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Couplings between quantum systems are frequently less robust and harder to implement than controls on individual systems; thus, constructing quantum gates with minimal interactions is an important problem in quantum computation. In this paper we study the optimal synthesis of two-qubit quantum gates for the tunable coupling scheme of coupled superconducting qubits and compute the minimal interaction time of generating any two-qubit quantum gate.

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I. INTRODUCTION

Superconducting systems are among the leading candidates for the implementation of quantum information processing applications [1]. Due to the ubiquitous bath degrees of freedom in the solid-state environment, the time over which quantum coherence can be maintained remains limited, although significant progress in lengthening that time has been achieved. A key challenge is how to produce accurate quantum gates and how to minimize their duration such that the number of operations within $T2$ meets the error correction threshold. Concomitantly, progress has been made in applying optimal control techniques to steer quantum systems in a robust, relaxation-minimizing [2,3], or time optimal way [4–6].

In order to perform multiqubit operations, one needs a reliable method to perform switchable coupling between the qubits, i.e., a coupling mechanism that can be easily turned on and off. Over the past few years, there has been considerable interest in this question, both theoretically [7–11] and experimentally [12–22]. As a practically relevant and illustrative example, we consider the tunable coupling scheme for flux qubits and considering building two-qubit gates using minimal coupling time. This is an extension of previous studies of time optimal control [4], which considers constructing quantum gates using one coupling Hamiltonian, to the problem of constructing quantum gates using two or more coupling Hamiltonians.

II. SYNTHESIZATION OF TWO-QUBIT GATES

In this paper, we will take the tunable coupling scheme proposed in [23] as our main model for superconducting quantum computing, but the methods can be easily generalized to the other schemes. The tunable coupling scheme in [23] uses an extra high-frequency qubit to obtain the parametric ac-modulated coupling, as shown in Fig. 1. As discussed in [23], this coupling scheme has some useful features including an optimal point for the effective coupling energy. At such a point the effective coupling energy is insensitive to low-frequency flux noise, so two-qubit oscilla-

tions can be expected to be long lasting. This coupling scheme has been recently realized experimentally [24].

The effective Hamiltonian of this system is [23]

$$H = -\frac{1}{2}\sum_{j=1}^{2}\left[\Delta_j \sigma_z^j - u_j(t)\sigma_x^j\right] - J_{12}(t)\sigma_x^1\sigma_x^2. \quad (1)$$

In this Hamiltonian, the coupling $J_{12}(t)$ is modulated sinusoidally at the angular frequency $\omega_\pm = (\Delta_2 \pm \Delta_1)$. The essence of the coupling scheme is seen by considering a general modulation of the form

$$J_{12}(t) = g_0 + g_+(t)\cos(\omega_+ t) + g_-(t)\cos(\omega_- t). \quad (2)$$

To perform single-qubit operations we use Rabi oscillations driven by a resonant microwave control field,

$$u_j(t) = 2\Omega_j(t)\cos[\Delta_j t + \phi_j(t)]. \quad (3)$$

In this setup all the temporal dependence of the Hamiltonian is assumed to arise from the time-dependent flux of the applied fields. The rotating wave approximation, which is also valid if cross couplings are taken into account, results in a rotating frame Hamiltonian of the form
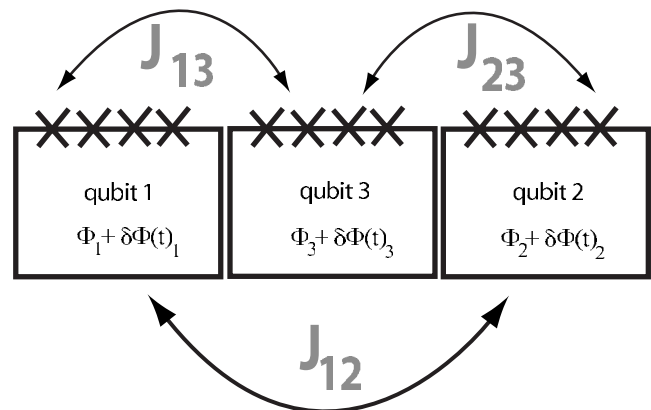

FIG. 1. By modulating external magnetic field on qubit 3, we can turn on and off the effective coupling between qubit 1 and qubit 2.

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$$H^{\mathrm{rot}} = \frac{1}{2}\sum_{j=1}^{2}\Omega_j(t)[\cos\,\phi_j(t)\sigma_x^j - \sin\,\phi_j(t)\sigma_y^j]$$

$$-\frac{g_+(t)}{4}(\sigma_x^1\sigma_x^2 - \sigma_y^1\sigma_y^2) - \frac{g_-(t)}{4}(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2).$$

Here, $g_+(t)$ and $g_-(t)$ are the effective coupling strengths in the rotating frame when we apply magnetic field with frequencies ω_+ and ω_-, respectively, on the auxiliary qubit.

The problem of using one given coupling Hamiltonian and local operations to simulate another Hamiltonian has been extensively studied in [4,5]. What is different here is that instead of one given coupling Hamiltonian, we have two tunable coupling Hamiltonians:

$$H_1 = \sigma_x^1\sigma_x^2 - \sigma_y^1\sigma_y^2,$$

$$H_2 = \sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2. \qquad (4)$$

Combining with local controls, we want to find out the minimal coupling time synthesizing any given two-qubit gate. Note that our result can be easily generalized to simulating Hamiltonian using two general coupling Hamiltonians or multiple coupling Hamiltonians.

We first review some background materials.

Proposition 1 (Canonical decomposition [4,25]). Any two-qubit nonlocal Hamiltonian $H = \sum_{i,j}M_{ij}(t)\sigma_i \otimes \sigma_j, i,j \in \{x,y,z\}$ can be written in the form

$$H = (A \otimes B)^\dagger(\theta_1^H\sigma_x \otimes \sigma_x + \theta_2^H\sigma_y \otimes \sigma_y + \theta_3^H\sigma_z \otimes \sigma_z)(A \otimes B) \qquad (5)$$

and any two-qubit unitary $U \in \mathrm{SU}(4)$ may be written in the form

$$U = (A_1 \otimes B_1)e^{-i(\theta_1^U\sigma_x\otimes\sigma_x + \theta_2^U\sigma_y\otimes\sigma_y + \theta_3^U Z\otimes Z)}(A_2 \otimes B_2). \quad (6)$$

Here A, A_1, A_2, B, B_1, and B_2 are single-qubit unitaries and

$$\theta_1^H \geq \theta_2^H \geq |\theta_3^H|,$$

$$\frac{\pi}{4} \geq \theta_1^U \geq \theta_2^U \geq |\theta_3^U|. \qquad (7)$$

We call $\theta_1^H\sigma_x \otimes \sigma_x + \theta_2^H\sigma_y \otimes \sigma_y + \theta_3^H\sigma_z \otimes \sigma_z$ and $e^{-i(\theta_1^U\sigma_x\otimes\sigma_x + \theta_2^U\sigma_y\otimes\sigma_y + \theta_3^U\sigma_z\otimes\sigma_z)}$ the canonical form of H and U, respectively, and $\vec{\theta}^H$ and $\vec{\theta}^U$ the canonical parameters of H and U, respectively.

For an element $x = (x_1, x_2, x_3)^T$ of $\mathbb{R}^3$, introduce the vector $\hat{x} = (|x_1|, |x_2|, |x_3|)^T$ and define the *s*-order version x^s of x by setting $x_1^s = \hat{x}_1^\downarrow$, $x_2^s = \hat{x}_2^\downarrow$, and $x_3^s = \mathrm{sgn}(x_1x_2x_3)\hat{x}_3^\downarrow$, where $(\hat{x}_1^\downarrow, \hat{x}_2^\downarrow, \hat{x}_3^\downarrow)$ is a permutation of $\hat{x}$ which arranges the entries in decreasing order, i.e., $\hat{x}_1^\downarrow \geq \hat{x}_2^\downarrow \geq \hat{x}_3^\downarrow$.

Definition 1 [26–30]. The vector $x \in \mathbb{R}^3$ is *s* majorized by $y \in \mathbb{R}^3$ (denoted $x <_s y$) if

$$x_1^s \leq y_1^s,$$

$$x_1^s + x_2^s + x_3^s \leq y_1^s + y_2^s + y_3^s,$$

$$x_1^s + x_2^s - x_3^s \leq y_1^s + y_2^s - y_3^s. \qquad (8)$$

Remark 1. $x <_s y$ means that x lies in the convex hull of $\{(\epsilon_1 y_{\pi(1)}, \epsilon_2 y_{\pi(2)}, \epsilon_3 y_{\pi(3)}) | \epsilon_{1,2,3} = \pm 1, \ \Pi_{i=1}^3\epsilon_i = 1, \pi$ is permutation on $\{1,2,3\}\}$.

From now on, when we say constructing a gate in time T, we mean constructing the gate with coupling time less or equal to T, aided by local operations.

Theorem 1 [5]. A unitary gate $U \in \mathrm{SU}(4)$ can be generated in coupling time T if and only if we can simulate a Hamiltonian $\tilde{H}$ in coupling time T, such that $\vec{\theta}^U <_s \vec{\theta}^{\tilde{H}}$ or $\vec{\theta}^U + \frac{\pi}{2}(-1,0,0) <_s \vec{\theta}^{\tilde{H}}$. For a given Hamiltonian $H(t)$, the Hamiltonians that can be simulated within time T using $H(t)$ and local control are the Hamiltonians with the canonical forms of the following:

$$\left\{ \begin{array}{l} \theta_1\sigma_x \otimes \sigma_x + \theta_2\sigma_y \otimes \sigma_y + \theta_3\sigma_z \otimes \sigma_z| \\[2mm] (\theta_1, \theta_2, \theta_3) <_s \int_0^T \vec{\theta}^H(t)dt, \quad \theta_1 \geq \theta_2 \geq |\theta_3| \end{array} \right\}.$$

This theorem reduces our problem to the integration of the canonical form of

$$\begin{aligned} H(t) &= g_+(t)(\sigma_x^1\sigma_x^2 - \sigma_y^1\sigma_y^2) + g_-(t)(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2) \\ &= (g_+(t) + g_-(t))\sigma_x^1\sigma_x^2 + (g_-(t) - g_+(t))\sigma_y^1\sigma_y^2 \quad (9) \end{aligned}$$

while we absorbed the constant $\frac{1}{4}$ into $g_+(t)$ and $g_-(t)$ and here $g_+(t)$ and $g_-(t)$ are both non-negative. We need to find out the boundary of $\int_0^T\vec{\theta}^H(t)dt$, as all the Hamiltonians that can be simulated within time T is s majorized by one of the points on the boundary. We study the problem in two cases:

Case 1. The magnetic fields of frequencies ω_+ and ω_- can be generated power independently, i.e., g_+ and g_- are constrained independently. Let us say they take values independently in the ranges of $[0,A]$ and $[0,B]$.

Using Proposition 1, we get the canonical parameter of the Hamiltonian (9), which is

$$(g_+(t) + g_-(t), |g_-(t) - g_+(t)|, 0).$$

We will see that it is actually *s* majorized by

$$(A + B, |A - B|, 0)$$

since

$$g_+(t) + g_-(t) \leq A + B,$$

$$g_+(t) + g_-(t) + |g_-(t) - g_+(t)| = \max\{2g_+(t), 2g_-(t)\}$$

$$\leq \max\{2A, 2B\}$$

$$= A + B + |A - B|,$$

so $(g_+(t)+g_-(t), |g_-(t)-g_+(t)|, 0) <_s (A+B, |A-B|, 0)$ for all t. Then given a unitary matrix $U \in \mathrm{SU}(4)$, the minimal coupling time needed to generate U is the minimal T such that

$$\vec{\theta}^U <_s ((A+B)T, |A-B|T, 0)$$

or

$$\vec{\theta}^U + \frac{\pi}{2}(-1,0,0) \prec_s ((A+B)T, |A-B|T, 0)$$

holds.

For example, suppose we want to generate a controlled-NOT (CNOT) gate,

$$\text{CNOT} = \exp\left[-i\frac{\pi}{4}\left(\sigma_z \otimes \sigma_x - \sigma_z \otimes I - I \otimes \sigma_x + \frac{1}{2}I\right)\right].$$

The canonical parameter of CNOT is $(\frac{\pi}{4}, 0, 0)$. The two inequalities coincide in this case, so the minimal coupling time needed is the minimal T such that

$$\left(\frac{\pi}{4}, 0, 0\right) \prec_s ((A+B)T, |A-B|T, 0),$$

which is $\frac{\pi}{4(A+B)}$.

Case 2. The magnetic fields of frequencies $\omega_\pm$ are generated power correlated, i.e., g_+ and g_- are jointly constrained, $g_+^2 + g_-^2 \leq M^2$. In this case the situation is more subtle as there is no single point, like $(A+B, |A-B|, 0)$, that s majorizes all the points in the region.

Since $\vec{x} \prec_s D\vec{x}$ when $D \geq 1$, to be at the boundary, $g_+(t)$ and $g_-(t)$ should take maximal values. Let $g_+(t) = M \sin \alpha(t)$ and $g_-(t) = M \cos \alpha(t)$, where $\alpha(t) \in [0, \frac{\pi}{4}]$ (as α and $\frac{\pi}{2} - \alpha$ give the same canonical parameter, we will just consider $\alpha \in [0, \frac{\pi}{4}]$), then integration of these canonical parameters give a set of boundary points,

$$\left(\int_0^T g_+(t) + g_-(t)dt, \int_0^T |g_-(t) - g_+(t)|dt, 0\right).$$

In the case that α is constant, we obtain

$$\left(\int_0^T g_+(t) + g_-(t)dt, \int_0^T |g_-(t) - g_+(t)|dt, 0\right)$$

$$= TM(\sin \alpha + \cos \alpha, \cos \alpha - \sin \alpha, 0)$$

$$= TM\left(\sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right), \sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right), 0\right). \quad (10)$$

In the general case where α varies in time, we get

$$\left(\int_0^T g_+(t) + g_-(t)dt, \int_0^T |g_-(t) - g_+(t)|dt, 0\right)$$

$$= \left(\int_0^T M \sin(\alpha(t)) + M \cos(\alpha(t))dt,\right.$$

$$\left.\int_0^T M \cos(\alpha(t)) - M \sin(\alpha(t))dt, 0\right) \quad (11)$$

$$= \left(\sum_{i=1}^{T/\delta t} M(\sin(\alpha(t_i)) + \cos(\alpha(t_i)))\delta t,\right.$$

$$\left.\sum_{i=1}^{T/\delta t} M(\cos(\alpha(t_i)) - \sin(\alpha(t_i)))\delta t, 0\right) \quad (12)$$

$$= \sum_{i=1}^{T/\delta t} \frac{\delta t}{T} TM\left(\sqrt{2}\sin\left(\alpha(t_i) + \frac{\pi}{4}\right), \sqrt{2}\cos\left(\alpha(t_i) + \frac{\pi}{4}\right), 0\right).$$

$$(13)$$

This is just the convex combination of the points we obtained by constant α, and lies in the interior of those points, so all the boundary points are achieved by constant α.

So given a unitary matrix $U \in \text{SU}(4)$, the minimal time needed to generate U is the minimal T such that $\exists \alpha \in [0, \frac{\pi}{4}]$,

$$\vec{\theta}^U \prec_s TM\left(\sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right), \sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right), 0\right)$$

or

$$\vec{\theta}^U + \frac{\pi}{2}(-1,0,0) \prec_s TM\left(\sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right), \sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right), 0\right).$$

Again taking CNOT, for example, the minimal time needed to generate it is the minimal T such that

$$\left(\frac{\pi}{4}, 0, 0\right) \prec_s TM\left(\sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right), \sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right), 0\right)$$

holds for one $\alpha \in [0, \frac{\pi}{4}]$. It is obvious that $\alpha = \frac{\pi}{4}$ gives the minimal time $T = \frac{\pi}{4\sqrt{2}M}$. So to generate CNOT, we should take $g_+(t) = g_-(t) = \frac{\sqrt{2}M}{2}$, turning on the coupling for a time $\frac{\pi}{4\sqrt{2}M}$. This control sequence generates the term $\exp[-i\frac{\pi}{4}\sigma_x \otimes \sigma_x]$, which is locally equivalent to the CNOT gate.

III. CONCLUSION

To perform large-scale quantum information processing, it is necessary to control the interactions between individual qubits while retaining quantum coherence. To this end, superconducting circuits allow for a high degree of flexibility. In this paper, we found the minimal coupling time required to generate arbitrary two-qubit quantum gate in the superconducting quantum computing scheme. We reduced this problem to the problem of simulating a desired Hamiltonian using two Hamiltonians and single-qubit operations and derived explicit forms to compute the minimal time needed to control the system. The results of this work might be useful for guiding the design of pulses in superconducting quantum computing experiments. Minimizing the time taken to perform an operation is certainly helpful in the presence of finite decoherence times. If one were able to take into account the specific form and time dependence of the environmental coupling, one might be able to devise more robust schemes. While it lies outside the scope of the current paper, it might be interesting to combine the time optimal control techniques with methods that take advantage of correlations in the noise, e.g., implementing refocusing sequences to counteract the effects of $1/f$ noise [31,32]. Such optimization in the face of the combination of several different types of constraints is a challenging problem in both classical and quantum control theories.

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