Optimization of a Mixer-Ejector System for Supersonic Civilian Transport Aircraft

by

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Abstract

This thesis describes the development and implementation of a design methodology for optimizing a mixer-ejector nozzle for noise suppression on supersonic civilian transport aircraft. Parametric trade studies and multi-variable optimization studies were performed to define the impact of a mixer-ejector system on the vehicle take-off gross weight.

A fluid dynamic model was developed for these studies which is based on a quasi-ID compound compressible formulation, and includes a discrete vortex code to capture details of the mixing process within the nozzle. The model was assessed using experimental data for a range of designs. An integrated tool was constructed to predict mixer-ejector performance which consists of the fluid dynamic model, an acoustic code, and a nozzle weight code, the latter two codes were acquired from industry. The tool allows assessment of the impact of design variables on three figures of merit for supersonic mixer-ejector noise suppressors: net thrust coefficient, EPNLdB, and nozzle weight. Using system weightings, the three figures of merit are combined into a single cost function - effect on take-off gross weight.

Physical trade-offs were identified by conducting parametric trade studies, where each design variable was varied independently about a baseline design. The integrated tool was then embedded in a multi-variable optimization routine and used to conduct optimization studies. The multi-variable optimal design differed from the trade study results significantly indicating that parametric testing alone will not yield an optimal design. The multi-variable optimization process was repeated while varying various subcomponents of the fluid dynamic model in order to determine the sensitivities of the optimal design and performance to modeling uncertainties.

Design guidelines were developed by repeating the multi-variable optimization process for four different weighting systems: a baseline system and three others, each emphasizing a different figure of merit. The different weighting systems result in different optimal designs. As the relative weighting for either noise level or nozzle weight is increased, net thrust is sacrificed to achieve a reduction in the emphasized figure of merit. As weighting for noise relative to thrust increases, the trend is toward mixers with longer perimeters (higher penetration and larger number of lobes) and more streamwise circulation. There is also an increase in duct length to allow for more acoustic liner. The same trend in the mixer design (towards longer mixer perimeters and higher circulation) is seen as the relative weighting for the nozzle weight is increased.

Thesis Supervisor: Professor Ian A. Waitz
Title: Associate Professor of Aeronautics and Astronautics
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Nomenclature

Roman

\( A \)  	Cross-sectional area
\( \text{AR} \)  	Aspect ratio, \((W/H)_m\)
\( \text{CER} \)  	Chute expansion ratio, \(A_p/A_j\)
\( \text{EPNLdB} \)  	Estimated Perceived Noise Level in decibels
\( \text{HSCT} \)  	High Speed Civil Transport

\( h \)  	Lobe peak to peak height
\( M \)  	Mach number
\( \text{MAR} \)  	Mixing area ratio, \(A_0/A_m\)
\( P \)  	Pressure
\( NPR \)  	Nozzle pressure ratio
\( Pr \)  	Prandtl number

\( r \)  	Secondary to primary velocity ratio
\( s \)  	Secondary to primary density ratio
\( \text{SAR} \)  	Suppressor area ratio, \(A_m/A_j\)
\( Sc \)  	Schmidt number
\( \text{SNPR} \)  	Secondary nozzle pressure ratio
\( T \)  	Temperature
\( \bar{U} \)  	Mean axial velocity
\( u \)  	Axial velocity
\( w \)  	Lobe width

Greek

\( \alpha \)  	Lobe half angle
\( \delta \)  	Shear layer thickness
\( \gamma \)  	Ratio of specific heats
\( \Gamma \)  Streamwise-circulation shed from half-wavelength span
\( \lambda \)  Lobe wavelength
\( \tau \)  Primary to secondary total temperature ratio
\( \xi \)  Similarity variable

**Subscripts**

9  Duct exit quantity
amb  Ambient quantity
c  Convective
f  Flight quantity
j  Primary throat quantity
m  Mixer exit quantity
p  Primary stream quantity
s  Secondary stream quantity
t  Total quantity

**Superscripts**

*  Non-dimensional quantity

**Full quantities**

\( M_c \)  Convective Mach number
\( x^* \)  Distance from the trailing edge in wavelengths
\( Re_x \)  Reynold’s number based on \( x \)
Chapter 1

Introduction

1.1 Motivation

Market studies show that a supersonic civilian transport could capture more than half of the growing, long-range market[2]. Total passenger demand is forecast to be 315,000 passengers per day by the year 2000, and 600,000 per day by the year 2015 [1]. Boeing studies show that an economically competitive high-speed civil transport (HSCT) could capture as much as 75 percent of the long-haul international market [2]. This market share could support a fleet of 900 to 1200 aircraft valued at $250 billion if the aircraft are economically competitive with current subsonic designs. Increases in ticket prices from 20% over subsonic to 80% more than subsonic are estimated to reduce market share to only 25% as shown in Table 1.1 [1].

Table 1.1: Dependence of Market Share of HSCT on Ticket Prices [1]

<table>
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<tr>
<th>Market Share (%)</th>
<th>HSCT/Subsonic Cost</th>
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<td>25</td>
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<td>50</td>
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A major technological hurdle in developing a viable supersonic civilian transport is meeting noise regulations without sacrificing the performance required to be commercially competitive. Take-off noise levels are specified by the Federal Aviation Administration in the FAR 36 Stage III community noise regulations [1, 8]. Current engine designs will require over 20 EPNLdB of noise suppression in order to meet these regulations [25, 27] and this is not possible with current noise reduction technology. Using current noise reduction technology, it is estimated that the noise levels will exceed the FAR 36 Stage III requirements by at least 6 EPNLdB [2].

To meet the FAR 36 Stage III requirements, NASA’s High Speed Research (HSR) Program is
investigating noise suppression concepts for use on proposed supersonic civilian transport aircraft [11, 18, 32, 33, 35]. One concept being examined is an acoustically treated mixer-ejector system [1, 8, 25]. The mixer-ejector is designed to entrain ambient air and mix it with primary jet exhaust thereby lowering the average exit velocity as shown in Figure 1-1, and reducing in the intensity of the turbulent mixing which is responsible for the jet noise [32].

![Conceptual Mixer-Ejector Noise Suppression System](image)

**Figure 1-1:** Conceptual Mixer-Ejector Noise Suppression System [28].

However, the mixer-ejector also leads to an increase in engine weight and the potential for total pressure losses that impact the engine thrust. The goal is to design a mixer-ejector system that reduces noise levels while minimizing the weight and thrust penalties [2]. This thesis is motivated by the current lack of understanding of mixer-ejectors on the system level. The thesis discusses a systematic approach to design optimization based on a computationally-efficient, physically-based model of the system and the use of multi-variable optimization techniques.

### 1.2 Background

In this section, previous research on mixer-ejectors is summarized. The review is not exhaustive, rather it is presented to motivate the objectives of the current work and to provide insight into some of the underlying concepts on which the various models used in the study are based.

#### 1.2.1 Mixer-Ejectors

The mixer-ejector allows the primary exhaust to mix with entrained air thereby decreasing the exit jet velocity and hence the radiated noise. The ejector also provides surface area for acoustic treatment further reducing the jet noise.

In many mixer-ejector systems, lobed mixers as shown in Figure 1-2 are used to increase the mixing rate between primary exhaust flow and entrained air. The mixing rate is increased by two mechanisms: the increase in the initial interface length due to the convoluted trailing edge and the further stretching of this interface downstream of the trailing edge due to the presence of streamwise
vorticity. These two effects can be seen clearly in Navier-Stokes calculations showing the evolution of a scalar field performed by Qiu [23] which are shown in Figures 1-3 and 1-4. The different contours correspond to different axial locations. Initially (t=0), there is an increase in the initial interface length due to the convoluted trailing edge. As the interface convects downstream (t > 0), the interface is stretched and rolled up, increasing the interface length.

Manning was able to separate the mixing enhancement associated with these two mechanisms in a series of water tunnel tests [19]. For these tests, he used three types of mixers: a flat plate, a convoluted plate (a lobed mixer with little or no streamwise vorticity and hence little augmentation of interface length), and a forced mixer. Figure 1-5 shows molecular mixedness as a function of the axial position for the three types of mixers. By comparing the molecular mixing rates of the flat plate and the convoluted plate, the mixing rate increase due to the increased interface length can be defined. Comparing the molecular mixing rates of the convoluted plate and the forced mixer, shows the additional mixing rate increase with the addition of streamwise vorticity.

However, there are also losses associated with these two mechanisms. The wetted surface of
the mixer increases when using a convoluted trailing edge resulting in greater skin friction losses and there turning losses are associated with introducing streamwise vorticity. Both the mixing enhancement and the losses are tied to the geometry of the mixer, and it requires a system-level understanding of the mixer-ejector to optimize these trade-offs.

In addition to noise-suppressors, the design methodology described in this thesis could be applied to other lobed mixer applications where mixing enhancement is critical. These applications include: vortex generators for use in boundary layer control, core-bypass mixing for use in gas turbine nozzles, infrared signature reduction, low-emissions combustor technology, and scramjet fuel injectors [34]. Currently, the tools described in this thesis are being adapted to investigate some of these cases.

1.2.2 Design of Supersonic Mixer-Ejectors

Until recently, the principal method of mixer-ejector design has been through parametric testing looking for performance trends. The tests have ranged from examining the flow through a single lobe [4] to complete mixer-ejector systems [12]. The main emphasis has been on performance measures such as the thrust coefficient and the ratio of the entrained mass flow to the primary flow, which is commonly referred to as pumping ratio; higher pumping ratios lead to lower mass-averaged exit velocities.

The methods of analysis used in industry have embodied two extremes. Control volume approaches have been used for initial sizing and design while 3D Navier-Stokes solutions have been used to examine fine details such as flow separation. To bridge the gap between these two extremes, Clark developed a model [7] based on compound flow assumptions; although due to the proprietary nature of his work, it has not been used outside of the Boeing Company.

At MIT, an attempt has been made to develop basic understanding of the phenomena that are critical to mixer-ejector fluid flow. This physical understanding has been exploited to develop models that are faster than three-dimensional CFD yet capture the essential flow features. For example, a computational model for assessing the mixing augmentation due to shed streamwise vorticity was developed by Qiu [23] based on slender-body assumptions which allows the formulation of a 3-D steady problem as a 2-D unsteady problem (see Figure 1-6) allowing rapid solution techniques.

Another model, developed by Tew is a vortex code, again based on the slender-body approximations, to track the shed streamwise vortices [29]. The method can be used to determine the cross-planar interface length and hence estimate mixing rates. This code, which formed the basis for the mixing model used in the current studies, will be discussed in Section 3.4.4.

Indeed, using a simplified version of the fluid dynamic model described in the following chapters, Tew performed trade studies to investigate the impact of streamwise vorticity and lobe height-to-wavelength ratio (h*) on exit mixedness, thrust, and pumping. The effects of skin friction across the mixer and in the duct were not included in Tew's model. However, these studies demonstrate the
important role that a computationally-efficient model plays in enabling system-level trade studies. They also illustrate some of the important physical trade-offs discussed in Chapter 6. Figure 1-7 shows the exit mixedness increasing with both the lobe half-angle and the lobe height-to-wavelength ratio. However, Figure 1-8 shows the mixing augmentation at the duct exit and demonstrates that the effectiveness of the streamwise vorticity decreases as the lobe height-to-wavelength ratio increases [31]. As $h^*$ increases, the amount of stretching of the cross-planar interface due to the streamwise vorticity decreases as seen in Figure 1-9. This effect is caused by the closer proximity, and thus more rapid cancellation by diffusion, of vorticity of the opposite sign from neighboring lobes.

Figures 1-10 and 1-11 show the impact of lobe angle on pumping and thrust for four different lobe-to-wavelength ratios. The streamwise vorticity associated with the 15 degree mixer and a lobe height-to-wavelength ratio of 1.25 leads to a pumping augmentation, but a net thrust penalty. For the higher lobe height-to-wavelength ratio mixers, both the net thrust and pumping decrease with the addition of streamwise vorticity.

### 1.3 Objectives

The objective of this research was to develop the tools and understanding necessary to perform optimization studies for mixer-ejectors and to use these tool to:

- Define critical phenomena in mixer-ejector fluid dynamics.
- Develop a time-efficient model of the mixer-ejector fluid dynamics.
- Develop an integrated tool for predicting mixer-ejector performance.
- Determine dominant physical trade-offs represented by each design variable.
- Determine the design variables that are the primary drivers for mixer-ejector performance.
- Develop design guidelines based on optimization studies.
- Assess systems-level understanding using experimental data.

Although the focus of the optimization studies was on mixer-ejectors for a supersonic civilian transport, the methods and tools are general and can be used for a broad range of operating conditions and applications.

### 1.4 Approach

To carry out optimization studies, it was necessary to develop a system model and this involved a tradeoff between the fidelity of the physical models included and the processing time. Two extremes
can be used to illustrate this tradeoff. Full 3-D Navier-Stokes (3DFNS) calculations reflect the current state-of-the-art in fluid dynamics calculations, but can take weeks to compute, making this level of modeling impractical for optimization purposes. At the other extreme, a fully-mixed-out control volume can be solved in a fraction of a second, but leaves out many effects, such as incomplete mixing and skin friction which are important for most practical devices.

Defining the necessary level of modeling complexity is an important part of the intellectual framework of this thesis. It was carried out by comparing fluid dynamic model predictions to both 3DFNS and experiments as well as refining of the physical sub-models to increase the accuracy of the overall fluid dynamic model. Two aspects of the comparisons are important for optimization purposes: accuracy and the ability to predict trends. For a linearized performance model, the ability to predict trends is sufficient to calculate the optimal design.

Once confidence in the fluid dynamic model was developed, the next step was to link the fluid dynamics to the figures of merit, which for this study are: jet noise, net thrust coefficient, and nozzle weight. Each figure of merit was determined as a function of the geometry and fluid flow properties. The figures of merit were then combined into a single cost function, the change in the take-off gross weight (TOGW) of the vehicle. The impact of each figure of merit on the TOGW is based on system studies performed by industry [14]. TOGW is used as an indicator of the vehicle cost and the direct operating costs. The resulting integrated tool was used to predict overall mixer-ejector performance and carry out optimizations studies.

To identify tradeoffs at the system-level, design trades were performed where each design parameter was varied independently from a baseline design. A multi-variable optimization algorithm was used to calculate the optimal design which was used as a starting point for sensitivity studies. Sensitivities of performance to changes in the design were used to identify the primary drivers for mixer-ejector performance; sensitivities of the design and performance to modeling uncertainties and constraints were examined to provide information for future design efforts, and sensitivities to TOGW weightings were used to develop design guidelines. TOGW weightings differ depending on the measuring point used for noise certification and are expected to change as the design matures and as different vehicles are considered.

### 1.5 Contributions

The main contributions of this thesis are

1. Definition of fluid dynamic phenomena which are important to mixer-ejector performance.

2. Developed models for compound shocks and effects of total temperature on interstream shear stress.
3. Development of a time-efficient model of mixer-ejector fluid dynamics based on physical understanding.

4. Definition maximum entrainment condition based on pressure matching control volume.

5. Developed code for use by High Speed Research (HSR) team and supported HSR design efforts. (Note: work was conducted under limited exclusive rights (LER) agreement and is not discussed in the thesis.)


7. Determination of important system-level aspects of mixer-ejectors.
   (a) Determination of design variables to which the performance is most sensitive.
   (b) Identification of the tradeoffs for each design variable
   (c) Determination of sensitivities of optimal design and performance to modeling uncertainties, constraints, and TOGW weightings.
   (d) Determination of bounds on optimal design and performance.
   (e) Identification of possible constraints to be examined for vehicle trade studies.
   (f) Development of design guidelines for optimal design based on TOGW weightings.

1.6 Overview of thesis

The fluid dynamic model for the mixer-ejector is described in Chapter 2 with the details of the various components in Chapter 3. In Chapter 4, the accuracy and the ability of the model to capture trends was determined by comparison to experimental data. The development of an integrated tool to determine mixer-ejector performance based on the fluid dynamic model is discussed in Chapter 5. In Chapter 6, trade studies were conducted to determine the physical trade-offs represented by each design variable and the design was optimized for a baseline weighting system. The sensitivities to modeling uncertainties, constraints, and weightings were calculated and are given in Chapter 7. Finally, the conclusions and recommendations for future use are given in Chapter 8.

It is important to note that this work was carried out as part of NASA’s High Speed Research Program (HSR) design team. Information was acquired from industry under a Limited Exclusive Rights (LER) agreement, therefore discussions of some aspects of the research, such as experimental data and industry models, are limited by this agreement. Aspects of the optimization process were changed to protect LER designs.
Figure 1-3: Contours of Scalar Value at Different Time (t=0, t=0.4) [23].
Figure 1-4: Contours of Scalar Value at Different Time (t=1.0, t=2.9) [23].
Figure 1-5: Water Tunnel Measurements of Molecular Mixedness Downstream of a Flat Plate, Convoluted Plate, and Forced Mixer for a Velocity Ratio of 0.67.

Figure 1-6: Cartoon of Slender Body Concept.
Figure 1-7: Scalar Mixedness at Duct Exit (30λ) as a Function of $h^*$ and $\alpha$ ($\delta_{vel} = 0.10, \epsilon^* = 0.25$) [31].

Figure 1-8: Scalar Mixing Augmentation at Duct Exit (30λ) as a Function of $h^*$ and $\alpha$ ($\delta_{vel} = 0.10, \epsilon^* = 0.25$) [31].
Figure 1-9: Scalar Fields $2\lambda$ Downstream of 15° Forced Mixer at Four Different Lobe Height-to-Wavelength Ratios ($\delta_{ee}' = 0.10, \epsilon' = 0.25$) [31].
Figure 1-10: Pumping Ratio Variation with Half-Lobe Angle [31].

Figure 1-11: Gross Thrust Variation with Half-Lobe Angle [31].
Chapter 2

Fluid Dynamic Model

2.1 Overview

The goal in designing the fluid dynamic model was to capture the essential features of the mixer-ejector flow in a time-efficient manner. The tradeoff was between increasing fidelity and decreasing time to perform the calculations. Decisions on whether a feature was essential in the description of the flow situations of interest were based on comparisons to experimental data.

At the heart of the fluid dynamic model is a differential control volume based on the one-dimensional compound compressible flow theory developed by Bernstein et al. [6]. The compound flow analysis (CFA) represents the flow as two separate streams (primary and secondary) that exchange momentum and energy but not mass. The pressure is assumed constant across the duct and thus is assumed to vary in only the axial direction.

The components of the model are depicted in Figure 2-1. Operating conditions and system geometry are inputs. Losses in both the primary and secondary streams are computed up to the trailing edge of the mixer using semi-empirical scaling laws. The pressures in the two streams are assumed to equalize rapidly via a non-isentropic process modeled using a control volume analysis. A differential control volume based on the compound flow assumptions (CFA) [6] is then used to calculate the flow variables for both streams as a function of axial position through the duct. The cross-stream interfacial contact length between the two streams is calculated using a vortex code developed by Tew [30]. Two effects cause changes in the interface; the presence of streamwise circulation stretches the interface, while the merging of shear layers cause the interface to transition to that of a planar shear layer. The vortex code accounts for these effects by calculating an effective axial interface length distribution based on the rate of change of the scalar mixedness (see Equation G.12). Exchange of momentum and energy between the two streams is set using planar shear layer correlations and the effective interface length distribution calculated using the discrete vortex code.
(see Appendix G). The initial conditions are then iterated upon to match either the ambient pressure at the duct exit or choked flow conditions.

![Diagram of CFA Model](image)

**Figure 2-1: Components of CFA Model.**

## 2.2 Modes of Operation

The modes of operation of an ejector are analogous to the subsonic and supersonic modes of a converging-diverging nozzle with a single stream. Indeed a compound indicator ($\beta$) can be formulated for compound flows which is analogous to the Mach number in single stream flows [6]. Appendix A contains a more detailed explanation of the compound indicator and the analogy with the Mach number for single stream flows.

The modes of operation can be divided into two main categories: subsonic and supersonic primary nozzle flow. Each main category can be further subdivided into either compound subsonic or compound choked. Each case is discussed below with the exception of the case where the flow is compound supersonic at the mixing plane since this is of little practical interest for jet noise suppressors. For a supersonic civilian transport, the mixer-ejector would be utilized only at take-off and landing where the flight Mach number will be subsonic.

### 2.2.1 Subsonic Primary

When the primary nozzle is subsonic, the primary mass flow is determined by the static pressure of the two streams at the mixing plane. The code matches the static pressure at the duct exit by iterating on the primary Mach number; the corresponding secondary Mach number is selected to match the static pressures at the mixing plane.

According to compound flow theory, the maximum mass flow is limited by a compound choke condition describe in Appendix G. If this occurs, the primary Mach number is iterated upon to place the compound choke point at the compound aerodynamic throat. With the initial conditions fixed, the code attempts to match the ambient pressure at the duct exit by fitting a compound shock.
downstream of the aerodynamic throat. If it is not possible to match the ambient pressure at the duct exit via shock fitting, it is assumed that an oblique shock or expansion fan exists at the duct exit and the pressure is not matched.

2.2.2 Supersonic Primary

When the primary stream is supersonic, the primary mass flow is independent of the ambient pressure and only the secondary mass flow can be varied. For this branch of solutions, there are two possibilities.

1. The flow remains compound subsonic throughout the duct. For this case, the secondary Mach number is iterated upon to match the static pressures at the duct exit as shown in Figure 2-2.

\[
\text{Iterate upon } M_s \text{ to match ambient pressure}
\]

\[
\text{Secondary Loss Model} \rightarrow \text{Pressure Matching Control Volume} \rightarrow \text{CFA}
\]

\[
\text{Primary Loss Model}
\]

**Figure 2-2:** Flow Chart for Compound Subsonic Case.

2. The flow compound chokes at an aerodynamic throat. For typical supersonic mixer-ejector designs, the duct is either strictly diverging or converging. Therefore, the flow tends to compound choke at either the mixing plane or the duct exit. For the cases where the flow compound chokes at the mixing plane, the secondary Mach number is iterated upon to set the compound indicator equal to zero (compound sonic) for the flow exiting the pressure matching control volume. With the secondary Mach number fixed, the location of compound shock is iterated upon to match the static pressures at the duct exit as shown in Figure 2-3. For the cases where the flow compound chokes at the duct exit, the secondary Mach number is iterated upon to set the compound indicator equal to zero at the duct exit as depicted in Figure 2-4.
2.3 Loss Models

Losses in the primary nozzle and in the secondary inlet are calculated using correlations obtained from Pratt and Whitney under a limited exclusive rights (LER) agreement.

2.3.1 Secondary Inlet

Losses in the secondary stream from the freestream up to the mixing plane are calculated using an inlet recovery factor ($\eta_s$).

$$\eta_s = \frac{P_{s,mp}}{P_{s,fs}}$$  (2.1)
The inlet recovery factor has been correlated against the square of secondary Mach number for HSR experiments by Gamble [12] which is consistent with scaling the total pressure losses with the dynamic head.

\[ \eta_s = 1 - kM^2 \]  

(2.2)

The dominant loss source is the skin friction between the flow plenum and the mixer exit. This correlation works well for an experimental apparatus where the skin friction is the dominant loss source (such as the experiments from which the correlation was calculated).

For inlets, there are several mechanisms responsible for total pressure losses: boundary layer losses due to ingesting the external boundary layer, turning losses when the flow changes direction to enter the inlet, lip losses due to separations along the inside of the cowl lip, and skin friction losses across the mixer. These losses were estimated by Leland for an inlet for a supersonic civilian aircraft [17]. The boundary layer loss, turning loss, and lip loss are functions of the flight Mach number and geometric parameters and are independent of the secondary Mach number at the mixing plane [17]. The skin friction losses scale with the dynamic head and are estimated as proportional to the secondary Mach number squared. Therefore, the recovery factor can be expressed in the following form.

\[ \eta_s = 1 - f(M_f) - cM^2 \]  

(2.3)

If skin friction losses are dominant, one recovers the functional form used by Gamble.

For optimization purposes, the flight Mach number is considered fixed. For this case Equation 2.3 can be expressed as:

\[ \eta_s = 1 - c_1 - c_2 M^2 \]  

(2.4)

### 2.3.2 Primary Nozzle

The stream thrust coefficient was correlated as a function of the diameter ratio \( D_{he} = \text{Hydraulic Diameter/Equivalent Diameter} \) by Gamble [12] for a series of HSR tests.

\[ D_{he} = \frac{\sqrt{174A_p}}{\text{Perimeter}} \]  

(2.5)

The correlation is not given here in accordance with the LER agreement. The stream thrust coefficient which reflects the fraction of momentum across the primary nozzle and was used to calculate the losses due to skin friction. For uniform flow conditions, the stream thrust coefficient is given by
the following equation.

\[ C_s = \frac{\dot{m}u + PA}{(\dot{m}u + PA)_{ideal}} \]  \hspace{1cm} (2.6)

The ideal quantities are calculated using the isentropic (i.e. no losses) nozzle relationships.

<table>
<thead>
<tr>
<th>Specified</th>
<th>Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal flow variables</td>
<td>Specified Assumptions</td>
</tr>
<tr>
<td>P, ( u ), T</td>
<td>P, ( u ), T</td>
</tr>
<tr>
<td>( P_{id}, u_{id}, T_{id} )</td>
<td>( P_{id}, u_{id}, T_{id} )</td>
</tr>
</tbody>
</table>

**Figure 2-5:** Control Volume for Primary Losses.

Combining the definition of stream thrust with the equations of continuity, conservation of energy, and the state equation for an ideal gas; the control volume shown in Figure 2-5 can be reduced to a quadratic equation.

\[ \left[ \gamma^2 - k^2 \frac{\gamma - 1}{2} \right] M^4 + [2\gamma - k^2]M^2 + 1 = 0 \]  \hspace{1cm} (2.7)

where;

\[ \frac{1 + \gamma M^2}{M\sqrt{\theta}} = C_s \frac{1 + \gamma M_{id}^2}{M_{id}\sqrt{\theta_{id}}} = k \]  \hspace{1cm} (2.8)

After solving the quadratic equation for the Mach number, the total pressure recovery ratio can be calculated.

\[ \frac{P_t}{P_{tid}} = \frac{1}{C_s} \frac{1 + \gamma M_{id}^2}{1 + \gamma M^2} \left( \frac{\theta}{\theta_{id}} \right)^{\frac{\gamma - 1}{2}} \]  \hspace{1cm} (2.9)

The total pressure, total temperature (which is conserved), and Mach number are sufficient to determine the thermodynamic state and velocity of the primary stream at the mixer exit.

2.3.3 Losses Due to Off-Axis Vectoring of Flow

If the exit streams are vectored off-axis, total pressure losses result. These losses are accounted for in the pressure matching control volume discussed in the following section. The flow is broken down into two components going into the pressure matching control volume: axial and off-axis flow. The flow is assumed to be axial leaving the pressure matching control volume as depicted in Figure 2-6 (i.e. the off-axis kinetic energy is discarded at the mixer exit).
2.4 Pressure Matching Control Volume

If the flow is supersonic, the static pressures in the primary and secondary streams are not necessarily matched. The stream areas adjust over a finite length ($\Delta x$) to equalize the static pressures. Across this length, changes due to both pressure equalization and mixing occur. This pressure matching region is modeled assuming the pressure equalization occurs over an infinitely thin control volume of constant area as shown in Figure 2-7. Appendix B contains a more detailed description of the pressure matching control volume. Changes due to mixing during the equalization process are accounted for in the differential control volume, which uses the outputs of the pressure matching control volume as initial conditions. The two processes are shown schematically in Figure 2-8. The validity of this assumption was assessed by comparison with experimental data (see Chapter 4).

2.5 Differential Control Volume

The core of the compound flow analysis is a two stream differential control volume shown in Figure 2-9. Momentum and energy are exchanged between the two streams, but mass is not. The overall system is assumed adiabatic. The sum of individual cross-sectional areas equals the cross sectional
The pressure is taken as constant across the duct and thus varies only in the axial direction.

\[ P_p = P_s = P(x) \quad (2.11) \]

The differential control volume reduces to an initial value problem with seven differential equations (see Appendix A) in seven primary variables \((A_p, A_s, P, M_p, M_s, T_p, T_s)\). The differential equations are solved using a fourth order Runge-Kutta method with a variable stepsize. All other flow properties can be derived from the primary variables.
2.6 Compound Shocks

A control volume formulation to deal with compound flow shock waves is illustrated in Figure 2-10. In addition to the compound flow assumption that the pressure between the two streams is matched upstream and downstream of the control volume, energy is conserved in each stream, total momentum is conserved, and the subsonic stream is assumed isentropic. This results in a special case of the pressure matching control volume where the pressures are matched and the flow is compound supersonic going into the control volume and compound subsonic exiting the control volume. Appendix C contains a more detailed description of the compound shock model and its implementation. Assessment of the model is discussed in Chapter 4.

![Figure 2-10: Control Volume for Compound Shocks.](image-url)

<table>
<thead>
<tr>
<th>Specified Inflow conditions</th>
<th>Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>As, us, Ts, P, Ap, up, Tp</td>
<td>Mass conserved in each stream</td>
</tr>
<tr>
<td></td>
<td>Total momentum conserved</td>
</tr>
<tr>
<td></td>
<td>Constant area</td>
</tr>
<tr>
<td></td>
<td>Isentropic secondary stream</td>
</tr>
<tr>
<td></td>
<td>Ideal gases</td>
</tr>
<tr>
<td></td>
<td>Compound subsonic exiting C. V.</td>
</tr>
</tbody>
</table>
Chapter 3

Description of Components of the Fluid Dynamic Model

In Chapter 2, the general structure of the fluid dynamic model was described. The discussion focused on various components of this model which provide estimates for losses for the nozzle and inlet, the pressure matching process, the compound compressible flow, and the compound shocks. In this chapter, other components are described in detail: wall shear stress, interstream shear stress, interstream thermal conduction, and the effective interface length. Chapter 4 then discusses several sets of experimental data used to assess and refine the fluid dynamic model.

3.1 Wall Shear Stress

The wall shear stress is calculated using the local values for the skin-friction coefficient \( C_f \) and the dynamic head \( \frac{e u^2}{2} \).

\[
\tau_{\text{wall}} = C_f \frac{1}{2} \rho u^2
\]  

(3.1)

The local dynamic head is calculated directly using the local flow variables. The local skin friction coefficient is calculated using correlations detailed in White [36]. The wall shear stress is modeled for both laminar and turbulent flows and transition is taken to occur at a Reynolds number \( Re_x \) of 500,000 [36].

\[
Re_x = \frac{u x}{\nu}
\]  

(3.2)

\[
\nu = \frac{\mu}{\rho}
\]  

(3.3)
The correlations used for each flow regime are given in the following two sections. In these, the dependence of viscosity on static temperature is approximated using the Sutherland viscosity formula.

\[
\frac{\mu}{\mu_0} \approx \left( \frac{T}{T_0} \right)^{\frac{3}{2}} \frac{T_0 + S}{T + S}
\]  

(3.4)

For air, \( T_0 = 273 \text{ K}, S = 111 \text{ K}, \mu_0 = 1.716 \cdot 10^{-5} \text{ kg m}^{-1} \text{ s}^{-1} \). This approximation is accurate to within 2 percent for temperatures 170-1900 K [36].

### 3.1.1 Laminar Flow

For laminar flow, the skin friction coefficient is approximated using an equation for laminar flat plate flow modified for reference temperature \((T^*)\).

\[
C_f \approx \frac{0.664\sqrt{C^*}}{\sqrt{Re_x}}
\]

(3.5)

The Chapman-Rubesin parameter \((C^*)\) has the following general form.

\[
C^* \approx \left( \frac{T^*}{T} \right)^{N-1}
\]

(3.6)

For air, \( N \) has a value of two-thirds [36].

\[
C^* \approx \left( \frac{T^*}{T} \right)^{-1/3}
\]

(3.7)

The correlation for the reference temperature is given by the following equation [36].

\[
\frac{T^*}{T} \approx 0.5 + 0.039M^2 + 0.5\frac{T_w}{T}
\]

(3.8)

Wall temperature \((T_w)\) is calculated using the adiabatic wall temperature \((T_{aw})\),

\[
T_{aw} = T \left[ 1 + r \frac{\gamma - 1}{2} M^2 \right]
\]

(3.9)

where the recovery factor \((r)\) for laminar flow is well-approximated by the square root of the Prandtl number for Prandtl numbers between 0.1 and 3.0 [36].

\[
r_{laminar} \approx \sqrt{Pr}
\]

(3.10)

The Prandtl number for air is approximately 0.72 [36].
3.1.2 Turbulent

For turbulent, compressible flows, the skin friction coefficient is calculated using the method by White and Christoph [36].

\[ C_f \approx \frac{0.455}{S^2 \ln^2 \left( \frac{0.06}{S} Re_x \frac{\mu}{\mu_w} \sqrt{\frac{T}{T_w}} \right)} \quad (3.11) \]

\[ S = \sqrt{\frac{T_{aw} - 1}{(\sin^{-1} A + \sin^{-1} B)^2}} \quad (3.12) \]

The coefficients A and B are given by the following equations.

\[ A = \frac{2a^2 - b}{\sqrt{b^2 + 4a^2}} \quad (3.13) \]

\[ B = \frac{b}{\sqrt{b^2 + 4a^2}} \quad (3.14) \]

\[ a = \sqrt{\frac{\gamma - 1}{2} M^2 \frac{T}{T_{aw}}} \quad (3.15) \]

\[ b = \frac{T_{aw}}{T_w} - 1 \quad (3.16) \]

For an adiabatic wall, these coefficients reduce to:

\[ A = a \quad (3.17) \]

\[ B = 0 \quad (3.18) \]

The adiabatic wall temperature is calculated using Equation 3.9, with the recovery factor for turbulent flow approximated by the cube root of the Prandtl number [36].

\[ r_{turbulent} \approx P^{1/3} \quad (3.19) \]

The adiabatic wall temperature is also used to calculate the ratio of the viscosity in the free stream.
(\(\mu\)) to the viscosity near the wall (\(\mu_w\)) using Equation 3.4.

\[
\frac{\mu}{\mu_w} \approx \left( \frac{T}{T_{aw}} \right)^{\frac{3}{2}} \frac{T_{aw} + S}{T + S}
\]  

(3.20)

### 3.2 Interstream Shear Stress

If a velocity difference exists between the two steams of fluid exiting the lobed mixer, there will be interstream shear stresses such that the faster fluid accelerates the slower fluid.

#### 3.2.1 Derivation of Basic Model

The basic model for the interstream shear stress was suggested by Clark [7] based on an analysis by Goertler [24]. The derivation begins with the constant-pressure steady-state 2-D incompressible boundary layer equations.

\[
\frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial \tau}{\partial y}
\]  

(3.21)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

(3.22)

An alternative derivation by Qiu [23] begins instead with a diffusion equation for a frame convecting at the mean velocity (\(\bar{U}\)).

\[
\frac{\partial u}{\partial t^*} = \frac{\partial^2 u}{\partial y^2}
\]  

(3.23)

The convective time (\(t^*\)) is defined as follows.

\[
t^* = \frac{x}{\bar{U}}
\]  

(3.24)

Both derivations proceed similarly and have the same result, since the slender body approach taken by Qiu is based on the equivalence of a 3-D steady flow field and a 2-D unsteady flow field. The slender body approach is also used in the derivation of the vortex code and the interstream thermal conduction models.

The turbulent viscosity (\(\epsilon\)) is taken to scale with the velocity difference [24].

\[
\epsilon = \chi b(u_{max} - u_{min}) = \chi b(u_p - u_s)
\]  

(3.25)
For a fully turbulent shear layer, the width \( b \) grows linearly [21].

\[
b = cx \tag{3.26}
\]

From the solution of Equations 3.21, 3.22, 3.25, and 3.26, the velocity profile can be expressed in similarity form \( \phi(\xi) \) in terms of the free stream values and a spread rate parameter \( (\sigma) \).

\[
\phi(\xi) \equiv \frac{u - u_s}{u_p - u_s} = \frac{1}{2}[1 + \text{erf}(\xi - \xi_0)]
\tag{3.27}
\]

The similarity variable \( (\xi) \) has the following form,

\[
\xi = \sigma \frac{y}{x}
\tag{3.28}
\]

where the spread rate parameter \( (\sigma) \) is defined as:

\[
\sigma = \frac{1}{2}(\chi c \lambda)^{-\frac{1}{2}}
\tag{3.29}
\]

\[
\lambda = \frac{U_1 - U_2}{U_1 + U_2}
\tag{3.30}
\]

The turbulent shear stress \( (\tau) \) is related to the velocity gradient and the turbulent viscosity \( (\epsilon) \).

\[
\tau = \rho c \frac{\partial u}{\partial y}
\tag{3.31}
\]

The shear stress is evaluated at the symmetry point \( \xi = \xi_0 \), where the density is assumed to be the mean of the two free stream values \( (\bar{\rho}) \).

\[
\tau = \bar{\rho} \left( \frac{1}{4\sqrt{\pi} \sigma} \right) (u_p + u_s)(u_p - u_s)
\tag{3.32}
\]

Given the free stream velocity values, the only parameter necessary to calculate the interstream shear stress is the spread rate parameter \( (\sigma) \).

Using the velocity profile, the spread rate parameter can be expressed in terms of a velocity thickness growth rate. The velocity thickness \( (\delta_\phi) \) is defined as the thickness over which \( \phi \) varies between 0.05 and 0.95. For a fully turbulent shear layer, one expects it to grow linearly [21].

\[
\delta_\phi = \delta_\phi x
\tag{3.33}
\]
In terms of the similarity variable, the velocity thickness is constant

\[ \Delta \xi = 2.33 \tag{3.34} \]

and the spread rate parameter can be expressed in terms of the velocity thickness growth rate.

\[ \sigma = \frac{2.33}{\delta'_{p}} \tag{3.35} \]

The following sections detail how the spread rate parameter is linked to existing planar shear layer data.

### 3.2.2 Planar Shear Layer Data

We define the normalized pitot pressure as follows.

\[ \varphi = \frac{P_{\text{pitot}} - P_{\text{pitot},s}}{P_{\text{pitot},p} - P_{\text{pitot},s}} \tag{3.36} \]

The pitot thickness \( (\delta_{\text{pitot}}) \) is defined as the thickness over which the normalized pitot pressure \( (\varphi) \) varies between 0.05 and 0.95 \[21\]. Papamoschou developed correlations for the pitot thickness growth rate as a function of three parameters \[21\].

1. the velocity ratio \( (r) \),
2. the density ratio \( (s) \), and
3. the convective Mach number \( (M_c) \).

Dimotakis \[9\] modified the correlation developed by Papamoschou \[21\] for a temporally growing shear layer to account also for the spatial growth of a shear layer.

\[ \frac{\partial \delta'_{\text{pitot},inc}}{\partial x} = \delta'_{\text{pitot},inc} = 0.14 \frac{(1-r)(1+\sqrt{s})}{1+r\sqrt{s}} \left[ 1 - \frac{(1-\sqrt{s})(1+\sqrt{s})}{1+2.9(1+r)} \right] \tag{3.37} \]

Papamoschou found that the effects of compressibility could be correlated against a single parameter, the convective Mach number \( (M_c) \).

\[ M_c = \frac{u_p - u_c}{a_p} \tag{3.38} \]

The convective velocity \( (u_c) \) is defined as follows.

\[ u_c = \frac{u_p + r \sqrt{s}}{1 + \sqrt{s}} \tag{3.39} \]
The convective Mach number \( (M_c) \) can also be expressed in the following form.

\[
M_c = \frac{u_p}{a_p} \left[ 1 - \frac{1 + r \sqrt{s}}{1 + \sqrt{s}} \right] = M_p \sqrt{s} (1 - r) \tag{3.40}
\]

The ratio of the compressible pitot thickness growth to the incompressible pitot thickness growth is correlated against the convective Mach number. The compressible pitot thickness is roughly equal to the incompressible pitot thickness for \( M_c < 0.25 \), then drops off linearly to one-fifth of the incompressible value at \( M_c = 0.5 \) as shown in Figure 3-1 [21].

![Figure 3-1: Planar Shear Layer Dependency on Convective Number.](image)

### 3.2.3 Temperature Effects

To account for variations in stagnation temperature, we used the approximate Munk and Prim substitution principle. The original Munk and Prim formulation is for an inviscid flow and states that the Mach number and stagnation pressure profiles remain similar regardless of the stagnation temperature distributions. The approximate similarity principle states that for a viscous flow with heat transfer, stagnation pressure and Mach number profiles will be approximately similar if total pressure changes due to heat transfer are approximately equal and opposite to the total pressure changes due to viscous forces along a streamline [16]. The concept has been shown to be applicable for supersonic mixer-ejectors based on comparisons with experimental data [28]. Therefore, the pitot thickness growth rate should remain the same despite variations in total temperatures.
The correlations were compared only to cold flow data. To guarantee that the correlations satisfy the approximate similarity principle, the hot flow parameters \((r_h, s_h)\) were scaled to the equivalent cold flow parameters \((r_c, s_c)\) before using the correlations in the previous section. The similarity principle states that the Mach numbers should not change with total temperature ratio; therefore, the velocity ratio should scale with the square root of the total temperature ratio \((\tau)\).

\[
    r_c = r_h \sqrt{\tau} \tag{3.41}
\]

The additional assumption of uniform axial pressure is used to scale the density ratio.

\[
    s_c = \frac{s_h}{\tau} \tag{3.42}
\]

The velocity ratio and density ratio are scaled to a cold flow case which according to the similarity principle has the same pitot thickness growth rate which is determined using the correlations in the previous section.

### 3.2.4 Velocity Thickness

The final step in determining the interstream shear stress is to find a link between pitot thickness and velocity thickness. The velocity thickness can be determined from the error function velocity profile and for a given primary Mach number, secondary Mach number and a total temperature ratio \((\tau)\), the Mach number distribution can be determined and used to calculate the pitot pressure profile. With the two profiles, the ratio of velocity thickness to pitot thickness can be calculated and used to link the shear layer correlations to the interstream shear stress.

For \(P=P(x)\) and \(y=y(x)\), the pitot pressure can be determined from the Mach number profile. The normalized pitot pressure \((\varphi)\) can be expressed in the following form.

\[
    \varphi = \frac{P_{pitot} - P_{pitot,s}}{P_{pitot,p} - P_{pitot,s}} \tag{3.43}
\]

For subsonic Mach numbers, the ratio of the pitot pressure \((P_{pitot})\) to the static pressure \((P)\) is given by an isentropic relationship.

\[
    \frac{P_{pitot}}{P} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma+1}{\gamma-1}} \tag{3.44}
\]

For supersonic Mach numbers, the ratio of the pitot pressure \((P_{pitot})\) to the static pressure \((P)\), is given by the Rayleigh Pitot tube formula [3].

\[
    \frac{P_{pitot}}{P} = \left(\frac{(\gamma + 1)^2 M^2}{4\gamma M^2 - 2(\gamma - 1)}\right)^{\frac{1}{\gamma-1}} \frac{1 - \gamma + 2\gamma M^2}{\gamma + 1} \tag{3.45}
\]
The temperature distribution in Equation 3.52 derived in Section 3.3 can be used to calculate the corresponding Mach number distribution and in turn the pitot thickness. (Both \( \gamma \) and \( R \) are assumed to be uniform.)

\[
M = \frac{u}{u_p} \sqrt{\frac{T_p}{T} M_p}
\]

\[
\frac{T}{T_p} = \frac{1 - TR}{2} [1 + \text{erf} \left( T \right)] + TR
\]

\[
TR = \frac{T_s}{T_p}
\]

The velocity ratio \( r \) and static temperature ratio \( TR \) are calculated using the following equations.

\[
TR = \frac{1}{\tau} \frac{\theta_s}{\theta_p}
\]

\[
r = \frac{M_s}{M_p} \sqrt{TR}
\]

The velocity ratio and temperature ratio are then used to determine the velocity and temperature profiles. The velocity and temperature profiles are combined to give the Mach number profile which in turn yields the pitot pressure profiles. The ratio of velocity to pitot thickness is mapped out as a function of primary and secondary Mach numbers for three different total temperature ratios: \( \tau = 1 \) (Figure 3-2), \( \tau = 2 \) (Figure 3-3), and \( \tau = 3 \) (Figure 3-4).

There are singularities and numerical difficulties for areas near \( M_p = M_s \). These difficulties occur for two reasons. First, when the two far-field pitot pressures are the same, it results in infinite normalized pitot pressures. The second problem occurs when the pitot pressure profile has a wake-like defect and the normalized pitot pressure is no longer monotonic. However, for supersonic mixer-ejectors, usually the primary stream is supersonic and the secondary stream is subsonic and therefore operates in a region where these problems do not occur.

For a total temperature ratio of one, most supersonic mixer-ejectors operate where the ratio of velocity to pitot thickness is between 0.9 and 1.1; for a total temperature ratio of two, the ratio of velocity to pitot thickness is between 1.1 and 1.2; and for a total temperature of three, the ratio of velocity to pitot thickness is between 1.2 and 1.3. A constant ratio of 1.2 was chosen as giving the best pumping ratio comparisons for a set of tests performed at the different total temperature ratios.
3.3 Interstream Thermal Conduction

Consistent with the slender body assumptions, the temperature distribution can be approximated by a thermal diffusion equation for a frame of reference convecting at the mean velocity.

\[
\frac{\partial T}{\partial \tau^*} = \alpha \frac{\partial^2 T}{\partial y^2}
\]  

(3.51)

The resulting normalized temperature has an error function distribution.

\[
\frac{T - T_s}{T_p - T_s} = \frac{1}{2} [1 + \text{erf} (\xi_T - \xi_{T0})]
\]

(3.52)

The velocity and temperature profiles are related by the turbulent Prandtl number \(PR_{turb}\).

\[
PR_{turb} = \frac{\epsilon}{\alpha}
\]

(3.53)

Based on 2-D jet experiments [24], the turbulent Prandtl number was assumed to be 0.5. The similarity variable \((\xi_T)\) for the temperature profile is related to the similarity variable of the velocity...
profile \((\xi)\) by the turbulent Prandtl number.

\[
\xi_T = \sigma \sqrt{PR_{turb}} \frac{y}{x}
\]  

(3.54)

Since the turbulent Prandtl number is less than one, the temperature profile is wider than the velocity profile.

The amount of thermal energy transferred between the two streams is given by

\[
\dot{q} = \kappa \frac{\partial T}{\partial y}
\]  

(3.55)

The preceding equation is evaluated at \(y=0\).

\[
\dot{q} = \rho \bar{C}_p(T_p - T_s)(u_p + u_s) \frac{1}{4\sigma \sqrt{\pi PR_{turb}}}
\]  

(3.56)

### 3.4 Evolution of Interface Length

To calculate the shear work and the convected heat flow, it is necessary to calculate an interface length between the two streams. Because it is difficult to calculate the interface length directly, an
alternative approach was derived. For an incremental change in the shear layer thickness \((\partial \delta)\), there is an incremental change in the cross-sectional area of the shear layer \((\partial A_{sl})\) proportional to the effective interface length.

\[
\partial A_{sl} = L_{eff} \partial \delta
\]  

(3.57)

The scalar mixedness can be expressed as the ratio of the mixed-out area \((A_{mo})\) to the total area \((A)\).

\[
M = \frac{A_{mo}}{A}
\]  

(3.58)

The incremental change in shear layer thickness \((\partial \delta)\) can be expressed in terms of an incremental change in axial position \((\partial x)\).

\[
\partial \delta = \delta' \partial x
\]  

(3.59)
If one considers a fully mixed-out shear layer thickness analogous to the displacement thickness, the effective interface length can be expressed in the following form.

\[ L_{\text{eff}} = \frac{A}{\delta_{\text{mo}}} \frac{\partial M}{\partial x} \]  

(3.60)

For flows with negligible streamwise vorticity (i.e. flat plates and convoluted plates), analytical models can be used to predict the effective interface length as a function of the shear layer thickness. For flows with significant streamwise vorticity, a vortex model is necessary to predict the effective interface length.

3.4.1 Flat Plate

The simplest mixer conceptually is a flat plate. There is no shed streamwise vorticity and as long as the shear layer is much smaller than the duct height, the interface length is constant and equal to the duct width.

3.4.2 Convoluted Plate

For mixers with negligible streamwise vorticity, the interface length distribution can be approximated as a function of shear layer thickness based on mixer geometry. Initially, the interface length is equal to the perimeter of the mixer lobes. As the shear layer grows, the perimeter transitions to a planar shear layer as shown in Figure 3-6.

3.4.3 Forced Mixers

For flows with streamwise vorticity, the interface length increases due to the winding of the shear layer. A vortex method calculation developed by Tew [30] is used to find the interface length. The strength of the vorticity at the trailing edge is estimated as suggested by Barber et al [5]. The line integral of the velocity field is evaluated along the trailing edge based on three assumptions.
1. The streamwise component of the velocity is uniform across the lobe.

2. The flow at the lobe surface is tangent to the surface.

3. The horizontal portions of the integration contour do not contribute to the shed streamwise vorticity.

The result is an expression for the non-dimensional circulation ($\Gamma^*$) in terms of the velocity ratio ($r$), the primary and secondary lobe angles ($\alpha_p$ and $\alpha_s$ respectively), and a constant ($C_T$) which is a function of the lobe trailing edge geometry.

$$\Gamma^* = \frac{\Gamma_{le}}{U h} = C_T \left[ \frac{\tan \alpha_1 + r \tan \alpha_2}{1 + r} \right]$$  \hspace{1cm} (3.61)

$C_T = 2.0$ for square lobe geometries, $C_T = 1.23$ for sinusoidal lobes and $C_T = 1$ for triangular lobes.

In Euler computations Elliott demonstrated that over the range of Mach numbers he examined ($M < 2.0$), compressibility had only a marginal effect on the generation of streamwise circulation by the mixers, and hence he showed that the above scaling law can be used to estimate the shed streamwise circulation in compressible flow regimes [10]. However, in Navier-Stokes computations, O’Sullivan observed that boundary layer blockage can reduce the shed streamwise circulation by reducing the lobe angle seen by the free stream fluid [20].

### 3.4.4 Vortex Code

The viscous-vortex code developed by Tew [30] is based on an approach similar to that taken by Qiu [23]. A slender body approximation is used to model the steady three-dimensional flow field as an unsteady two-dimensional one. The shed streamwise vorticity is represented by a set of discrete vortices. The cross-flow velocity of the $k^{th}$ vortex at non-dimensional time ($t^* = x^* \Gamma^* h^*$) is found...
by superimposing the induced velocity associated with other vortices.

\[
v_k^* (\zeta^*_k, t^*) = \left( \frac{i}{2\pi} \right) \sum_{j=1}^{N} \Gamma_j^* \frac{\zeta_k^* - \zeta_j^*}{|\zeta_k^* - \zeta_j^*|^2} \left( 1 - e^{-\frac{t^*}{2|\zeta_k^* - \zeta_j^*|}} \right) . \tag{3.62}
\]

Vortex locations are tracked as they convect down the duct by integrating the cross-flow velocities.

Tew compared experimental Mie images to simulations of a 15° forced mixer at several axial positions for two different shear layer growth rates as shown in Figures 3-7 and 3-8. The model qualitatively captured the evolution of the interface length for both shear layer growth rates. At the higher shear layer growth rate, the streamwise vorticity does not stretch the interface length as much because it is diffused away from the interface more rapidly than at the lower shear growth rate.

A scalar mixedness (M) distribution is obtained based on the vortex locations and the shear layer thickness as described in Section G.2.2. Figure 3-9 shows the scalar mixedness distribution for four different mixers. There is a constant mixing rate for the flat plate, a mixing rate increase with the convoluted plate due to the convoluted trailing edge, and an additional mixing rate increase with the addition of streamwise vorticity.

The effective interface length \((L_{\text{eff}})\) from Equation 3.60 is shown in Figure 3-10 for the four mixers. Again, one sees the increase in interface length for the convoluted plate due to the convoluted trailing edge and an additional increase due to the winding of the interface caused by the streamwise vorticity. The effective interface length of all four mixers approach that of a flat plate as the shear layers merge.

Appendix G contains a more detailed description of the vortex code and its use.
Figure 3-7: Vortex model and Mie data cross-flow interface structure comparison. Cross-flow planes are 1\lambda, 2\lambda, and 3\lambda downstream of the 15° mixer with $M_p = 2.4$ and $M_s = 1.6$. $M_c = 0.22$, $r = 0.80$, $s = 0.70$. Model parameters: $\Gamma^* = 0.5$, $\delta_{cel}^* = 0.02$, and $\epsilon^* = 0.05$ [31].
Figure 3-8: Vortex model and Mie data cross-flow interface structure comparison. Cross-flow planes are 1λ, 2λ, and 3λ downstream of the 15° mixer with \( M_p = 1.6 \) and \( M_s = 0.3 \). \( M_c = 0.56, r = 0.21, s = 0.67 \). Model parameters: \( \Gamma^* = 0.5, \delta_{rel} = 0.10 \), and \( e^* = 0.25 \) [31].
Figure 3-9: Modeled Scalar Mixedness Distribution for Various Types of Mixers [31].

Figure 3-10: Evolution of Mixing Length for Various Types of Mixers [31].
Chapter 4

Validation of the Mixer-Ejector Model

4.1 Validation Methodology

Because the fluid dynamic model is composed of many separate components, an attempt was made to compare the model to experimental results that span a range of complexities. We start with the simplest experiments and proceed in increasing order of complexity.

1. Flat plates
   For flat plates, it was not necessary to have subcomponents for the evolution of the cross-planar interface (it was assumed constant) or the off-axis losses. The effects examined were the pressure matching control volume, the wall shear stress, the interstream thermal conduction, the interstream shear stress. Each was varied and the resulting entrainment ratios compared to experimental data.

2. Convoluted plates
   After confidence was built in the other subcomponents, the examination of the evolution of the cross-planar interface was started by comparing the fluid dynamic model predictions with experimental data from convoluted plate experiments. The vortex code was not used. Instead, the interface length was held constant until the shear layer was the same width as half the wavelength and then it was transitioned to the interface length of a flat plate. It was not necessary to account for the off-axis losses because the flow is assumed to leave the convoluted plate with no off-axis component of velocity.

3. Forced mixers
   The final step was to examine forced mixers where the effects of streamwise vorticity were
significant. The vortex model and the loss model for off-axis flow were incorporated and investigated.

4.2 Metrics Used for Validation

4.2.1 Entrainment Ratio

The pumping ratio (WSWP) is the ratio of the entrained mass flow ($\dot{m}_s$) to the primary mass flow ($\dot{m}_p$).

$$WSWP = \frac{\dot{m}_s}{\dot{m}_p}$$ (4.1)

For mixed-out flow, maximum entrainment corresponds to minimum jet velocity. Consequently, the pumping ratio is often used as a noise indicator with increased entrainment indicating decreased jet noise.

A variation of the pumping ratio that is often used is the temperature corrected pumping ratio (WSWPC).

$$WSWPC = \frac{\dot{m}_s}{\dot{m}_p} \sqrt{\frac{1}{\tau}}$$ (4.2)

The approximate Munk and Prim similarity principle implies the temperature corrected pumping ratio should be independent of the total temperature ratio ($\tau$) [28]; therefore, it can be used to compare mixers tested at different total temperature ratios.

4.2.2 Net Thrust Coefficient

A second figure of merit that could be compared directly between model results and experimental data was the net thrust coefficient ($C_{fn}$).

$$C_{fn} = \frac{F_{net}}{F_{ideal}}$$ (4.3)

The ideal thrust is calculated by isentropically expanding the primary nozzle flow to ambient pressure.

$$F_{ip} = \dot{m}_p \sqrt{\gamma_p R_p T_p} \sqrt{\frac{2}{\gamma_p - 1} \left[ 1 - NPR^{\frac{1-\gamma_p}{\gamma_p - 1}} \right]}$$ (4.4)
The net thrust for the experiments comes from thrust stand measurements, the calculated net thrust for the fluid dynamic model comes from the momentum equation.

\[ F = \sum (\dot{m}_i u_i)_e - \dot{m}_s u_0 + \Delta P A_e - D_{ext} \]  

Typically, the net thrust for the experiments was from static tests or was corrected to eliminate the external drag. Therefore, it was not necessary to account for external drag to make comparisons with the experimental data.

### 4.2.3 Mixing Rates

There are several measures of mixedness that have been used. Each is defined so it transitions between zero at the mixer trailing edge to one at fully mixed-out conditions.

1. **Scalar mixedness.**
   The scalar mixedness measures the diffusion of a passive scalar. It is calculated using the vortex code. Experimentally, the scalar mixedness can be determined using Mie imaging.

2. **Mixedness based on total temperatures.**

   \[ \phi = 1 - \frac{\int (T_i - \bar{T}_i) d\dot{m}}{\int |T_i - \bar{T}_i| d\dot{m}|_{T.E.}} \]  

   \[ \bar{T}_i = \frac{\int T_i d\dot{m}}{\int d\dot{m}} \]  

   Experimentally, total temperature profiles are taken using rakes of temperature probes. A mixedness based on total temperature can also be obtained from CFD results for comparison purposes. The total temperature mixedness can be approximated using the mass averaged values for the primary and secondary streams or it can be obtained from the vortex code by starting with the scalar mixedness field and assuming appropriate turbulent Schmidt and Prandtl numbers.

3. **Mixedness based on total pressures.**

   \[ \phi = 1 - \frac{\int (P_i - \bar{P}_i) d\dot{m}}{\int |P_i - \bar{P}_i| d\dot{m}|_{T.E.}} \]  

   Mixedness based on total pressure has been used for cold flow cases. One drawback to this measure is that it is sensitive to total pressure losses, especially those due to shocks.
4.3 Single Lobe Data

As shown in Figure 4-1, the single lobe tests were designed to represent a single lobe of a multi-lobed mixer. The tests were designed to allow for variation in nozzle pressure ratio (NPR), secondary nozzle pressure ratio (SNPR), total temperature ratio (τ), duct length, and mixer area ratio (MAR). A duct extension was available to increase the duct length, but few tests were run with it. All other design variables such as the chute expansion ratio (CER), suppressor area ratio (SAR), aspect ratio (AR), penetration (PEN) and flow angles were fixed by the design of the facility.

The mixer angles at the trailing edge were 0°, therefore the streamwise vorticity was assumed to be negligible and the vortex code was not used. The interface length was held constant at twice the duct width, equal to the mixer perimeter at the trailing edge. It was also not necessary to model the off-axis losses.

In the experiment, there was a contraction in the first inch after the mixer. The duct was hinged at that point to allow for variation in the mixer area ratio (MAR). A curve was fit to the area distribution of the single lobe experiments to ensure that the first derivative was continuous and eliminate numerical problems in the integration subroutines. The contraction was modeled using a parabolic fit.

\[ A_{\text{duct}} = ax^2 + bx + c \]  \hspace{1cm} (4.9)

The area at \((x = 0)\) is the initial duct height multiplied by the duct width.

\[ c = h_0 W \]  \hspace{1cm} (4.10)

The parabolic fit also matches the area at the hinge point \((x = x_c)\).

\[ ax_c^2 + bx_c + c = h_c W \]  \hspace{1cm} (4.11)

The third requirement is that the first derivative of the area be continuous. For axial positions past the hinge point \((x \geq x_c)\), the first derivative of the area is constant.

\[ \frac{dA}{dx} = 1 - MAR \]  \hspace{1cm} (4.12)

\[ b = 1 - 2x_a a - MAR \]  \hspace{1cm} (4.13)

Comparison of static pressure profiles from the model and pitot measurements are shown in Figures 4-2 and 4-3. Several observations can be made: the static pressure at the duct exit was
Lobed Mixer-Ejector

Single Lobe

Rotate 90°

Secondary

Primary

Secondary

Secondary

Primary

Secondary

Secondary

Mixing Duct
Figure 4-2: Static Pressure Comparison with TTR=1

Figure 4-3: Static Pressure Comparison with TTR=2
matched to ambient, (i.e. the mixer-ejectors were operating in compound subsonic mode); the area change affects the static pressure profile by causing an initial pressure drop equal to 20% of the overall pressure rise across the duct; pressure matching between the secondary and primary streams occurs approximately in the first 20% of the duct length; and while static pressures at a given axial location can differ, the general trends are captured. The static pressure profile does not change with total temperature ratio, in agreement with the approximate similarity principle.

Pumping ratio comparisons were made for three different total temperatures of approximately one, two, and three (Figures 4-4, 4-5, and 4-6) to investigate the impact of total temperature ratio on interstream shear stress and interstream thermal conduction. For a fixed velocity-to-pitot thickness ratio, the pumping ratios from the fluid dynamic model were within 5% of the experimental values. Variations in the pumping ratios for runs at the same nozzle pressure ratio (NPR) are due to differences in the operating conditions (SNPR, total temperature ratios, and ambient pressure). By accounting for these differences in operating conditions, the model captures the variations in pumping ratios. For a total temperature ratio of one, pumping ratios from the mixed-out control volume described in Appendix D are also included. Note that the mixed-out control volume shown if Figure 4-4 overpredicts the pumping ratio by 30% (six times the maximum difference using the fluid dynamic model) indicating that the flows are not fully mixed at the duct exit. This comparison

![Figure 4-4: Pumping Ratio Comparisons Between Single Lobe Data and Model for Total Temperature Ratios Near One.](image)

The static pressure profile does not change with total temperature ratio, in agreement with the approximate similarity principle.
Figure 4-5: Pumping Ratio Comparisons Between Single Lobe Data and Model for Total Temperature Ratios Near Two.

Figure 4-6: Pumping Ratio Comparisons Between Single Lobe Data and Model for Total Temperature Ratios Near Three.
also demonstrates the importance of accurately predicting the mixing rates.

A source of experimental error was leakage flow between the primary and secondary streams. Analysis showed that the leakage had a 2-4% effect on the pumping ratios. Nozzle and inlet losses were estimated using correlations derived from the experimental data. Sensitivity studies showed that changes in the wall shear stress, interstream shear stress, and the interstream thermal conduction components did not systematically improve the comparisons. For these reasons, it was concluded that those components needed no further refinement. The interstream shear stress was adjusted by changing the velocity-to-pitot thickness ratio to match the experimental data. The pumping ratios could be matched for interstream shear stresses between 8% less and 40% more than the baseline model. The velocity-to-pitot thickness ratio was held constant for all further calculations.

4.4 Convoluted Plate Experiments

To provide a database for mixer-ejector noise suppression systems, sub-scale testing was performed at Boeing’s Nozzle Test Facility (NTF). Hot flow tests of various designs supplied by Pratt and Whitney and General Electric were performed and thrust performance, pumping, and static pressure measurements were taken [12].

Pumping ratio and gross thrust coefficient ($C_{fg}$) comparisons were made between the fluid dynamic model and data from the experiments at three different values of MAR corresponding to a converging duct, a slightly converging duct, and a diverging duct. These are shown in Figures 4-7 through 4-12. For $MAR < 1$, the fluid dynamic model was able to capture the pumping ratio within 5% and the net thrust coefficient within 1.5%. For $MAR > 1$, the fluid dynamic model gave the pumping ratios within 15% and the net thrust coefficients within 7% of experiments. The decrease in accuracy occurs when modeling flows where compound shock fitting is necessary. This was not of prime concern since it was determined by industry that this 'mode shift' was undesirable due to increased mixing noise.

4.5 Forced Mixers

Pumping and net thrust comparisons for CFD and experiment were made for three mixers with different penetrations. The fluid dynamic model had comparable accuracy to the CFD and captured the trends shown in the experimental data as shown in Figure 4-13. Pumping and net thrust comparisons between the fluid dynamic model predictions and the experimental data were also made for NPR sweeps with each of the three mixers. The comparisons are shown in Figures 4-14 through 4-19. Comparisons from the mixed-out control volume are also included for the lowest penetration case. The model captures the experimental data to within 10% for pumping ratios.
Figure 4-7: Pumping Ratio Comparisons Between Convoluted Plate Data and Model with Converging Duct for an NPR Sweep.

Figure 4-8: Thrust Comparisons Between Convoluted Plate Data and Model with Converging Duct for an NPR Sweep.
**Figure 4-9:** Pumping Ratio Comparisons Between Convoluted Plate Data and Model with Slightly Converging Duct for an NPR Sweep.

**Figure 4-10:** Thrust Comparisons Between Convoluted Plate Data and Model with Slightly Converging Duct for an NPR Sweep.
Figure 4-11: Pumping Ratio Comparisons Between Convoluted Plate Data and Model with Diverging Duct for an NPR Sweep.

Figure 4-12: Thrust Comparisons Between Convoluted Plate Data and Model with Diverging Duct for an NPR Sweep.
Figure 4-13: Comparisons Between Forced Mixer Data and Model and 3-D Full Navier-Stokes Calculations
Figure 4-14: Temperature Corrected Pumping Ratio Comparisons Between Forced Mixer Data and Model with PEN1 for an NPR Sweep.

Figure 4-15: Thrust Comparisons Between Forced Mixer Data and Model with PEN1 for an NPR Sweep.
**Figure 4-16:** Temperature Corrected Pumping Ratio Comparisons Between Forced Mixer Data and Model with PEN2 for an NPR Sweep.

**Figure 4-17:** Thrust Comparisons Between Forced Mixer Data and Model with PEN2 for an NPR Sweep.
Figure 4-18: Temperature Corrected Pumping Ratio Comparisons Between Forced Mixer Data and Model with PEN3 for an NPR Sweep.

Figure 4-19: Thrust Comparisons Between Forced Mixer Data and Model with PEN3 for an NPR Sweep.
and 2% for thrust coefficients, while the mixed-out control volume overpredicts the pumping ratio by 30% and the thrust coefficient by 12%.

The sensitivity of the fluid dynamic model to the vorticity distribution was determined by comparing the predictions made by the fluid dynamic model using a vorticity distribution from a 3DNFS calculation and one from an analytical approximation as shown in Figure 4-20. Because streamwise

![Figure 4-20: Comparison Between Forced Mixer Data and Model with CFD Vorticity Distribution and Model with an Analytical Vorticity Distribution](image)

vorticity is most effective for low lobe height-to-wavelength ratios, it was to be expected that the lowest penetration mixer would be most sensitive to variations in the vorticity distribution. The higher lobe height-to-wavelength ratio mixers are barely affected by changes in the vorticity distribution. While the two distributions result in different pumping and thrust, the accuracy of each compared to experimental data is the same.

Other comparisons were made with 3-D CFD calculations but are omitted here because they involve LER information. Velocity profiles from the model showed qualitative agreement with 3-D CFD. Comparisons of axial distributions of mixedness based on total pressures were made. Differences between the distributions were comparable to differences in 3-D CFD computations using different turbulence models.
4.5.1 Summary

Comparisons with experimental data were made for three types of mixers: flat plates (single lobe), convoluted plates, and forced mixers. Pumping comparisons were within 5% of the experimental values for the single lobe and convoluted plate tests and within 10% of the experimental values for forced mixer tests. Thrust comparisons were within 2% for cases where compound shocks did not occur. For cases where compound shocks did occur, the thrust comparisons were within 7%. All the trends were captured.

Comparisons were also made with a mixed-out control volume analysis and 3-D CFD computations. The fluid dynamic model was significantly more accurate than the control volume which was only within 30% of the pumping ratios and 12% of the net thrust coefficient and had a comparable accuracy to 3-D CFD. This accuracy obtained with a computationally-efficient model makes it ideal for carrying out optimization studies.
Chapter 5

Development of Integrated Tool to Determine Mixer-Ejector Performance

Chapters 2 and 3 discuss the structure and components of the fluid dynamic model. In Chapter 4, comparisons between model and experimental data were made to illustrate the ability to compute accurately the aerodynamic measures of performance such as entrainment and net thrust.

There are two important links to understanding the mixer-ejector system that are examined in this chapter. The first involves linking the fluid dynamic performance to the figures of merit: thrust, acoustics, and nozzle weight. The second is to link the figures of merit into a single cost function. To combine the figures of merit, system weightings (relative weightings for each figure of merit) are used to determine the impact on take-off gross weight (TOGW). The end result is the integrated tool described in this chapter that links the design variables to the figures of merit and the take-off gross weight.

With the integrated tool, optimization studies become possible. Trade studies involving variations about a baseline design are discussed in Chapter 6. The final step was to embed the integrated tool in an multi-variable optimization algorithm. The multi-variable optimization studies are discussed at the end of Chapter 6 and in Chapter 7.

5.1 Overview

An important step in optimization is to form a cost function, a measure to compare designs. The cost function for a mixer-ejector should account for noise, thrust, and nozzle weight. As shown schematically in Figure 5-1, the integrated tool links modules designed to predict these three figures...
of merit based on the operating conditions and design variables. Noise and thrust are functions of

**Figures of Merit:**

![Flowchart of Integrated Tool Modules.

both flow properties and geometry. Weight is a function of geometry only. Each of the figures of

One way to form a single cost function is to assign a weighting to each figure of merit. This is

what is done when designers express a desired noise reduction in terms of dB per percentage point
drop in thrust coefficient (Cfn). The integrated tool weights each of the figures of merit based on

their impact on the take-off gross weight of the vehicle (TOGW). This weighting system is detailed

at the end of the chapter.

The fluid dynamic model is coded in C, while the other modules are programmed in Matlab

script. All of the code, input files, and output files, are detailed in Appendix E.

5.2 Acoustics

Estimates of the perceived noise level in decibels (EPNLdB) can be made based on the outputs of the

fluid dynamic model. Currently, a Simplified Acoustic Tool [37] acquired from Pratt and Whitney is

used to make acoustic estimates. The Simplified Acoustic Tool includes correlations based on HSR

acoustic experiments to account for external noise, internal noise, profile noise, and acoustic liner

attenuation. While future efforts may require more advanced acoustic codes, the modular form of

the integrated tool allows the replacement of any module without affecting other modules.

5.2.1 External Noise

The external noise (in decibels) is correlated using a linear relationship.

\[
\text{dB} = a + bV_{j,mix}
\]  

(5.1)
\( V_{j,mix} \) is the mass averaged velocity if the duct exit static pressure is matched to the ambient pressure.

\[
V_{j,mix} = \frac{F}{m}
\]  

(5.2)

5.2.2 Internal Noise

The incremental increase in noise due to internal mixing is given by:

\[
\Delta dB_{\text{Internal}} = dB_{\text{int}} - dB_{\text{ext}} = 10 \log \left( \frac{W_{\text{int}}}{W_{\text{ext}}} \right) + 60 \log \left( \frac{V_{\text{int}}}{V_{\text{ext}}} \right)
\]

(5.3)

The mass flow ratio is related to the pumping ratio \((W_s/W_p)\).

\[
\frac{W_{\text{int}}}{W_{\text{ext}}} = \frac{1}{1 + W_s/W_p}
\]

(5.4)

\[
V_{\text{int}} = V_p \left( 1 - \frac{V_s}{V_p} \right)^{2/3}
\]

(5.5)

\[
V_{\text{ext}} = V_{j,mix} \left( 1 - \frac{V_0}{V_{j,mix}} \right)
\]

(5.6)

5.2.3 Profile Noise

The effect of non-uniform exit profile on the radiated noise is modeled using a hot streak analogy (see Figure 5-2). The hot streak analogy models the flow at the duct exit in a very similar manner to the fluid dynamic model. A fraction of the mass flow \( \left( \frac{W_{\text{hot}}}{W_{\text{Total}}} \right) \) is assumed to leave at a velocity \( (V_{\text{Hot}}) \) greater than \( V_{j,mix} \). The noise due to the hot streak \( dB_{\text{HotStreak}} \) is also related to the freestream velocity \( (V_0) \) and the noise level if the flow were uniform \( (dB_{\text{Reference}}) \).

\[
dB_{\text{HotStreak}} = 10 \log \left( \frac{W_{\text{Hot}}}{W_{\text{Total}}} \right) + 70 \log \left( \frac{V_{\text{Hot}}(1 - \frac{V_0}{V_{\text{Hot}}})^{2/3}}{V_{j,mix}(1 - \frac{V_0}{V_{j,mix}})^{2/3}} \right) + dB_{\text{Reference}}
\]

(5.7)
In a similar manner, the noise due to the remaining flow \( dB_{\text{Remainder}} \) is also calculated.

\[
dB_{\text{Remainder}} = 10 \log \frac{W_{\text{Rem}}}{W_{\text{Total}}} + 70 \log \frac{V_{\text{Rem}}(1 - V_0/V_{\text{Rem}})^{2/3}}{V_{j,mix}(1 - V_0/V_{j,mix})^{2/3}} + dB_{\text{Reference}} \tag{5.8}
\]

The profile noise \( \Delta dB_{\text{HotStreak}} \) is then calculated by logarithmically adding the noise of the two streams and subtracting out the reference noise level.

\[
\Delta dB_{\text{HotStreak}} = 10 \log \left[ \frac{10^{dB_{\text{HotStreak}}/10} + 10^{dB_{\text{Remainder}}/10}}{10} \right] - dB_{\text{Reference}} \tag{5.9}
\]

The primary stream flow values are used to approximate the hot streak values, while the secondary stream flow values are used to approximate the remainder flow.

### 5.2.4 Acoustic Liner Attenuation

The attenuation due to the presence of acoustic liner is modeled as a function of \( V_{j,mix} \) and \( Atam \), with the liner effectiveness correlated as a quadratic function of \( V_{j,mix} \).

\[
\frac{\Delta dB}{\Delta Atam} = a - bV_{j,mix} + cV_{j,mix}^2 \tag{5.10}
\]

\( Atam \) is the ratio of treated area to the cross-sectional area at the mixing plane.

\[
Atam = \frac{A_{\text{eff}}}{A_{mix}} \tag{5.11}
\]

The treated area includes an efficiency factor, \( C_{\text{treat}} \), to account for supports not covered by the liner.

\[
A_{\text{eff}} = C_{\text{treat}}(A_{\text{treat,S/W}} + A_{\text{treat,flaps}}) \tag{5.12}
\]

### 5.2.5 Residual

A residual term was correlated as a function of \( V_{j,mix} \). With this residual term added, the Simplified Acoustic Tool estimates the noise within 1 dB of the experimental measurements [37].

### 5.2.6 Logarithmic Addition

The noise source components are summed logarithmically.

\[
dB_{\text{tot}} = 10 \log \left[ 10^{dB_{\text{ext}}/10} + 10^{dB_{\text{int}} + dB_{\text{lin}} - \Delta dB_{\text{liner}}/10} + 10^{dB_{\text{ext}} + \Delta dB_{\text{profile}}/10} \right] \tag{5.13}
\]
5.3 Net Thrust Coefficient

The net force exerted by the mixer-ejector can be expressed as the sum of momentum flux, ram drag, pressure forces, and external drag.

\[ F = \sum (\dot{m}_i u_i) e - \dot{m}_s u_0 + \Delta P A_e - D_{ext} \] (5.14)

The thrust coefficient is the ratio of the actual thrust to the ideal thrust.

\[ C_{TD} = \frac{F}{F_{ip}} \] (5.15)

The ideal thrust is the force exerted if the primary nozzle was isentropically expanded to ambient pressure.

\[ F_{ip} = \dot{m}_p \sqrt{\gamma_p R_p T_{t_p}} \sqrt{\frac{2}{\gamma_p - 1} \left[ 1 - NPR \frac{1 - \gamma_p}{\gamma_p} \right]} \] (5.16)

The mass flow for a supersonic primary nozzle is a function of the stagnation conditions and the cross-sectional area at the throat \((A^*)\).

\[ \dot{m} = \frac{P_0 A^*}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R} \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{\gamma - 1}}} \] (5.17)

5.4 Nozzle Weight

The weight of the mixer-ejector system is computed using a model developed at Pratt and Whitney, with nozzle weight calculated based on the duct length, SAR, and AR. This model was also used to calculate ATAM. Further details of the nozzle weight model are omitted here because it was acquired under the limited exclusive rights (LER) agreement.

5.5 Take-off Gross Weight

The model used to approximate the change in take-off gross weight due to changes in net thrust, noise, and nozzle weight is

\[ \Delta TOGW = K_{C_{fn}} \Delta C_{fn} + K_W \Delta W + K_{EPNL} \Delta EPNL dB \] (5.18)

These weightings are derived from vehicle sizing studies performed by industry [14]. Further details of these studies are omitted here because they were acquired under the limited exclusive rights (LER) agreement.
Chapter 6

Design Optimization

6.1 Approach

Two approaches to design optimization were taken: trade studies and multi-variable optimization. Trade studies involve starting with a baseline design and varying each design variable independently. Multi-variable optimization allows a number of the design variables to be varied simultaneously.

Before performing the design optimization, operating conditions and a baseline design were chosen. Alternative weighting systems were also defined in order to examine the impact of different relative weightings of the figures of merit on the trade studies and the multi-variable optimization results.

The trade studies were used to identify and analyze the physical trade-offs represented by each design variable. Multi-variable optimization was used to obtain the optimal set of design variables. The usefulness of parametric testing was examined by comparing the results of the trade studies to the results of the multi-variable optimization.

Sensitivity of the multi-variable optimization process to modeling uncertainties, constraints, and TOGW weightings were also determined as described in Chapter 7.

6.2 Operating Conditions

The ambient atmospheric conditions were taken to be standard pressure and temperature. The ratio of specific heats of the entrained air was evaluated at the standard temperature. The ambient conditions are given in Table 6.1.

For a given flight Mach number \( (M_f) \), the secondary nozzle pressure ratio (SNPR) and the
Table 6.1: Ambient Conditions

<table>
<thead>
<tr>
<th>Ambient Condition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{amb}$</td>
<td>101325 Pa</td>
</tr>
<tr>
<td>$T_{amb}$</td>
<td>300 K</td>
</tr>
<tr>
<td>$\gamma_s$</td>
<td>1.41</td>
</tr>
</tbody>
</table>

secondary total temperature ($T_{ts}$) can be found using the following equations.

$$SNPR = \frac{P_{ts}}{P_{amb}} = \left(1 + \frac{\gamma_s - 1}{2} M_f^2\right)^{\frac{\gamma_s}{\gamma_s - 1}} \quad (6.1)$$

$$T_{ts} = \left(1 + \frac{\gamma_s - 1}{2} M_f^2\right) \quad (6.2)$$

A take-off flight Mach number $M_f = 0.3$ was used. The resulting SNPR and $T_{ts}$ are given in Table 6.2.

Table 6.2: Flight Conditions

<table>
<thead>
<tr>
<th>Flight Condition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_f$</td>
<td>0.3</td>
</tr>
<tr>
<td>SNPR</td>
<td>1.06</td>
</tr>
<tr>
<td>$T_{ts}$</td>
<td>305.5</td>
</tr>
</tbody>
</table>

There are three inputs to the fluid dynamic model that are dependent on the engine cycle: primary nozzle pressure ratio (NPR), total temperature ratio ($\tau$), and the ratio of specific heats for the primary stream ($\gamma_p$). However, the engine operating conditions can vary during take-off. For certification, acoustic measurements are taken at two reference points as the vehicle takes-off, sideline reference and take-off reference. To reduce the measured noise levels at the take-off reference point, the engine power is commonly throttled back, or cutback. Typical engine cycle values for sideline and cutback are given in Table 6.3.

Table 6.3: Engine Cycles

<table>
<thead>
<tr>
<th>Operating Condition</th>
<th>Sideline</th>
<th>Cutback</th>
</tr>
</thead>
<tbody>
<tr>
<td>NPR</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$\tau$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$\gamma_p$</td>
<td>1.33</td>
<td>1.35</td>
</tr>
</tbody>
</table>
6.3 Baseline Design

For the primary nozzle, values for the cross-sectional area of the throat $A_j = 1m^2$, discharge coefficient $C_D = 0.97$, and primary Mach number $M_p = 1.5$ were used. The discharge coefficient accounts for boundary layer blockage in the throat by defining an effective throat area ($A^*$).

\[ C_D = \frac{A^*}{A_j} \]  \hspace{1cm} (6.3)

The effective chute expansion ratio ($CER_{eff}$) can be expressed as a function of the primary Mach number ($M_p$).

\[ CER_{eff} = \frac{A_p}{A^*} = \frac{1}{M_p} \left[ \frac{2(1 + \frac{\gamma_p - 1}{2}M_p^2)}{\gamma_p + 1} \right]^{\frac{\gamma_p + 1}{2(\gamma_p - 1)}} \]  \hspace{1cm} (6.4)

The chute expansion ratio can be calculated from the discharge coefficient ($C_D$) and the effective chute expansion ratio ($CER_{eff}$).

\[ CER = CER_{eff}C_D \]  \hspace{1cm} (6.5)

Table 6.4 includes the baseline design values for the primary nozzle.

<table>
<thead>
<tr>
<th>Design Parameter</th>
<th>Design Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_j$</td>
<td>1 m$^2$</td>
</tr>
<tr>
<td>$C_D$</td>
<td>0.97</td>
</tr>
<tr>
<td>$A^*$</td>
<td>0.97 m$^2$</td>
</tr>
<tr>
<td>$M_p$</td>
<td>1.5</td>
</tr>
<tr>
<td>$CER_{eff}$</td>
<td>1.185</td>
</tr>
<tr>
<td>CER</td>
<td>1.15</td>
</tr>
</tbody>
</table>

For a given effective throat area and operating conditions, the primary mass flow ($\dot{m}_p$) is given by the following equation.

\[ \dot{m} = \frac{P_{tp}A^*}{\sqrt{T_{tp}}} \sqrt{\gamma_p \left( \frac{2}{\gamma_p - 1} \right)^{\frac{\gamma_p + 1}{2(\gamma_p - 1)}}} \]  \hspace{1cm} (6.6)

The ideal thrust ($F_{id}$) is obtained by isentropically expanding the primary flow to match the ambient pressure.

\[ F_{id} = \dot{m}_p \sqrt{\gamma_p R_p T_{tp}} \sqrt{\frac{2}{\gamma_p - 1} \left[ 1 - NPR \frac{\gamma_p - 1}{\gamma_p} \right]} \]  \hspace{1cm} (6.7)
For both the primary and secondary streams, a gas constant $R=287.04 \text{ J/kgK}$ was used. Table 6.5 shows the primary mass flow and ideal thrust for the baseline design at sideline and cutback engine operating conditions.

**Table 6.5:** Primary Mass Flows and Ideal Thrust at Sideline and Cutback

<table>
<thead>
<tr>
<th>Design Parameter</th>
<th>Sideline</th>
<th>Cutback</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{m}(\text{kg/s})$</td>
<td>386.7</td>
<td>317.4</td>
</tr>
<tr>
<td>$F_{id}(\text{N})$</td>
<td>275,000</td>
<td>149,700</td>
</tr>
</tbody>
</table>

For the duct, baseline values for the secondary-to-primary area ratio $\frac{A_s}{A_p} = 2$, mixing area ratio $\text{MAR}=1$, duct aspect ratio $\text{AR}=1$, and duct length $L_{duct} = 3\text{ m}$ were used. The suppressor area ratio (SAR) can be expressed in terms of the chute expansion ratio (CER) and the secondary-to-primary area ratio.

$$SAR = \frac{A_m}{A_j} = CER\left(1 + \frac{A_s}{A_p}\right) \quad (6.8)$$

Duct baseline values are given in Table 6.6.

**Table 6.6:** Baseline Duct

<table>
<thead>
<tr>
<th>Design Parameter</th>
<th>Baseline Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{A_s}{A_p}$</td>
<td>2</td>
</tr>
<tr>
<td>MAR</td>
<td>1</td>
</tr>
<tr>
<td>Aspect Ratio</td>
<td>1</td>
</tr>
<tr>
<td>$L_{duct}$</td>
<td>3 m</td>
</tr>
<tr>
<td>SAR</td>
<td>3.45</td>
</tr>
</tbody>
</table>

Many of the characteristics of the baseline mixer were chosen to be the same as the 15 degree forced mixer tested by Tew [30] and is shown in Figure 6-1. The mixer has a top and bottom rack with 10 lobes. For a mixer with $N$ lobes, the duct width-to-wavelength ratio ($W^*$) is $N/2$.

$$W^* = \frac{N}{2} \quad (6.9)$$

The duct aspect ratio (AR) is defined as the ratio of duct width (W) to duct height (H).

$$AR = \frac{W}{H} = \frac{W^*}{H^*} \quad (6.10)$$
The normalized duct height \( H^* \) is solved for using the definition of the duct aspect ratio.

\[
H^* = \frac{N}{2AR} \quad (6.11)
\]

The wavelength can be determined using the definition for the suppressor area ratio (SAR) in Equation 6.8.

\[
\lambda = \sqrt{\frac{A_jSAR}{H^*W^*}} \quad (6.12)
\]

The baseline value of the secondary-to-primary lobe width ratio \( \frac{w_s}{w_p} = 1 \) is the same as the 15° mixer used by Tew. The non-dimensional primary lobe width \( (w_p^*) \) can be found using the following equation.

\[
w_p^* = \frac{1}{1 + \frac{N}{w_p}} \quad (6.13)
\]

The baseline mixer lobe height-to-wavelength ratio \( h^* = 1.25 \) is the same as the 15° mixer used by Tew. The primary area is equal to the area of the primary lobes and the area between the upper and lower rack.

\[
\frac{A_p}{\lambda^2} = Nh^*w_p^* + h_1 \left( \frac{N}{2} - 1 + w_p^* \right) \quad (6.14)
\]

For a given primary area, the gap between the upper and lower rack \( (h_1) \) can be found by rearranging the previous equation.

\[
h_1 = \frac{\lambda}{\frac{N}{2} - 1 + w_p^*} \left[ \frac{A_p}{\lambda^2} - Nh^*w_p^* \right] \quad (6.15)
\]
The penetration (PEN) of the mixer is defined as the ratio of the mixer height \((h_i + 2h)\) to the duct height \((H)\).

\[
PEN = \frac{h_i + 2h}{H}
\]  

(6.16)

The initial interface length \((L_{mix})\) is equal to the perimeter of the mixer.

\[
L_{mix} = 2h_i + 2Nh + (N - 2)\lambda + 2w_p
\]  

(6.17)

The fraction of the primary flow in the lobes \((f_p)\) and the fraction of the secondary flow in the lobes \((f_s)\) are expressed in the following equations.

\[
f_p = \frac{N w_p h^* \lambda^2}{A_p}
\]  

(6.18)

\[
f_s = \frac{(N - 2)(1 - w^*_p) h^* \lambda^2}{A_s}
\]  

(6.19)

The baseline values for the primary mixer angle \((\alpha_{mp})\) and the secondary mixer angle \((\alpha_{ms})\) were chosen to be 15 degrees. The effective angles for the primary \((\alpha_p)\) and secondary \((\alpha_s)\) streams were found using the following equations (see Appendix F for a more complete explanation).

\[
\alpha_p = \cos^{-1} \sqrt{(1 - f_p) + f_p \cos^2 \alpha_{mp}}
\]  

(6.20)

\[
\alpha_s = \cos^{-1} \sqrt{(1 - f_s) + f_s \cos^2 \alpha_{ms}}
\]  

(6.21)

The non-dimensional circulation for a square lobe with equal mixer angles \((\alpha_{mp} = \alpha_{ms} = \alpha)\) can be approximated using the following equation.

\[
\Gamma^* = 2 \tan \alpha
\]  

(6.22)

Finally, the non-dimensional boundary layer thickness \(\epsilon^* = 0.05\) is the same as the 15° used by Tew. was chosen based [30]. Table 6.7 contains all the design values for the baseline mixer.
6.4 Weightings

By manipulating Equation 5.18, the fractional change in take-off gross weight can be expressed in the following form.

\[
\frac{\Delta TOGW}{TOGW} = \frac{K_W}{TOGW} \left[ \Delta W + \frac{K_{Cf_n}}{K_w} \Delta C_{f_n} + \frac{K_{EPNL}}{K_w} \Delta EPNLdB \right]
\]  

(6.23)

The fractional change in weight is characterized by the two weighting ratios \( \frac{K_{Cf_n}}{K_w} , \frac{K_{EPNL}}{K_w} \). The baseline weighting system uses 100 for each weighting ratio. By decreasing the weighting ratios simultaneously, the weighting system places relatively more emphasis on the nozzle weight. By increasing only the first weighting ratio \( \frac{K_{Cf_n}}{K_w} \), the weighting system places more emphasis on the net thrust coefficient \( C_{f_n} \). Similarly, by increasing the second weighting ratio \( \frac{K_{EPNL}}{K_w} \), the weighting system places more emphasis on EPNLdB.

Four weighting systems are given in Table 6.8. The baseline weighting system is based on the Boeing vehicle sizing studies. Because these studies are proprietary, the weightings were rounded to the nearest order of magnitude. The other weighting systems were chosen to emphasize each figure of merit and to encompass the design point system weightings.

I. Baseline Weighting System

II. Relative Weighting of Nozzle Weight Increased by an Order of Magnitude

III. Relative Weighting of \( C_{f_n} \) Increased by an Order of Magnitude

---

Table 6.7: Baseline Mixer

<table>
<thead>
<tr>
<th>Design Parameter</th>
<th>Baseline Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>10</td>
</tr>
<tr>
<td>( H^* )</td>
<td>5</td>
</tr>
<tr>
<td>( W^* )</td>
<td>5</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.3714 m</td>
</tr>
<tr>
<td>( \frac{w_a}{w_p} )</td>
<td>1</td>
</tr>
<tr>
<td>( w_p^* )</td>
<td>0.5</td>
</tr>
<tr>
<td>( h^* )</td>
<td>1.25</td>
</tr>
<tr>
<td>( h_1^* )</td>
<td>0.463</td>
</tr>
<tr>
<td>PEN</td>
<td>0.593</td>
</tr>
<tr>
<td>( L_{niz}^* )</td>
<td>34.93</td>
</tr>
<tr>
<td>( f_p )</td>
<td>0.75</td>
</tr>
<tr>
<td>( f_s )</td>
<td>0.3</td>
</tr>
<tr>
<td>( \alpha_p )</td>
<td>12.95 deg</td>
</tr>
<tr>
<td>( \alpha_s )</td>
<td>8.15 deg</td>
</tr>
<tr>
<td>( \Gamma^* )</td>
<td>0.5359</td>
</tr>
<tr>
<td>( \epsilon^* )</td>
<td>0.05</td>
</tr>
</tbody>
</table>
IV. Relative Weighting of EPNLdB Increased by an Order of Magnitude

Table 6.8: Weighting Systems

<table>
<thead>
<tr>
<th>Weighting Ratio</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{K_{\text{CFm}}}{K_{\text{W}}}$</td>
<td>100</td>
<td>10</td>
<td>1000</td>
<td>100</td>
</tr>
<tr>
<td>$\frac{K_{\text{EPNL}}}{K_{\text{W}}}$</td>
<td>100</td>
<td>10</td>
<td>100</td>
<td>1000</td>
</tr>
</tbody>
</table>

6.5 Trade Studies

Trade studies involve independently varying each design variable starting from the baseline design. A set of design variables was chosen that was consistent with parametric testing performed on supersonic mixer-ejectors, so parametric data could be used to validate the trends observed in the trade studies.

One variable was used to characterize the primary nozzle.

- Chute expansion ratio (CER)

Four variables are commonly used to define the duct geometry.

- Suppressor area ratio (SAR)
- Aspect ratio (AR)
- Mixing area ratio (MAR)
- Length

For the mixers examined, five variables are commonly used to define the mixer geometry.

- Penetration (PEN)
- Number of lobes (N)
- Secondary-to-primary lobe width ratio ($\frac{w_2}{w_1}$)
- Primary mixer angle
- Secondary mixer angle

For the mixer-ejector system, this gives a total of ten degrees of freedom which were each explored in a trade study. Table 6.9 contains the baseline values for the ten design variables.
Table 6.9: Baseline Values for Trade Studies

<table>
<thead>
<tr>
<th>Design Parameter</th>
<th>Baseline Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.15</td>
</tr>
<tr>
<td>SAR</td>
<td>3.45</td>
</tr>
<tr>
<td>Aspect Ratio</td>
<td>1</td>
</tr>
<tr>
<td>MAR</td>
<td>1</td>
</tr>
<tr>
<td>$L_{duct}$</td>
<td>3 m</td>
</tr>
<tr>
<td>PEN</td>
<td>0.593</td>
</tr>
<tr>
<td>$N$</td>
<td>10</td>
</tr>
<tr>
<td>$\frac{w_p}{w_r}$</td>
<td>1</td>
</tr>
<tr>
<td>$\alpha_{mp}$</td>
<td>15 degrees</td>
</tr>
<tr>
<td>$\alpha_{ms}$</td>
<td>15 degrees</td>
</tr>
</tbody>
</table>

For variables that do not affect the nozzle weight, the following relationship can be used to find the optimal value for the trade study.

$$\frac{dEPNLdB}{dC_{fn}} = \frac{K_{C_{fn}}}{K_{EPNLdB}}$$ (6.24)

Therefore, for these variables, the trade-offs can be summarized with a single noise-thrust line. For a given weighting system, the optimal value for the trade study can be solved by finding the point on the noise-thrust line with the proper slope and selecting the corresponding design value. For example, for the baseline weighting system, optimal values for the design variables that do not affect the nozzle weight will occur where the slope of the noise thrust line is $\frac{EPNLdB}{%Thrust}$.

6.5.1 Link Between Design Variables and Fluid Dynamic Model Inputs

To use the fluid dynamic model, the design variables must be used to solve for the corresponding set of fluid dynamic model inputs described in Appendix F. Since the primary throat area ($A_j$) is fixed, the primary area at the mixer exit is found using the definition of the chute expansion ratio (CER).

$$A_p = A_j CER$$ (6.25)

The non-dimensional duct width ($W^*$) is related to the number of lobes using Equation 6.9 and the non-dimensional duct height ($H^*$) can be found using the definition of the duct aspect ratio (AR) via Equation 6.11. The definition of the suppressor area ratio (SAR) is used to determine the wavelength ($\lambda$) using Equation 6.12 and the non-dimensional primary lobe width ($w_p^*$) is found from the secondary-to-primary lobe width ratio using Equation 6.13. An equation for the mixer lobe...
height-to-wavelength ratio \((h^*)\) is obtained by combining Equation 6.14 and Equation 6.16.

\[
h^* = \frac{\frac{A_p}{\lambda} - PENH^* (\frac{N}{2} - 1 + w^*_p)}{N \cdot w^*_p - 2 (\frac{N}{2} - 1 + w^*_p)} \tag{6.26}
\]

The non-dimensional gap height between the upper and lower rack \((h^*_i)\) can be solved using the definition of penetration.

\[
h^*_i = PENH^* - 2h^* \tag{6.27}
\]

Once the non-dimensional lobe height is solved for, the fraction of the primary flow in the lobes \((f_p)\) and the fraction of the secondary flow in the lobes \((f_s)\) can be found using Equation 6.18 and Equation 6.19, respectively. A discharge coefficient \((C_D)\) of 0.97 is used to calculate the effective chute expansion ratio \((CER_{eff})\).

\[
CER_{eff} = \frac{CER}{C_D} \tag{6.28}
\]

The perimeter of the mixer normalized by the wavelength is calculated using the following equation.

\[
L_{mix}^* = 2h^*_i + 2Nh^* + (N - 2) + 2w^*_p \tag{6.29}
\]

The non-dimensional circulation is calculated using the following equation.

\[
\Gamma^* = 2 \frac{\tan \alpha_{mp} + r \tan \alpha_{ms}}{1 + r} \tag{6.30}
\]

The secondary-to-primary velocity ratio \((r)\) is not known beforehand, but reasonable approximations can be made. These approximations are discussed in the section on flow angularity. The boundary layer thickness is assumed to be constant, therefore the non-dimensional boundary layer is inversely proportional to the wavelength \((\lambda)\).

\[
\epsilon^* = 0.05 \frac{\lambda_0}{\lambda} \tag{6.31}
\]

The primary effective flow angle \((\alpha_p)\) and the secondary effective flow angle \((\alpha_s)\) can be calculated using Equation 6.20 and Equation 6.21, respectively.

### 6.5.2 Duct Length

The duct length affects all three components of the TOGW model: the nozzle weight, the net thrust coefficient, and the EPNLdB. By increasing the duct length, the nozzle weight is increased as shown in Figure 6-2.
Changes in duct length affect the net thrust coefficient in two ways. By increasing the duct length, the amount of mixing is increased which leads to an increase in the pumping ratio as shown in Figure 6-3. The increased entrainment tends to increase the net thrust coefficient. However, the increase in duct length also leads to an increase in duct skin friction losses because of the greater amount of wetted surface. The two effects are equal and opposite and therefore result in a maximum net thrust coefficient at 77% of the baseline duct length as shown in Figure 6-4. For shorter ducts, the increase in thrust due to increased entrainment is greater than the decrease in thrust due to the increase in wetted surface. For longer ducts, the skin friction effect dominates and the net thrust coefficient drops with increased duct length.

Increasing the duct length decreases the noise level in four ways: by allowing for more acoustic liner, by decreasing $V_{j,mix}$ due to increased entrainment, by decreasing $V_{j,mix}$ due to additional skin friction losses, and by increasing the exit mixedness. The dominant effect is the increase in acoustic liner treatment due to increased duct length. Figure 6-5 shows the increase in ATAM, which is the ratio of the treated area to the cross-sectional area of the duct, with duct length. The decrease in noise due to the presence of acoustic liner is modeled as being proportional to ATAM. (A more detailed description of the modeling of the liner attenuation is given in Section 5.2.4.) The other three effects are negligible in comparison; thus as shown in Figure 6-6, the decrease in EPNLdB is linear.

The effect of the different weighting systems on the optimal duct length is shown in Figure 6-7.
Sideline Conditions

Figure 6-3: Pumping Ratio Variation with Duct Length.

Sideline Conditions

Figure 6-4: Net Thrust Coefficient Variation with Duct Length.
Figure 6-5: ATAM Variation with Duct Length.

Figure 6-6: EPNLdB Variation with Duct Length.
For weighting system II, which emphasizes nozzle weight, the optimal duct length is driven to a very short duct of around 2.05 m. The optimal duct length for the baseline weighting system is around 2.3 m, which corresponds to the maximum net thrust coefficient. The optimal duct length for weighting system III, which emphasizes the net thrust coefficient also occurs at the same point as the maximum net thrust coefficient. For weighting system IV, which emphasizes EPNLdB, the optimal duct length increases to near 4.5 m where the thrust and nozzle weight penalties offset any additional decrease in EPNLdB.

6.5.3 Suppressor Area Ratio

The suppressor area ratio (SAR) is another of the three design variables that affects the nozzle weight. As shown in Figure 6-8, the nozzle weight increases linearly with SAR. As SAR is increased, the secondary-to-primary area ratio is also increased. As shown in Figure 6-9, the pumping ratio is proportional to the secondary-to-primary area ratio. Increased entrainment tends to increase the net thrust and decrease the EPNLdB.

As SAR increases, the height and width of the duct also increase. Because the penetration is kept constant at the baseline value, the lobe height also increases, resulting in a longer mixer perimeter as shown in Figure 6-10. The lobe height can only increase to the point where all of the primary flow is in the lobes, i.e. the gap height is zero, providing an upper limit for the trade study. Conversely, as SAR decreases, the lobe height decreases until the mixer is rectangular, i.e. the lobe height is zero.
Figure 6-8: Normalized Nozzle Weight Variation with SAR.

Figure 6-9: Pumping Ratio as a Function of Area Ratio for Changes in SAR.
near SAR of 2.2, providing a minimum value for the trade study. Increasing the mixer perimeter leads to an increase in the mixing rate and an increase in the nozzle losses. Additionally, as the lobe height increases, more of the flow leaves at the mixer angle; thereby increasing the off-axis losses. Also, as the dimensions of the duct increase, the duct surface area increases, causing greater losses due to skin friction. All these effects outweigh the increase in thrust due to the increased entrainment; thus there is a decrease in thrust with increasing SAR as shown in Figure 6-11.

ATAM can increase due to two reasons: an increase in surface area and an increase in the initial mixing rate allowing the liner treatment to begin earlier in the duct. The variation in ATAM with SAR is shown in Figure 6-12. The nominal ATAM corresponds to the ATAM if the treatment for all the cases started at the same axial location. The nominal ATAM decreases for geometric reasons. The nominal treated area scales linearly with the duct perimeter which scales with the square root of SAR. The cross-sectional area scales linearly with SAR. Therefore, the nominal ATAM is inversely proportional to the square root of SAR. However, as the SAR is increased, the mixer perimeter increases and consequently so does the initial mixing rate. This allows the liner treatment to begin earlier in the duct; thereby increasing ATAM. This effect levels off near a SAR of 2.3 resulting in a peak ATAM of about 6.1. However, the decrease in noise due to increased entrainment dominates, causing the noise level to decrease with increasing SAR as shown in Figure 6-13.

The effect of SAR on TOGW for the four weighting systems is shown in Figure 6-14. For the baseline weighting system, the optimal SAR occurs at a value of 2.41. For both the nozzle

![Figure 6-10: Normalized Mixer Perimeter Variation with SAR.](image-url)
Figure 6-11: Net Thrust Coefficient Variation with SAR.

Figure 6-12: ATAM Variation with SAR.
Figure 6-13: EPNLdB Variation with SAR.

Figure 6-14: Trade Study on Effect of SAR.
weight emphasized and the net thrust emphasized weighting systems, the optimal SAR (2.2 and 2.24 respectively) occurs near the minimum value of SAR, corresponding to the maximum thrust point. For the EPNLdB emphasized weighting system, the weight and thrust penalties do not outweigh the noise benefits until a value of 3.47 is reached.

6.5.4 Duct Aspect Ratio

The duct aspect ratio (AR) is defined as the ratio of the duct width to the duct height.

\[ AR = \frac{W^*}{H^*} \]  

(6.32)

The duct aspect ratio is the last of the three design variable that affects the nozzle weight as shown in Figure 6-15. However, the effect of AR on nozzle weight is very small, only about 4% maximum variation. The trade study was limited to AR greater than 0.9 because of limitations in the nozzle weight module resulting in non-physical values.

For a fixed penetration, the mixing perimeter slightly decreases as AR is increased as shown in Figure 6-16. The lobe height-to-wavelength ratio \( (h^*) \) decreases significantly as shown in Figure 6-17. Tew found that the effectiveness of streamwise vorticity increased with decreasing lobe height-to-wavelength ratios mixers. Therefore, there is a slight increase in the mixing rate with increasing
Figure 6-16: Normalized Mixer Perimeter Variation with Duct Aspect Ratio.

duct aspect ratio despite a slight decrease in the mixer perimeter. The increase in mixing rate results in an increase in the pumping ratio as shown in Figure 6-18. Discontinuities such as the one seen in the pumping ratio are due to discretizations used in the vortex code and can be decreased by increasing the resolution, resulting in increased run times. Associated with the decrease in mixing perimeter, there is a decrease in the primary nozzle losses. Another consequence of changing the duct aspect ratio is that it affects the amount of wetted surface in the duct. The minimum skin friction occurs for a duct aspect ratio of one and increases for both increasing and decreasing values of AR. All of these effects on the net thrust are small and result in a nearly constant net thrust coefficient with varying values of AR as shown in Figure 6-19.

There is a small decrease in EPNLdB associated with the increase in entrainment as shown in Figure 6-20. An important effect that is not included in this analysis is the effect of aspect ratio on EPNLdB. The aspect ratio affects the directivity of the jet noise and one would expect decrease in sideline noise for higher duct aspect ratios.

The effect of AR on the TOGW for the four weighting systems is shown in Figure 6-20. The optimal point for the weighting system emphasizing the EPNLdB level occurs at the minimum aspect ratio. The curve generated using the weighting system emphasizing net thrust is fairly flat due to the nearly constant thrust variation with AR. For the same reason, the optimal point for the weighting system emphasizing net thrust is very close to that of the baseline weighting system. The maximum variation of the TOGW with AR is the smallest of all the design variables.
Figure 6-17: Lobe Height-to-Wavelength Ratio Variation with Duct Aspect Ratio.

Figure 6-18: Pumping Ratio Variation with Duct Aspect Ratio.
Figure 6-19: Net Thrust Coefficient Variation with Duct Aspect Ratio.

Figure 6-20: EPNLdB Variation with Duct Aspect Ratio.
6.5.5 Mixing Area Ratio

As the mixing area ratio (MAR) is increased, the entrainment ratio increases up to the compound choked point at the balance plane, or the exit of the pressure matching control volume. Because the effects of compound shocks on the noise levels are not modeled and because mode shift is considered undesirable for operation, only mixing area ratios resulting in compound subsonic flow were considered for the trade study.

The noise-thrust trade-off as a function of MAR is shown in Figure 6-23. MAR increases as the curve is followed starting from the upper left corner. There are two competing effects of MAR on the net thrust. As MAR increases, there is a thrust benefit due to increased entrainment, but there is also a thrust penalty due to an increase in skin friction from the increased wall surface area. At the point of maximum thrust (MAR of 1.02), the first derivatives of these two effects with respect to MAR are equal. For lower MAR, the first derivative of the thrust penalty due to the increased skin friction is greater, resulting in decreasing thrust coefficients for decreasing MAR. For MAR greater than 1.02, the benefits in thrust due to increased entrainment is greater than the penalty due to increased skin friction.

The variation of TOGW with MAR for the four weighting systems is shown in Figure 6-24. Because the EPNLdB drops and the thrust increases up to MAR of 1.02, the optimal point will be greater or equal to this value regardless of the weighting system. Since MAR does not affect the
Figure 6-22: Pumping Ratio Variation with MAR.

Figure 6-23: Noise-Thrust Trade-Off with Variations in MAR.
nozzle weight, the optimal value for the baseline weighting system and the nozzle weight emphasized weighting system are equal. The optimal value of MAR has a value of 1.06 where the thrust penalty outweighs any benefit in noise level for further increasing MAR. For the noise emphasized weighting system, this point occurs for a value of MAR of 1.13. The net thrust emphasized weighting system has an optimal MAR at the maximum thrust point.

6.5.6 Penetration

There are geometric limits to the mixer penetration if the other design variables are fixed. As penetration increases, more of the flows are in the lobes, increasing the losses due to flow angularity. Minimum penetration corresponds to a flat plate and the maximum penetration corresponds to a mixer with the maximum amount of flow in the lobes. For the baseline case, the penetration is bounded by $0.39 \leq \text{PEN} \leq 0.67$.

Increasing the penetration also increases the mixer perimeter as shown in Figure 6-25.

The same trends in the pumping ratio and thrust are observed as for the 15 degree mixer in Tew’s trade studies for increasing height-to-wavelength as shown in Figures 1-10 and 1-11. The pumping ratio increases with increasing penetration as shown in Figure 6-26, but the thrust coefficient decreases with increasing penetration as shown in Figure 6-27. The net thrust decreases more rapidly than in Tew’s trade studies because Tew did not account for the mixer skin friction which increases as the penetration increases.
Figure 6-25: Normalized Mixer Perimeter Variation with Penetration.

Figure 6-26: Normalized Mixer Perimeter Variation with Penetration.
The noise-thrust line for varying penetration is shown in Figure 6-28 with the penetration increasing from the upper right corner to the lower left corner of the plot. The slope of the linear part of the noise-thrust line has a slope very close to \(1 \times \frac{EPNL_{dB}}{\%Thrust}\). Because this is the critical slope for the baseline weighting system and the nozzle weight emphasized system, this results in a very flat variation in TOGW for these two weighting systems as shown in Figure 6-29. The optimal value of penetration for the net thrust weighting system is rapidly driven towards the maximum thrust, i.e. minimum penetration; while the optimal value of the penetration for the noise emphasized weighting system is rapidly driven towards the minimum thrust, i.e. maximum penetration.

6.5.7 Number of Lobes

The number of lobes is constrained to an even integer value of four or greater. As the number of lobes increases, so does the mixer perimeter. Figure 6-30 shows how the mixer perimeter normalized by the baseline mixer perimeter varies with the number of lobes. Increases in the mixer perimeter lead to an increase in the pumping ratio as shown in Figure 6-31.

The mixer perimeter also affects the mixing rate, the primary nozzle losses, and the base drag. The noise-thrust line for varying number of lobes is shown in Figure 6-32. The number of lobes starts with four in the upper right corner and increases along the curve. The bend in the curve corresponds to between ten to sixteen lobes. Regardless of the weighting system, the optimal number of lobes
Figure 6-28: Noise-Thrust Trade-Off for Variations in Penetration.

Figure 6-29: Trade Study on Effect of Penetration.
Figure 6-30: Normalized Mixer Perimeter Variation with Number of Lobes.

Figure 6-31: Pumping Ratio Variation with Number of Lobes.
will be less than eighteen because the noise level rises and the thrust decreases for number of lobes greater than sixteen.

Figure 6-32: Noise-Thrust Trade-Off for Variations in Number of Lobes.

The effect of the number of lobes on the TOGW is shown in Figure 6-33. The optimal number of lobes for the net thrust emphasized weighting system is ten. As the relative importance of the EPNLdB is increased the optimal number of lobes increases. For the baseline weighting system and the nozzle weight emphasized weighting system, the optimal number of lobes increases to fourteen lobes. For the noise emphasized weighting system, the optimal number of lobes is sixteen.

6.5.8 Lobe Width Ratio

The lobe width ratio is defined as the ratio of the secondary lobe width to the primary lobe width. As the lobe width ratio is increased, the lobe height-to-wavelength ratio decreases causing the mixer perimeter to decrease as shown in Figure 6-34.

The lobe width ratio also affects the off-axis losses. As the lobe width ratio is increased, less of the primary flow is in the lobes. Also, as the lobe width ratio is increased, the lobe height decreases and less of the secondary flow is in the lobes. The change in the fraction of the flow in the lobes (f) for both the primary and secondary streams is shown in Figure 6-35.

The noise-thrust line for variations in lobe width ratio is shown in Figure 6-36. The lobe width ratio starts with the minimum value of 0.8, corresponding to a gap height of zero, in the upper right corner of the graph and increases along the curve towards the lower left corner of the graph.
Figure 6-33: Trade Study on Effect of Number of Lobes.

Figure 6-34: Normalized Mixer Perimeter Variation with Lobe Width Ratio.
Figure 6-35: Fraction of Flow in Lobes for Variations in Lobe Width Ratio.

Figure 6-36: Noise-Thrust Trade-Off for Variations in Lobe Width Ratio.
The effect of lobe width ratio on TOGW for the four weighting systems is shown in Figure 6-37. Since the slope of the noise-thrust line is near one over most of the range of lobe width ratios, the curves for the baseline weighting system and the nozzle weighting system are flat. (The two curves are actually scaled versions of each other.) The optimal lobe width ratio for the noise emphasized weighting system is driven to the minimum EPNLdB corresponding to the minimum lobe width ratio. The optimal lobe width ratio for the thrust emphasized weighting system occurs at a lobe width ratio of 4.45 where the noise penalty starts to outweigh the thrust benefit.

6.5.9 Primary Mixer Angle

There are two ways to remove flow angularity.

1. Lower $h/\lambda$ which also lowers the initial perimeter.

2. Add extensions to the lobes to turn the flow axially.

For the trade studies, it was assumed that the penetration and the flow angularity could be varied independently. This can be done to a degree by adding extensions. Figure 6-38 shows how the circulation drops off as extensions are added to the forced mixer. Most, but not all, of the flow angularity can be removed. However, there is a penalty associated with decreasing the flow angularity.
Figure 6-38: Axial Forced Mixer and Convoluted Plate Circulation Distribution [20].
by adding extensions, increased skin friction drag. This effect is not accounted for in the following trade studies. If it were, it would drive the solution to higher flow angles.

The primary mixer angle affects the amount of streamwise vorticity and the form drag losses. Increased flow angularity results in more streamwise vorticity which increases the rate of mixing. This increase in the mixing rate allows the acoustic treatment to begin earlier in the duct, increasing ATAM as shown in Figure 6-39. The mixing rate increase results in a greater exit mixedness and therefore an increase in the entrainment as indicated in Figure 6-40. The same trend is observed in Figure 1-10 by Tew for the $h^* = 1.25$ mixer where the pumping ratio increases and begins to level off near a lobe angle of 15 degrees. The trends in the net thrust coefficient are similar, but slightly different than, those observed by Tew in Figure 1-11 for the $h^* = 1.25$ mixer. In the calculations by Tew, the is a slight initial increase of 0.3% in net thrust coefficient with the addition of flow angularity, while there is an initial decrease of 1.2% in net thrust coefficient with primary mixer angle as shown in Figure 6-41. The differences are due to different fractions of the primary flow in the lobes. For the baseline design, 75% of the primary flow is in the lobes as compared to 25% for the mixer used by Tew. This difference results in a greater thrust penalty for increasing the primary flow angle. For lobe angles greater than 10 degrees, both trade studies indicate a declining thrust coefficient.
Figure 6-40: Pumping Ratio Variation with Primary Mixer Angle.

Figure 6-41: Net Thrust Coefficient Variation with Primary Mixer Angle.
Increased flow angularity also leads to increased form drag. Figure 6-42 shows the noise-thrust line for variations in the primary mixer angle. The absence of a maximum thrust indicates that the form losses are always greater than the increase in thrust due to increased pumping. As the primary mixer angle is increased, both noise and thrust decrease.

![Figure 6-42: Noise-Thrust Trade-Off for Variation in Primary Mixer Angle.](image)

The impact of the primary lobe angle on the TOGW is shown in Figure 6-43 for the four weighting systems. The optimal primary lobe angle for the baseline weighting system is 5.9 degrees. By increasing the relative weighting of the net thrust coefficient, the optimal primary lobe angle is lowered to 2.1 degrees. There is no benefit to further reducing the primary lobe angle because the slope of the noise-thrust line rapidly increases at this point. The increasing the relative weighting of the noise level, the optimal lobe angle is increased to the maximum flow angle of 15 degrees. This will occur whenever the ratio of net thrust sensitivity to the noise sensitivity exceeds the slope of the noise-thrust line, approximately 0.41.

### 6.5.10 Secondary Mixer Angle

The same trade-offs occur when varying the secondary mixer angle as varying the primary mixer angle. However, the secondary mixer angle has much smaller leverage on the streamwise vorticity and the form drag losses. As shown in Equation 3.61, the impact of the secondary mixer angle on the circulation is smaller than the impact of the primary mixer angle by a factor of \( r \), the ratio of the secondary to primary velocities. The form drag losses are proportional to the off-axis kinetic energy.
Sideline Conditions

Baseline Weighting
- Nozzle Weight Emphasized
- Net Thrust Emphasized
- Noise Emphasized

Figure 6-43: Trade Study on Effect of Primary Mixer Angle.

(as described in Section 2.3.3), which is much less in the secondary stream than in the primary stream. The amount of treated area increases as shown in Figure 6-44.

The pumping increases with increased secondary mixer angle as shown in Figure 6-45. The same trend is observed in Figure 1-10 by Tew where the pumping ratio increases and begins to level off near a lobe angle of 15 degrees. The trends in the net thrust coefficient are also similar to those observed by Tew in Figure 1-11. There is a slight increase in the net thrust coefficient at first, but as the lobe angle increases the form drag effects begin to dominate as shown in Figure 6-46.

Unlike the primary area ratio, there is a maximum thrust point on the noise-thrust line. This maximum is the point where the increase in thrust due to additional pumping is exactly counterbalanced by the increase in form drag. The maximum thrust occurs at a secondary lobe angle of 3.6 degrees. Also, since EPNLdB is monotonic, the optimal secondary lobe angle will never be less than the 3.6 degrees, regardless of the weighting system. The reason why there is a different behavior between the primary and secondary noise-thrust trade-off can be traced back to the different ratios of lobe areas to total area, $f_p$ and $f_s$. Because $f_p$ (0.75) is greater than $f_s$ (0.3), a greater amount of the primary leaves at the mixer angle. This does not affect the non-dimensional circulation, but it does affect the off-axis losses.

The optimal secondary mixer angle for the baseline weighting system is 13.7 degrees. At this point, the trade-off between thrust and EPNLdB is equal to one EPNLdB per point of thrust loss.
Figure 6-44: ATAM Variation with Secondary Mixer Angle.

Figure 6-45: Pumping Ratio Variation with Secondary Mixer Angle.
Figure 6-46: Net Thrust Coefficient Variation with Secondary Mixer Angle.

Figure 6-47: Noise-Thrust Trade-Off for Variations in Secondary Mixer Angle.
By increasing the relative weighting of the net thrust, the optimal point moves to 4.1 degrees which is much closer to the maximum thrust point. For an increased relative weighting of EPNLdB, the optimal secondary lobe angle increases to the maximum of 15 degrees. This means that for the range of 0 to 15 degrees, the trade-off between thrust and EPNLdB never exceeds 10 EPNLdB per point of thrust loss.

![Sideline Conditions](image)

**Figure 6-48**: Trade Study on Effect of Secondary Mixer Angle.

### 6.5.11 Chute Expansion Ratio

As the chute expansion ratio is increased, the ratio of secondary to primary area is decreased. As shown in Figure 6-49, this results in a decrease in the pumping ratio. The maximum entrainment occurs at a CER value of 1.13.

\[
\frac{A_s}{A_p} = \frac{SAR}{CER} - 1
\]  

(6.33)

As CER increases, the primary area increases. Because the penetration is kept constant at the baseline value, the lobe height decreases and the gap height increase in order to accommodate the increase in the primary area. This leads to a decrease in the mixing perimeter as shown in Figure 6-50.
Figure 6-49: Pumping Ratio Variation with Chute Expansion Ratio.

Figure 6-50: Normalized Mixer Perimeter Variation with Chute Expansion Ratio.
For CER less than that corresponding to the peak net thrust coefficient, there is an approximately linear relationship between noise and thrust as shown in Figure 6-51. The average slope is approximately $0.1 \frac{EPNL_{dB}}{\% \text{Thrust}}$. Because this is the critical slope for the noise emphasized weighting system, the resultant TOGW plot for that weighting system is nearly constant over the range of CER. All other weighting systems are driven towards the maximum net thrust coefficient which occurs at a CER of approximately 1.2 as shown in Figure 6-52.

### 6.5.12 Summary of Trade Studies

Six physical effects have been examined in the trades studies: effects of mixer perimeter, effect of area ratio, liner attenuation, flow angularity losses, effects of entrainment, and skin friction. Mixer perimeter is an important characteristic for optimization because it affects the figures of merit in many ways. Increasing the mixer perimeter; increases the initial mixing rate allowing for more liner treatment, increases the total mixing thereby increasing entrainment, increases the nozzle losses due to an increase in the hydraulic diameter, and increases the base drag due to an increase in the base area. Increasing the area ratio increases entrainment, but increases nozzle weight. Increasing flow angularity tends to decrease EPNLdB at the expense of net thrust. Increasing entrainment both decreases jet noise by lowering the average exit velocity and increases the net thrust. Increasing the wetted surface of the duct increases the skin friction losses.
Table 6.10 provides a summary of the main physical effects involved in each of the trade studies. The effect of each parameter of each of six physical effects is given: the mixer perimeter (P), the area ratio ($\frac{A_e}{A_p}$), the amount of liner attenuation (ATAM), the effective flow angularity losses ($\alpha_{eff}$), the pumping ratio (WSWP), and the skin friction (S). Each parameter affects these physical parameters in one of four ways: increase (↑), decrease (↓), no effect (=), or initially increases then decreases (<>).

The optimal points from the trade studies are summarized in Table 6.11. For the baseline weighting system, there is a balance between noise, thrust, and nozzle weight. As nozzle weight is
emphasized, only those design variables which affect the nozzle weight (duct length, SAR, and AR) are affected. For the net thrust emphasized weighting system, the trade studies indicate designs with less mixer perimeter, characterized by lower number of lobes and lower penetration. The net thrust emphasized weighting system trade studies also indicate designs with less (but non-zero) mixer angles. For the EPNLdB emphasized weighting system, the trends are the opposite. Trade studies for the noise emphasized weighting system indicate designs with increased mixing perimeter (higher lobe count and higher penetration) and increased flow angularity.

However, optimizing each variable individually does not necessarily result in an optimal design. For instance, the designs resulting from the trade study results are not even physically possible for all but the noise emphasized weighting system.

### 6.6 Optimization

The next step was to perform a multi-variable optimization of the mixer-ejector to see if the the optimal design variables are driven to a different region of the design space than the baseline design.

#### 6.6.1 Formulation of Problem

The design optimization was formulated as a constrained optimization problem. The cost function minimized was the take-off gross weight (TOGW), subject to inequality constraints detailed in the following section.

\[
\min_{\alpha_{\text{mixer},p}, \alpha_{\text{mixer},s}} \text{TOGW} \quad \text{s.t.} \quad d_{\text{min}} \leq d \leq d_{\text{max}}
\]  

(6.34)
6.6.2 Constraints

Two of the constraints pertain to realizable geometries. It is possible to select a set of design variables which is non-physical. In order to avoid this, the non-dimensional gap height \( h^*_1 \) and the lobe height-to-wavelength ratio \( h^* \) are constrained to be positive.

\[
h^*_1 \geq 0 \quad (6.35)
\]

\[
h^* \geq 0 \quad (6.36)
\]

Because the system weightings can only be approximated as linear for values near the design point, the net thrust coefficient is constrained to be greater than 0.91 and the EPNLdB is constrained to be less than 101.1.

\[
C_{fn} \geq 0.91 \quad (6.37)
\]

\[
EPNLdB \leq 101.1 \quad (6.38)
\]

There are also upper and lower bounds on individual design variables. MAR was constrained to be less than one to avoid compound shocks. The chute expansion ratio (CER) was constrained to be greater than one to maintain a supersonic nozzle and the primary lobe angle was limited to 20 degrees to avoid separation.

6.6.3 Numerical Method

The optimization algorithm was formulated as a sequential quadratic programming (SQP) problem with the Hessian updated using the BFGS (Broyden, Fletcher, Goldfarb, and Shanno) method. It is commercially available and is detailed in Reference [15]. The algorithm requires a continuous function and constraints which were calculated using the integrated tool.

6.6.4 Optimal Design

First, the multi-variable optimization was performed allowing the number of lobes to take positive real numbers. After finding the optimal real value of N, the optimization was run using the next lowest even number for N and again using the next highest even number for N. The lowest TOGW for these two cases was used as the optimal design.

Useful information can be derived by comparing the optimal design variable values from the multi-variable optimization (Table 6.12) and the optimal design variable values from the trade
Table 6.12: Optimal Design Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Optimal Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.182</td>
</tr>
<tr>
<td>SAR</td>
<td>2.504</td>
</tr>
<tr>
<td>AR</td>
<td>1.532</td>
</tr>
<tr>
<td>MAR</td>
<td>1.000</td>
</tr>
<tr>
<td>$L_{duct}$</td>
<td>2.430 m</td>
</tr>
<tr>
<td>PEN</td>
<td>0.746</td>
</tr>
<tr>
<td>N</td>
<td>16</td>
</tr>
<tr>
<td>$\frac{w_a}{w_p}$</td>
<td>1.048</td>
</tr>
<tr>
<td>$\alpha_{mizer,p}$</td>
<td>12.50</td>
</tr>
<tr>
<td>$\alpha_{mizer,s}$</td>
<td>15.50</td>
</tr>
</tbody>
</table>

studies (Table 6.11). The most striking difference is the value of the penetration. The optimal penetration occurs at a value not even possible starting from the baseline design because it would result in a primary lobe area greater than the total primary area. The flow angles are also greater than the trade study counterparts. The optimal lobe width ratio is almost half the value of the trade study counterpart. Increasing the penetration and flow angles increases the mixing rates, but increases the losses. Reducing the lobe width ratio mitigates the losses by reducing the amount of primary flow leaving at the mixer angle and thereby reducing the off-axis losses. This supports the view that the optimal design will not be reached by examining each design variable independently as is done with parametric testing.

6.6.5 Optimal Performance

The optimal performance is given in Table 6.13. The constraint on EPNLdB is active. This means

Table 6.13: Optimal Performance Compared to Baseline Mixer-Ejector Performance

<table>
<thead>
<tr>
<th>Figure of Merit</th>
<th>Optimal Value</th>
<th>Baseline Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{fn}$</td>
<td>0.966</td>
<td>0.947</td>
</tr>
<tr>
<td>EPNLdB</td>
<td>101.1</td>
<td>97.6</td>
</tr>
<tr>
<td>Nozzle Weight*</td>
<td>0.8312</td>
<td>1</td>
</tr>
<tr>
<td>$\Delta TOGW^*$</td>
<td>-0.2585</td>
<td>1</td>
</tr>
</tbody>
</table>

that the TOGW could be lowered if the noise level were allowed to exceed 101.1 EPNLdB. The nozzle weight is about 17% less than the baseline design. The optimal thrust is 1.9% higher and the noise level is 3.5 EPNLdB higher. This means that for the baseline weighting system, it is beneficial to increase the noise level in order to increase thrust and decrease the nozzle weight.
6.6.6 Sensitivity Analysis

To determine the primary drivers, a sensitivity study was performed to determine the change in TOGW for changes in the optimal design. One measure of sensitivity is to compute the partial derivative of the TOGW with respect to each design variable. If a variable is unconstrained, the partial derivative will be zero and an alternative measure of sensitivity is necessary. The method chosen is similar to the partial derivative, but uses a finite difference (equivalent to a partial derivative for a difference approaching zero.) For this study, a difference of 10 percent in the design variable was used to calculate the sensitivities. Both measures were normalized so that the resulting numbers represent a fraction change in $\Delta TOGW$ for a fraction change in the design variable.

$$S_i = \frac{D_{i,\text{opt}}}{TOGW_{\text{opt}}} \frac{\Delta TOGW}{\Delta D_i} \bigg|_{\Delta D_i = \pm 10\%} \quad (6.39)$$

The resulting partial derivatives are shown in Figure 6-53 and the sensitivities for a 10% change in the design variables are given in Table 6-54.

Near the optimal design, TOGW is most sensitive to changes in SAR, MAR, and primary flow angle. For a 1% increase in SAR, there is a 3.5% increase in the optimal $\Delta TOGW$. Conversely, for a 1% decrease in SAR, there would be a 3.5% decrease in the optimal $\Delta TOGW$, except this would lead to a violation of the maximum EPNLdB constraint. Similarly, the $\Delta TOGW$ could be
decreased by decreasing the penetration, lobe width ratio, or the primary mixer angle, but they are all constrained by the maximum EPNLdB.

![Normalized Sensitivities of TOGW for Finite Design Changes.](image)

**Figure 6-54:** Normalized Sensitivities of TOGW for Finite Design Changes.

For design variables with a positive $+S_i$ and $-S_i$, the TOGW increases as the design variable is changed in either direction and correspond to partial derivatives of zero (or small values due to numerical precision) at the design point. For design variables with one sensitivity positive and one negative, the TOGW would decrease in the direction of the negative value. For example, the TOGW would decrease for a decrease in SAR as indicated by the negative value of $-S_i$ for SAR. However, for every case this would also lead to an increase in EPNLdB violating the constraint on the maximum EPNLdB. This occurs for SAR, PEN, the lobe width ratio, and the primary mixer angle. The partial derivative of MAR is negative because it is constrained. The take-off gross weight could be further decreased for an increase in MAR, as long as compound shocks do not occur.

In addition, the shadow costs, or increases in the cost function due to the presence of an active constraint, are given in Table 6.14. The two hard constraints: a positive gap height and a positive lobe height are not active, but it would be meaningless to assign a shadow cost even of zero. The only active constraint is on the maximum EPNLdB. The optimal $\Delta TOGW$ could be reduced an additional 22% if this constraint were relaxed by 1 EPNLdB.
Table 6.14: Shadow Costs of Constraints

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Shadow Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum EPNLdB</td>
<td>0.22</td>
</tr>
<tr>
<td>Minimum Cfn</td>
<td>0</td>
</tr>
<tr>
<td>Positive Gap Height</td>
<td>NA</td>
</tr>
<tr>
<td>Positive Lobe Height</td>
<td>NA</td>
</tr>
</tbody>
</table>

6.6.7 Primary Drivers

The design variables that are the primary drivers in mixer-ejector performance near the optimal design are SAR, MAR, and the primary flow angle. SAR and the primary flow are constrained by the maximum allowable noise and MAR to avoid compound shocks. By relaxing these constraints, a corresponding increase in mixer-ejector performance can be obtained by changing these design variables.
Chapter 7

Sensitivity Studies

7.1 Modeling Uncertainties

For each component of the fluid dynamic model, there is a degree of uncertainty. In this section, the sensitivity of the optimal design and the optimal performance to these uncertainties is assessed by making changes in each subcomponent and repeating the optimization. The sensitivities establish how uncertainties in the modeling process are propagated into the optimization process and become uncertainties in the optimal design and performance. These sensitivities are also used to determine which components, and therefore which effects, have the most impact on the optimal design and performance and thus should be focused on.

There are also inherent uncertainties in the optimization process. The multi-variable optimization continues until the solution meets prescribed tolerances on the optimal design variables, constraint violations, and the optimal value of the cost function. The convergence criterion on the design variables is 0.01 of the optimal value and the convergence criterion on the optimal ΔTOGW is 5.0, corresponding to roughly 2.4% of the optimal value. If changes in a model change the optimal performance by less than 2.4%, it can not be determined if the difference is due to modeling uncertainty or numerical precision of the optimization algorithm.

7.1.1 Shear Layer Growth Rate

The shear layer growth rate, used to calculate the interstream shear stress and also used as an input to the vortex code, represents one of the largest uncertainties. There is uncertainty in the pitot thickness growth rate correlation, especially for convective Mach numbers in the range between 0.25 and 0.5 as shown in Figure 3-1 and in the velocity thickness to pitot thickness ratio as described in Section 3.2.4.

To examine the sensitivity of the optimal design and performance to the shear layer growth rate
sub-model, the optimization process was repeated with values for the shear layer growth rate of 75% and 125% of the nominal values, roughly corresponding to the range of values calculated to match the single lobe experiments discussed in Chapter 4. The resulting optimal designs and the percentage change ($\Delta\%$) from the nominal optimal design are given in Table 7.1. Increasing the shear layer growth rate affects the mixing rate in two ways. It increases the interstream shear stress, but it decreases the effective interface length by causing the shear layers to merge earlier in the duct. The two effects roughly counteract each other as shown by the relatively small changes in the design and performance. Most of the changes in the design variables are by a few percent, not much greater than the prescribed tolerance for the design variables used in the optimization process, thus can be considered insignificant.

The resulting figures of merit and the percentage change in the $\Delta$TOGW are given in Table 7.2. The nozzle weight is normalized by the nozzle weight of the baseline design used for the trade studies.

### Table 7.1: Sensitivity of Optimal Design to Uncertainty in Shear Layer Correlations

<table>
<thead>
<tr>
<th>Variable</th>
<th>75%</th>
<th>$\Delta%$</th>
<th>125%</th>
<th>$\Delta%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.148</td>
<td>-2.9</td>
<td>1.169</td>
<td>-1.1</td>
</tr>
<tr>
<td>SAR</td>
<td>2.491</td>
<td>-0.5</td>
<td>2.515</td>
<td>0.4</td>
</tr>
<tr>
<td>AR</td>
<td>1.490</td>
<td>-2.7</td>
<td>1.447</td>
<td>-5.5</td>
</tr>
<tr>
<td>MAR</td>
<td>1.000</td>
<td>0</td>
<td>1.000</td>
<td>0</td>
</tr>
<tr>
<td>$L_{duct}$</td>
<td>2.474</td>
<td>1.8</td>
<td>2.483</td>
<td>2.2</td>
</tr>
<tr>
<td>PEN</td>
<td>0.765</td>
<td>2.6</td>
<td>0.707</td>
<td>-5.3</td>
</tr>
<tr>
<td>N</td>
<td>14</td>
<td>-12.5</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{w_{a}}{w_{p}}$</td>
<td>1.104</td>
<td>5.4</td>
<td>1.070</td>
<td>2.1</td>
</tr>
<tr>
<td>$\alpha_{mixture,p}$</td>
<td>13.445</td>
<td>7.6</td>
<td>13.001</td>
<td>4.0</td>
</tr>
<tr>
<td>$\alpha_{mixture,s}$</td>
<td>15.766</td>
<td>1.7</td>
<td>17.235</td>
<td>11.2</td>
</tr>
</tbody>
</table>

7.1.2 Skin Friction

The presence of acoustic liner introduces the major source of uncertainty in skin friction. Acoustic liners increase the roughness factor of the walls, thereby increasing the skin-friction coefficient.
Assuming a constant skin friction coefficient throughout the duct, Gamble estimated the increase in skin friction coefficient due to the presence of acoustic liner using data from sub-scale experiments [13] and calculated the roughness factor necessary to result in the calculated skin friction coefficient. The roughness factor was then used to estimate the increase in skin friction due to acoustic liner for the projected product size. His estimations show that the presence of acoustic liner could increase the skin friction by a factor of two for a product-scale mixer-ejector system.

To examine the sensitivity of the optimal design and performance to uncertainties in the skin friction, the optimization process was repeated with values for the skin friction coefficient \( C_f \) of 50% and 200% of the nominal value. The factor of one-half roughly corresponds to the decrease in skin friction between sub-scale and product scale due to Reynolds number effects. The factor of two roughly corresponds to the projected increase in the skin friction due to the presence of acoustic liner calculated by Gamble. The resulting optimal designs and the percentage change (\( \Delta \% \)) from the nominal optimal design are given in Table 7.3. There is little change in the design variables as a result of the change in the amount of skin friction. As expected, as the skin friction is increased, the amount of duct surface area for the optimal design decreases. The amount of duct surface area for the 50%, nominal, and 200% cases are 16.14 \( m^2 \), 15.73 \( m^2 \), and 15.71 \( m^2 \) respectively.

The resulting figures of merit and the percentage change in the \( \Delta \)TOGW are given in Table 7.4. As expected, there is a slight thrust increase in the net thrust coefficient with a decrease in the

<table>
<thead>
<tr>
<th>Variable</th>
<th>50% A%</th>
<th>200% A%</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.186</td>
<td>0.4</td>
</tr>
<tr>
<td>SAR</td>
<td>2.494</td>
<td>-0.4</td>
</tr>
<tr>
<td>AR</td>
<td>1.458</td>
<td>-4.8</td>
</tr>
<tr>
<td>MAR</td>
<td>1.000</td>
<td>0</td>
</tr>
<tr>
<td>L_{duct}</td>
<td>2.510</td>
<td>3.3</td>
</tr>
<tr>
<td>PEN</td>
<td>0.723</td>
<td>-3.1</td>
</tr>
<tr>
<td>N</td>
<td>14</td>
<td>-12.5</td>
</tr>
<tr>
<td>( \omega_s )</td>
<td>0.959</td>
<td>-8.4</td>
</tr>
<tr>
<td>( \alpha_{mixture,p} )</td>
<td>12.438</td>
<td>-0.5</td>
</tr>
<tr>
<td>( \alpha_{mixture,s} )</td>
<td>16.235</td>
<td>4.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Figure of Merit</th>
<th>50%</th>
<th>200%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{f_n} )</td>
<td>0.971</td>
<td>0.961</td>
</tr>
<tr>
<td>EPNLdB</td>
<td>101.1</td>
<td>101.1</td>
</tr>
<tr>
<td>Nozzle Weight*</td>
<td>0.833</td>
<td>0.830</td>
</tr>
<tr>
<td>% Change in ( \Delta )TOGW</td>
<td>-12.7</td>
<td>25.2</td>
</tr>
</tbody>
</table>
skin friction and a slight decrease in the net thrust coefficient with an increase in the skin friction coefficient.

### 7.1.3 Off-Axis Losses

Off-axis losses are associated with the loss of the off-axis kinetic energy. If some fraction \( f \) of the off-axis kinetic energy is recovered isentropically, losses associated with flow angularity will decrease. Isentropic recovery of the off-axis kinetic energy can be modeled by modifying the effective angles. The off-axis kinetic energy that is lost scales with the sine of the angle squared. Therefore, the fraction of the off-axis kinetic energy satisfies the following relationship involving the effective angle \( \alpha_{\text{eff}} \) and the modified, or new effective angle \( \alpha_{\text{neff}} \).

\[
1 - f = \frac{\sin^2 \alpha_{\text{neff}}}{\sin^2 \alpha_{\text{eff}}}
\]  

(7.1)

By rearranging Equation 7.1, one arrives at the following relationship for the new effective angle based on the previous effective angle and the value of \( f \).

\[
\alpha_{\text{neff}} = \sin^{-1}(\sqrt{1 - f \sin \alpha_{\text{eff}}})
\]  

(7.2)

To examine the sensitivity of the optimal design and performance to uncertainty in the off-axis losses, the optimization process was repeated for the case where half of the kinetic energy is recovered isentropically \( (f = 0.5) \). The resulting optimal designs and the percentage change (\( \Delta\% \)) from the nominal optimal design are given in Table 7.5. As expected, if some of the off-axis flow is turned isentropically, the optimal design increases in the amount of flow angularity in both the primary and secondary streams.

<table>
<thead>
<tr>
<th>Variable</th>
<th>( f = 0.5 )</th>
<th>( \Delta% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.175</td>
<td>-0.6</td>
</tr>
<tr>
<td>SAR</td>
<td>2.501</td>
<td>-0.1</td>
</tr>
<tr>
<td>AR</td>
<td>1.470</td>
<td>-4.0</td>
</tr>
<tr>
<td>MAR</td>
<td>1.000</td>
<td>0</td>
</tr>
<tr>
<td>( L_{\text{duct}} ) (m)</td>
<td>2.532</td>
<td>4.2</td>
</tr>
<tr>
<td>PEN</td>
<td>0.765</td>
<td>2.6</td>
</tr>
<tr>
<td>( \omega )</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_p )</td>
<td>1.102</td>
<td>5.2</td>
</tr>
<tr>
<td>( \alpha_{\text{mizer}_{,p}} )</td>
<td>14.432</td>
<td>15.5</td>
</tr>
<tr>
<td>( \alpha_{\text{mizer}_{,s}} )</td>
<td>17.108</td>
<td>10.4</td>
</tr>
</tbody>
</table>

Table 7.5: Sensitivity of Optimal Design to Uncertainty in Form Drag

The resulting figures of merit and the percentage change in the \( \Delta\text{TOGW} \) are given in Table 7.6.
As a fraction of the off-axis kinetic energy is recovered, there is an increase in the net thrust of the optimal design and consequently a decrease in the optimal TOGW.

### 7.1.4 Inlet Losses

To examine the sensitivity of the optimal design to the inlet losses, the optimization process was repeated for values of the inlet total pressure loss of one-half and twice the nominal values. The resulting optimal designs and the percentage change (Δ%) from the nominal optimal design are given in Table 7.7. The variation in inlet losses has little effect on the optimal design. The reason for this can be traced back to the modeling of the inlet losses. The total pressure recovery (ηₚ) correlation is composed of two parts: a constant part (a) independent of the design variables and the skin friction term (bMₛ²) which is only affected through the secondary Mach number.

\[ ηₚ = a + bMₛ² \]  

(7.3)

The inlet total recovery factor has little dependency on the design variables, so there is little change in the optimal design variables as the magnitude of the inlet losses are varied.

However, this does not imply that the inlet losses are insignificant. The resulting figures of merit and the percentage change in the ΔTOGW given in Table 7.8. Because the inlet losses are not

### Table 7.6: Sensitivity of Optimal Performance to Uncertainty in Form Drag

<table>
<thead>
<tr>
<th>Figure of Merit</th>
<th>f = 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cᵢ₀</td>
<td>0.973</td>
</tr>
<tr>
<td>EPNLdB</td>
<td>101.1</td>
</tr>
<tr>
<td>Nozzle Weight*</td>
<td>0.835</td>
</tr>
<tr>
<td>% Change in ΔTOGW</td>
<td>-19.6</td>
</tr>
</tbody>
</table>

### Table 7.7: Sensitivity of Optimal Design to Uncertainty in Inlet Losses

<table>
<thead>
<tr>
<th>Variable</th>
<th>50%</th>
<th>Δ%</th>
<th>200%</th>
<th>Δ%</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.190</td>
<td>0.7</td>
<td>1.175</td>
<td>-0.6</td>
</tr>
<tr>
<td>SAR</td>
<td>2.481</td>
<td>-0.9</td>
<td>2.497</td>
<td>-0.3</td>
</tr>
<tr>
<td>AR</td>
<td>1.539</td>
<td>0.5</td>
<td>1.465</td>
<td>-4.4</td>
</tr>
<tr>
<td>MAR</td>
<td>1.000</td>
<td>0</td>
<td>1.000</td>
<td>0</td>
</tr>
<tr>
<td>L_duct</td>
<td>2.562</td>
<td>5.4</td>
<td>2.518</td>
<td>3.6</td>
</tr>
<tr>
<td>PEN</td>
<td>0.742</td>
<td>-0.5</td>
<td>0.746</td>
<td>0.0</td>
</tr>
<tr>
<td>N</td>
<td>16</td>
<td>0</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td>w₆/wₚ</td>
<td>1.118</td>
<td>6.7</td>
<td>1.106</td>
<td>5.5</td>
</tr>
<tr>
<td>αₘixer,p</td>
<td>13.032</td>
<td>4.3</td>
<td>13.736</td>
<td>9.9</td>
</tr>
<tr>
<td>αₘixer,s</td>
<td>16.424</td>
<td>6.0</td>
<td>16.465</td>
<td>6.2</td>
</tr>
</tbody>
</table>
Table 7.8: Sensitivity of Optimal Performance to Uncertainty in Inlet Losses

<table>
<thead>
<tr>
<th>Figure of Merit</th>
<th>50%</th>
<th>200%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{fn}$</td>
<td>0.975</td>
<td>0.949</td>
</tr>
<tr>
<td>EPNLdB</td>
<td>101.0</td>
<td>101.1</td>
</tr>
<tr>
<td>Nozzle Weight*</td>
<td>0.833</td>
<td>0.834</td>
</tr>
<tr>
<td>% Change in $\Delta$TOGW</td>
<td>-40.5</td>
<td>91.4</td>
</tr>
</tbody>
</table>

sensitive to changes in the design variables, it is not possible to mitigate increases in the inlet losses by changing the design. Changes in the inlet losses thus translate directly to changes in the optimal net thrust coefficient. As expected, the net thrust coefficient at the optimal design point is greater for decreased inlet losses and less for increased inlet losses.

7.1.5 Nozzle Losses

To examine the sensitivity of the optimal design to the primary nozzle losses, the optimization process was repeated with values for the stream thrust coefficient ($C_s$) of one-half and twice the nominal values.

The resulting optimal designs and the percentage change (Δ%) from the nominal optimal design are given in Table 7.9.

Table 7.9: Sensitivity of Optimal Design to Uncertainty in Nozzle Losses

<table>
<thead>
<tr>
<th>Variable</th>
<th>50%</th>
<th>Δ%</th>
<th>200%</th>
<th>Δ%</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.157</td>
<td>-2.1</td>
<td>1.189</td>
<td>0.7</td>
</tr>
<tr>
<td>SAR</td>
<td>2.472</td>
<td>-1.3</td>
<td>2.525</td>
<td>0.8</td>
</tr>
<tr>
<td>AR</td>
<td>1.480</td>
<td>-3.4</td>
<td>1.442</td>
<td>-5.9</td>
</tr>
<tr>
<td>MAR</td>
<td>1.000</td>
<td>0</td>
<td>1.000</td>
<td>0</td>
</tr>
<tr>
<td>$L_{duct}$</td>
<td>2.485</td>
<td>2.2</td>
<td>2.487</td>
<td>2.4</td>
</tr>
<tr>
<td>PEN</td>
<td>0.747</td>
<td>0.1</td>
<td>0.761</td>
<td>2.0</td>
</tr>
<tr>
<td>$N$</td>
<td>14</td>
<td>-12.5</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{S_{i_a}}{w_p}$</td>
<td>1.083</td>
<td>3.3</td>
<td>1.094</td>
<td>4.4</td>
</tr>
<tr>
<td>$\alpha_{m, p}$</td>
<td>12.635</td>
<td>1.1</td>
<td>12.658</td>
<td>1.3</td>
</tr>
<tr>
<td>$\alpha_{m, s}$</td>
<td>17.173</td>
<td>10.8</td>
<td>17.644</td>
<td>13.9</td>
</tr>
</tbody>
</table>

The resulting figures of merit and the percentage change in the $\Delta$TOGW are given in Table 7.10. As expected, the net thrust coefficient at the optimal design point is greater for decreased nozzle losses and less for increased nozzle losses.
7.1.6 Base Drag

To examine the sensitivity of the optimal design to the base drag, the optimization process was repeated with values for the base pressure coefficient ($C_B$) of 0.1 and 0.4 (one-half and twice the nominal value).

The resulting optimal designs and the percentage change ($\Delta\%$) from the nominal optimal design are given in Table 7.11. The optimal design is not very sensitive to changes in the base drag. As the base drag is decreased, the mixer perimeter of the optimal design increases slightly.

The resulting figures of merit and the percentage change in the $\Delta\text{TOGW}$ are given in Table 7.12. Varying the base drag also has little impact on the optimal performance because the it has less impact on the thrust ($0.07\%$) than other effects such as the nozzle losses ($2.1\%$), inlet losses ($1.8\%$), etc.

### Table 7.10: Sensitivity of Optimal Performance to Uncertainty in Nozzle Losses

<table>
<thead>
<tr>
<th>Figure of Merit</th>
<th>50%</th>
<th>200%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{fn}$</td>
<td>0.971</td>
<td>0.958</td>
</tr>
<tr>
<td>EPNLdB</td>
<td>101.1</td>
<td>100.9</td>
</tr>
<tr>
<td>Nozzle Weight*</td>
<td>0.830</td>
<td>0.837</td>
</tr>
<tr>
<td>% Change in $\Delta\text{TOGW}$</td>
<td>-24.3</td>
<td>50.6</td>
</tr>
</tbody>
</table>

**Table 7.11: Sensitivity of Optimal Design to Uncertainty in Base Drag Coefficient**

<table>
<thead>
<tr>
<th>Variable</th>
<th>50%</th>
<th>$\Delta%$</th>
<th>200%</th>
<th>$\Delta%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.189</td>
<td>0.6</td>
<td>1.183</td>
<td>0.1</td>
</tr>
<tr>
<td>SAR</td>
<td>2.477</td>
<td>-1.0</td>
<td>2.498</td>
<td>-0.2</td>
</tr>
<tr>
<td>AR</td>
<td>1.492</td>
<td>-2.6</td>
<td>1.496</td>
<td>-2.3</td>
</tr>
<tr>
<td>MAR</td>
<td>1.000</td>
<td>0</td>
<td>1.000</td>
<td>0</td>
</tr>
<tr>
<td>$L_{dual}$</td>
<td>2.449</td>
<td>0.8</td>
<td>2.478</td>
<td>2.0</td>
</tr>
<tr>
<td>PEN</td>
<td>0.750</td>
<td>0.5</td>
<td>0.761</td>
<td>2.0</td>
</tr>
<tr>
<td>N</td>
<td>16</td>
<td>0</td>
<td>14</td>
<td>-12.5</td>
</tr>
<tr>
<td>$\omega_{w}$</td>
<td>0.961</td>
<td>-8.3</td>
<td>1.099</td>
<td>4.9</td>
</tr>
<tr>
<td>$\alpha_{\text{mizer,p}}$</td>
<td>12.778</td>
<td>2.3</td>
<td>12.291</td>
<td>-1.6</td>
</tr>
<tr>
<td>$\alpha_{\text{mizer,s}}$</td>
<td>15.502</td>
<td>0.0</td>
<td>15.942</td>
<td>2.9</td>
</tr>
</tbody>
</table>

### Table 7.12: Sensitivity of Optimal Performance to Uncertainty in Base Drag Coefficient

<table>
<thead>
<tr>
<th>Figure of Merit</th>
<th>50%</th>
<th>200%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{fn}$</td>
<td>0.966</td>
<td>0.966</td>
</tr>
<tr>
<td>EPNLdB</td>
<td>101.1</td>
<td>101.1</td>
</tr>
<tr>
<td>Nozzle Weight*</td>
<td>0.829</td>
<td>0.833</td>
</tr>
<tr>
<td>% Change in $\Delta\text{TOGW}$</td>
<td>-2.6</td>
<td>5.6</td>
</tr>
</tbody>
</table>
off-axis losses (1.2%), and skin friction losses (0.6%). The changes in the optimal performance are of the same magnitudes as the proscribed convergence criterion for the optimal TOGW.

### 7.1.7 Acoustic Liner

Nominally, the integrated tool allows for the acoustic liner to begin when the mixedness has reached 0.4. In order to examine the sensitivity of the optimal design to the value of the mixedness threshold to begin liner treatment, the optimization process was repeated for values of the mixedness threshold of 0.3 and 0.5.

The resulting optimal designs and the percentage change ($\Delta\%$) from the nominal optimal design are given in Table 7.13. As the mixedness threshold for the acoustic liner is increased, the increase in mixing due to streamwise vorticity becomes more important. Therefore, the optimal amount of streamwise vorticity increases with mixedness threshold for the acoustic liner.

The resulting figures of merit and the percentage change in the ATOGW are given in Table 7.14. Lowering the mixedness threshold means the liner can begin at a earlier axial position. This allows for an increase in treated area over the nominal case. As expected, there is a decrease in $\Delta \text{TOGW}$ as the mixedness threshold is decreased and an increase in $\Delta \text{TOGW}$ as the mixedness threshold is increased.

<table>
<thead>
<tr>
<th>Variable</th>
<th>0.3</th>
<th>$\Delta%$</th>
<th>0.5</th>
<th>$\Delta%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.174</td>
<td>-0.7</td>
<td>1.188</td>
<td>0.5</td>
</tr>
<tr>
<td>SAR</td>
<td>2.485</td>
<td>-0.8</td>
<td>2.497</td>
<td>-0.3</td>
</tr>
<tr>
<td>AR</td>
<td>1.517</td>
<td>-0.9</td>
<td>1.477</td>
<td>-3.6</td>
</tr>
<tr>
<td>MAR</td>
<td>1.000</td>
<td>0</td>
<td>1.000</td>
<td>0</td>
</tr>
<tr>
<td>$L_{duct}$</td>
<td>2.436</td>
<td>0.3</td>
<td>2.458</td>
<td>1.2</td>
</tr>
<tr>
<td>PEN</td>
<td>0.756</td>
<td>1.4</td>
<td>0.715</td>
<td>-4.2</td>
</tr>
<tr>
<td>N</td>
<td>14</td>
<td>-12.5</td>
<td>16</td>
<td>0.2</td>
</tr>
<tr>
<td>$\frac{w_A}{w_p}$</td>
<td>1.061</td>
<td>1.2</td>
<td>0.934</td>
<td>-10.8</td>
</tr>
<tr>
<td>$\alpha_{mizer,p}$</td>
<td>12.348</td>
<td>-1.2</td>
<td>13.581</td>
<td>8.7</td>
</tr>
<tr>
<td>$\alpha_{mizer,s}$</td>
<td>15.296</td>
<td>-1.3</td>
<td>16.314</td>
<td>5.3</td>
</tr>
</tbody>
</table>

Table 7.13: Sensitivity of Optimal Design to Uncertainty in Mixedness Threshold for Acoustic Liner

<table>
<thead>
<tr>
<th>Figure of Merit</th>
<th>0.3</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{fn}$</td>
<td>0.968</td>
<td>0.965</td>
</tr>
<tr>
<td>EPNLdB</td>
<td>101.1</td>
<td>101.1</td>
</tr>
<tr>
<td>Nozzle Weight*</td>
<td>0.830</td>
<td>0.832</td>
</tr>
<tr>
<td>$% \text{ Change in } \Delta \text{TOGW}$</td>
<td>-10.9</td>
<td>11.2</td>
</tr>
</tbody>
</table>

Table 7.14: Sensitivity of Optimal Performance to Uncertainty in Mixedness Threshold for Acoustic Liner
7.1.8 Bounds on Optimal Design and Performance

The greatest uncertainty displayed in the previous sensitivity studies is in the flow angles. For the other design variables, typical uncertainty bounds would roughly be ±5%. For the flow angles, the typical uncertainty bounds would be ±15%. The minimum and maximum values for each design variable from all of the sensitivity studies for modeling uncertainties are listed in Table 7.15.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.148</td>
<td>1.190</td>
</tr>
<tr>
<td>SAR</td>
<td>2.472</td>
<td>2.525</td>
</tr>
<tr>
<td>AR</td>
<td>1.442</td>
<td>1.532</td>
</tr>
<tr>
<td>MAR</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>$L_{duct}$ (m)</td>
<td>2.43</td>
<td>2.562</td>
</tr>
<tr>
<td>PEN</td>
<td>0.707</td>
<td>0.776</td>
</tr>
<tr>
<td>N</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>$\frac{w_2}{w_p}$</td>
<td>0.959</td>
<td>1.118</td>
</tr>
<tr>
<td>$\alpha_{mixture}^p$</td>
<td>12.413</td>
<td>14.432</td>
</tr>
<tr>
<td>$\alpha_{mixture}^s$</td>
<td>15.50</td>
<td>17.235</td>
</tr>
</tbody>
</table>

The minimum and maximum values for each figure of merit and the percentage change in the ΔTOGW from all of the sensitivity studies for modeling uncertainties are listed in Table 7.16. There is no real variation in the EPNLdB level because it is an active constraint. There is less than 1% variation in the optimal nozzle weight. The only significant variation is in the figures of merit is in the net thrust coefficient.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_f^*$</td>
<td>0.949</td>
<td>0.975</td>
</tr>
<tr>
<td>EPNLdB</td>
<td>100.9</td>
<td>101.1</td>
</tr>
<tr>
<td>Nozzle Weight*</td>
<td>0.830</td>
<td>0.837</td>
</tr>
<tr>
<td>% Change in ΔTOGW</td>
<td>-40.5</td>
<td>91.4</td>
</tr>
</tbody>
</table>

7.2 Constraints

One reason to establish the constraint on the maximum EPNLdB is to avoid 'virtual throttling'. This can occur because the EPNLdB decreases with additional losses because they result in a lowering of the average exit velocity. For weighting systems that emphasize noise levels, the optimization
process can be driven towards a maximum loss system because it is also a minimum noise system. This is similar to throttling back, thereby decreasing the thrust levels, in order to decrease the noise levels.

The maximum noise constraint is enforced to ensure that the design falls in a region of the design space where the baseline weightings are valid. One way to avoid needing this constraint is to use system weightings that have a functional dependency on the figures of merit. Vehicle trade studies show that system weightings for the net thrust coefficient and EPNLdB increase the farther the figure of merit is from meeting the design goals. As the figures of merit meet or exceed the design goals, the system weightings decrease. For example, there is little benefit to decreasing the EPNLdB beyond the FAR III requirements. For such a weighting system, virtual throttling would not occur because as the net thrust starts to drop, the relative weighting of the net thrust would increase relative to the weighting on the EPNLdB, driving the optimal design towards the design goal.

Another active constraint is the upper limit of MAR=1 in order to avoid mode shift. The normalized partial derivative indicates that an additional 1.5% in the ∆TOGW could be achieved for each 1% increase in the maximum allowable MAR. With an acoustic code capable of predicting the noise for mode shift, this constraint could be eliminated.

### 7.3 TOGW Weightings

The optimization process was repeated for the other three weighting systems (see Table 6.8). The resulting optimal designs are given in Table 7.17.

<table>
<thead>
<tr>
<th>Variable</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>CER</td>
<td>1.182</td>
<td>1.173</td>
<td>1.151</td>
<td>1.315</td>
</tr>
<tr>
<td>SAR</td>
<td>2.504</td>
<td>2.112</td>
<td>2.647</td>
<td>2.983</td>
</tr>
<tr>
<td>AR</td>
<td>1.532</td>
<td>1.620</td>
<td>1.322</td>
<td>0.850</td>
</tr>
<tr>
<td>MAR</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>L_{duct} (m)</td>
<td>2.430</td>
<td>2.438</td>
<td>2.762</td>
<td>4.822</td>
</tr>
<tr>
<td>PEN</td>
<td>0.746</td>
<td>0.941</td>
<td>0.601</td>
<td>0.773</td>
</tr>
<tr>
<td>N</td>
<td>16</td>
<td>16</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>w_{wa}</td>
<td>1.048</td>
<td>0.757</td>
<td>1.146</td>
<td>0.906</td>
</tr>
<tr>
<td>α_{mixer,p}</td>
<td>12.50</td>
<td>20</td>
<td>10.417</td>
<td>17.136</td>
</tr>
<tr>
<td>α_{mixer,s}</td>
<td>15.50</td>
<td>23.895</td>
<td>13.839</td>
<td>18.526</td>
</tr>
</tbody>
</table>

The resulting figures of merit at the optimal design point for each weighting system are given in Table 7.18.

Drawings of each optimal design at the mixer plane are depicted in Figure 7-1.

For the weighting system emphasizing nozzle weight, net thrust is sacrificed until the minimum
Table 7.18: Variation in Optimal Figures of Merit for Different TOGW Weightings

<table>
<thead>
<tr>
<th>Variable</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{fn}$</td>
<td>0.967</td>
<td>0.911</td>
<td>0.978</td>
<td>0.910</td>
</tr>
<tr>
<td>EPNLdB</td>
<td>101.1</td>
<td>101.1</td>
<td>101.1</td>
<td>93.6</td>
</tr>
<tr>
<td>Nozzle Weight*</td>
<td>0.832</td>
<td>0.786</td>
<td>0.864</td>
<td>1.204</td>
</tr>
</tbody>
</table>

Baseline

Nozzle Weight Emphasized

Thrust Emphasized

Noise Emphasized

**Figure 7-1**: Drawings of Optimal Designs at the Mixer Plane for the Four Different Weighting Systems.

net thrust constraint is activated. Both the mixer perimeter, via the penetration, and the flow angularity are increased in order to allow for a smaller duct without violating the maximum noise constraint. Increasing the mixer perimeter and the flow angularity result in a faster initial mixing rate enabling the liner treatment to begin earlier in the duct. The largest change in the design variables that affect the nozzle weight is in SAR, indicating the decreasing SAR is the most efficient way to decrease nozzle weight.

For the weighting system emphasizing the net thrust, the opposite trends are visible. To offset the decrease in mixing rate due to the decrease in mixer perimeter and flow angularity, the noise constraint is met by increasing the amount of acoustic liner. The most thrust efficient way of accomplishing this is to increase the duct dimensions allowing for additional treatment area. The maximum noise constraint can then be met with a lower penetration, lower flow angle, mixer which has less thrust penalty associated with it.
For the weighting system emphasizing EPNLdB, the optimal design is pushed towards a combination of higher penetration and larger duct dimensions allowing for the greatest amount of noise attenuation. For this case, the design is constrained by the minimum allowable net thrust.

7.4 Design Guidelines

The purpose of repeating the design optimization was to establish design guidelines for how the optimal design varies with system weightings. The weightings are different for each acoustic measuring point for certification: sideline, cutback, and approach. General trends in the design with changes in the system weightings have been observed. These trends are limited by the constraints placed on the system. The maximum noise constraint is active for all but the noise-emphasized weighting system, in which the noise is reduced until the minimum thrust constraint is met. The optimal MAR is set by the upper limit constraint for all of the weighting systems.

The different weighting systems result in different optimal designs. As weighting for noise relative to thrust increases, the optimal mixers have longer perimeters (higher penetration and larger number of lobes) and more streamwise circulation. As the relative weighting for the nozzle weight is increased, the optimal mixer have longer perimeters and higher flow angularity in order to meet the noise constraint while reducing SAR thus reducing the nozzle weight. In either case, net thrust is sacrificed to achieve a reduction in the emphasized figure of merit. As the relative system weighting of EPNLdB is increased, there is also an increase in duct length.
Chapter 8

Summary, Conclusions, and Recommendations

8.1 Summary

A design methodology was set forth and implemented for optimizing a mixer-ejector noise suppression system. First, a time-efficient model for mixer-ejectors was developed by combining a discrete vortex code, a pressure matching control volume analysis, a differential control volume analysis based on compound flow, and a compound shock analysis. Subcomponents of the model account for inlet losses, nozzle losses, base drag, off-axis losses, wall friction, interstream shear stress, and interstream thermal conduction. The model was compared against experimental data for a range of designs and was found to have an accuracy comparable to Navier-Stokes calculations when predicting net thrust and entrainment ratios.

Next, an integrated tool was developed to assess the impact of design variables on three figures of merit for supersonic mixer-ejector noise suppressors: net thrust coefficient, EPNLdB, and nozzle weight. The integrated tool consists of the model described above, an acoustic code, and a nozzle weight code. The acoustic and nozzle weight codes were acquired from industry. The impact of the figures of merit on the take-off gross weight was determined using system weightings.

The integrated tool was then used to carry out two kinds of optimization studies: trade studies and multi-variable optimization. The trade studies were used to identify the physical trade-offs represented by each design variable by varying each one independently about a baseline design. The multi-variable optimization allowed all the design variables to be changed simultaneously subject to certain constraints. Uncertainty bounds on the optimal design variables were established by varying subcomponents of the fluid dynamic model and repeating the optimization process to determine the sensitivities of the optimal design and performance to modeling uncertainties.
Finally, the multi-variable optimization was repeated for weighting systems which emphasized different figures of merit. These calculations were used to establish design guidelines for trends in design for different weighting systems.

8.2 Conclusions

The major conclusions of this work are:

1. The fluid dynamic model has improved accuracy over a mixed-out control volume and captures performance parameters derived from experimental data as well as 3-D CFD computations. The code is currently being used by several members of NASA’s High Speed Research Program Nozzle Design Team.

2. From the trade studies, it was concluded that,
   - All of the physical trade-offs included one or more of the following effects: changes in mixer perimeter, area ratio, acoustic liner treatment area, flow angularity, pumping ratio, and skin friction.
   - For variables that do not affect the nozzle weight, the trade study can be characterized by a single noise-thrust line.
   - Almost all of the trade-studies yielded a minimum in TOGW where the physical trade-offs balance each other.
   - For the weighting system emphasizing noise, the trade studies shifted towards designs with greater flow angularity, greater mixer perimeters and longer ducts.
   - For the weighting system emphasizing the net thrust coefficient, the trade studies shifted to mixers with fewer number of lobes, lower penetration, and low, but non-zero flow angularity.

3. Using the integrated tool in a multi-variable optimization routine led to the following results.
   - The multi-variable optimal design varied by as much as a factor of two from the trade study results indicating that parametric testing alone will not yield an optimal design. Specifically, the multi-variable optimization resulted in a higher penetration and greater flow angularity but a lower secondary-to-primary lobe width ratio.
   - The primary drivers for optimal performance are SAR, MAR, and the primary flow angle. The sensitivities range from 1.5% to 3.5% change in the optimal $\Delta TOGW$ for a % change in the design variable.

4. The multi-variable optimization process was repeated while varying various subcomponents of the fluid dynamic model to determine the sensitivities of the optimal design and performance to modeling uncertainties.
   - The optimal design and the optimal performance are not sensitive to the uncertainty in the
base drag because the base drag is small compared to the other mixer losses.

- As the mixedness threshold is increased, the flow angularity of the optimal design increases in order to allow the liner treatment to begin earlier in the duct. In other words the relative importance of this physical effect increases with increasing mixedness threshold.
- The optimal performance was most sensitive to changes in the inlet and nozzle losses.

5. The multi-variable optimization process was repeated for four different weighting systems: a baseline system and three others, each emphasizing a different figure of merit. This was done to establish design guidelines for lobed mixer design for changes in the system weightings.
- The different weighting systems result in different optimal designs.
- As the relative weighting for either the noise level or the nozzle weight is increased, net thrust is sacrificed to achieve a reduction in the emphasized figure of merit.
- As weighting for noise relative to thrust increases, the trend is toward mixers with longer perimeters (5% higher penetration, 50% decrease in aspect ratio) and more streamwise circulation (40% increase in secondary mixer angle and 20% increase in primary mixer angle). There is also an increase in duct length of 100% to allow for more acoustic liner.
- As the relative weighting for the nozzle weight is increased, noise is reduced by increasing the mixer perimeter (20% increase in penetration) and the streamwise circulation (65% increase in secondary mixer angle and 50% increase in primary mixer angle). This allows the design to meet the noise constraint while reducing SAR 16% resulting in a 5% decrease in nozzle weight.

8.3 Recommendations for Future Work

There are several suggestions for the improvement and extension of the current work.

1. Develop tools for calculating vorticity distributions for arbitrary chute shapes. One possibility is to calculate vorticity distributions for a baseline chute based on Navier-Stokes calculations and scale the vorticity distribution for similar chute shapes based on analytical arguments.

2. Allow for non-constant area duct geometries in the vortex code by allowing for non-steady boundary conditions.

3. Adapt the current fluid dynamic model to handle a broader range of applications. - Add the ability to handle axisymmetric duct geometries.
- Adapt subcomponents for subsonic primary nozzle flow.
- Include industry ‘stack-up’ models to predict inlet and nozzle losses for a broader range of inlet and nozzle designs.

4. Develop or acquire an improved, more physically-based acoustic code. Improvements should include the effects of directivity especially the dependency on duct aspect ratio.
5. Repeat the optimization process using a non-linear take-off gross weight model. By allowing
the system weightings to vary as a function of the figures of merit, problems such as virtual
throttling could be avoided. It would also eliminate the shadow costs on constraints such as
the maximum noise level and the minimum net thrust coefficient.

6. The cutback and approach conditions should be examined with an acoustic code capable of
predicting noise levels at these measuring points and with appropriate system weightings. The
optimal design minimizes the maximum $\Delta TOGW$ for the three measuring points.

7. The same methodology and the fluid dynamic model could be utilized to optimize the design
of lobed mixers for uses other than noise suppression. Other uses include, but are not limited
to, lobed mixers for fan-core mixing and for low observability emissions.

There are several uses for the integrated tool that are planned or are already in progress. To this
end, a users’ manual is currently being written at Boeing Company, St. Louis. There are also plans
to refine the programming and the user interface.
Appendix A

Compound Flow

A.1 Differential Control Volume

Specified Initial Conditions
\[ \frac{d(A_p + A_s)}{dx} \]

Assumptions
- Mass in each stream is constant
- Adiabatic system
- Total momentum conserved
- \( P = P(x) \)
- Ideal gases

Figure A-1: Differential Control Volume.

The differential control volume can be expressed as seven ordinary differential equations in seven primary variables \([A_p, A_s, \ln P, M_p^2, M_s^2, T_{tp}, T_{ts}]\). The derivation of these equations is similar to the derivation outlined by Clark [7] based on compound flow theory [6]. The only difference is the addition of the heat conduction terms in order to account for interstream heat exchange.

\[
\frac{1}{A_p} \frac{dA_p}{dx} = \frac{1 - M_p^2}{\gamma_p M_p^2} \frac{d\ln P}{dx} + \frac{1 + (\gamma_p - 1)M_p^2}{\gamma_p M_p^2} \frac{\dot{\tau}_p}{Pdx} + \frac{\theta_p}{T_{tp}} \frac{dT_{tp}}{dx} \quad (A.1)
\]
\[ \frac{1}{A_s} \frac{dA_s}{dx} = \frac{1 - M_p^2}{\gamma_s M_s^2} \frac{d \ln P}{dx} + \frac{1 + (\gamma_s - 1)M_s^2}{\gamma_s M_s^2} \frac{\dot{\tau}_s}{\dot{T}_s} dx + \frac{\theta_s}{T_s} \frac{dT_{ts}}{dx} \] \tag{A.2}

\[ \frac{d \ln P}{dx} = \frac{1}{\beta} \left[ \frac{dA}{dx} - \frac{1 + (\gamma_p - 1)M_p^2}{\gamma_p M_p^2} \frac{\dot{\tau}_p A_p}{Pdx} - \frac{1 + (\gamma_s - 1)M_s^2}{\gamma_s M_s^2} \frac{\dot{\tau}_s A_s}{Pdx} - \theta_p \frac{A_p}{T_p} \frac{dT_{tp}}{dx} + \theta_s \frac{A_s}{T_s} \frac{dT_{ts}}{dx} \right] \] \tag{A.3}

The Mach number gradients are identical to those of a one stream flow with arbitrary shear stresses, heat transfer, and area variation.

\[ \frac{1}{M_p^2} \frac{dM_p^2}{dx} = 2 \theta_p \left[ \frac{\dot{\tau}_p}{Pdx} + \frac{1}{2} (1 + \gamma_p M_p^2) \frac{dT_{tp}}{dx} \frac{1}{T_{tp}} - \frac{1}{A_p} \frac{dA_p}{dx} \right] \] \tag{A.4}

\[ \frac{1}{M_s^2} \frac{dM_s^2}{dx} = 2 \theta_s \left[ \frac{\dot{\tau}_s}{Pdx} + \frac{1}{2} (1 + \gamma_s M_s^2) \frac{dT_{ts}}{dx} \frac{1}{T_{ts}} - \frac{1}{A_s} \frac{dA_s}{dx} \right] \] \tag{A.5}

Changes in the total temperatures are the result of heat conduction and work done on the stream by shear forces.

\[ \dot{m}_p C_{pp} \frac{dT_{tp}}{dx} = \left[ q_p - \tau_{mp} U \right] \ell_m \] \tag{A.6}

\[ \dot{m}_s C_{ps} \frac{dT_{ts}}{dx} = \left[ q_s - \tau_{ms} U \right] \ell_m \] \tag{A.7}

The shear stress term accounts for both wall friction and interstream shear stress.

\[ \dot{\tau} = \left( \tau_{wall} \frac{l_{wall}}{A} + \tau_{mixing} \frac{l_{mixing}}{A} \right) dx \] \tag{A.8}

The shear stress exerted on the primary stream by the secondary stream must be equal and opposite to the shear stress exerted on the secondary stream by the primary stream.

\[ \tau_{mp} = -\tau_{ms} \] \tag{A.9}

If the duct is adiabatic, then the sum of the heat fluxes must be zero.

\[ q_p = -q_s \] \tag{A.10}
A.1.1 Singularities in Equations

There are three apparent singularities in the compound flow equations.

1. $\beta \to 0$ This corresponds to compound choking situation. This is only realizable when the numerator is zero (a compound throat).

2. $M_p \to 1$ This corresponds to choked flow in the primary stream. This apparent singularity in Equation A.4 does not result in an infinite gradient because the numerator approaches zero as the denominator does. In physical terms, the primary area adjusts such that an aerodynamic throat coincides with the point where the primary stream chokes.

3. $M_s \to 1$ This corresponds to choked flow in the secondary stream. This apparent singularity in Equation A.5 does not result in an infinite gradient because the numerator approaches zero as the denominator does. In physical terms, the secondary area adjusts such that an aerodynamic throat coincides with the point where the secondary stream chokes.

Near the critical values ($\beta = 0$, $M_p = 1$ and $M_s = 1$) numerical difficulties arise in the corresponding equations (Equation A.3, Equation A.4, and Equation A.5). This problem is solved by holding the derivative in the corresponding equation constant near ($\pm 0.0001$) the critical value. This approximation was tested by examining the individual mass flows and the total energy flow at axial locations before and after the approximation was made. All were conserved to within numerical accuracy.

A.2 Analogy with Single-Stream Flow

An important parameter that characterizes compound flow is the compound flow indicator ($\beta$).

$$\beta = \frac{1 - M^2_p}{\gamma_p M^2_p A_p} + \frac{1 - M^2_s}{\gamma_s M^2_s A_s}$$  \hspace{1cm} (A.11)

Berstein et al. [6] showed that the compound wave velocity (positive in the upstream direction) has the same sign as the compound flow indicator.

Therefore, the compound flow indicator plays the same role as the Mach number does in single-stream flows. This relationship becomes obvious when the two streams are identical ($M_p = M_s$ and $\gamma_p = \gamma_s$).

$$\beta = \frac{1 - M^2}{\gamma M^2 A}$$  \hspace{1cm} (A.12)

For this case, the sign of the compound flow indicator depends only on the Mach number as shown in Table A.1.

In general, for the flow to be compound sonic, one stream must be supersonic and the other
stream must be subsonic. Also note that it is not necessary for both streams to be supersonic in order for the flow to be compound supersonic.

A.2.1 **Aerodynamic Throat for Compound Flows**

When the compound indicator equals zero, the flow is choked. Analytically, this would result in an infinite pressure gradient. Since infinite pressure gradients are non-physical, a compound indicator of zero can only occur where the numerator of Equation A.3 also equals zero corresponding to a compound aerodynamic throat. This is handled by iterating upon the initial conditions until the compound choke point is placed at the compound aerodynamic throat. The flow then becomes compound supersonic and shockfitting is necessary to match the exit static pressure to ambient.

A.3 **Solution Technique**

The differential equations are solved using a fourth-order, Runge-Kutta method with variable step-size [22].

<table>
<thead>
<tr>
<th>Flow Regime</th>
<th>Compound Indicator</th>
<th>Corresponding Mach Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compound Subsonic</td>
<td>&gt; 0</td>
<td>&lt; 1</td>
</tr>
<tr>
<td>Compound Sonic</td>
<td>= 0</td>
<td>= 1</td>
</tr>
<tr>
<td>Compound Supersonic</td>
<td>&lt; 0</td>
<td>&gt; 1</td>
</tr>
</tbody>
</table>
Appendix B

Pressure Matching Control Volume

B.1 Derivation of Model

<table>
<thead>
<tr>
<th>Specified Inflow conditions</th>
<th>Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>As</td>
<td>Mass conserved in each stream</td>
</tr>
<tr>
<td>us</td>
<td>Energy conserved in each stream</td>
</tr>
<tr>
<td>Ts</td>
<td>Total momentum conserved</td>
</tr>
<tr>
<td>Ps</td>
<td>Constant area</td>
</tr>
<tr>
<td>Ap</td>
<td>Isentropic secondary stream</td>
</tr>
<tr>
<td>up</td>
<td>Pressures matched exiting C. V.</td>
</tr>
<tr>
<td>Tp</td>
<td>Ideal gases</td>
</tr>
<tr>
<td>Pp</td>
<td></td>
</tr>
<tr>
<td>As'</td>
<td></td>
</tr>
<tr>
<td>us'</td>
<td></td>
</tr>
<tr>
<td>Ts'</td>
<td></td>
</tr>
<tr>
<td>P'</td>
<td></td>
</tr>
<tr>
<td>Ap'</td>
<td></td>
</tr>
<tr>
<td>up'</td>
<td></td>
</tr>
<tr>
<td>Tp'</td>
<td></td>
</tr>
</tbody>
</table>

Figure B-1: Pressure Matching Control Volume.

The static pressures are matched coming out of the control volume.

\[ P_i' = P' \]  \hspace{1cm} (B.1)

The pressure equalization is assumed to occur over a negligible length; therefore the total area does not change.

\[ \sum A_i = \sum A_i' = A \] \hspace{1cm} (B.2)
The secondary stream is assumed to be isentropic.

\[ P_{ts} = P'_{ts} \quad \text{(B.3)} \]

Mass is conserved in each stream.

\[ \dot{m}_i = \rho_i u_i A_i = \rho'_i u'_i A'_i \quad \text{(B.4)} \]

Energy is conserved in each stream.

\[ T_{ti} = T_i + \frac{u_i^2}{2C_{pi}} = T'_i + \frac{u'_i^2}{2C_{pi}} \quad \text{(B.5)} \]

The energy equation can alternatively be expressed as follows.

\[ T_{ti} = T_i \theta_i = T'_i \theta'_i \quad \text{(B.6)} \]

Total momentum is conserved.

\[ \sum (P_i A_i + \dot{m}_i u_i) = P'A + \sum \dot{m}_i u'_i \quad \text{(B.7)} \]

Ideal Gas

\[ P' = \rho'_i R_i T'_i \quad \text{(B.8)} \]

**B.2 Solutions**

For given inflow conditions, there are three possibilities. If the secondary Mach number exceeds a critical Mach number, there is no solution to the pressure matching control volume. If the secondary Mach number is below the critical Mach number, there is both a compound subsonic and a compound supersonic solution. If the secondary Mach number equal the critical Mach number, there is only one solution and it is compound sonic. This is due the fact that compound choking at the exit of the control volume limits the entrained mass flow and therefore limits the secondary Mach number.
Appendix C

Compound Shocks

Compound shocks are treated using a control volume identical to the pressure matching control volume. As was mentioned in Appendix B, there are two solutions to the pressure matching control volume; a compound subsonic solution and a compound supersonic solution. If the two flows are identical, the normal shock equations are recovered.

Figure C-1: Example of Compound Shock.
Appendix D

Mixed Out Control Volume

D.1 Derivation of Model

The simplest model of the mixer-ejector is the fully mixed case. The control volume is depicted in Figure D-1. The resulting equations can be reduced to a single quadratic equation in terms of the

Specified Inflow Conditions

\[ P_p, u_p, T_p, A_p \]
\[ P_s, u_s, T_s, A_s \]

Assumptions

Uniform exit flow
Mass conserved
Momentum conserved
Energy conserved
Constant area
Ideal gases

Figure D-1: Mixed Out Control Volume.

The exit velocity.

\[ \frac{\gamma + 1}{2(\gamma - 1)} u^2 - \frac{\gamma}{\gamma - 1} \frac{M}{\dot{m}} u + \frac{E}{\dot{m}} = 0 \]  

(D.1)

The coefficients in Equation D.1 involve the total momentum flow (M),

\[ M = P_p A_p + P_s A_s + \dot{m}_p u_p + \dot{m}_s u_s \]  

(D.2)

the total energy flow (E),

\[ E = \dot{m}_p C_{pp} T_p + \dot{m}_s C_{ps} T_s \]  

(D.3)
the total mass flow ($\dot{m}$),

$$\dot{m} = \dot{m}_p + \dot{m}_s$$  \hspace{1cm} (D.4)

and the mass averaged value of the ratio of specific heats ($\gamma$).

$$\gamma = \frac{\dot{m}_p \gamma_p + \dot{m}_s \gamma_s}{\dot{m}_p + \dot{m}_s}$$  \hspace{1cm} (D.5)

Once the exit velocity is known, the exit static temperature is solved using the conservation of energy equation.

$$T = \frac{1}{C_p} \left[ \frac{E}{\dot{m}} - \frac{u^2}{2} \right]$$  \hspace{1cm} (D.6)

The exit coefficient of specific heat ($C_p$) is taken to be the mass average of the primary and secondary values.

$$C_p = \frac{\dot{m}_p C_{pp} + \dot{m}_s C_{ss}}{\dot{m}}$$  \hspace{1cm} (D.7)

The exit static density is solved using the continuity equation.

$$\rho = \frac{\dot{m}}{uA}$$  \hspace{1cm} (D.8)

Finally, the exit static pressure is found using the ideal gas equation.

$$P = \rho RT$$  \hspace{1cm} (D.9)

### D.2 Solution Technique

For a supersonic nozzle, the primary conditions are fixed. For subsonic nozzles, the primary and secondary flows must have equal static pressures. In either case, the secondary Mach number is iterated upon (with the primary Mach number fixed for supersonic nozzles and chosen so as to match the static pressures for subsonic nozzles) in order to match the duct exit static pressure to the ambient static pressure.

For certain inflow conditions, there is no real solution to Equation D.1. Mathematically, this occurs when the quadratic equation has complex roots ($b^2 < 4ac$). Physically, the total mass flow exceeds the choked mass flow ($\dot{m} > \dot{m}_{M=1}$). It can be shown that the two inequalities are identical.
D.3 Adaptation to Variable Area Ducts

A non-constant area control volume can be used to model a variable area duct. One way to solve the resulting control volume is to break it into two successive control volumes: the constant area control volume described in Section D.1 and an isentropic area change. The solution to the constant area control volume is then used as the inflow conditions to the isentropic variable area control volume.
Appendix E

Description of MIT Noise Suppressor Code

E.1 Overview

This appendix describes how the integrated tool is broken down into programming files. Figure E-1 is a programming version of Figure 5-1. This appendix is meant for those readers who are attempting to compile and run the MIT Noise Suppressor Code.

![Flow Chart for MIT Noise Suppressor Code](image)

**Figure E-1:** Flow Chart for MIT Noise Suppressor Code.

E.2 Fluid Dynamic Model

E.2.1 Code

The fluid dynamic model is broken down into six files.

1. **compound.c** - This file contains the main section of the fluid dynamic code.
2. effective.c - This file contains the vortex code.

3. nrutil.c - This file contains utility programs from Reference [22].

4. nrutil.h - This file contains the prototype declarations for subroutines in nrutil.c.

5. rk.c - This file contains the Runge-Kutta integration subroutines [22].

6. complex.c - This file contains the subroutines for handling complex numbers.

E.2.2 Compiling

In order to compile the fluid dynamic model, type:

```
cc compound.c -lm
```

The fluid dynamic model has been successfully compiled on IBM RS/6000, Dec Alpha, Sun, Silicon Graphics, and Hewlett-Packard 7xx workstations.

E.2.3 Input Files

1. input - This file contains the operating conditions and design variables. The file is composed of two columns. The first column contains the numerical inputs and the second contains a description of the preceding input. The descriptions are restricted to continuous strings (no blank spaces or tabs). A sample input is included in Figure E-2. See Appendix F for a complete description of the individual inputs.

```
101325  --Ambient_Static_Pressure-Pa(PSI*6894)
305.535  --Secondary_Flow_Stagnation_Temp.-K(5/9*R)
0.3714417  --Lobed-Mixer_Wavelength(m)--->m=inch*0.0254
3.0  --Primary_Nozzle_Pressure_Ratio_(NPR)
1.06489  --Secondary_Nozzle_Pressure_Ratio_(SNPR)
3.0  --Stagnation_Temperature_Ratio
1.33  --Primary_Flow_Ratio_of_Specific_Heats
1.41  --Secondary_Flow_Ratio_of_Specific_Heats
8.0766376  --Duct_Length--wavelengths
5.0  --Height_of_Duct--wavelengths
5.0  --Width_of_Duct--wavelengths
1.0  --Mixing_Duct_Area_Ratio_(MAR)
1.1853  --Effective_CER_(CER/Cd)
2.0  --Ratio_of_Initial_Areas_(As/As)
34.926  --Initial_Mixing_Interface_Length--wavelengths
0.5359  --Non_Dimensional_Circulation
1.0  --Secondary/Primary_Trough_Width_Ratio
1.25  --Aspect_Ratio=Height/Width_Ratio
0.05  --Boundary_Layer_Thickness--wavelengths
0.2260645  --Flow_Angle_of_Primary_radian(deg*0.01745)
0.1422402  --Flow_Angle_of_Secndry_radian(deg*0.01745)
```

Figure E-2: Sample Input File.
2. **Inputv** - This file contains the initial discrete vorticity locations and fractional circulations in three columns. For a half wavelength section as shown in Figure E-3, the three columns, in order, are:

(a) $y$-coordinates of vortices normalized by wavelength ($y^*$).
(b) $z$-coordinates of vortices normalized by wavelength ($z^*$).
(c) Fractional circulation of each vortex ($\Gamma_i^*$).

![Figure E-3: Axis Used for Inputv File.](image)

See Appendix G for a description of the vortex code and the necessary inputs.

### E.2.4 Output Files

1. **Output** This file contains eight columns of output.

   (a) $x[m]$  
   (b) $A_p[m^2]$  
   (c) $A_s[m^2]$  
   (d) $P[Pa]$  
   (e) $M_p$  
   (f) $M_s$  
   (g) $T_{tp}[K]$  
   (h) $T_{ts}[K]$  

2. **outputv** This file contains three columns of output.

   (a) $x^*$ Normalized $x$ coordinate.  
   (b) $M_{sc}$ Scalar Mixedness
E.3  Net Thrust Model

teeple.m - This Matlab script contains the equations used to postprocess the fluid dynamic outputs. If the user types help teeple in Matlab, the message in Figure E-4 will be displayed.

>> help teeple

>> TEEPLE('run_name')

TEEPLE loads an input file (run_name.in) and an output file (run_name.out) and calculates a variety of parameters based on the inputs and outputs. If the parameter varies along the duct, it is stored in a column array corresponding to the column array of x values.

To see the list of parameters, type WHO at the prompt.

The TEEPLE function puts matlab in the keyboard mode in order to enable the user to manipulate the calculated parameters. In order to exit the keyboard mode, type RETURN.

Figure E-4: Help Screen for Matlab Function teeple.

E.4  Simplified Acoustic Tool

actool.m - This Matlab script contains the Simplified Acoustic Tool obtained from Pratt and Whitney [37]. If the user types help actool in Matlab, the message in Figure E-5 will be displayed.

E.5  Nozzle Weight Model

nozzleweight.m - This Matlab script contains the model obtained from Pratt and Whitney used to predict the nozzle weight based on duct length \( L_{mix} \), suppressor area ratio (SAR), and aspect ratio (AR). In addition, the model calculates ATAM which is used as an input to the Simplified Acoustic Tool. If the user types help nozzleweight in Matlab, the message in Figure E-6 will be displayed.
>> help actool

actool(VJMIX, VPRI, VSEC, WSWP, VINF, VHOT, VREM, WHOT, WIDTH, HEIGHT, ATAM)

actool calculates the EPNLdB based on:

- **VJMIX**: Thrust / mass flow (ft/s)
- **VPRI**: Primary velocity at mixing plane (ft/s)
- **VSEC**: Secondary velocity at mixing plane (ft/s)
- **WSWP**: Ratio of secondary mass flow to primary mass flow
- **VINF**: Freestream velocity (ft/s)
- **VHOT**: Area averaged velocity of flow above VJMIX expanded to ambient pressure (ft/s)
- **VREM**: Area averaged velocity of flow below VJMIX expanded to ambient pressure (ft/s)
- **WHOT**: Fraction of area represented by VHOT
- **WIDTH**: Width of duct (in)
- **HEIGHT**: Height of duct (in)
- **ATAM**: ATEFF/AMIX

Where,

- **ATEFF**: Effective treated area
- **AMIX**: Mixing plane area

Figure E-5: Help Screen for Matlab Function actool.

>> help nozzleweight

[WEIGHT, ATAM] = nozzleweight(Lmix, SAR, AR)

nozzleweight calculates the weight (lbs) and the ATAM

ATAM = Ateff/Amix

Where,

- **Ateff**: Surface area of acoustic liner (Assumes 93% efficiency)
- **Amix**: Area of mixing plane

of the nozzle based on:

- **LMIX**: Length of duct (in)
- **SAR**: Amix/A8
- **AR**: Aspect Ratio (Width/Height @ mixing plane)

Figure E-6: Help Screen for Matlab Function nozzleweight.

### E.6 TOGW Model

**deltaTOGW.m** - This Matlab script contains the linearized cost model used to determine the impact of the mixer-ejector on the take-off gross weight (TOGW) of the vehicle. If the user types `help deltaTOGW` in Matlab, the message in Figure E-7 will be displayed.
>> help deltaTOGW

deltaTOGW(nozzle_weight,Cfn,EPNLdb,WEIGHTS)

deltaTOGW calculates the deviation (lbs) from the baseline Take-Off Gross Weight based on:

- Nozzle weight (lbs)
- Net thrust coefficient
- Effective Noise Level (dB)

<table>
<thead>
<tr>
<th>WEIGHTS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Preliminary</td>
<td>1</td>
</tr>
<tr>
<td>Average</td>
<td>2</td>
</tr>
<tr>
<td>Design</td>
<td>3</td>
</tr>
<tr>
<td>Current</td>
<td>4</td>
</tr>
<tr>
<td>Cutback</td>
<td>5</td>
</tr>
</tbody>
</table>

Figure E-7: Help Screen for Matlab Function deltaTOGW.
Appendix F

Inputs and Outputs for Fluid Dynamic Model

F.1 Inputs

The inputs for the fluid dynamic model are separated into two categories for optimization purposes. Operating conditions are considered fixed during the optimization process. These inputs are determined by atmospheric conditions at takeoff, flight Mach number, and engine cycle. Design variables are the inputs that are allowed to vary during the optimization process. The design variables can be further broken down into two categories: mixer geometry and duct geometry.

All lengths are scaled by the mixer lobe wavelength. The ambient pressure, secondary stagnation temperature, and the wavelength of the mixer lobe give the minimum set of dimensional inputs needed to define the inputs for the fluid dynamic model. All other inputs are non-dimensional. An effort was made to use non-dimensional parameters in common with previous mixer-ejector research [12].

F.1.1 Operating Conditions

1. Ambient static pressure in Pascals ($P_{\text{amb}}$).

2. Stagnation temperature of secondary (entrained) flow in Kelvin. For a vehicle moving at a flight Mach number, $M_f$, the stagnation temperature can be related to the ambient static temperature.

$$T_{ts} = T_{\text{amb}} \left[ 1 + \frac{\gamma - 1}{2} M_f^2 \right]$$

(F.1)
3. Nozzle pressure ratio (NPR).

\[ NPR = \frac{P_{tp}}{P_{amb}} \] (F.2)

4. Secondary nozzle pressure ratio (SNPR).

\[ SNPR = \frac{P_{ts}}{P_{amb}} \] (F.3)

For a vehicle moving at a flight Mach number, \( M_f \), SNPR can be found using an isentropic relation.

\[ SNPR = \left[ 1 + \frac{\gamma - 1}{2} M_f^2 \right]^{\frac{-\gamma}{\gamma - 1}} \] (F.4)

5. Total temperature ratio (\( \tau \)).

\[ \tau = \frac{T_{tp}}{T_{ts}} \] (F.5)

6. Ratio of specific heats for primary stream (\( \gamma_p \)).

For actual devices, \( \gamma_p \) can be expressed as a function of the primary static temperature and consequently as a function of the primary stagnation temperature. This dependence was not explicitly modeled in order to allow for comparisons with experiments utilizing different gases (with different ratios of specific heats).

7. Ratio of specific heats for secondary stream (\( \gamma_s \)).

For purposes of a supersonic transport, \( \gamma_s \) can be assumed to be constant at a value of roughly 1.41. It is only included as an input to allow for comparisons with experiments utilizing different gases (with different ratios of specific heats).

F.1.2 Design Variables

1. Wavelength (\( \lambda \)) of mixer lobe in meters. The wavelength is defined as the horizontal distance between two adjoining peaks. For a square lobe (see Fig. F-1);

\[ \lambda = w_s + w_p \] (F.6)

For a sinusoidal lobe;

\[ z = \frac{h}{2} \sin \frac{2\pi y}{\lambda} \] (F.7)
Figure F-1: Trailing Edge of Mixer.

For a flat plate, the wavelength is taken as equal to the width of the duct.

2. Ratio of initial areas ($A_s/A_p$). These cross-sectional areas are measured at the mixing plane.

3. Initial mixing interface length normalized by mixer lobe wavelength. The initial mixing interface length equals the perimeter of the trailing edge of the mixer.

4. Flow angularity in primary stream ($\alpha_p$). The flow angularity in the primary stream is used to calculate the component of the momentum in the axial direction for use in the pressure matching control volume. This component of the axial momentum can be expressed in terms of the local flow angle.

$$\int ud\bar{m} = \int p u^2 dA = \int p V^2 \cos \alpha^2 dA$$  \hspace{1cm} (F.8)

In general, an effective flow angle can be defined.

$$\alpha_{eff} = \cos^{-1} \left( \frac{\int V \cos \alpha d\bar{m}}{\int V d\bar{m}} \right)$$  \hspace{1cm} (F.9)

For comparison with CFD results, Equation F.9 was used to calculate the effective flow angle. If the flow is assumed to be uniform in each stream with respect to pressure and temperature and therefore density and velocity, the density and velocity can be brought outside the integral and an effective angle can be defined.

$$\alpha_{eff} = \cos^{-1} \sqrt{\frac{\int \cos^2 \alpha dA}{A}}$$  \hspace{1cm} (F.10)

$$\alpha_{p,eff} = \cos^{-1} \sqrt{(1 - f_p) + f_p \cos^2 \alpha_p}$$  \hspace{1cm} (F.11)
See Figure F-2.

5. Flow angularity in secondary stream ($\alpha_s$). The flow angularity in the secondary stream is also used to calculate the component of the momentum in the axial direction. If similar assumptions about the flow angularity in the secondary stream are made (see Figure F-3, then a similar equation for the effective flow angle is obtained.

$$\alpha_{s,eff} = \cos^{-1} \left( 1 - f_s \right) + f_s \cos^2 \alpha_s$$

(F.13)

$$f_s = \frac{A_{\alpha=\alpha_s}}{A_s}$$

(F.14)

If a CFD simulation is used to calculate the inputs for the fluid dynamic model, Equation F.9 can also be used to calculate the effective flow angle in the secondary stream.

6. Ratio of secondary to primary trough width ratio ($w_s/w_p$). This input is only used by the vortex code to generate the vortex locations for a square lobe. If an arbitrary lobe shape is input, then this parameter is not used.
7. Non-dimensional circulation ($\Gamma^*$).

$$\Gamma^* = \frac{\Gamma}{U_h}$$  \hspace{1cm} (F.15)

The non-dimensional circulation for a given lobe shape can be expressed in terms of the flow angles in the primary and secondary lobes and the velocity ratio [5].

$$\Gamma^* = C_{\Gamma} \tan \alpha_p + r \tan \alpha_s \over 1 + r$$  \hspace{1cm} (F.16)

For $\alpha_p = \alpha_s$, the non-dimensional circulation is only a function of the lobe half angle ($\alpha$).

$$\Gamma^* = C_{\Gamma} \tan \alpha$$  \hspace{1cm} (F.17)

For square lobes,

$$C_{\Gamma} \approx 2$$  \hspace{1cm} (F.18)

The circulation coefficient can be calculated for arbitrary lobe shapes using the method proposed by Barber et. al. [5]. The general method and the extension to arbitrary lobe shapes is detailed in Appendix G. This method is also used to calculate the vorticity distribution along the lobe which is used as an input to the vortex code. Experiments performed by Skebe [26] show these equations to be accurate to within 5% for lobes with aspect ratios typical of those used in mixer-ejector systems.

8. Mixer height-to-wavelength ratio ($h^*$).

9. Boundary layer thickness-to-wavelength ratio ($\epsilon^*$).

10. Effective chute expansion ratio.

$$CER_{eff} = CER \over C_D$$  \hspace{1cm} (F.19)

The chute expansion ratio (CER) is the ratio of the primary cross-sectional area at the mixing plane ($A_p$) to the cross-sectional area of the primary throat ($A_j$).

$$CER = A_p \over A_j$$  \hspace{1cm} (F.20)

The effective chute expansion ratio is used to determine the ideal (no losses) primary Mach number.

11. Duct length to wavelength ratio ($L_{duct}^*$).
12. Duct width to wavelength ratio ($W_{duct}$).

13. Duct height to wavelength ratio ($H_{duct}$).

14. Mixing duct area ratio (MAR).

$$MAR = \frac{A_{exit}}{A_{mix}}$$  \hspace{1cm} (F.21)

For a constant width duct;

$$MAR = \frac{H_{exit}}{H_{mix}}$$  \hspace{1cm} (F.22)

### F.2 Outputs from Differential Control Volume

The outputs from the differential control volume are the seven primary variables at different axial locations. At each axial location, the cross-sectional area of each stream, static pressure, Mach number of each stream, and the total temperature of each stream is recorded. The thermodynamic state is completely determined by the inputs and outputs. There are also outputs from the vortex code which are given in detail in Appendix G.
Appendix G

Vortex Code

G.1 Inputs

There are three parameters required as inputs to the vortex code subroutine: the non-dimensional circulation \((\Gamma^*)\), the non-dimensional boundary layer thickness \((\epsilon^*)\) and the shear layer growth rate \((\delta')\). The first two are user inputs, while the shear layer growth rate is calculated using the planar shear layer correlations described in Chapter 2.

1. Non-dimensional circulation

\[ \Gamma^* = \frac{\Gamma}{UH} \]  \hspace{1cm} (G.1)

2. Boundary layer thickness

\[ \epsilon^* = \frac{\epsilon}{\lambda} \]  \hspace{1cm} (G.2)

3. Shear layer growth rate - Calculated using compound choked conditions at mixer trailing edge because this only requires one iteration of the vortex code.

G.1.1 Vortex Locations

The vortex locations represent points on the trailing edge of the mixer. In order to reduce computational time, only a half wavelength is modeled. The coordinates are scaled and translated so that the y coordinate varies from 0 to 0.5.

\[ y_{trans} = min(y) \]  \hspace{1cm} (G.3)
\[ z_{trans} = \frac{\max(z) + \min(z)}{2} \quad (G.4) \]

\[ \text{scale} = \frac{0.5}{\max(y) - \min(y)} \quad (G.5) \]

\[ y_N = \text{scale}(y - y_{trans}) \quad (G.6) \]

\[ z_N = \text{scale}(z - z_{trans}) \quad (G.7) \]

For square lobes, the vortex locations and distributions are automatically generated from the inputs to the fluid dynamic model.

**G.1.2 Vorticity Distributions**

In order to determine the effect of boundary conditions on the circulation distribution of the lobe, full Navier Stokes computations (FNS CFD) were compared with the analytical predictions based on the lobe shape. The boundary conditions are the duct wall on the top of region and a symmetry condition at the bottom of the region corresponding to the duct centerline. Both require the flow to turn axially, decreasing the circulation near the boundaries, causing an increase in the fractional circulation at the middle of the lobe. (The sum of the fractional circulations equals one.) As the penetration increases, this effect becomes more pronounced. The comparisons are not included because they involve LER information.

\[ \frac{\Gamma_i}{Uh} = \frac{2 \tan \alpha_p + r \tan \alpha_s}{h} \ell_i \cos \theta_i \quad (G.8) \]

\[ \Gamma^* = \frac{\sum \Gamma_i}{Uh} = \frac{2 \tan \alpha_p + r \tan \alpha_s}{h} \frac{1 + r}{1 + r} \sum \ell_i \cos \theta_i \quad (G.9) \]
G.2 Outputs

G.2.1 Vortex Locations

1. \texttt{yvort} - Contains the normalized y coordinate of the discrete vortices. \(y^*(i,j)\) corresponds to the \(i\)th vortex at the \(j\)th \(x^*\) location.

2. \texttt{zvort} - Contains the normalized z coordinate of the discrete vortices. \(z^*(i,j)\) corresponds to the \(i\)th vortex at the \(j\)th \(x^*\) location.

G.2.2 Scalar Mixedness Fields

\[ \eta = 1 - \text{erf} \left( \frac{r_{\min}}{\delta_c} \right) \]  \hspace{1cm} (G.10)

3. \texttt{exit} - Contains the scalar mixedness field at the duct exit. With a small modification, the code can output the scalar mixedness field at the other axial positions.

G.2.3 Effective Interface Length Distributions

4. \texttt{outputv} -

(a) \(x^*\)

Normally, the number of integration steps is equal to the length of duct divided by the wavelength rounded up ([L\textsubscript{duct}]).

(b) Scalar Mixedness

\[ M_{sc} = \frac{1}{A} \int \eta dA \]  \hspace{1cm} (G.11)

(c) Effective Interface Length Ratio

\[ \frac{L}{L_0} = \frac{dM_{\text{scalar}}}{d\delta^*} \]  \hspace{1cm} (G.12)
Bibliography


