Viscous effects on Bragg scattering of water waves by an array of piles

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We examine the effects of viscous damping in the boundary layers on the Bragg resonance of surface water waves by a two-dimensional array of vertical cylinders standing across the depth of an open sea. The theory was motivated by possible construction of offshore airports consisting of a platform supported above water by vertical piles. Under the assumptions of small cylinders and large spacing comparable to a typical wavelength, a linearized theory for infinitesimal waves was studied. In particular, the cylinder spacing was assumed to be comparable to the wave length, but the cylinder radius to be small: $\mu = ka \ll 1$. Although scattering by one cylinder is weak, $O(\mu^2)$, the accumulated effects of many cylinders over a large region of length scale $O(1/k\mu^2)$ become significant when Bragg condition is nearly met. For a given lattice, the directions of the Bragg-resonated waves were first found by Ewald’s construction. The evolution equations of the wave amplitudes were derived by the asymptotic method of multiple scales and were then solved for waves scattered by an infinite array of cylinders in a strip of finite width. Both normal and oblique incidence were studied. Analytical and numerical results were obtained for two and three resonant waves. Effects of band gaps on scattering characteristics were analyzed in detail.

In this article we wish to assess the effects of viscous damping in the boundary layers on the Bragg resonance of surface water waves by a periodic two-dimensional array of vertical cylinders standing across the depth of an open sea. For cylinders of small radius relative to the wavelength, we first derive an effective boundary condition for the radial derivative of the velocity potential to account for the viscous forces. Coupled-mode equations are then rederived by an asymptotic method for the envelopes of multiply resonated waves inside the array. Effects of viscosity on band gaps and scattering coefficients due to a plane incident wave are examined analytically for an infinitely long array of finite width surrounded by open water. For normal incidence the envelope physics is one dimensional. The transmission and reflection properties are studied first. Oblique incidence can in principle excite several wave trains in different directions. Explicit solutions are given and discussed when there are only two wave trains inside the array. Results are compared with recent theories where viscosity is not taken into account. The asymptotic theory can be modified for two-dimensional sound scattering by a cylinder array.

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I. INTRODUCTION

In [1,2], a theory was given for the resonant scattering of water waves by a periodic two-dimensional array of vertical cylinders standing across the depth of an open sea. The theory was motivated by possible construction of offshore airports consisting of a platform supported above water by vertical piles. Under the assumptions of small cylinders and large spacing comparable to a typical wavelength, a linearized theory for infinitesimal waves was studied. In particular, the cylinder spacing was assumed to be comparable to the wave length, but the cylinder radius to be small: $\mu = ka \ll 1$. Although scattering by one cylinder is weak, $O(\mu^2)$, the accumulated effects of many cylinders over a large region of length scale $O(1/k\mu^2)$ become significant when Bragg condition is nearly met. For a given lattice, the directions of the Bragg-resonated waves were first found by Ewald’s construction. The evolution equations of the wave amplitudes were derived by the asymptotic method of multiple scales and were then solved for waves scattered by an infinite array of cylinders in a strip of finite width. Both normal and oblique incidence were studied. Analytical and numerical results were obtained for two and three resonant waves. Effects of band gaps on scattering characteristics were analyzed in detail.

In this article we wish to assess the effects of viscous damping in the boundary layers around the cylinders, which are unavoidable in laboratory experiments and in the field. In particular, we shall show that viscosity blurs the boundary of band gaps and hence the distinction between propagation and evanescence. In Appendix A, we give estimates to show that nonlinear effects of vortex shedding, vital for wave forces on piles in sufficiently strong waves, are not important for the weak waves considered herein. Vortex damping in large waves is likely much more important than resonant scattering and hence is a separate topic which requires a much more empirical treatment.

II. POTENTIAL FORMULATION MODIFIED FOR BOUNDARY-LAYER EFFECTS

We consider the diffraction of plane monochromatic incident waves from the open sea by a two-dimensional array of bottom-mounted vertical cylinders. The sea depth $H$ is assumed to be constant and the radius $a$ of the cylinders much smaller than the incident wavelength $2\pi/k$ so that $\mu = ka \ll 1$ is a small parameter. Assuming irrationality in most of the fluid, the velocity potential outside the viscous boundary layers next to the cylinders must satisfy

$$\nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0, \quad -H \leq z \leq 0, \quad (2.1)$$

where $\nabla$ is the gradient operator in the horizontal plane $(x, y)$. On the sea surface, the atmospheric pressure is assumed to be constant. Restricting to infinitesimal waves, the kinematic and dynamic free surface conditions can be combined to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = 0, \quad z = 0. \quad (2.2)$$

On the sea bed of constant depth, we have

$$\frac{\partial \Phi}{\partial z} = 0, \quad z = -H. \quad (2.3)$$

Let $R(m_1, m_2) = m_1 a_1 + m_2 a_2$ denote the lattice coordinate of the center of cylinder, where $(a_1, a_2)$ are the primitive lattice vectors in the horizontal $(x, y)$ plane and $(m_1, m_2)$ are integers. Let $(r', z)$ be the local cylindrical polar coordinate centered at the lattice node $R(m_1, m_2)$. The position of any point in space is $(r, z)$ with $r = R(m_1, m_2) + r'$. If dissipation near the cylinders is totally ignored, the boundary condition on the cylinder surface is simply
\[
\frac{\partial \Phi}{\partial r'} = 0, \quad \forall r' = a. \tag{2.4}
\]

It is well known for sinusoidal waves that the linearized problem can be reduced from three to two dimensions by the substitution

\[
\Phi(x, y, z, t) = \phi(x, y, z) e^{-i\omega t} = \eta(x, y) Z(z) e^{-i\omega t}, \tag{2.5}
\]

where \( \eta \) is proportional to the free-surface displacement according to

\[
\zeta = \frac{i \omega \eta}{g}, \tag{2.6}
\]

and

\[
Z(z) = -\frac{i g \cosh k(z + H)}{\omega \cosh kH}, \tag{2.7}
\]

with \( k = |k| \) satisfying the dispersion relation \([3]\)

\[
o^2 = gk \tanh kH. \tag{2.8}
\]

For brevity, we omit the symbol “Re” (real part of) for all complex expressions. The surface displacement amplitude \( \eta \), which represents the horizontal pattern of \( \phi(x, y, z) \), then satisfies the Helmholtz equation in the horizontal plane

\[
\nabla^2 \eta + k^2 \eta = 0. \tag{2.9}
\]

Hence for the present geometry the water-wave scattering problem is mathematically equivalent to that in two-dimensional acoustics.

The condition (2.4) is inadequate to account for viscosity, which is important in a thin boundary layer on the cylinder walls. We now find a fictitious condition within the frame of inviscid potential theory that predicts the correct force on the cylinder.

First, for a sufficiently small cylinder in relatively long waves such that \( \mu = k a \ll 1 \), it is well known \([4]\) that the scattering amplitude by any one cylinder is of \( O(k^2 a^2) \), meaning that the disturbance to waves is weak. Outside the boundary layer, but still near a cylinder the ratio of two terms in (2.9) is \( k^2 \eta / \nabla^2 \eta = O(k^2 a^2) \); hence, \( \eta \) is approximately a potential—i.e.,

\[
\nabla^2 \eta = 0 + O(k^2 a^2), \quad r' > a. \tag{2.10}
\]

Without the cylinder the local velocity at the center of the cylinder and at depth \( z \) is approximately uniform horizontally. The corresponding \( \eta \) can be represented by

\[
\eta = U r' \cos(\theta - \theta_0), \tag{2.11}
\]

where \( \theta_0 \) is the direction and \( U(x, y) \) the complex amplitude of \( \bar{U} \), which must be solved from the analysis of waves. With the cylinder the total solution outside the boundary layer is

\[
\eta = U \left( r' + \frac{a^2}{r} \right) \cos(\theta - \theta_0). \tag{2.12}
\]

Inside the oscillatory boundary layer, the tangential and transverse velocity components \((u, v)Z(z)\) are, according to Stokes theory,

\[
u(r', \theta) = \frac{2U}{a} \cos(\theta - \theta_0) \left[ \frac{\sigma - \frac{\delta}{1 - i e^{-i(\delta+\delta)}}}{a} \right], \tag{2.13a}
\]

\[
 v(r', \theta) = \frac{2U}{a} \sin(\theta - \theta_0) \left[ \frac{1 - e^{-i(\delta+\delta)}}{a} \right], \tag{2.13b}
\]

where \( \sigma = r' - a \) and \( \delta = \sqrt{V^2} \omega \) is, the Stokes boundary layer thickness. By integrating the viscous stresses obtained from (2.13a) and (2.13b), and pressure from Bernoulli’s law \((p = \rho \rho \omega^2)\) and (2.12), the total horizontal force per unit height of the cylinder at depth \( z \) can be found:

\[
 F(z) = \frac{2\pi \alpha a^2 U}{1 + (1 + i \delta) a} \frac{\delta}{a} Z(z), \tag{2.14}
\]

which is in the direction of \( \bar{U} \). The first term in the square brackets is due to inviscid pressure and the second to viscosity.

For the convenience of later analysis, let us represent the boundary layer effects by the effective (fictitious) boundary condition \([5]\)

\[
\frac{\partial \Phi}{\partial r'} + \bar{\alpha} \Phi = 0, \quad r' = a, \tag{2.15}
\]

which yields the same force (2.14) on the cylinder. The approximate potential satisfying (2.10) near and (2.15) on the cylinder, and approaching (2.11) at \( r' \gg a \), is readily found to be

\[
\Phi \equiv U \left( r' + \frac{a^2 + 1 + \bar{\alpha} a}{r' - 1 - \bar{\alpha} a} \right) \cos(\theta - \theta_0) Z(z). \tag{2.16}
\]

It will be shown shortly that \( \bar{\alpha} a \ll 1 \), so that

\[
\Phi \equiv U \left( r' + \frac{a^2}{r} (1 + 2 \bar{\alpha} a) \right) \cos(\theta - \theta_0) Z(z). \tag{2.17}
\]

With this result the dynamic pressure on the cylinder wall is

\[
 p'_{\text{wall}} = i \omega \rho \Phi |_{r' = a} \approx 2\pi \rho \omega^2 U(1 + \bar{\alpha} a) \cos(\theta - \theta_0), \tag{2.18}
\]

which gives the following amplitude of the horizontal force on the cylinder:

\[
 F = \int_{0}^{2\pi} \frac{1}{\omega} \cos \theta \, d\theta \approx -2\pi \rho \omega^2 U(1 + \bar{\alpha} a). \tag{2.19}
\]

By equating (2.14) and (2.19), the dimensionless coefficient \( \bar{\alpha} a \) is found:

\[
\bar{\alpha} a = (\bar{\alpha} - i \bar{\alpha} a) a = (1 + i) \frac{\delta}{2a} = (1 + i) \frac{1}{a} \sqrt{\frac{\nu}{2\omega}}, \tag{2.20}
\]

which is a complex constant. Let us estimate the rough magnitudes of \( \bar{\alpha} a \) for water waves \([6]\). In the field, the following values are representative: \( a \sim 1-5 \text{ m} \) and \( ka \sim 0.1 \). Let the boundary layer be turbulent and the eddy viscosity be 100
times that of the molecular viscosity—i.e., $\nu=10^{-4}$ m$^2$/s; then, $\Lambda a \sim \lambda, a \sim \lambda/\nu \approx 0.01$–0.002 and is small. For laboratory experiments, $a \sim 2.5$ cm, $\omega \sim 5$ rad/s, and the kinematic viscosity at 20 °C $\nu=1\times10^{-2}$ cm$^2$/s, we get $\Lambda a \sim 0.0125$, which is also small. In view of these estimates, we shall write in subsequent analysis $\Lambda=\mu^2\Lambda$, with $\mu=ka \ll 1$ and $\Lambda=O(1)$. From here on, we shall solve (2.1) subject to the boundary conditions (2.2) and (2.3) and
\[
\frac{\partial \phi}{\partial r} + \mu^2 \Lambda \phi = 0, \quad r' = a, \quad \Lambda a = (\Lambda_1 + i \Lambda_2) \alpha = (1 + i) \frac{\delta}{2 \mu^2 a}.
\]
(2.21)

Returning to the wave problem, it is known that scattering by each small cylinder is weak $[O(\mu^3)]$. However, when the number of cylinders in any direction is large $[O(1/\mu^2)]$, the accumulated effects may become of $O(1)$ if the Bragg resonance condition is met. In that case there are two contrasting length scales: $1/k$ and $1/\mu^2 k$. If small frequency detuning exists, there are also two contrasting time scales: $O(1/\omega)$ and $1/\mu^2 \omega$. It is therefore natural to employ the asymptotic method of multiple scales. Perturbation equations are derived for the short-scale variations in each unit cell. Solvability of a higher-order cell problem will lead to the equations governing large-scale dynamics. Since the full analysis is quite similar to the inviscid theory of [2], many details are omitted here and only new parts will be described.

III. MULTIPLE-SCALE ANALYSIS AND THE FIRST-ORDER CELL PROBLEM

We shall consider a periodic array over an area much greater than the typical wavelength. Except in the immediate neighborhood of the cylinders, the wave (outer) problem is one of two contrasting scales. As in [2], we introduce fast and slow variables
\[
x, y, z, t; \quad (X, Y, T) = \mu^2(x, y, t),
\]
(3.1)
so that $x$, $y$, $z$, and $t$ describe the fast motion characterized by the length and time scales of $1/k$ and $1/\omega$, while $X$, $Y$, and $T$ describe the slow variation of the envelope over the whole array. On the short scale of a unit (periodic) cell or a wavelength, the total array is practically infinite in extent. We shall apply Bloch theorem [7,8] so that for any lattice vector $R=R(m_1, m_2)$,
\[
\eta(r) = e^{iK \cdot r} \eta(r+R) \quad \text{or} \quad \phi(r, z) = e^{iK \cdot r} \phi(r+R, z)
\]
(3.2)
with $\eta$ and $\phi$ being horizontally periodic.

Let us expand the outer potential as follows:
\[
\Phi = [\phi_1 + \mu^2 \phi_2 + O(\mu^4)]e^{-i\omega t},
\]
(3.3)
where $\phi_1$ and $\phi_2$ are functions of $(x, y, z; X, Y, T)$. Substituting (3.3) into the governing equations (2.1)–(2.3), we obtain the governing equations for the perturbation potentials $\phi_1$ and $\phi_2$.

In particular, $\phi_1$ only satisfies the homogeneous conditions on the fast scale in a unit cell $V$ as sketched in Fig. 1:

![FIG. 1. A periodic cell around a vertical cylinder. $a_1$ and $a_2$ are the primitive lattice vectors.](image)

\[
\nabla^2 \phi_1 + \frac{\partial^2 \phi_1}{\partial z^2} = 0, \quad \text{in} \ V,
\]
(3.4)
\[
\frac{\partial \phi_1}{\partial z} = \frac{\omega^2}{8} \phi_1 = 0, \quad z = 0,
\]
(3.5)
\[
\frac{\partial \phi_1}{\partial z} = 0, \quad z = -H.
\]
(3.6)

Note that these differential equations describe variations over the fast (short) scale and are the same from one periodic cell to another. The total domain $(kX, kY)=O(1)$ being very large relative to the fast coordinates, we impose Bloch condition (3.2) on $\phi_1$.

Because the cylinders are so small, condition (2.21) is not effective at this order.

We assume that $N$ progressive plane waves satisfy (or nearly satisfy) the Bragg condition of resonance. Let $k_j$ denote the incident wave vector and $k_j=k(\cos \beta_j, \sin \beta_j), \ j = 2, 3, \ldots, N$, the resonantly scattered wave vectors, where $\beta_j$ denotes the direction of $k_j$ with respect to the $x$ axis. Let $K_{1,j}$ be the reciprocal lattice vector pointing from the tip of $k_1$ to the tip of $k_j$. The Bragg condition [7,8] reads
\[
K_j = k_1 + K_{1,j}, \quad j = 1, 2, 3, \ldots, N.
\]
(3.7)

For a given lattice and incident wave vector, $k_1$ and $k_j$, $j \neq 1$, can be found by Ewald’s geometrical construction [7,8].

Formally the solution for $\phi_1$ is the sum of all $N$ mutually resonating progressive waves of amplitudes $A_j(X, Y, T)$:
\[
\phi_1 = \sum_{j=1}^{N} A_j \psi_j(x, y, z) = \sum_{j=1}^{N} A_j Z(z)e^{iK_j \cdot r}, \quad \psi_j = Z(z)e^{iK_j \cdot r}.
\]
(3.8)

The slow variations of $A_j(X, Y, T)$ remain to be found at the next order.

IV. SECOND-ORDER PROBLEM

At second order, $\phi_2$ is governed by
\[ \nabla^2 \phi_2 + \frac{\partial^2 \phi_2}{\partial z^2} = -2 \nabla \cdot \nabla \phi_1, \quad \text{in } V, \]  
(4.9) where \( \nabla = (\partial_x, \partial_y) \),

\[ \frac{\partial \phi_2}{\partial z} - \frac{\omega^2}{g} \phi_2 = \frac{2i\omega}{g} \frac{\partial \phi_1}{\partial T}, \quad z = 0, \]  
(4.10) and

\[ \frac{\partial \phi_2}{\partial z} = 0, \quad z = -H. \]  
(4.11)

Bloch condition (3.2) also applies to \( \phi_2 \). On the cylinder surface, condition (2.21) now requires

\[ \frac{\partial \phi_2}{\partial \rho} = -\frac{1}{\mu^2} \frac{\partial \phi_1}{\partial \rho} - \Lambda \phi_1, \quad \rho' = |\rho'| = a. \]  
(4.12)

These inhomogeneous equations govern \( \phi_2 \) over the short coordinates in a periodic equations.

Equation (4.12) requires that the \( O(1) \) gradient \( \partial \phi_1/\partial \rho' \) on the small cylinder must be canceled by the large gradient of a small \( O(\mu^2) \) potential. Thus \( \phi_2 \) must change quickly in the neighborhood of \( \rho' = O(a) \), which is much smaller than the cell size \( \sim O(1/k) \). This requirement is outside the realm of (4.9)–(4.11) and (3.2) We now let \( \rho_0 \) be the sum of the outer and inner solutions: \( \phi_{2,\text{in}}^\text{in} \) and \( \phi_{2,\text{out}}^\text{in} \). The outer solution is dominant in the far field, \( \phi_2 \equiv \phi_{2,\text{out}}^\text{in}, k r = O(1) \), and satisfies (4.9)–(4.11) and (3.2). On the other hand, the inner solution is of local importance and dominant in the near field only: \( \phi_2 = \phi_{2,\text{in}}^\text{in}, r' = O(a) \). It must vanish at \( O(1/k) \gg r' \gg a \), while its radial gradient must be so large as to satisfy (4.12). In this small neighborhood, \( \phi_{2,\text{in}}^\text{in} \) needs only satisfy the horizontal Laplace equation instead of (4.9), with an error of \( \mu^2 \). By using the approximation of \( \partial \phi_1/\partial \rho' \) on the cylinder described in detail in [2], (4.12) becomes, approximately,

\[ \frac{\partial \phi_{2,\text{in}}^\text{in}}{\partial \rho'} = -Z(z) \sum_{\rho_0} A_{m_0} e^{ik \rho_0} \left\{ \frac{i \cos(\varphi - \beta_0)}{\mu^2} \right. \]

\[ - \frac{1 + \cos 2(\varphi - \beta_0)}{2 \mu} + \frac{\Lambda a}{\mu} + O(\mu^3) \left. \right\}, \quad r' = a, \]  
(4.13) where the last term involving \( \Lambda \) is new. It is now easy to find the approximate solution

\[ \phi_{2,\text{in}}^\text{in} = Z(z) \sum_{j=1}^N A_j e^{ik r_j} \left\{ \frac{\ln kr_j}{2} (1 - 2 \Lambda a) \right. \]

\[ + \frac{i a}{\rho' \mu^2} \cos(\varphi - \beta_j) - \frac{i a^2}{4 r_j^2} \cos 2(\varphi - \beta_j) \left. \right\} + O(\mu). \]  
(4.14)

From this, the value of \( \phi_{2,\text{in}}^\text{in} \) on the cylinder \( r' = a \) can be obtained. Together with the outer solution, the sum

\[ \phi_2 = \phi_{2,\text{out}}^\text{in} + \phi_{2,\text{in}}^\text{in} \]  
(4.15)

is uniformly valid everywhere in the unit cell surrounding the cylinder.

We now derive the envelope equations for \( A_j \) by examining the solvability of \( \phi_2 \), without solving for \( \phi_{2,\text{out}}^\text{in} \) explicitly.

A. Solvability of \( \phi_2 \) and envelope equations

Refering to (3.8), we apply Green’s identity to \( \psi^{j(8)} \) and \( \phi_2 \) over the unit cell shown in Fig. 1:

\[ \int \int_V \left\{ \phi_2 \left( \nabla^2 + \frac{\partial^2}{\partial z^2} \right) \psi^{j(8)} - \psi^{j(8)} \left( \nabla^2 + \frac{\partial^2}{\partial z^2} \right) \phi_2 \right\} dV = \int \int_{\partial V} \left( \phi_2 \frac{\partial \psi^{j(8)}}{\partial n} - \psi^{j(8)} \frac{\partial \phi_2}{\partial n} \right) dS \]  
(4.16)

where \( \psi^{j(8)} = Z(z) e^{-ik \rho_j} \) denotes the complex conjugate of the leading-order potential \( \psi^{j(8)} \). The bounding surface of \( V \), denoted by \( \partial V \), consists of the free surface \( S_F \), the cylinder surface \( S_B \), the vertical surfaces \( S_V \), and the sea bottom at \( z = -H \). Since the governing equations for \( \phi_2 \) (dominated by \( \phi_{2,\text{out}}^\text{in} \)) are known away from the cylinders in terms of \( A_j \), the above identity amounts to the solvability condition for the inhomogeneous boundary value problem and should give the evolution equations for the wave envelopes. Using the governing equations and the explicit solution of \( \phi_{2,\text{out}}^\text{in} \) near the cylinders, all the integrals can be evaluated, leading to the evolution equations for \( A_j \). Most of these integrals have already derived in [2]. The surface integral over the cylinder wall is slightly different. By following the steps of [29] we get

\[ I_B = \sum_{h=1}^N \pi \bar{A}_h (1 - 2 \Lambda a - 2 \cos(\beta_j - \beta_h)) \int_{-H}^0 |Z(z)|^2 dz \]

\[ + O(\mu). \]  
(4.17)

With this modification the new envelope equations are found. Letting

\[ C^{(j)}_g = \frac{k_j}{\kappa_j}, \quad j = 1, \ldots, N, \]  
(4.18)

denote the group velocity of wave \( j \) and

\[ \Omega_0 = \pi C_g k A \]  
(4.19)

coupling coefficient, where \( A \) denotes the cross-sectional area of the cell, we obtain

\[ \frac{\partial A_j}{\partial T} + C_g^{j(8)} \cdot \nabla A_j = -\frac{i}{2} \Omega_0 \sum_{h=1}^N [(1 - 2 \Lambda a) - 2 \cos(\beta_j - \beta_h)] A_h, \]

\[ j = 1, \ldots, N. \]  
(4.20)

Returning to natural coordinates, the envelope equations read

which couple \( N_j \) mutually resonating waves in the array. The coupling coefficient \( (ka)^2 \Omega_0 = k C_g (\pi a / A) \) on the right-hand side of (4.21) is proportional to volume density of the cylinders in water. The effect of boundary layers is represented by the complex factor \( \Lambda a \). In open waters outside the area of cylinders, \( a = 0 \), (4.20) are uncoupled and reduce to
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\[ \frac{\partial A_j}{\partial t} + \mathbf{C}_g^{(j)} \cdot \nabla A_j = 0, \quad j = 1, \ldots, N. \] (4.22)

With viscous effects, the law of energy conservation becomes

\[ \frac{\partial}{\partial t} \sum_{j=1}^{N} |A_j|^2 + \sum_{j=1}^{N} (\mathbf{C}_g^{(j)} \cdot \nabla |A_j|^2) = -\Lambda_0 \alpha \alpha_0 \sum_{j=1}^{N} \sum_{k=1}^{N} (A_k \alpha_j)^* . \] (4.23)

As expected, the total energy is damped in the cylinder array by viscous dissipation whose rate increases with \( \Lambda_0 \alpha_0 \) and \( N \).

In principle the system (4.20) can be used to study resonant diffraction by a large array in an area of any plan form, for which numerical techniques are needed in general. We consider a long and straight strip of many rows of cylinders parallel to the rows of a rectangular array. The incidence is along the \( X \) direction. This problem of one-directional propagation is equivalent to Bragg scattering by a linear array along the centerline of a long channel of width \( a_2 \).

Letting \( k = n \pi/a_1 \), \( \beta_1 = 0 \), and \( \beta_2 = \pi \) in (4.20), the following pair of equations for the envelopes of the incident (\( A_1 \)) and reflected (\( A_2 \)) waves are found:

\[ \frac{\partial A_1}{\partial t} + C_g \frac{\partial A_1}{\partial X} = - \frac{1}{2} i \Omega_0 (-A_1 + 3A_2) + i \Omega_0 \Lambda_0 \alpha_0 (A_1 + A_2), \] (5.5a)

\[ \frac{\partial A_2}{\partial t} - C_g \frac{\partial A_2}{\partial X} = - \frac{1}{2} i \Omega_0 (3A_1 - A_2) + i \Omega_0 \Lambda_0 \alpha_0 (A_1 + A_2), \] (5.5b)

where

\[ \Omega_0 = \frac{\pi C_g}{k a_1}, \quad \alpha = \frac{\pi C_g}{a_1 a_2}, \quad \frac{C_g}{\alpha_0} . \] (5.6)

The energy equation (4.23) takes the following form:

\[ \frac{\partial}{\partial t} (|A_1|^2 + |A_2|^2) + C_g \frac{\partial}{\partial X} (|A_1|^2 - |A_2|^2) = -2 \Lambda_0 \alpha_0 |A_1 + A_2|^2 . \]

1. Envelope dispersion in an infinite domain

Equations (5.5a) and (5.5b) can be combined by eliminating either \( A_1 \) or \( A_2 \) to yield the complex Klein-Gordon equation

\[ \left[ \frac{\partial^2}{\partial t^2} + i \Omega_0 (1 + 2 \Lambda_0 \alpha_0) \frac{\partial}{\partial T} - C_g^2 \frac{\partial^2}{\partial X^2} + 2 \Omega_0^2 (1 - 2 \Lambda_0 \alpha) \right] \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 . \] (5.7)

Consider the following solution in an infinite domain:

\[ A_1 = A_0 e^{i\mu_2 \mu X - i\Omega t}, \] (5.8)

where \( \mu^2 \mu \) and \( \mu^2 \Omega \) correspond to detunings of wave number and frequency. Equation (5.7) gives the dispersion relation

\[ K^2 = \left( \frac{\Omega}{C_g} \right)^2 + \left( \frac{\Omega}{\Omega_0} + 2 \right) \left( \frac{\Omega}{\Omega_0} + 2 \Lambda_0 - 1 \right) . \] (5.9)

For a range of real \( \Omega \), the real and imaginary parts of the complex parameter \( K \) are plotted in Fig. 2 for different values of the complex parameter \( \Lambda_0 \). Without dissipation, \( \Lambda_0 = 0.0 \), it is known \([1]\) that \( K \) is purely imaginary within the band gap \(-2 < \Omega/\Omega_0 < 1 \), where propagation is forbidden; outside the band gap, \( K \) is purely real, so the envelopes propagate without attenuation as dispersive waves. See Fig. 2(a). In con-
uncoupled equations: 

\[ X = \text{Im}(KCg/\Omega_0). \]

Thus the wave numbers are different from that in open waters. At the entrance and exit, we require the boundary conditions

\[ T(0) = 1, \quad R(L) = 0. \]

The solution can be readily found:

\[ T(X) = T_1 e^{i mX} + T_2 e^{-i mX}, \]
\[ R(X) = R_1 e^{i mX} + R_2 e^{-i mX}, \]

where

\[ T_1 = \frac{B + C}{(B + C) - (B - C)e^{2i mL}}, \]
\[ T_2 = -\frac{(B - C)e^{2i mL}}{(B + C) - (B - C)e^{2i mL}}, \]
\[ R_1 = \left( \frac{3}{2} - \Lambda a \right) \frac{1}{(B + C) - (B - C)e^{2i mL}}, \]
\[ R_2 = -\left( \frac{3}{2} - \Lambda a \right) \frac{e^{2i mL}}{(B + C) - (B - C)e^{2i mL}}, \]

with

\[ B = \frac{\Omega}{\Omega_0} + \frac{1}{2} + \Lambda a, \]
\[ C = \frac{mC}{\Omega_0} = \sqrt{\frac{\zeta_r + i\zeta_i}{2}}, \]
\[ = \sqrt{\frac{\zeta_r^2 + \zeta_i^2 + \zeta_r}{2}}, \]

and

\[ \zeta_r = \left( \frac{\Omega}{\Omega_0} + 2 \right) \left( \frac{\Omega}{\Omega_0} + 2 \Lambda a - 1 \right), \quad \zeta_i = \left( \frac{\Omega}{\Omega_0} + 2 \right) (2 \Lambda a). \]

In particular, the transmission coefficient at the exit is

\[ T(L) = e^{-i mL} + \frac{(B + C)(e^{i mL} - e^{-i mL})}{(B + C) - (B - C)e^{2i mL}}, \]

and the reflection coefficient at the entrance is

\[ R(0) = \left( \frac{3}{2} - \Lambda a \right) \frac{1 - e^{2i mL}}{(B + C) - (B - C)e^{2i mL}}. \]

The controlling parameters are \( \Lambda a \) (viscosity), \( \Omega/\Omega_0 \) (detuning), and \( \Omega_0 L / C_g \) (array width). For different viscosity parameters, \( \Lambda a \), we display in Fig. 3 the spatial variation of the reflected energy intensity \( |R(X)|^2 \) as a function of dimensionless width of the cylinder array \( \Omega_0 L/C_g \). Two detuning fre-
FIG. 3. Reflection intensity $|R(X)|^2$ along the lattice for various $\Omega_0L/C_g$ with detuning parameter $\Omega/\Omega_0=0.5$ (left figures) and 2.0 (right figures) for different values of the parameter $\Lambda a$: (a) $\Lambda a=0.0$ (no dissipation), (b) $\Lambda a=0.1(1+i)$, (c) $\Lambda a=0.5(1+i)$, and (d) $\Lambda a=2.0(1+i)$. Thick solid curve: $\Omega_0L/C_g=1$. Thin solid curve: $\Omega_0L/C_g=4$. Dashed curve: $\Omega_0L/C_g=8$.

FIG. 4. Dependence of the reflection intensity at inlet $X=0$, $|R(0)|^2$, on array width $\Omega_0L/C_g$ for various detuning parameters $\Omega/\Omega_0$ for (thin solid curve), −0.5 (dotted curve), 0.5 (dash-dotted curve), 1 (dashed curve), and 2 (thick solid curve) for different values of the parameter $\Lambda a$: (a) $\Lambda a=0.0$ (no dissipation), (b) $\Lambda a=0.1(1+i)$, (c) $\Lambda a=0.5(1+i)$, and (d) $\Lambda a=2.0(1+i)$.

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Fig. 5. Dependence of the reflection intensity at $X=0$, $|R(0)|^2$, on detuning parameter $\Omega/\Omega_0$ for various array widths $\Omega_0L/C_g = 0.5$ (thick solid curve), 1.0 (thin solid curve), 4.0 (dashed curve), 8.0 (dashdot curve) for different values of the parameter $\Lambda\alpha$: (a) $\Lambda\alpha=0.0$ (no dissipation), (b) $\Lambda\alpha=0.1(1+i)$, (c) $\Lambda\alpha=0.5(1+i)$, and (d) $\Lambda\alpha=2.0(1+i)$.

$\alpha=-0.5$ [see Fig. 5(a)], similar to the dispersion relation shown in Fig. 2(a). When $\Lambda\alpha \neq 0.0$, symmetry is lost. The highest reflection occurs at $\Omega/\Omega_0=-2$. Also $|R(0)|^2$ does not vanish for any values of $\Omega/\Omega_0$ and oscillates with increasing $\Lambda\alpha$. In all cases, reflection in general decreases with increasing $|\Omega/\Omega_0|$.

VI. TWO-DIMENSIONAL SCATTERING: OBLIQUE INCIDENCE BY A LONG ARRAY

In this section we shall reexamine the simpler cases studied by [2] where only one new wave is resonantly scattered in the array—i.e., $N=2$. The case of two or more new waves ($N=3, 4, \ldots$) is algebraically more complex and is not treated here.

Without loss of generality we let the direction of the incident wave be $0<\beta_1<\pi/2$, so that $k_1$ points to the north-east.Limiting to a square lattice, one finds by Ewald’s construction that four possibilities exist for the scattered waves, as shown in Figs. 6(a)–(d) in [2]. They are the following: (i) Forward scattering: (a) $0<\beta_2<\pi/2$ and (b) $-\pi/2<\beta_2<0$. There is in general no reflection on the left of the array, but two transmitted waves on the right. (ii) Backward scattering: (c) $\pi/2<\beta_2<\pi$ and (d) $\beta_2<\beta_2<3\pi/2$. There is in general reflection (hence two waves) on the left and only transmission hence one wave) on the right of the array.

A. Solutions for wave envelopes

Inside the array, $0<X<L$, the envelopes are denoted by $A_j$, $j=1, 2$, which are governed by (4.20). We denote the envelopes in the open waters left of the array by $A_j^\uparrow$ and right of the array by $A_j^\downarrow$, $j=1, 2$, which are governed by (4.22) instead. Let the incident wave envelope be

$$A_1 = \exp[iK(X \cos \beta_1 + Y \sin \beta_1) - i\Omega T],$$

$$\Omega = C_gK, \quad X < 0.$$  \hspace{1cm} (6.1)

We assume solutions of the form

$$[A_j^\uparrow, A_j^\downarrow] = A_0[2\pi B_j(X), B_j(X)]e^{i(\kappa \sin \beta_Y - \Omega T)}, \quad j=1, 2.$$  \hspace{1cm} (6.2)

Then, inside the array, $0<X<L$, we have

$$\frac{dB_1}{dX} = \frac{i\Omega_0}{C_g} \left[ \frac{1 + 2\Lambda\alpha}{2 \cos \beta_1} \frac{\Omega}{\Omega_0} \cos \beta_1 \right] B_1 + \frac{2 \cos(\beta_1 - \beta_2) - 1 + 2\Lambda\alpha}{2 \cos \beta_1} B_2, \quad (6.3a)$$

$$\frac{dB_2}{dX} = \frac{i\Omega_0}{C_g} \left[ \frac{2 \cos(\beta_1 - \beta_2) - 1 + 2\Lambda\alpha}{2 \cos \beta_2} B_1 + \left[ \frac{1 + 2\Lambda\alpha}{2 \cos \beta_2} \frac{\Omega}{\Omega_0} \frac{1 - \sin \beta_1 \sin \beta_2}{\cos \beta_2} \right] B_2 \right]. \quad (6.3b)$$

In the open water on the left ($X<0$), the envelopes are uncoupled:

$$\frac{dB_1^\uparrow}{dX} = iK \cos \beta_1 B_1^\uparrow,$$  \hspace{1cm} (6.4a)

$$\frac{dB_2^\uparrow}{dX} = \frac{iK(1 - \sin \beta_1 \sin \beta_2)}{\cos \beta_2} B_2^\uparrow. \hspace{1cm} (6.4b)$$

On the transmission side ($X>L$), we have instead

$$\frac{dB_1^\downarrow}{dX} = iK \cos \beta_1 B_1^\downarrow,$$  \hspace{1cm} (6.5a)

$$\frac{dB_2^\downarrow}{dX} = \frac{iK(1 - \sin \beta_1 \sin \beta_2)}{\cos \beta_2} B_2^\downarrow. \hspace{1cm} (6.5b)$$

Use is made of the relation $\Omega=C_gK$. Inside the array of cylinders, the envelopes are coupled:

Fig. 6. Forward scattering by a square lattice of spacing $a_1$. The incident and scattered wave vectors $k_1$ and $k_2$ are inclined at $\beta_1 = \pi/3$ and $\beta_2 = -\pi/3$. 026314-8
The solution is a linear combination of exponential terms

$$
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\
 b_{21} & b_{22}
\end{bmatrix} \begin{bmatrix} E_1 \\
E_2
\end{bmatrix},
$$

where

$$
E_1 = \exp \left( \frac{i \xi_1 \Omega_0 X}{C_g} \right), \quad E_2 = \exp \left( \frac{i \xi_2 \Omega_0 X}{C_g} \right).
$$

The eigenvalues $\xi_1$ and $\xi_2$ are the roots of a quadratic equation

$$
d \begin{bmatrix} B_1 \\
B_2
\end{bmatrix} = \frac{i \Omega_0 M}{C_g} \begin{bmatrix} B_1 \\
B_2
\end{bmatrix} = 0,
$$

where $M$ is the matrix with elements

$$
M_{11} = \frac{1 + 2 \Lambda \alpha}{2 \cos \beta_1} + \left( \frac{\Omega}{\Omega_0} \right) \cos \beta_1,
$$

$$
M_{12} = \frac{2 \cos(\beta_1 - \beta_2) - 1 + 2 \Lambda \alpha}{2 \cos \beta_1},
$$

$$
M_{21} = \frac{2 \cos(\beta_1 - \beta_2) - 1 + 2 \Lambda \alpha}{2 \cos \beta_2},
$$

$$
M_{22} = \frac{1 + 2 \Lambda \alpha}{2 \cos \beta_2} + \left( \frac{\Omega}{\Omega_0} \right) \frac{1 - \sin \beta_1 \sin \beta_2}{\cos \beta_2}.
$$

The eigenvalues $\xi_1$ and $\xi_2$ are the roots of a quadratic equation

$$
\xi_{1,2} = \frac{(M_{11} + M_{22}) \pm \Delta^{1/2}}{2},
$$

where $\Delta$ is the discriminant:

$$
\Delta = (M_{11} + M_{22})^2 - 4(M_{11}M_{22} - M_{12}M_{21})
= (M_{11} - M_{22})^2 + 4M_{12}M_{21}
= \left[ 1 + 2 \Lambda \alpha \left( \frac{1}{\cos \beta_1} - \frac{1}{\cos \beta_2} \right) + \left( \frac{\Omega}{\Omega_0} \right) \frac{\cos(\beta_1 - \beta_2) - 1}{\cos \beta_2} \right]^2
+ \frac{2 \cos(\beta_1 - \beta_2) - 1 + 2 \Lambda \alpha}{\cos \beta_1 \cos \beta_2}.
$$

Since $\Lambda \alpha$ is complex, both eigenvalues are always complex, implying spatial attenuation or amplification along $X$ in addition to the oscillatory behavior. For a given lattice, we first find the direction $\beta_2$ of the scattered wave for a given incident wave $k_1$ by Ewald's construction. The discriminant $\Delta$ and the eigenvalues then depend on the inclinations $(\beta_1, \beta_2)$ and the detuning frequency $\Omega/\Omega_0$ as given by (6.10) and (6.11), which are affected by viscosity. We now discuss two cases.

**B. Forward scattering: $\cos \beta_2 > 0$**

We shall only study one of the two cases where the scattered wave is directed to the south-east, as shown in Fig. 6.
For these \( \beta_1, \beta_2 \), the real and imaginary parts of the two eigenvalues \( \xi_1 \) and \( \xi_2 \) are displayed in Fig. 7 for different values of the dissipation parameter \( \Delta \alpha \) and for a range of the detuning parameter \( \frac{\Delta \omega}{\Omega_0} \). In the absence of viscosity, both eigenvalues are real [see Fig. 7(a)], but complex otherwise. The imaginary part is larger for greater viscosity.

The boundary conditions are

\[
B_1^* = B_1, \quad X = 0 \quad \text{and} \quad B^+ = B^*, \quad B_2 = B_2^*, \quad X = L.
\]

(6.12)

These are formally the same as Eqs. (72), (73a) and (73b) in [2] with the new \( \xi_i \) and \( M_{ij} \) given above.

The wave intensities \( |B_1(X)|^2 \) and \( |B_2(X)|^2 \) across the array are shown in Fig. 8 for \( \Omega_0 L/C_g = 8 \). As the detuning parameter \( \Omega/\Omega_0 \) increases, the scattered wave becomes weaker. Spatial oscillations of both \( |B_1(X)|^2 \) and \( |B_2(X)|^2 \) become more intense as detuning increases, but disappear as \( \Delta \alpha \) increases. Without dissipation one finds, at certain \( X_m/L, B_1(X_m) = 1 \) and \( B_2(X_m) = 0 \). Therefore transmission is perfect if the array width is exactly \( X_m \). On the other hand, at certain other \( X_m/L, B_1(X_m) = 1 \) and \( B_2(X_m) = 0 \); only the scattered wave is seen at the exit if the array width is exactly \( X_m \). Now on the transmission side only the scattered wave emerges, which can be viewed as a transmitted wave inclined at the angle \( \beta_2 = -\pi/3 \). In this particular case the array behaves like a piece of transparent glass, but like a mirror transverse to the array and reflects the incident waves. With increasing viscosity, this occurrence is weakened.

To examine the effect of detuning on the waves at the exit edge, we fix the array width \( L \). The dependence of \( |B_1(L)|^2 \) on detuning is shown in Fig. 8 for \( \Delta \alpha = 0.0 \) (no dissipation), and \( \Delta \alpha = 0.1 \) (dashed curve), and \( \Delta \alpha = 0.5 \) (dash-dotted curve).
and $|B_2(L)|^2$ on the detuning frequency $\frac{\Omega}{\Omega_0}$ is plotted for two array widths $\Omega_0 L/C_g = 2$ and 4 in Fig. 9. Without dissipation, transmission is small, but scattering is strong in the inviscid band gap centered around $\Omega = 0$. With dissipation, the band gap disappears. For small $\Lambda a = 0.1(1+i)$, transmission and scattering are both weakened. With larger dissipation, the two forward waves become equally small and significant only near $\Omega = 0$.

### C. Backward scattering: $\cos \beta_2 < 0$

As another example, we consider a square lattice of spacing $a_1$ and choose the incident wave vector $k_1$ such that the scattered wave vector $k_2$ is as shown in Fig. 10—i.e., $\beta_1 = \pi/6$—so that $\beta_2 = 5\pi/6$. The incident wave number is $k = 2\pi/\sqrt{3}a_1$. Now the boundary conditions at the edges of the cylinder array are

$$B_1(0) = 1, \quad B_2(L) = 0.$$  \hspace{1cm} (6.15)

The solutions are formally the same as Eqs. (98)–(100) in [2].

For these angles ($\beta_1, \beta_2$), the real and imaginary parts of the complex $\xi_1$ and $\xi_2$ are shown for a range of detuning $\Omega/\Omega_0$ and different values of $\lambda a$ in Fig. 11. Understandably, the qualitative features are similar to the one-dimensional case of normal incidence and reflection. Without viscosity $\Lambda a = 0.0$, there is a band gap. With finite dissipation, both eigenvalues are complex for all $\Omega/\Omega_0$. The band gap disappears.

Figure 12 shows the spatial variation of the transmission intensity $|B_1(X)|^2$ and the reflection intensity $|B_2(X)|^2$ across the strip for various detunings $\frac{\Omega}{\Omega_0} = 0.5, 2, -3$. When $\Lambda a = 0.0$, the wave intensities are oscillatory in $X$ for $\Omega/\Omega_0 = -3$ and 2 which are outside the band gap, and attenuate monotonically for $\Omega/\Omega_0 = 0.5$, which is inside the band gap. Understandably, the solutions are qualitatively similar to the one-dimensional case of simple reflection. When $\Lambda a \neq 0.0$,
dissipation damps out the oscillations outside the band gap and makes them disappear as $\Lambda a$ becomes larger. Within the band gap, wave intensities attenuate monotonically, but faster as $\Lambda a$ increases.

Figure 13 shows the dependence of transmission intensity at the exit edge $X=L$ (left) and the reflection intensity at the entry edge $X=0$ (right) on the detuning $\Omega/\Omega_0$ for different array widths: $\Omega L/C_g=1, 2, 8$. Without viscosity, weak transmission and strong backscattering prevail inside a clear band gap. With increasing viscosity the band gap shrinks to the immediate neighborhood of $\Omega/\Omega_0=\pm 2$. The transmission intensity diminishes more rapidly than reflection.

VII. CONCLUDING REMARKS

We have examined the effects of viscosity on the propagation of small-amplitude water waves through a periodic array of vertical cylinders. Under the assumptions likely realistic for future offshore airports—i.e., small cylinders and large spacing—we have considered the phenomenon of Bragg resonance. The asymptotic approach of [1,2] is followed. Boundary layer effects due to viscosity are represented in terms of the velocity potential by using a fictitious boundary condition on the cylinders. The effective coefficient is chosen so as to give the correct dynamical effect on the cylinders. The model should be directly applicable to laboratory tests where molecular viscosity is relevant. In the field the boundary layer is likely turbulent so that a much larger (empirical) value of eddy viscosity must be used instead. We also reason in the Appendix that vortex shedding is important only for large-amplitude waves. Since the mathematical problem treated here is similar to two-dimensional scattering of sound by a periodic array of parallel wires, the present theory may be modified to examine the effects of dissipation on multiple scattering including band gaps, etc.
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APPENDIX: REMARKS ON VORTEX SHEDDING

As is well known, flow around a smooth vertical circular cylinder in waves is characterized by the Reynolds number \( \text{Re} = \frac{U_0 D}{\nu} \) and the Keulegan-Carpenter number \( \text{KC} = \frac{U_0 T}{D} \), where \( U_0 \) is the horizontal fluid velocity at the center of the cylinder in the absence of the cylinder, \( T \) the wave period, \( D \) the cylinder diameter, and \( \nu \) kinematic viscosity (see, for example, [10,11]).

To examine the role of shedding vortices in gentle waves considered here, we focus attention on the free surface \( z = 0 \) where the horizontal fluid velocity attains its maximum value.

In view of (2.5) and (2.7), the order of magnitude of the horizontal velocity at the free surface can be estimated by

\[
|U(x,y,z=0,t)| \approx |\nabla \eta(0)| = \frac{\rho}{\omega} |\text{Re} \nabla \eta| = \epsilon \frac{\rho}{\omega}, \tag{A1}
\]

where \( \epsilon = O(\nabla \eta) \) is the measure of local wave steepness. Making use of (2.8) and \( \mu = ka \), the maximum of Reynolds and Keulegan-Carpenter numbers at the free surface are

\[
\text{Re}_{\text{max}} = \frac{U(z=0)|_{\text{max}}}{\nu} = \frac{2a \frac{\rho}{\omega}}{\nu \omega} \epsilon, \tag{A2a}
\]

\[
(\text{KC})_{\text{max}} = \frac{U(z=0)|_{\text{max}}}{D} = \frac{\pi}{\rho a \omega} \frac{\rho}{\omega} \epsilon \frac{\epsilon}{\mu \tanh kH}. \tag{A2b}
\]

The linearized asymptotic theory in [1,2] is valid if the wave steepness is small enough such that

\[
\epsilon \leq O(\mu^2) = k^2 a^2 \tag{A3}
\]

(see [1]). Within the realm of the linearized theory, we have the following:

\[
\text{Re}_{\text{max}} \leq \frac{2a}{\nu \omega} O(\mu^2), \quad (\text{KC})_{\text{max}} \leq \frac{\pi}{\tanh kH} O(\mu), \tag{A4}
\]

in view of (A2) and (A3). Note that the maximum of Keulegan-Carpenter number is \( O(\mu) \).

Now let us estimate \( \text{Re}_{\text{max}} \) and \( (\text{KC})_{\text{max}} \) in reality. In the field, the typical values are \( a \approx 1-5 \text{ m}, ka \approx 0.1, \) and \( kH = 1, \) which for \( \nu = 10^{-6} \text{ m}^2/\text{s} \) leads to the following:

\[
\text{Re}_{\text{max}} \leq (0.23 - 2.5) \times 10^6, \quad (\text{KC})_{\text{max}} \leq 0.41. \tag{A5}
\]
In laboratory experiments, on the other hand, we take $a \sim 5–25$ cm, $ka=0.1$, and $kH=1$ so that

$$Re_{max} \leq (0.25 - 2.8) \times 10^4, \quad (K_C)_{max} \leq 0.41. \quad \text{(A6)}$$

According to typical field data (see, e.g., Fig. 3.16 of [11]), for Reynolds numbers in the above range for the field, $(0.23–2.5) \times 10^6$, the Keulegan-Carpenter number has to be greater than about 5.4 for vortex shedding to occur. From laboratory tests (see Fig. 3.15 of [11]), for Reynolds numbers in the range $(0.25–2.8) \times 10^4$, the Keulegan-Carpenter number has to be above 7 to trigger vortex shedding. Both values of the Keulegan-Carpenter number are higher than the estimated values in (A5) and (A6). Hence, within the bounds of linearized theory, vortex shedding is ineffective. For very strong waves, nonlinearity and vortex shedding can of course be important and even overwhelm Bragg scattering. A very different theory is then needed.

[6] This fictitious condition can also be applied to two-dimensional sound scattering by thin wires. Take, for example, the sound frequency $f=\omega/2\pi=10$ kHz, so that $\omega=2\pi \times 10^3$ rad/s. The Stokes boundary layer thickness is $\delta=0.707 \times 10^{-3}$ cm. For a fiber radius of 0.1 cm, we get $\tilde{a} \sim 0.01$ which is also small.
[9] In Sec. IV A and Appendix B of [2], $z^2$ everywhere should be changed to $|Z|^2$.