this model and the analysis in this thesis is completely general for Josephson junction in all superconductors. The time and space constants will change according to the particular superconducting material and the geometry of the junction.
APPENDICES

A. Lateral Inductance Derivation

The purpose of the ERSJ model is to solve the sine-Gordon equation (2.2), which has only two independent parameters, $\lambda_j$ and $\tau_j$. Since the circuit model has three parameters, $I_c$, $R_n$ and $L_n$, we know that these three parameters cannot be independent. To insure that this circuit model is equivalent to the sine-Gordon equation, we must correctly fix the relationships between these parameters and insure that the relationships are in terms of measurable quantities. The $\tau_j$ is related to $I_c$ and $R_n$, by Eqn. 2.8. These three variables can be experimentally determined by dc current-voltage measurements. The following derivation is used to determine the correct relationship for $\lambda_j$ and $L_n$.

We assume that there is a single correct lateral inductivity $L_L$, defined as the lateral inductance per unit width. For our junction this is given by
If we assume uniqueness, then if we determine $L_L$ for one solution to the sine-Gordon equation, that value will be correct for all solutions. Thus we will analyze the simplest nontrivial solution, the stationary vortex in an infinite junction, which was presented in §2.1. This is specifically a solution to the time-independent sine-Gordon equation,

$$J(x) = a \frac{2}{\mu_0 J_c} \frac{\partial B_z(x)}{\partial x}. \quad (6.2)$$

where I have included the current density and magnetic field relationships from Ampere's law.

We make the connection between the circuit model and the differential equation (Eqn. 6.2) by considering the lateral current $I_L$ in an ERSJ model in the limit where $\Delta W$ becomes infinitesimally small. Eqn. 2.15 gives the current density distribution for the current crossing the junction in the $y$-direction. Previously we looked at the current density only within the junction, which is all in the $y$-direction and is a function only of $x$. To completely describe the physical vortex we must also consider the current outside the junction, which is a vector with $x$ and $y$ components ($J_x(x, y)$ and $J_y(x, y)$). The current density in the superconductor falls off exponentially like $\lambda_x$. The vortex has a circular current flow about the $z$-axis. We see from Fig. 2.3 that all of the current which crosses the junction on the positive-$x$ half of the vortex returns on the negative-$x$ half. Thus using current continuity around the vortex we can calculate the lateral
current distribution $I(x)$ on each side of the vortex from the current distribution crossing the vortex as follows

$$I(x) = h \int_{y}^{\infty} J_y(x, y) \, dy = h \int_{x}^{\infty} J_y(x', 0) \, dx' ,$$  \hspace{1cm} (6.3)$$

where $I(x)$ is defined as the total current through the negative-y half-plane that is parallel to the y-z axis and crossing the x-axis at $x$. The lateral current on the opposite side of the junction is equal and opposite. If we take the ERSJ model to have an infinite number of infinitesimally small junctions then this lateral current is also the lateral current through the lateral inductor located at $x$.

Inside the junction, Ampere's Law reduces to

$$\frac{\partial B_z}{\partial x} = -\mu_0 J_y(x) .$$ \hspace{1cm} (6.4)$$

Solving this differential equation for the magnetic field we get

$$B_z(x) = \mu_0 \int_{x}^{\infty} J_y(x') \, dx' ,$$ \hspace{1cm} (6.5)$$

noting that $B(\infty) = 0$ and $J(\infty) = 0$. This fluxon contains one flux quantum. The LI product of the lateral inductors in the circuit model is equivalent to the magnetic flux of the fluxon. Therefore,
the sum of the LI products of all of the lateral inductors is equal to one flux quantum $\Phi_0$. This is expressed as

$$\Phi_0 = 4 L_l \int_{0}^{\infty} J_l(x') \, dx' = 2 d_m \int_{0}^{\infty} B_{l}(x') \, dx' \quad (6.6)$$

where we have made use of the fact that both functions, $I_l(x)$ and $B_l(x)$, are symmetric in $x$ and we have included the lateral current on both sides of the junction. Substituting Eqns. 6.3 and 6.5 into 6.6 we get

$$\Phi_0 = 4 L_l h \int_{0}^{\infty} J_j(x,0) \, dx = 2 d_m \mu_0 \int_{0}^{\infty} J_j(x,0) \, dx \quad (6.7)$$

By canceling the integrals in the two terms in Eqn. 6.7 and then substituting Eqns. 2.7 for $\lambda_j$ and $I_c = J_j h W$, we get the lateral inductivity,

$$L_l = \frac{\mu_0 d_m}{2h} \frac{\Phi_0}{4 \pi J_j h \lambda_j^2} = \frac{\Phi_0 W}{4 \pi I_c \lambda_j^2} \quad (6.8)$$

Finally, we have the lateral inductance of the circuit,

$$L_l = L_L W = \frac{\mu_0 d_m W}{2h} \frac{\Phi_0}{4 \pi I_c \left( \frac{W}{\lambda_j} \right)^2} \quad (6.9)$$
Experimentally all of these quantities can be determined from $I_c$ and $\lambda_i^{26}$. The third term in Eqn. 6.9 is identical to the inductance of a parallel-plate transmission line.

A careful analysis of this derivation shows that the result does not depend on the particular solution used, nor on the initial assumptions. In Eqn. 6.7 we integrated over all space for convenience, to get a single fluxon. We could have chosen to evaluate the flux over any interval in $x$ from any solution to the sine-Gordon equation. Therefore, we can be confident that this relationship is correct in general.

Ultimately, I verified this result by comparing the $\lambda_i$ in Eqn. 6.9 with the $\lambda_i$ that I measured graphically from many calculated vortices. It is clear from the analytical vortex solution in Fig. 2.3 that the distance between the maximums in the absolute value of the current density is approximately $2 \lambda_i$. The two values for $\lambda_i$ were always well within the margin of error associated with the graphical measurement.

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26 D. M. Sheen, op. cit.
B. Table of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>Magnetic Field: The magnetic field in the JJ is in the z-direction. Given by Eqn. 2.5.</td>
</tr>
<tr>
<td>$C$</td>
<td>Speed of Light in the JJ. See Eqn. 3.9.</td>
</tr>
<tr>
<td>$C_j$</td>
<td>Junction Capacitance: The capacitance of the JJ. $C_j = \frac{\epsilon h W}{d_e}$.</td>
</tr>
<tr>
<td>$d$</td>
<td>Interlayer Distance: The distance across the JJ in the y-direction. See Fig. 2.1.</td>
</tr>
<tr>
<td>$d_e$</td>
<td>Effective Electrical Thickness of the Junction: This is equal to $d$ for a barrier junction such as in Fig. 2.1. $V(x) = E(x)d_e$.</td>
</tr>
<tr>
<td>$d_m$</td>
<td>Effective Magnetic Thickness of the junction: $d_m = 2\lambda_i + d_e$.</td>
</tr>
<tr>
<td>$E$</td>
<td>Electric Field: The electric field in the JJ is in the y-direction and given by Eqn. 2.4.</td>
</tr>
<tr>
<td>$f_j$</td>
<td>Josephson Frequency: The plasma frequency of a JJ is proportional to $I_c R_n$ and given by Eqn. 2.8.</td>
</tr>
<tr>
<td>$f_l$</td>
<td>Resonant Mode Frequency: The frequency of the $l$-th resonant mode of the resonator. See Eqn. 4.1.</td>
</tr>
<tr>
<td>$f_{l,m}$</td>
<td>The $m$-th harmonic of the $l$-th resonant mode: See Eqn. 4.2.</td>
</tr>
<tr>
<td>$f_m$</td>
<td>Harmonic Frequency: The frequency of the $m$-th harmonic. See Eqn. 2.32.</td>
</tr>
<tr>
<td>$f_{rf}$</td>
<td>rf Frequency: The frequency of the driving or applied current source. $\tau_{rf} f_{rf} = 1$.</td>
</tr>
<tr>
<td>$h$</td>
<td>Film Thickness: The thickness (z-direction) of both the stripline and the JJ. See Fig. 2.1.</td>
</tr>
<tr>
<td>${I}$</td>
<td>The set of circuit elements within a unit cell $n$ of the ERSJ circuit model. The set includes: $n$: resistor, $j$: JJ, $l1$ &amp; $l2$: two inductors.</td>
</tr>
<tr>
<td>$I(n, t)$</td>
<td>The current through each of the two current sources of the $n$-th unit cell of the ERSJ circuit model. See Eqn. 2.21.</td>
</tr>
<tr>
<td>$I(t)$</td>
<td>Total Current: The net current, corresponding to $J$, crossing the JJ. $I = I_s + I_n$. Total current is equal to the rf current in the over-damped case. See Eqn. 2.10.</td>
</tr>
<tr>
<td>$I_{rf}$</td>
<td>rf Current Amplitude: Defined by Eqn. 2.10.</td>
</tr>
</tbody>
</table>
$I_c$ Critical Current of the JJ: $I_c = J_h W$.

$I_{c(n)}$ The critical current of the $n$-th JJ in the ERSJ circuit model. See Eqn. 2.23.

$I_{cf}$ Critical Current of the Stripline: $I_{cf} = J_{pair} h W$ where $J_{pair}$ is the depairing critical current density.

$I_e$ The particular rf current amplitude at which the maximum current density of the stripline is equal to the critical current density of the junction. Defined by Eqn. 3.1.

$I_{ef}$ The particular rf current amplitude at which the stripline current density distribution obtained from a linear assumption would result in a peak current density equal to $J_{pair}$. See Eqn. 4.3.

$I_f$ Fluxon Nucleation Current: The particular rf current amplitude that marks the onset of Josephson vortices, and separates the quasilinear regime from the vortex regime.

$I_{FR}$ Reduced Fluxon Nucleation Current: $I_{FR} = \frac{I_f}{I_c}$.

$I(n, t)$ The current of the $I$-th element of the $n$-th unit cell of the ERSJ circuit model. See {I}.

$I_n$ Normal Current: Corresponds to $J_n$ and the sum of the currents through all of the resistors in the ERSJ circuit model.

$I_R$ Reduced rf Current Amplitude: See Eqn. 5.2

$I_{rms}$ Root Mean Squared Current: The square root of the average of the square of the total current. $I_{rms} = \frac{I_a}{\sqrt{2}}$.

$I_s$ Supercurrent: Corresponds to $J_s$ and the sum of the currents through all of the JJ elements in the ERSJ circuit model.

$J$ Total Current Density: The current density crossing the JJ in the $y$-direction. $J = J_s + J_n$.

$J_{avg}$ Average Current Density: Defined to be the spatial average of $J_f(x)$. $J_{avg}$ is the denominator in Eqn. 2.11.

$J_c$ Critical Current Density of the JJ: The maximum supercurrent density allowed in the JJ. See Eqn. 2.3.

$J_f(x)$ Stripline Current Density: The spatial current density distribution of the stripline and driving function. $J_f(x)$ reflects $\lambda_i$ and the geometry of the stripline. See Fig 2.2.
$J(x, t)$ Total Driving Current Density: $J(x, t)$ is the complete driving function and gives the space and time dependence of the current density distribution of the stripline. See Eqn. 2.12.

$j(x)$ Reduced Driving Current Density: The dimensionless spatial current density distribution of the stripline and driving function, defined by Eqn. 2.11.

$J_n$ Normal Current Density: The normal-electron current crossing the junction in the $y$-direction. Normal current exists whenever there is an electric field present. $E = J_n \rho_n$

$J_{\text{pair}}$ Depairing Critical Current Density: The maximum supercurrent density allowed in the stripline, due to the breaking of Cooper pairs.

$J_s$ Supercurrent Density: The Josephson current or Cooper pair current crossing the junction in the $y$-direction. See Eqn. 2.3.

$L$ The length of the stripline in the $y$-direction.

$L_j$ Junction Inductance: The time varying inductance of an ideal JJ. See Eqn. 3.7.

$L(n)$ The inductance of each of the two inductors in the $n$-th unit cell of the ERSJ circuit model. See Eqn. 2.24.

$N$ The number of ideal JJ circuit elements used in the ERSJ circuit model.

$n$ The index for the unit cells of the ERSJ circuit array or lattice. Each unit cell includes $\{I\}$ and two current sources. When used as a subscript, $n$ always means the normal channel.

$P(t)$ Total Power: The total power in the ERSJ circuit model is given by the sum of the IV products of all circuit elements. See Eqn. 2.29.

$p(t)$ Power Density: Power per unit area is given by Eqn. 2.27.

$p_n$ The power density of the normal channel. See Eqn. 3.3.

$R_m$ The $m$-th in-phase Fourier component given by Eqn. 2.30, which is the in-phase component of the $m$-th harmonic.

$R_n$ Normal Resistance: The normal state resistance associated normal electrons.  
\[ R_n = \rho_n \frac{d_e}{\hbar W} \]

$R_n(n)$ The resistance of the $n$-th resistor in the ERSJ circuit model. See Eqn. 2.22.
\( t \) Time Coordinate.

\( t_R \) Reduced Time Coordinate: See Eqn. 5.2.

\( U_J \) JJ Energy: Stored energy of a JJ due to kinetic inductance. See Eqn. 3.6.

\( v \) Vortex Velocity: See Eqn. 3.9.

\( V(t) \) Effective Voltage: The effective voltage of the entire JJ defined by \( P(t) = I(t)V(t) \).

\( V(x) \) Voltage: The electric potential difference across the JJ at \( x \). \( V(x) = E(x)d_e \).

\( V(n, t) \) The voltage across the \( l \)-th element of the \( n \)-th unit cell of the ERSJ circuit model. The index \( l \) may be dropped to indicate the voltage across the JJ-resistor pair, since this is essentially the unit cell \( n \).

\( W \) Junction Width: The width or long dimension (x-direction) of the JJ and the width of the stripline. See Fig. 2.1.

\( x \) Position Coordinate: The long dimension of the JJ. See Fig. 2.1.

\( X_m \) The \( m \)-th out-of-phase Fourier component given by Eqn. 2.31, which is the out-of-phase component of the \( m \)-th harmonic.

\( x_R \) Reduced Position Coordinate: \( x_R = x/W \).

\( y \) Position Coordinate: The cross-JJ dimension, the long dimension of the Stripline, and the direction of current flow. See Fig. 2.1.

\( z \) Position Coordinate: The dimension of the JJ height, the film thickness, and the YBCO c-axis. See Fig. 2.1.

\( \Delta L(n) \) The distance represented by \( L_i(n) \) which is the distance between the centers of the regions \( \Delta W(n) \) and \( \Delta W(n + 1) \).

\( \Delta W(n) \) Junction Spacing: The spatial region (x-direction) represented by the \( n \)-th unit cell of the ERSJ circuit model. The index \( n \) may be dropped when the junction spacing is uniform.

\( \Delta t \) Time Step: Used in the computation of the ERSJ solutions.

\( \Delta \phi \) The difference between the Josephson phases at positions separated by \( \Delta x \). See Eqn. 3.4.
Josephson Penetration Depth: The length scale of a JJ and of Josephson vortices. See Eqn. 2.7.

London Penetration Depth: The length scale applicable in the stripline. Determines the current density distribution of the stripline.

Reduced Josephson Penetration Depth: \( \lambda_R = \lambda/W \).

Normal Resistivity: The normal state resistivity associated with normal electrons.

Reduced Capacitive Time Constant. See Eqn. 5.5.

Josephson Time Constant: The time scale of a JJ, which is inversely proportional to \( I_c R_n \). See Eqn. 2.8.

Reduced Josephson Time Constant. See Eqn. 5.2.

Capacitive Time Constant: See Eqn. 2.9.

rf Period: The length of time necessary for one complete cycle of the driving or applied current source. \( \tau_{rf} f_{rf} = 1 \).

Josephson Phase: The Gauge Invariant Phase Difference of the superconducting wave function across the JJ. Obeys Eqn. 2.1.

The Josephson Phase of the JJ in the \( n \)-th unit cell of the ERSJ circuit model.

Magnetic Flux Quantum: The quantized unit of magnetic flux. Each vortex contains one fluxon or \( \Phi_0 \) where \( \Phi_0 = \frac{\hbar \pi}{e} = 2.068 \times 10^{-15} \, \text{T m}^2 \).

Total Magnetic Flux: All of the magnetic flux in the junction. The equivalent ERSJ circuit quantity is the sum of the LI products of all the lateral inductors.

Angular rf Frequency: The angular frequency of the driving or applied current source. \( \tau_{rf} \omega_{rf} = 2\pi \).