Brief Announcement: Minimum Spanning Trees and Cone-Based Topology Control

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ABSTRACT

Consider a setting where nodes can vary their transmission power thereby changing the network topology, the goal of topology control is to reduce the transmission power while ensuring the communication graph remains connected. Wattenhofer et al. [6] introduced the distributed cone-based topology control algorithm with parameter \( \alpha \) \((CBTC(\alpha))\) and proved it correct if \( \alpha \leq \frac{2\pi}{3} \). Li et al. [4] and Bahramgiri et al. [1] proved that when applied separately these optimizations preserve connectivity. Bahramgiri et al. [1] proved that the algorithm produces some power assignment \( \sigma_\alpha \) : \( V \to [p_{\text{min}}, p_{\text{max}}] \). Using \( \sigma_\alpha \) we define the graph \( G^*_\alpha = (V, E^*_\alpha) \) where \( E^*_\alpha = \{(u, v) \mid p(u, v) \leq \sigma_\alpha(u) \lor p(v, u) \leq \sigma_\alpha(v) \} \), and the symmetric version \( G^\alpha = (V, E^\alpha) \) where \( E^\alpha = \{(u, v) \mid p(u, v) \leq \sigma_\alpha(u) \land p(v, u) \leq \sigma_\alpha(v) \} \).

We describe the core of the \( CBTC(\alpha) \) algorithm informally; for a detailed description we refer the reader to Wattenhofer et al. [6]. The algorithm proceeds in synchronous rounds, at the beginning of a round every node \( u \) broadcasts a HELLO message. Each receiving node replies with an ACK message, and node \( u \) collects the replies along with the direction from which they came from. Initially nodes transmit with minimum power \( p_{\text{min}} \) and increase (i.e. double) the transmission power when going into the next round. Node \( u \) terminates the algorithm when either it reaches maximum power or every cone with apex at \( u \) of aperture \( \alpha \) contains a neighbor.

1. DEFINITIONS AND ALGORITHM

Consider a set \( V \) of \( n \) nodes where \( \|uv\| \) is the distance from \( u \) to \( v \) and \( p(u, v) : V \times V \to \mathbb{R}^+ \) is the minimum power required to reach node \( v \) from node \( u \). The model assumes all nodes can transmit with the same maximum power \( p_{\text{max}} \) and \( p(u, v) \leq p(u, w) \) iff \( \|uv\| \leq \|uw\| \); in other words the power function is symmetric and a non-decreasing function of distance. Furthermore a node has the ability to accurately determine the direction from which another node is transmitting.

Let \( G = (V, E) \) be the maximum power communication graph, so \( E = \{(u, v) \mid p(u, v) \leq p_{\text{max}} \} \). Running \( CBTC \) with parameter \( \alpha \) produces some power assignment \( \sigma_\alpha : V \to [p_{\text{min}}, p_{\text{max}}] \). Using \( \sigma_\alpha \) we define the graph \( G^*_\alpha = (V, E^*_\alpha) \) where \( E^*_\alpha = \{(u, v) \mid p(u, v) \leq \sigma_\alpha(u) \lor p(v, u) \leq \sigma_\alpha(v) \} \), and the symmetric version \( G^\alpha = (V, E^\alpha) \) where \( E^\alpha = \{(u, v) \mid p(u, v) \leq \sigma_\alpha(u) \land p(v, u) \leq \sigma_\alpha(v) \} \).

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2. RESULTS

Assuming the nodes are embedded in the Euclidean plane and \( \alpha \leq \frac{2\pi}{3} \), Wattenhofer et al. [6] proved that \( G^*_\alpha \) is a spanning subgraph of \( G \); under the same assumptions Li et al. [4] described the asymmetric edge removal procedure to obtain \( G^\alpha \) and proved it was a also spanning subgraph of \( G \); Bahramgiri et al. [1] allowed the nodes to be in three-dimensional space and proved \( G^*_\alpha \) preserved connectivity.

We start with a succinct proof of a special case of Yao’s lemma using the Euclidean metric; for the general lemma see Yao [7]. An Euclidean minimum spanning tree is a minimum spanning tree were the weight of an edge \( (u, v) \) is the Euclidean distance \( \|uv\| \).

Categories and Subject Descriptors
C.2.1 [Computer-Communication Networks]: Network Architecture and Design—Network topology, Wireless communication; G.2.2 [Discrete Mathematics]: Graph Theory—Graph algorithms

General Terms
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Lemma 1 (Yao’s Lemma). If an edge \((u,v)\) belongs to the Euclidean minimum spanning tree, then \(v\) is \(u\)'s closest neighbor in every cone with apex at \(u\) and aperture \(\frac{\pi}{2}\) which contains \(v\).

Proof. Suppose not, then there exists an edge \((u,v)\) which belongs to the Euclidean minimum spanning tree \(T\). \(u\) has a neighbor \(w\) where \(\|uw\| < \|uv\|\), and there is a cone with apex at \(u\) and aperture \(\frac{\pi}{2}\) which contains both \(v\) and \(w\).

Removing the edge \((u,v)\) from \(T\) creates two disjoint connected components \(P\) and \(Q\) where \(u \in P\) and \(v \in Q\). If \(w \in Q\) then the joining \(P\) and \(Q\) with the edge \((u,w)\) creates a tree of smaller weight – a contradiction.

Hence suppose \(w \in P\), by the cosine law we have \(\|uv\|^2 = \|uw\|^2 + \|wv\|^2 - 2 \|uw\| \|wv\| \cos \theta\) where \(\theta = \angle vw\). By assumption we have \(\|uw\| < \|uv\|\) and since \(v\) and \(w\) are contained in a cone with apex at \(u\) of aperture \(\frac{\pi}{2}\) then clearly \(\cos \theta \leq \frac{\pi}{2}\). Hence \(\cos \theta \in \left(\frac{1}{2}, 1\right]\) and thus \(\|wv\|^2 < \|wv\|^2\), therefore joining \(P\) and \(Q\) with the edge \((v,w)\) creates a tree of smaller weight – a contradiction.

Let \(\text{MST}(H)\) be the minimum spanning tree of \(H\), now we are ready to prove the main theorem.

Theorem 2. If \(\alpha \leq \frac{2\pi}{3}\) then \(\text{MST}(G) \subseteq G_\alpha^\infty\).

Proof. To prove this lemma its sufficient to show that if \(\alpha \leq \frac{2\pi}{3}\) then \(e \notin G_\alpha^\infty \Rightarrow e \notin \text{MST}(G)\).

Fix some edge \((u,v) \notin G_\alpha^\infty\); without loss of generality we assume \(p(u,v) > \sigma_\alpha(u)\). Consider a cone with apex at \(u\) and aperture \(\alpha \leq \frac{2\pi}{3}\) with its axis passing through \(v\). By construction of \(\text{CBTC}(\alpha)\) this cone contains some vertex \(w\) and moreover \(\|uw\| < \|uv\|\). Since the cone has aperture \(\leq \frac{2\pi}{3}\) and \(v\) is at its axis, it follows that \(\angle vw\leq \frac{\pi}{2}\). Hence there exists a cone with apex at \(u\) of aperture \(\frac{\pi}{2}\) which contains both \(v\) and \(w\), and by Lemma 1 it follows that \((u,v) \notin \text{MST}(G)\).

Now as promised the corollaries trivially follow.

Corollary 1. If \(\alpha \leq \frac{2\pi}{3}\) then \(\text{CBTC}(\alpha)\) enhanced with asymmetric edge removal preserves connectivity of \(G\) (Li et al. [4]).

Proof. By Theorem 2, \(\text{MST}(G) \subseteq G_\alpha^\infty\) and by definition \(\text{MST}(G)\) is a spanning subgraph of \(G\).

Corollary 2. If \(\alpha \leq \frac{2\pi}{3}\) then \(\text{CBTC}(\alpha)\) preserves connectivity of \(G\) (Wattenhofer et al. [6]).

Proof. The asymmetric edge removal procedure only removes edges hence \(G_\alpha^\infty \subseteq G_\alpha^\infty\), therefore as before we have \(\text{MST}(G) \subseteq G_\alpha^\infty\).

Corollary 3. If \(\alpha \leq \frac{2\pi}{3}\) then running \(\text{CBTC}(\alpha)\) where nodes are in three-dimensional space using three-dimensional cones of aperture \(\alpha\) preserves connectivity of \(G\) (Bahramgiri et al. [1]).

Proof. We prove something stronger; if \(\alpha \leq \frac{2\pi}{3}\) then \(\text{CBTC}(\alpha)\) enhanced with asymmetric edge removal preserves connectivity when nodes are embedded in \(n\)-dimensional space using \(n\)-dimensional cones of aperture \(\alpha\). This follows since the proof of Theorem 2 only required Yao’s Lemma which holds for any number of dimensions (and under different metrics).

3. REFERENCES


