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by

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Abstract

Variational inequalities have often been used as a mathematical programming tool in modeling various equilibria in economics and transportation science. The behavior of such equilibrium solutions as a result of the changes in problem data is always of concern. In this paper, we present an approach for conducting sensitivity analysis of variational inequalities defined on polyhedral sets. We introduce the notion of differentiability of a point-to-set mapping and derive continuity and differentiability properties regarding the perturbed equilibrium solutions, even when the solution is not unique. As illustrated by several examples, the assumptions made in this paper are in a certain sense the weakest possible conditions under which the stated properties are valid. We also discuss applications to some equilibrium problems, such as the traffic equilibrium problem.

Key words. Sensitivity Analysis, Variational Inequalities, Perturbed Solution, Complementarity.

1. Introduction

In this paper we consider a perturbed version of variational inequalities defined on polyhedral sets. As is well-known, a number of equilibrium problems in economics and transportation science can be cast as a variational inequality problem with an underlying polyhedral set. Examples include spatial market equilibrium problems, Nash equilibrium games, oligopolistic equilibrium models and traffic equilibrium problems. Common practical applications include energy planning, urban transit system analysis and design, and prediction of intercity freight flows. The purpose of sensitivity analysis for these problems is threefold. First, since estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis can sometimes help, to certain degree, to predict the future changes of the equilibria as a result of the changes in the governing system. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. In addition, from a theoretical point of view, sensitivity properties of a mathematical programming problem can provide new insight concerning the problem being studied and can sometimes stimulate new ideas for problem-solving.

A number of authors have addressed sensitivity and stability issues of variational inequalities with special linear structures. The methodologies suggested so far vary with the problem settings being studied. Assuming the differentiability of the perturbed solution, Irwin and Yang [1982] provided an iterative method for computing the derivatives of the perturbed solution of a spatial price equilibrium problem on a bipartite graph. Chao and Friesz [1984], who considered the same problem over a transshipment network that has an equivalent nonlinear programming formulation, applied nonlinear programming sensitivity analysis results developed by Fiacco [1984]. Dafermos and Nagurney [1984] derived a continuity property of the perturbed solution for the traffic equilibrium problem as well as for the spatial market equilibrium problem.

In a recent paper, Kyparisis [1986] considered a general form of variational inequalities defined on polyhedral sets. He extended Robinson's work [1985] on generalized equations and derived sufficient conditions for differentiability of the perturbed solution.

All of these sensitivity analyses either assumed or finally showed that the perturbed solution is locally unique. However, in this paper the conditions we impose do not imply the local uniqueness of the perturbed solution. For this reason, we generalize the usual definition of differentiability to a point-to-set mapping. We also show that these conditions are in a certain sense the weakest possible ones needed to ensure the differentiability of the perturbed solutions.

Typically, the development of variational inequality sensitivity analysis for equilibrium problems like those mentioned previously involves several technical difficulties. The traffic equilibrium problem provides one illustration. Due to the problem's special structure, the variational inequality formulation of the problem (see Section 4) usually includes path flow variables. However, the fact that the path flow pattern is usually not unique at equilibrium prohibits the direct application of the variational inequality sensitivity analysis to this problem. Since the methodology suggested in this paper does not require uniqueness of the equilibrium solution, it can be used to derive sensitivity properties for a number of equilibrium problems including this traffic equilibrium application.

A couple of authors have also considered the sensitivity analysis of variational inequalities defined on nonpolyhedral sets. Assuming strict complementary slackness condition, Tobin [1986] applied the nonlinear programming sensitivity analysis results of Fiacco [1983] to variational inequalities. In the absence of strict complementary slackness, Kyparisis [1985] extended the continuity results of Robinson [1980] on generalized equations to obtain sufficient conditions for differentiability of the perturbed solution. In a subsequent paper, we will extend the results of this paper to variational inequalities defined on nonpolyhedral sets.

The next section defines the problem being considered and gives the key assumptions we make throughout the paper. It also introduces the notion of Lipschitz continuity and directional differentiability of the perturbed solution set. Four instructive examples show the necessity of these assumptions. In Section 3, we describe the suggested approach in detail. We first establish the Lipschitz continuity property, and then the differentiability property of the perturbed solution set. Section 4, which considers the application of the method to traffic equilibrium problems, introduces a more general form of the underlying ground set to accommodate the special features of applications like the traffic equilibrium problem. Finally in Section 5, we provide a numerical example to illustrate the procedure for computing the directional derivative of traffic equilibria — the derivative is determined as a unique solution to certain linear variational inequality over the network.

We believe that the approach adopted in this paper, via the development of the intermediate Lipschitz continuity property, not only permits us to establish the current results, but has the potential to be a general proof technique for establishing a variety of differentiability results.

2. Formulation

In this and next section, we consider the perturbed variational inequality problem of the following form :

VI (ε): find $x \in P$ satisfying $F(x, \varepsilon)^T (y - x) \geq 0$ for any $y \in P$

where $F(\cdot, \cdot)$ is a point-to-point mapping from $P \times R^m$ to R^n, $\varepsilon \in R^m$ is a perturbation parameter, and P is a polyhedron defined by $P = \{ x \in R^n \mid Ax \geq b \}$. Let $S(\varepsilon)$ denote the solution set of VI (ε), and let $x(\varepsilon)$ be any vector in $S(\varepsilon)$. Also, suppose x^* solves the problem VI (ε^*).

In our development, we do not assume that $S(\varepsilon) \cap U$ is necessarily a singleton for any ε in any neighborhood U of x^*, i.e., $\varepsilon \to S(\varepsilon) \cap U$ is generally a point-to-set mapping. Therefore, apart from the usual notion of continuity and semicontinuity of a point-to-set mapping, we define the Lipschitz continuity and directional differentiability of point-to-set mapping $S(\varepsilon) \cap U$ at (x^*, ε^*) as follows.

Definition 2.1. The perturbed local solution set $S(\varepsilon) \cap U$ is said to be *Lipschitz continuous* at (x^*, ε^*) if for some neighborhood V of ε^* and some number $L > 0$, $\| x(\varepsilon) - x^* \| \leq L \| \varepsilon - \varepsilon^* \|$ for any $x(\varepsilon) \in S(\varepsilon) \cap U$ and $\varepsilon \in V$.

Definition 2.2. The perturbed local solution set $S(\varepsilon) \cap U$ is said to be *directionally differentiable* at (x^*, ε^*) in the direction $\varepsilon_0 \in R^m$, if there is a vector $d(\varepsilon_0) \in R^n$ satisfying the property that for any $x(\varepsilon^* + t\varepsilon_0) \in S(\varepsilon^* + t\varepsilon_0) \cap U$,

$$\lim_{t \to 0^+} \frac{1}{t} \left[x(\varepsilon^* + t\varepsilon_0) - x^* \right] = d(\varepsilon_0).$$

The perturbed local solution set is said to be *directionally differentiable* at (x^*, ε^*) if it is directionally differentiable in every direction $\varepsilon_0 \in R^m$.

These definitions are natural extensions of the same notions for point-to-point mappings and have clear geometrical meanings — when the mapping is single valued, these definitions are exactly the usual ones for functions. By our definition, differentiability is a strong property that requires all points in $S(x^* + t\varepsilon_0) \cap U$ converge to a common point along same direction and with same rate. For example, $S(\varepsilon) = \{\varepsilon\}$ is differentiable at (0, 0) while $S(\varepsilon) = [0, \varepsilon]$ is not. In general, even when a point-to-set mapping $S(\cdot)$ is not differentiable along direction ε_0 at (x^*, ε^*), we let $D(\varepsilon_0) = \{ d \mid d$

is the limit of some convergent sequence of the form $[\, x(\varepsilon^* + t_k \varepsilon_0) - x^* \,] / t_k \}$. To be more precise, we first let $S_d(t, \varepsilon_0) = \{ [\, x(\varepsilon^* + t\varepsilon_0) - x^* \,] / t \mid \text{any } x(\varepsilon^* + t\varepsilon_0) \in S(\varepsilon^* + t\varepsilon_0) \cap U \}$ for $t > 0$. Then we define

$$D(\varepsilon_0) = \overline{\lim_{t \to 0}}\; S_d(t, \varepsilon_0) \equiv \{ d \mid \exists\, d(t_k) \in S_d(t_k, \varepsilon_0) \textit{ such that } d(t_k) \to d \textit{ as } t_k \downarrow 0 \}.$$

Clearly, $D(\varepsilon_0)$ also contains first order information regarding the limiting behavior of $S(\varepsilon)$ at (x^*, ε^*). For example, $S(\varepsilon) = \{ x \in R^2 \mid x_1^2 + x_2^2 = \varepsilon^2 \}$ for $\varepsilon \geq 0$ is not differentiable at $(0, 0)$, but with $\varepsilon_0 = 1$, $D(1) = \{ x \in R^2 \mid x_1^2 + x_2^2 = 1 \}$, which means the set $S(\varepsilon)$ converges to 0 along all directions with the same rate (that is, all limiting d have the same norm). Note that $D(\varepsilon_0)$ is a singleton if and only if $S(\varepsilon)$ is differentiable in the direction ε_0. When $S(\varepsilon)$ is Lipschitz continuous at (x^*, ε^*) with Lipschitz constant L, $D(\varepsilon_0) \subseteq \{ x \mid \| x \| \leq L \}$.

We next summerize the key assumptions invoked in this paper, which all concern local properties of the function $F(\cdot, \cdot)$. Let $(A_1 \;\; b_1)$ be the submatrix of $(A \;\; b)$ that corresponds to the binding constraints at x^*. Also, let $P^{\perp} = \{ y \in R^n \mid F(x^*, \varepsilon^*)^T y = 0, A_1 y \geq 0 \}$. Note that when $F(x^*, \varepsilon^*) \neq 0$, $x^* + P^{\perp}$ contains the part of the feasible region that lies on the supporting hyperplane defined by $F(x^*, \varepsilon^*)$.

Assumption 2.1. (Continuity condition) For some neighborhoods U of x^* and V of ε^*, $F(\cdot, \cdot)$ is continuous over $U \times V$.

Assumption 2.2. (Convergence condition) For some neighborhoods U of x^* and V of ε^* and some number $L > 0$, $\| F(x, \varepsilon) - F(x, \varepsilon^*) \| \leq L \| \varepsilon - \varepsilon^* \|$ for any $x \in (x^* + P^{\perp}) \cap U$, $\varepsilon \in V$.

Assumption 2.3. (Differentiability condition) $F(\cdot, \cdot)$ is differentiable at (x^*, ε^*), i.e., for any $x_0 \in R^n$ and $\varepsilon_0 \in R^m$,

$$\lim_{t \to 0^+} \frac{1}{t} \left[F(x^* + t x_0 + o(t), \varepsilon^* + t\varepsilon_0 + o(t)) - F(x^*, \varepsilon^*) \right] = \nabla_x F(x^*, \varepsilon^*) x_0 + \nabla_\varepsilon F(x^*, \varepsilon^*) \varepsilon_0.$$

Finally, we make an assumption on the limiting function $F(\cdot, \varepsilon^*)$,

Assumption 2.4. $F(\cdot, \varepsilon^*)$ is differentiable at x^* and $\nabla_x F(x^*, \varepsilon^*)$ is positive definite on span $(P^{\perp})$.

These assumptions are the weakest possible in the sense that if any one of them fails, then the perturbed solutions need not satisfy the differentiability property. We use four simple, but instructive one-dimentional examples to illustrate this point. In each example, $P = \{ x \in R^1 \mid x \geq 0 \}$, $0 \leq \varepsilon < 1$, $(x^*, \varepsilon^*) = (0, 0)$, and the function F violates only one of the assumptions.

Example 2.1. This example shows if the continuity condition is not satisfied, then the perturbed solution set may be empty. Consider function F of the form (See Figure 1.):

$$F(x, \varepsilon) = \begin{cases} -\varepsilon^2 & 0 \leq x < \varepsilon^2 \\ x & \varepsilon^2 \leq x < \infty. \end{cases}$$

In this case, $S(0) = \{0\}$ and $S\{\varepsilon\} = \{\varnothing\}$ for $0 < \varepsilon < 1$. It is also easy to verify that this example satisfies Assumptions 2.2, 2.3 and 2.4.

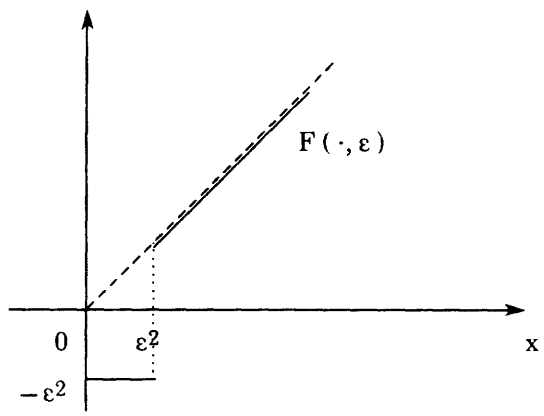

Figure 2.1. Continuity is Violated

Example 2.2. The convergence condition is violated by this example (let $x = 2\varepsilon^{1/2}$) while the rest of the conditions are still satisfied. F is chosen as follows (See Figure 2.2):

$$F(x, \varepsilon) = \begin{cases} 0 & 0 \leq x < \varepsilon^2 \\ (x - \varepsilon^2)/(1 - \varepsilon^{3/2}) & \varepsilon^2 \leq x < \varepsilon^{1/2} \\ -x + 2\varepsilon^{1/2} & \varepsilon^{1/2} \leq x < 2\varepsilon^{1/2} \\ 3(x - 2\varepsilon^{1/2}) & 2\varepsilon^{1/2} \leq x < 3\varepsilon^{1/2} \\ x & 3\varepsilon^{1/2} \leq x < \infty. \end{cases}$$

Clearly, $S(0) = \{0\}$ and $S(\varepsilon) = [0, \varepsilon^2] \cup \{2\varepsilon^{1/2}\}$ for $0 < \varepsilon < 1$. Thus $S(\varepsilon) \cap U$ is not differentiable in any neighborhood U of 0.

Example 2.3. This example shows that the perturbed solution set need not satisfy the differentiability property if F is not differentiable at (x^*, ε^*). Here F is defined by (See Figure 2.3.):

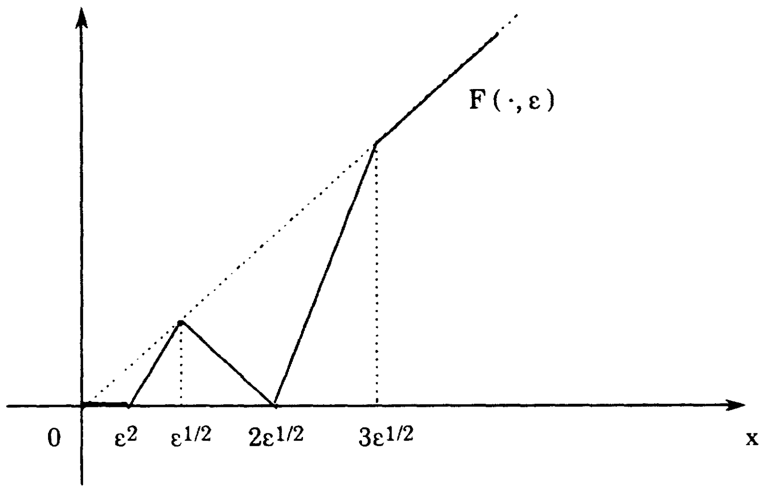

Figure 2.2. Convergence is Violated

$$F(x, \varepsilon) = \begin{cases} 0 & 0 \leq x < \varepsilon \\ 2(x - \varepsilon) & \varepsilon \leq x < 2\varepsilon \\ x & 2\varepsilon \leq x < \infty. \end{cases}$$

In this case, $S(\varepsilon) = [0, \varepsilon]$, which is not differentiable at (0, 0). Notice that F satisfies the other three conditions.

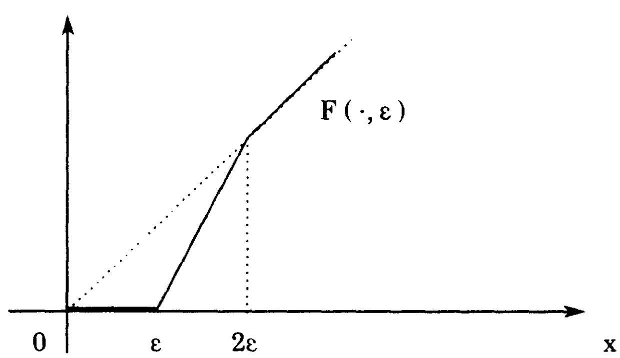

Figure 2.3. Differentiability is Violated

Example 2.4. In this example, $\nabla_x F(0, 0)$ does not satisfy the positive definiteness property. As a result, the perturbed solution set is not differentiable at (0, 0). F is specified as follows (See Figure 2.4.):

$$F(x, \varepsilon) = \begin{cases} 0 & 0 \le x < \varepsilon \\ 4\varepsilon(x - \varepsilon) & \varepsilon \le x < 2\varepsilon \\ x^2 & 2\varepsilon \le x < \infty. \end{cases}$$

Notice that function F satisfies Assumptions 2.1, 2.2 and 2.3 and that $S(\varepsilon) = [0, \varepsilon]$.

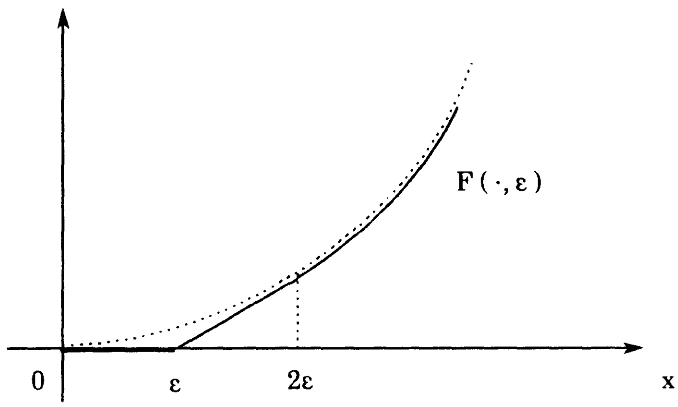

Figure 2.4. Positive Definiteness is Violated

Finally, in the example shown in Figure 2.5, F satisfies all the four assumptions and the perturbed solution set is indeed differentiable.

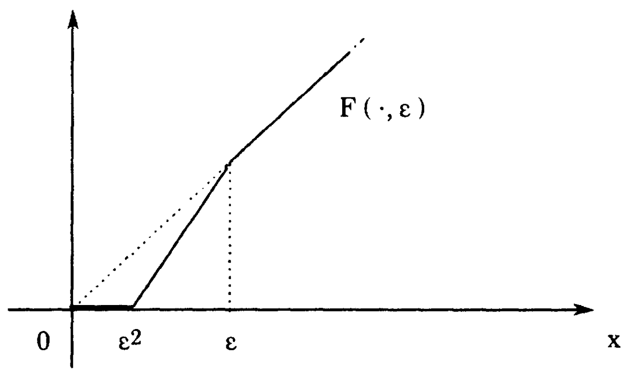

Figure 2.5. All Assumptions are Satisfied

As we will see in the next section, Assumptions 2.1-2.4 are weaker than those suggested by Kyparisis [1986], who assumed that F is once continuously differentiable around (x^*, ε^*) and that

∇_x F (x*, ε*) is positive definite on the subspace spanned by $P^{\perp}$. Under those conditions, Kyparisis showed that the perturbed solution set S (ε) is a singleton in a neighborhood of ε* and is directionally differentiable.

If we replace Assumption 2.4 with the following weaker condition,

Assumption 2.4.' $\nabla_x F(x^*, \varepsilon^*)$ is positive definite on $P^{\perp}$.

then the perturbed local solution set may not be directionally differentiable. This fact is illustrated by the following three-dimentional example. Let $P = \{ x \in R^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 = 0 \}$, $0 \leq \varepsilon < \infty$, and $F(x, \varepsilon) = (x_1 + x_2 - \varepsilon, x_1 + x_2 - \varepsilon, 1)$. Note that $x^* = (0, 0, 0)$ is the unique solution to VI (0) and that Assumptions 2.1, 2.2, 2.3 and 2.4' are satisfied. In this example. the perturbed solution set is given by $S(\varepsilon) = \{ x \in R^3 \mid x_1 + x_2 = \varepsilon, x_1 \geq 0, x_2 \geq 0, x_3 = 0 \}$ for $0 < \varepsilon < \infty$, which is not differentiable at (x*, ε*). However, for this example, we have $D(\varepsilon_0) = \{ x \in R^3 \mid x_1 + x_2 = \varepsilon_0, x_1 \geq 0, x_2 \geq 0, x_3 = 0 \}$.

In the next section, we establish the following results:

(i) Assumptions 2.1, 2.2 and 2.4' imply that the perturbed local solution set is Lipschitz continuous.

(ii) Assumptions 2.1, 2.2,2.3 and 2.4' imply that $D(\varepsilon_0)$ is bounded and contained in the solution set of a certain linear variational inequality.

(iii) Assumptions 2.1-2.4 imply that the perturbed local solution set is directionally differentiable for any direction ε_0 and the derivative uniquely solves a certain linear variational inequality.

3. Description of Method

This section consists of two parts. In the first part, imposing Assumptions 2.1, 2.2 and 2.4', we derive the Lipschitz continuity property of the perturbed local solution set. We show for small perturbations the perturbed local solution set is contained in $x^* + P^{\perp}$. The second part establishes the directional differentiability property of the perturbed local solution set. Imposing Assumptions 2.1, 2.2, 2.3 and 2.4', we prove any vector in $D(\varepsilon_0)$ is a solution to a certain linear variational inequality. Then we show that Assumption 2.4 implies this linear variational inequality has a unique solution.

Therefore, Assumptions 2.1-2.4 implies that the perturbed local solution set is directionally differentiable.

Throughout this section, we will use a simple reformulation of variational inequalities which we summerize in the following lemma, whose proof is immediate from the definitions.

Lemma 3.1. x^* is a solution to the variational inequality problem

$$F(x)^T(y - x) \geq 0 \qquad \text{for any } y \in P$$

if and only if x^* solves the linear programming problem

$$\min\{ F(x^*)^T x \mid x \in P \}$$

or, equivalently when $F(x^*) \neq 0$, if and only if $H = \{ x \mid F(x^*)^T(x - x^*) = 0 \}$ is a supporting hyperplane of P at x^* with $P \subseteq \{ x \mid F(x^*)^T(x - x^*) \geq 0 \}$.

This linear programming reformulation provides a natural approach (see Tobin [1986]) for evaluating the directional derivatives of $x(\varepsilon)$ by using a common method for conducting sensitivity analysis of nonlinear programs. Suppose we formulate this linear program as a set of inequalities defined by primal feasibility, dual feasibility, and complementary slackness. Then assuming *strict complementary slackness* would permit us to reformulate these conditions as a set of equations. And finally, by making appropriate assumptions, we could invoke an implicit function theorem to characterize (and consequently provide a means to compute) the derivatives of the perturbed solution.

This approach has the disadvantage of imposing conditions on the derived primal-dual optimality conditions rather than the problem data itself. Indeed, it is the lack of strict complementary slackness that has led Kyparisis to adopt the generalized equation approach and considerably complicates the analysis. In our approach, we first show that the linear programming primal solution and (an appropriately chosen) dual solution satisfy the Lipschitz condition. This fact permits us to show that the primal and dual " derivatives " satisfy an auxiliary linear complementarity problem (which can be restated as an equivalent linear variational inequality).

3.1. Lipschitz continuity of the perturbed solutions

We will first consider a locally restricted variational inequality problem that has a Lipschitz continuous solution set near (x^*, ε^*). Then we show the solutions of this local problem are exactly the local solutions of VI (ε).

Let U and V be chosen to satisfy Assumption 2.1 and 2.2. Suppose $U_1 \subseteq U$ is a neighborhood (which we choose as an open n-cube) of x^*. Consider the following locally restricted variational inequality problem:

VI '(ε): Find $x' \in P \cap Cl(U_1)$ satisfying $F(x', \varepsilon)^T (y - x') \geq 0$ for any $y \in P \cap Cl(U_1)$.

Let S' (ε) denote the solution set of VI' (ε). Since for each $\varepsilon \in V$, $F(\cdot, \varepsilon)$ is continuous over the compact convex set $P \cap Cl(U_1)$, S' (ε) is nonempty for all $\varepsilon \in V$. Also notice that $S(\varepsilon) \cap U^1 \subseteq S'(\varepsilon)$. As an immediate result of the next lemma we will show that the set S' (ε) is contained in $x^* + P^\perp$ for all ε in a neighborhood of ε^*. Then we prove that S' (ε) satisfies certain Lipschitz continuity property near ε^*. Finally, we point out that $S'(\varepsilon) = S(\varepsilon) \cap U_1$ for ε near ε^*.

Lemma 3.1.1. Some neighborhoods $U_1 \subseteq U$ of x^* and $V_1 \subseteq V$ of ε^* satisfy the following properties:

(a) For some $L_1, L_2 > 0$, $\| F(x, \varepsilon) - F(x^*, \varepsilon^*) \| \leq L_1 \| x - x^* \| + L_2 \| \varepsilon - \varepsilon^* \|$ $\forall x \in U_1, \varepsilon \in V_1$

(b) For some $\alpha > 0$, $[F(x, \varepsilon^*) - F(x^*, \varepsilon^*)]^T (x - x^*) \geq \alpha \| x - x^* \|^2$ for any $x \in (x^* + P^\perp) \cap U_1$

(c) For any $x^0 \in Cl(U_1)$ and $\varepsilon^0 \in V_1$, the solution set of the linear programming problem $\min \{ F(x^0, \varepsilon^0)^T x \mid x \in P \cap Cl(U_1) \}$ is contained in $x^* + P^\perp$.

Proof. See Appendix A.

□

Note that property (a) which follows from the differentiability of F (·, ·) at (x^*, ε^*), is not used until next subsection. In view of the fact stated in Lemma 3.1, any $x'(\varepsilon) \in S'(\varepsilon)$ solves the linear programming problem $\min \{ F(x'(\varepsilon), \varepsilon)^T x \mid x \in P \cap Cl(U_1) \}$. Then by Lemma 3.1.1. (c), $S'(\varepsilon) \subseteq x^* + P^\perp$ for $\varepsilon \in V_1$. (Note: Suppose we replace Assumption 2.4' with the following stronger assumption:

Assumption 3.1.1. For some neighborhoods U of x^* and V of ε^*, F (·, ε) is strictly monotone over the set $(x^* + P^\perp) \cap U$ for any $\varepsilon \in V$.

Then $S'(\varepsilon)$ is a singleton for all $\varepsilon \in V_1$).

Lemma 3.1.2. $S'(\varepsilon)$ is Lipschitz continuous at (x^*, ε^*).

Proof. For any $x'(\varepsilon) \in S'(\varepsilon)$ and $\varepsilon \in V_1$, since $x'(\varepsilon)$ solves $VI'(\varepsilon)$ and x^* solves $VI(\varepsilon^*)$, we have

$$F(x'(\varepsilon), \varepsilon)^T (x^* - x'(\varepsilon)) \geq 0, \text{ and}$$

$$F(x^*, \varepsilon^*)^T (x'(\varepsilon) - x^*) \geq 0.$$

Adding these two inequalities, we obtain $F(x'(\varepsilon), \varepsilon)^T (x'(\varepsilon) - x^*) \leq F(x^*, \varepsilon^*)^T (x'(\varepsilon) - x^*)$.

Then by Lemma 3.1.1. (b) and Assumption 2.2,

$$\begin{aligned} &\alpha \| x'(\varepsilon) - x^* \|^2 \\ &\leq [F(x'(\varepsilon), \varepsilon^*) - F(x^*, \varepsilon^*)]^T (x'(\varepsilon) - x^*) \\ &\leq [F(x'(\varepsilon), \varepsilon^*) - F(x'(\varepsilon), \varepsilon)]^T (x'(\varepsilon) - x^*) \\ &\leq L \| \varepsilon - \varepsilon^* \| \| x'(\varepsilon) - x^* \| \end{aligned}$$

Thus $\| x'(\varepsilon) - x^* \| \leq (L/\alpha) \cdot \| \varepsilon - \varepsilon^* \|$.

□

The next theorem establishes the Lipschitz continuity property at (x^*, ε^*) for the perturbed local solution set of $VI(\varepsilon)$.

Theorem 3.1.1. $S(\varepsilon) \cap U_1$ is Lipschitz continuous at (x^*, ε^*).

Proof. Since $x^* \subseteq U_1$, by Lemma 3.1.2, some neighborhood $V_2 \subseteq V_1$ of ε^* has the property that for any $\varepsilon \in V_2$, $S'(\varepsilon) \subseteq U_1$. Consequently, the supporting hyperplane $\{ x \mid F(x'(\varepsilon), \varepsilon)^T (x - x'(\varepsilon)) = 0 \}$ of $P \cap U_1$ at $x'(\varepsilon)$ is also a supporting hyperplane of P at $x'(\varepsilon)$. By Lemma 3.1, this result implies $x'(\varepsilon)$ is also a solution to $VI(\varepsilon)$. Thus, $S'(\varepsilon) = S(\varepsilon) \cap U_1$ for $\varepsilon \in V_2$, and hence $S(\varepsilon) \cap U_1$ is Lipschitz continuous at (x^*, ε^*).

□

It is possible to show (by examples like those in Section 2) that Assumptions 2.1, 2.2 and 2.4' are in a sense the weakest possible conditions under which the Lipschitz continuity property is valid.

3.2. Directional differentiability of the perturbed solutions

In this subsection, we establish the directional differentiability properties of the perturbed local solution set $S(\varepsilon) \cap U_1$. So far we have shown that $\| x(\varepsilon) - x^* \| / \| \varepsilon - \varepsilon^* \|$ is uniformly bounded by L/α for any $x(\varepsilon) \in S(\varepsilon) \cap U_1$ and $\varepsilon \in V_2$. Let ε_0 be a nonzero vector in R^m. Then we have $D(\varepsilon_0) \subseteq \{x \mid \|x\| \leq L/\alpha\}$. Now suppose x^L is an arbitrary vector in $D(\varepsilon_0)$, i.e., x^L is the limit of some convergent sequence $[x(\varepsilon^* + t_n \varepsilon_0) - x^*]/t_n$ with $t_n \to 0$, $n \in N$. We will show x^L is a solution of a certain linear variational inequality.

By Lemma 3.1, any $x(\varepsilon) \in S(\varepsilon)$ solves the linear program

$$\begin{array}{ll} \text{minimize} & F(x(\varepsilon), \varepsilon)^T x \\ \text{subject to} & A x \geq b. \end{array} \qquad (3.2.1)$$

Writing $x = x^1 - x^2$, $x^1 \geq 0$, $x^2 \geq 0$, we have the following equivalent linear programming formulation of (3.2.1) which has the same optimal dual solutions as (3.2.1):

$$\begin{array}{ll} \text{minimize} & F(x(\varepsilon), \varepsilon)^T x^1 - F(x(\varepsilon), \varepsilon)^T x^2 \\ \text{subject to} & A x^1 - A x^2 \geq b \\ & x^1 \geq 0, x^2 \geq 0. \end{array} \qquad (3.2.2)$$

The optimal dual solution of (3.2.2) may not be unique. However, we now show that some sequence of dual solutions $\pi(\varepsilon^* + t_k \varepsilon_0)$, $k \in K \subseteq N$ of (3.2.2) has the Lipschitz continuity property at $\pi(\varepsilon^*)$ (where $\pi(\varepsilon^*)$ is also appropriately chosen). Let $S(x(\varepsilon), \varepsilon)$ be the polyhedral solution set of the linear program (3.2.2). As is well known, $S(x(\varepsilon), \varepsilon)$ contains at least one basic solution and $S(\cdot, \cdot)$ is upper semicontinuous. Therefore, some neighborhood $V_3 \subseteq V_2$ of ε^* has the property that for any $\varepsilon \in V_3$ and $x(\varepsilon) \in S(\varepsilon) \cap U_1$, there is a basic solution $(x^1, x^2) \in S(x(\varepsilon), \varepsilon) \subseteq S(x^*, \varepsilon^*)$. Since there are only a finite number of basic solutions, for some fixed basic solution (x^1, x^2) and subsequence $K \in N$, $(x^1, x^2) \in S(x(\varepsilon^* + t_k \varepsilon_0), \varepsilon^* + t_k \varepsilon_0) \subseteq S(x^*, \varepsilon^*)$ for $k \in K$. Let B denote the basis corresponding to the basic solution (x^1, x^2), and let $C_B(x(\varepsilon), \varepsilon)$ be the corresponding subvector of the objective function of the linear program (3.2.2). We then choose the corresponding simplex multipliers as our optimal dual variables. Thus,

$$\pi(\varepsilon^*)^T = C_B(x^*, \varepsilon^*) B^{-1}, \text{ and}$$

$$\pi(\varepsilon^* + t_k \varepsilon_0)^T = C_B(x(\varepsilon^* + t_k \varepsilon_0), \varepsilon^* + t_k \varepsilon_0) B^{-1} \qquad \text{for } k \in K.$$

Now by Lemma 3.1.1. (a) and Theorem 3.1.1,

$$\| F(x(\varepsilon^* + t_k \varepsilon_0), \varepsilon^* + t_k \varepsilon_0) - F(x^*, \varepsilon^*) \|$$

$$\leq L_1 \| x(\varepsilon^* + t_k \varepsilon_0) - x^* \| + L_2 \| \varepsilon - \varepsilon^* \|$$

$$\leq (L_1 L / \alpha + L_2) \| \varepsilon_0 \| t_k \qquad \text{for } k \in K.$$

Notice that $C_B(x(\varepsilon), \varepsilon)$ is a subvector of $(F(x(\varepsilon), \varepsilon)^T, -F(x(\varepsilon), \varepsilon)^T)$, hence for some $M > 0$,

$$\| \pi(\varepsilon^* + t_k \varepsilon_0) - \pi(\varepsilon^*) \| \leq M t_k \qquad \text{for } k \in K.$$

So the sequence of dual solutions we choose has the desired property. Now obviously, some subsequence $K' \subseteq K \subseteq N$ and vectors π^L satisfy

$$[\pi(\varepsilon^* + t_k \varepsilon_0) - \pi(\varepsilon^*)] / t_k \rightarrow \pi^L \qquad \text{as } k \rightarrow \infty, k \in K'.$$

The following lemma gives the set of constraints that the two vectors x^L and π^L must satisfy simultaneously. We need the following notation to state the lemma. Let

$$K_1 = \{ k \mid \pi_k(\varepsilon^*) > 0, \sum_{i=1}^{n} a_{ki} x_i(\varepsilon^*) - b_k = 0 \}$$

$$K_2 = \{ k \mid \pi_k(\varepsilon^*) = 0, \sum_{i=1}^{n} a_{ki} x_i(\varepsilon^*) - b_k = 0 \}$$

$$K_3 = \{ k \mid \pi_k(\varepsilon^*) = 0, \sum_{i=1}^{n} a_{ki} x_i(\varepsilon^*) - b_k > 0 \}.$$

Lemma 3.2.1. The two vectors x^L and π^L satisfy the following linear complementarity constraints:

$$A^T \pi^L - \nabla_x F(x^*, \varepsilon^*) x^L - \nabla_\varepsilon F(x^*, \varepsilon^*) \varepsilon_0 = 0$$

$$\sum_{i=1}^{n} a_{ki} x_i^L = 0 \quad \textit{for } k \in K_1,\ k \in K_2 \textit{ and } \pi_k^L > 0 \qquad (3.2.3)$$

$$\sum_{i=1}^{n} a_{ki} x_i^L \geq 0 \quad \text{for } k \in K_2 \text{ and } \pi_k^L = 0$$

$$\pi_k^L \text{ UIS for } k \in K_1, \quad \pi_k^L \geq 0 \text{ for } k \in K_2, \quad \pi_k^L = 0 \text{ for } k \in K_3.$$

Proof. See Appendix B.

□

Now any vector in D (ε_0) satisfies system (3.2.3), which involves the auxiliary variable π^L. The following theorem gives a partial characterization of the set D (ε_0) in terms of the original data. We prove that D (ε^*) is contained in the solution set of a certain linear variational inequality.

Theorem 3.2.1. Suppose x^* solves VI (ε^*) and Assumptions 2.1, 2.2, 2.3 and 2.4' are satisfied. Then any $x^L \in D(\varepsilon_0)$ solves the following linear variational inequality problem:

$VI^\perp$: find $x \in P^\perp$ satisfying $[\nabla_x F(x^*, \varepsilon^*) x + \nabla_\varepsilon F(x^*, \varepsilon^*) \varepsilon_0]^T (y - x) \geq 0$ for any $y \in P^\perp$.

Proof. Notice that system (3.2.3) is the complementary slackness conditions of the following linear program:

$$\text{minimize} \quad [\nabla_x F(x^*, \varepsilon^*) x^L + \nabla_\varepsilon F(x^*, \varepsilon^*) \varepsilon_0] x$$

$$\text{subject to} \quad \sum_{i=1}^{n} a_{ki} x_i = 0 \quad \text{for } k \in K_1$$

$$\sum_{i=1}^{n} a_{ki} x_i \geq 0 \quad \text{for } k \in K_2.$$

Therefore, by Lemma 3.1.1, x^L satisfies the following variational inequality:

find $x \in P^0$ satisfying $[\nabla_x F(x^*, \varepsilon^*) x + \nabla_\varepsilon F(x^*, \varepsilon^*) \varepsilon_0]^T (y - x) \geq 0$ for any $y \in P^0$

$$\text{where } P^0 = \{x \mid \sum_{i=1}^{n} a_{ki} x_i = 0 \text{ for } k \in K_1, \text{ and } \sum_{i=1}^{n} a_{ki} x_i \geq 0 \text{ for } k \in K_2\}.$$

Now in order to show that x^L satisfies the variational inequality $VI^\perp$, it suffices to show that $P^0 = P^\perp \equiv \{x \mid F(x^*, \varepsilon^*) x = 0, A_1 x \geq 0\}$. Notice that the binding constraints at x^* are those in $K_1 \cup K_2$.

$P^0 \subseteq P^\perp$: Suppose $x \in P^0$. Then we have $A_1 x \geq 0$, and $F(x^*, \varepsilon^*)^T x = [\pi(\varepsilon^*)^T A] x = 0$. Thus, $x \in P^\perp$.

$P^\perp \subseteq P^0$: Suppose $x \in P^\perp$. Then

$$\sum_{i=1}^{n} a_{ki} x_i \geq 0 \quad for\, k \in K_1 \cup K_2, \quad and$$

$$0 = F(x^*, \varepsilon^*)^T x = [\pi(\varepsilon^*)^T A] x = \pi(\varepsilon^*)^T A x$$

If $k \in K_1$, then $\pi_k(\varepsilon^*) > 0$, which implies

$$\sum_{i=1}^{n} a_{ki} x_i = 0.$$

Thus, $x \in P^0$.

□

Actually, we conjecture that the set $D(\varepsilon_0)$ is equal to the solution set of the linear variational inequality problem $VI^{\perp}$ assuming the hypotheses of Theorem 3.2.1. In a subsequent paper, we will show that this is true for the case where the problem $VI^{\perp}$ satisfies the strict complementary slackness condition.

Corollary 3.2.1. Suppose x^* solves $VI(\varepsilon^*)$ and Assumptions 2.1-2.4 are satisfied. Then for some neighborhood U of x^*, $S(\varepsilon) \cap U$ is directionally differentiable at (x^*, ε^*) for any direction ε_0. Furthermore, the derivative $d(\varepsilon_0)$ uniquely solves the variational inequality $VI^{\perp}$.

Proof. Let x' and x'' be any two vectors in $D(\varepsilon_0)$. Then we have

$$[\nabla_x F(x^*, \varepsilon^*) x' + \nabla_\varepsilon F(x^*, \varepsilon^*) \varepsilon_0]^T (x'' - x') \geq 0, \quad \text{and}$$

$$[\nabla_x F(x^*, \varepsilon^*) x'' + \nabla_\varepsilon F(x^*, \varepsilon^*) \varepsilon_0]^T (x' - x'') \geq 0.$$

Adding these two inequalities, we obtain

$$(x' - x'')^T \nabla_x F(x^*, \varepsilon^*)(x' - x'') \leq 0.$$

Since $\nabla_x F(x^*, \varepsilon^*)$ is positive definite on span $(P^{\perp})$, the previous inequality implies $x' = x''$. Therefore, $D(\varepsilon_0)$ is a singleton, or in another words, for some neighborhood U of x^*, $S(\varepsilon) \cap U$ is directional differentiable at (x^*, ε^*).

□

Remarks.

(i) In view of its linear structure, the variational inequality problem $VI^{\perp}$, as in the case of linear complementarity problems, can be solved by certain pivoting algorithms. On the other hand, with the use of a diagnalization method to solve $VI^{\perp}$, in each step the subproblem is a quadratic optimization problem that is relatively easy to solve.

(ii) Note that for an underlying set of the form $P = \{ x \in R^n \mid A x \geq b, C x = d \}$, the results obtained so far remain valid if we just change $P^{\perp}$ to be $P^{\perp} = \{ x \in R^n \mid A_1 x \geq 0, C x = 0 \}$.

(iii) Notice that since the (linear) mapping of the variational inequality $VI^{\perp}$ satisfies Assumptions 2.1-2.4, the directional derivative $d(\varepsilon_0)$ is Lipschitz continuous and directionally differentiable with respect to the perturbation direction ε_0. In particular, if we let λ_{min} denote the minimum eigenvalue of the symmetric matrix $[\nabla_x F(x^*, \varepsilon^*) + \nabla_x F(x^*, \varepsilon^*)^T] / 2$ projected on to the subspace span ($P^{\perp}$), then it is possible to show that

$$\| d(\varepsilon_0') - d(\varepsilon_0'') \| \leq [\| \nabla_\varepsilon F(x^*, \varepsilon^*) \| / \lambda_{min}] \| \varepsilon_0' - \varepsilon_0'' \|.$$

Corollary 3.2.2. Suppose x^* solves VI (ε^*) and Assumptions 2.1-2.4 and 3.1.1 are satisfied. Then for some neighborhood U of x^*, $S(\varepsilon) \cap U$ is a singleton for ε near ε^*. Let $S(\varepsilon) \cap U = \{ x(\varepsilon) \}$. Then the directional derivative of function $x(\varepsilon)$ exists for any direction ε_0 and uniquely solves the variational inequality $VI^{\perp}$.

In the following lemma, we give a condition that implies Assumption 3.1.1.

Lemma 3.2.2. Suppose x^* solves VI (ε^*). Assume $F(\cdot, \varepsilon)$ is differentiable for each ε near ε^* and $\nabla_x F(x, \varepsilon)$ is continuous near (x^*, ε^*). Then $\nabla_x F(x^*, \varepsilon^*)$ being positive definite on span ($P^{\perp}$) implies Assumption 3.1.1.

Proof. Suppose Assumption 3.1.1 is not satisfied. Then some sequences $\{ x^n \}$, $\{ y^n \}$ and $\{ \varepsilon^n \}$ satisfy $x^n, y^n \in x^* + P^{\perp}$, $x^n \neq y^n$, and $x^n, y^n \to x^*$, $\varepsilon^n \to \varepsilon^*$, and $[F(x^n, \varepsilon^n) - F(y^n, \varepsilon^n)]^T (x^n - y^n) < \| x^n - y^n \|^2 / n$. Therefore by the mean value theorem, we have

$$\frac{(x^n - y^n)^T \nabla_x F(z^n, \varepsilon^n)(x^n - y^n)}{\| x^n - y^n \|^2} < \frac{1}{n}$$

where $z^n \in [x^n, y^n]$, the line segment joining x^n and y^n. Notice that $(x^n - y^n) / \| x^n - y^n \|$ is a vector of unit length. Let z be a limit vector of $\{ (x^n - y^n) / \| x^n - y^n \| \}$. Then $z \in$ span ($P^{\perp}$), $\| z \| = 1$, and

$z^T \nabla_x F(x^*, \varepsilon^*) z \leq 0$. This conclusion contradicts the positive definiteness of $\nabla_x F(x^*, \varepsilon^*)$ on span $(P^\perp)$.

□

One immediate implication of Corollary 3.2.2 and Lemma 3.2.2 is the result by Kyparisis [1986] (his Theorem 2.1), which we paraphrase in the following theorem.

Theorem 3.2.2. Suppose x^* solves VI (ε^*). Assume $F(\cdot, \cdot)$ is once continuously differentiable around (x^*, ε^*) and $\nabla_x F(x^*, \varepsilon^*)$ is positive definite on span $(P^\perp)$. Then for some neighborhood V of ε^*, the variational inequality VI (ε) has a unique solution $x(\varepsilon)$. Furthermore, the directional derivative of $x(\varepsilon)$ at ε^* exists for any direction ε_0 and uniquely solves the variational inequality problem $VI^\perp$.

4. Perturbed solutions of traffic equilibrium problems

In this section, we investigate the behavior of the perturbed solutions of traffic equilibrium problems and other problems in which the mapping F depends upon only a subset of the variables defining the polyhedron P. The VI sensitivity analysis results obtained in the previous section do not apply directly to these problems since the formulations and the assumptions for this model no longer fit the standard format we considered previously. However, it is possible to conduct the same type of analysis using the approach suggested in the previous section. In Subsection 4.1, we consider a perturbed version of the traffic equilibrium model with fixed demands. We suggest and analyze a version of the perturbed variational inequality with a more general form of the underlying set P that includes traffic equilibrium problems as special cases. Then in Subsection 4.2, we analyse the perturbation problem for a general traffic equilibrium model with elastic demands.

4.1. Cost perturbation for a traffic equilibrium problem with fixed demands

Consider the following traffic equilibrium model. Let $G = (N, A)$ represent a transportation network with N as the set of nodes and A the set of directed links. To describe the problem, we also introduce the following notation:

W = the set of origin-destination pairs in the network

P_w = the set of paths connecting O-D pair w

d_w = the demand associated with O-D pair w

Δ = the link-path incidence matrix

Γ = the O-D pair-path incidence matrix

f_a = the amount of flow on link a

h_p = the amount of flow on path p

$c_a(f)$ = the travel cost function of each user on link a

$\Omega(d) = \{ f \mid f = \Delta h, \Gamma h = d, h \geq 0 \}$, the set of feasible link flows.

With this notation, the traffic equilibrium problem can be stated as follows: find $f \in \Omega(d)$ satisfying

$$\sum_{a \in A} \delta_{ap} c_a(f) = min \left\{ \sum_{a \in A} \delta_{ap'} c_a(f) \mid p' \in P_w \right\} \quad if\, h_p > 0 \qquad for\, any\, p \in P_w,\, w \in W.$$

That is, in equilibrium, every path p carrying positive flow (i.e., with $h_p > 0$) must be a shortest path with respect to the link costs $c_a(f)$. Writing this problem as a variational inequality (see Smith [1979] or Dafermos [1980]) in terms of the vector $C(f) = (c_a(f), a \in A)$, we obtain

$$\text{find } f \in \Omega(d) \text{ satisfying} \qquad C(f)^T (f' - f) \geq 0 \qquad \text{for any } f' \in \Omega(d).$$

Practically, the underlying polyhedron $\Omega(d)$ cannot be written in the standard form we considered previously (as a polyhedron defined solely in terms of the variables f). Fourier elimination would do so, but possibly by requiring an exponential number of inequalities to specify the projected (on to f) polyhedron. By eliminating the link flow variables from the formulation, we obtain the following variational inequality in terms of path flows

$$\text{find } h \in \{ h \mid \Gamma h = d, h \geq 0 \} \text{ satisfy} \quad (\Delta^T C(\Delta h))^T (h' - h) \geq 0 \qquad \text{for any } h' \in \{ h \mid \Gamma h = d, h \geq 0 \}.$$

Now the underlying set can be written in the standard form but the path cost function generally does not satisfy our assumptions. Moreover, the equilibrium path flow pattern is usually not unique since the problem assumptions are usually imposed on the link cost function rather than the path cost function (and several path flow solutions might correspond to the same (unique) link flow).

In the following discussion, we first consider a class of more general variational inequalities. Then we interpret the results in terms of traffic equilibrium problems. Consider the perturbed variational inequality of the form:

VI (ε): $\quad F(x, \varepsilon)^T (y - x) \geq 0 \quad$ for any $y \in P$

where $P = \{ x \mid A x + B z \geq b \}$, A is a $l \times n$ matrix and B is a $l \times m$ matrix. This problem has special feature: the variable z does not appear as an argument of the function F and we are interested only in the changes in x as a function of ε. In the context of traffic equilibrium problem, variable z becomes the path flows h and x becomes the link flows f. Now suppose x^* solves VI (ε^*), and z^* satisfies the constraints $A x^* + B z \geq b$. Let A_i denote the ith row of the matrix A and B_i the ith row of the matrix B. Let $I = \{ i \mid A_i x^* + B_i z^* = b_i \}$ and $P^\perp = \{ x \mid F(x^*, \varepsilon^*)^T x = 0, A_i x + B_i z \geq 0 \text{ for } i \in I \}$. We now make the same assumptions on the function F as in Section 2 — the only difference is that $P^\perp$ has changed. Notice that the development in Section 3.1 did not use the explicit form of P or $P^\perp$ to establish the Lipschitz continuity property of the perturbed local solution set. Therefore, using the same approach we can derive the Lipschitz continuity property for the current problem. We summerize this result in the following lemma.

Lemma 4.1.1. For some neighborhoods U_1 of x^* and V_1 of ε^*, the perturbed local solution set $S(\varepsilon) \cap U_1$ is contained in $x^* + P^\perp$ for $\varepsilon \in V_1$ and is Lipschitz continuous at (x^*, ε^*), i.e.,

$$\| x(\varepsilon) - x^* \| \leq (L/\alpha) \| \varepsilon - \varepsilon^* \| \qquad \text{for any } x(\varepsilon) \in S(\varepsilon) \cap U_1 \text{ and } \varepsilon \in V_1$$

(α is defined in Lemma 3.1.1).

We notice that the soluton to the linear system $B z \geq b - A x(\varepsilon)$ might not be unique for each $x(\varepsilon) \in S(\varepsilon) \cap U_1$ and $\varepsilon \in V_1$. For the purpose of our analysis, we define $z(\varepsilon)$ to be the unique solution of the following convex programming problem:

$$\begin{aligned} &\text{minimize} \quad \| z - z^* \|^2 \\ &\text{subject to} \quad B z \geq b - A x(\varepsilon). \end{aligned}$$

The next lemma shows that $z(\varepsilon)$ is Lipschitz continuous at z^*.

Lemma 4.1.2. For some number $L_1 > 0$, $\| z(\varepsilon) - z^* \| \leq L_1 \varepsilon$ for any $x(\varepsilon) \in S(\varepsilon) \cap U_1$ and $\varepsilon \in V_1$.

Proof. Let D be a closed n-cube centered at x^* that satisfies the condition $S(\varepsilon) \cap U_1 \subseteq D$ for $\varepsilon \in V_1$. Then $D \cap P$ is a bounded polyhedron. Let $x^1, \cdots, x^r$ be an enumeration of its extreme points. Now consider the following quadratic problem:

$$\begin{aligned} &\text{minimize} \quad \| z - z^* \|^2 \\ &\text{subject to} \quad z \in Z(x) \end{aligned}$$

where $Z(x) = \{ z \mid Bz \geq b - Ax \}$. Let $z_0(x)$ denote the unique solution of this problem as a function of x over $D \cap P$. According to our definition, $z(\varepsilon) = z_0(x(\varepsilon))$. First notice that $\| z_0(x) - z^* \|^2$ is uniformly bounded over $D \cap P$. To see this result, suppose x is any vector in $D \cap P$. Then x is a convex combination of the extreme points of $D \cap P$, i.e.,

$$x = \sum_{i=1}^{r} \alpha_i x^i \quad \textit{for some } \alpha_i \geq 0, \ \sum_{i=1}^{r} \alpha_i = 1.$$

Since $z = \sum_{i=1}^{r} \alpha_i z_0(x^i) \in Z(x)$,

$$\| z_0(x) - z^* \|^2 \leq \| z - z^* \|^2 \leq \sum_{i=1}^{r} \alpha_i \| z_0(x^i) - z^* \|^2 \leq \max_i \{ \| z_0(x^i) - z^* \|^2 \}.$$

Since $D \cap P$ is a polyhedron, some number $\rho > 0$ has the property that for any $x \in D \cap P$, x is a convex combination of x^* and some $x' \in D \cap P \cap \{ x \mid \| x - x^* \| \geq \rho \}$. Now let

$$M^2 = \sup \left\{ \frac{\| z_0(x) - z^* \|^2}{\| x - x^* \|^2} : x \in D \cap P \cap \{ x \mid \| x - x^* \| \geq \rho \} \right\} < +\infty.$$

For any $x(\varepsilon) \in S(\varepsilon) \cap U_1$ and $\varepsilon \in V_1$, $x(\varepsilon) \neq x^*$, we have $x(\varepsilon) = \alpha x_\varepsilon + (1 - \alpha) x^*$ for some $x_\varepsilon \in D \cap P \cap \{ x \mid \| x - x^* \| \geq \rho \}$ and some $0 < \alpha \leq 1$. Let $z_\varepsilon = \alpha z_0(x_\varepsilon) + (1 - \alpha) z^*$. Then

$$\| x(\varepsilon) - x^* \|^2 = \alpha^2 \| x_\varepsilon - x^* \|^2, \text{ and}$$

$$\| z_\varepsilon - z^* \|^2 = \alpha^2 \| z_0(x_\varepsilon) - z^* \|^2.$$

Note that $z_\varepsilon \in Z(x(\varepsilon))$. So

$$\| z(\varepsilon) - z^* \|^2 \leq \| z_\varepsilon - z^* \|^2 = \alpha^2 \| z_0(x_\varepsilon) - z^* \|^2 = \| x(\varepsilon) - x^* \|^2 \| z_0(x_\varepsilon) - z^* \|^2 / \| x_\varepsilon - x^* \|^2$$

$$\leq M^2 \| x(\varepsilon) - x^* \|^2.$$

Therefore, $\| z(\varepsilon) - z^* \| \leq M \| x(\varepsilon) - x^* \| \leq (ML/\alpha) \cdot \| \varepsilon - \varepsilon^* \|$.

□

We now establish the directional differentiability property of the perturbed solution. Again let ε_0 be a perturbation direction. Notice that $x(\varepsilon)$ and $z(\varepsilon)$ solve the following linear program:

$$\begin{aligned} &\text{minimize} \quad F(x(\varepsilon),\varepsilon)^T x \\ &\text{subject to} \quad A x + B z \geq b. \end{aligned} \tag{4.1.1}$$

Let $u(\varepsilon) \geq 0$ denote any optimal dual solution of (4.1.1). Now we consider any convergent sequence $[x(\varepsilon^* + t_n \varepsilon_0) - x^*]/t_n$ with $t_n \to 0$, $n \in N$. Using the same argument as in Section 3.2, we can show that some subsequence $K \subseteq N$ and some sequence of optimal dual solutions $\{u(\varepsilon^* + t_k \varepsilon_0)\}_{k \in K}$ satisfy

$$[x(\varepsilon^* + t_k \varepsilon_0) - x^*]/t_k \to x^L,$$

$$[z(\varepsilon^* + t_k \varepsilon_0) - z^*]/t_k \to z^L, \text{ and}$$

$$[u(\varepsilon^* + t_k \varepsilon_0) - u^*]/t_k \to u^L.$$

Now in order to show the perturbed local solution set is directional differentiable, it suffices to show that x^L is uniquely determined independent of the choice of the convergent sequence. Let

$$I_1 = \{i \mid u_i^* > 0, A_i x^* + B_i z^* = b_i\}$$

$$I_2 = \{i \mid u_i^* = 0, A_i x^* + B_i z^* = b_i\}$$

$$I_3 = \{i \mid u_i^* = 0, A_i x^* + B_i z^* > b_i\}.$$

Then following the proof of Lemma 3.2.1, we obtain the next result.

Lemma 4.1.3. The three vectors x^L, z^L and u^L satisfy the following constraints:

$$\begin{aligned} &A^T u^L - \nabla_x F(x^*,\varepsilon^*)\, x^L - \nabla_\varepsilon F(x^*,\varepsilon^*)\,\varepsilon_0 = 0 \\ &B^T u^L = 0 \\ &A_i x^L + B_i z^L = 0 \qquad \text{for } i \in I_1, \text{ and } i \in I_2 \text{ and } u_i^L > 0 \\ &A_i x^L + B_i z^L \geq 0 \qquad \text{for } i \in I_2 \text{ and } u_i^L = 0 \\ &u_i^L \text{ UIS for } i \in I_1,\ u_i^L \geq 0 \text{ for } i \in I_2,\ u_i^L = 0 \text{ for } i \in I_3. \end{aligned} \tag{4.1.2}$$

Theorem 4.1.1. Suppose (x^*, z^*) solves VI (ε^*) and Assumption 2.1-2.4 are satisfied. Then for some neighborhood U of x^*, $S(\varepsilon) \cap U$ is directionally differentiable at x^* for any direction ε_0. Furthermore, the derivative $d(\varepsilon_0)$ uniquely solves the variational inequality:

$\mathrm{VI}^\perp$: find $x \in P^\perp$ satisfying $\quad [\nabla_x F(x^*,\varepsilon^*)\, x + \nabla_\varepsilon F(x^*,\varepsilon^*)\,\varepsilon_0]^T (y - x) \geq 0 \qquad$ for any $y \in P^\perp$.

Proof. Notice that system (4.1.2) is the complementary slackness conditions of the following linear program:

$$\text{minimize} \quad [\nabla_x F (x^*, \varepsilon^*) x^L + \nabla_\varepsilon F (x^*, \varepsilon^*) \varepsilon_0]^T x$$

$$\text{subject to} \quad A_i x + B_i z = 0 \quad \text{for } i \in I_1$$

$$A_i x + B_i z \geq 0 \quad \text{for } i \in I_2$$

with u^L as the optimal dual variable. Therefore, x^L solves the linear variational inequality

$$[\nabla_x F (x^*, \varepsilon^*) x + \nabla_\varepsilon F (x^*, \varepsilon^*) \varepsilon_0]^T (y - x) \geq 0 \qquad \text{for any } y \in P^0$$

where $P^0 = \{ x \mid A_i x + B_i z = 0 \ \text{ for } i \in I_1, A_i x + B_i z \geq 0 \ \text{ for } i \in I_2 \}$.To show x^L solves $VI^\perp$, we need to show only that $P^0 = P^\perp$. Notice that $I = I_1 \cup I_2$.

$P^0 \subseteq P^\perp$: Suppose $x \in P^0$. Then $A_i x + B_i z \geq 0$ for $i \in I$, and

$$F (x^*, \varepsilon^*)^T x = u (\varepsilon^*)^T A x = u (\varepsilon^*)^T (A x + B z) = 0.$$

Thus, $x \in P^\perp$.

$P^\perp \subseteq P^0$: Suppose $x \varepsilon P^\perp$. Then $A_i x + B_i z \geq 0$ for $i \in I_1 \cup I_2$, and

$$0 = F (x^*, \varepsilon^*)^T x = u (\varepsilon^*)^T A x = u (\varepsilon^*)^T (A x + B z)$$

If $i \in I_1$, then $u (\varepsilon^*) > 0$. Therefore $A_i x + B_i z = 0$. Thus, $x \in P^0$.

Now by Assumption 2.4, $\nabla_x F (x^*, \varepsilon^*)$ is positive definite on span ($P^\perp$). Therefore, $VI^\perp$ has a unique solution for the x variable, which completes the proof.

□

Now let's reconsider the traffic equilibrium problem. Writing it in our general form, we have the following correspondence:$(x, z) = (f, h)$

$$F (x, \varepsilon) = C (f, \varepsilon)$$

$$A = [I \ \ -I \ \ O \ \ O \ \ O]^T$$

$$B = [-\Delta^T \ \ \Delta^T \ \ -\Gamma^T \ \ \Gamma^T \ \ I]^T$$

$$b = [0 \ \ 0 \ \ -d \ \ d \ \ 0]$$

$$P = \Omega (d)$$

where I is an identity matrix and O is a matrix with all zero entries. Suppose f^* is an equilibrium link flow pattern for $\varepsilon = \varepsilon^*$. Let h^* be any corresponding feasible path flow pattern and let $J = \{ p \mid h_p^* = 0 \}$ be the index set of paths that carry no flow at equilibrium. It is not hard to see that in this case

$$\Omega^{\perp} = \{ f \mid C(x^*, \varepsilon^*)^T f = 0, f = \Delta h, \Gamma h = 0, h_p \geq 0 \quad \text{for } p \in J \}.$$

Imposing Assumptions 2.1-2.4 with respect to the function $C(\cdot, \cdot)$ and the set $\Omega^{\perp}$, we have the following properties regarding the perturbed equilibrium traffic flow patterns,

(i) For any small perturbation ε around ε^*, in a neighborhood of f^*, the perturbed equilibrium link flow pattern $f(\varepsilon)$ exists and satisfies the property that $\| f(\varepsilon) - f^* \| \leq M_1 \| \varepsilon - \varepsilon^* \|$ for some $M_1 > 0$.

(ii) For each perturbed equilibrium link flow pattern $f(\varepsilon)$, there exists a equilibrium path flow pattern $h(\varepsilon)$ satisfying $\| h(\varepsilon) - h^* \| \leq M_2 \| \varepsilon - \varepsilon^* \|$ for some $M_2 > 0$.

(iii) The perturbed local equilibrium link flow pattern $f(\varepsilon)$ is directionally differentiable. Furthermore, the derivative in the direction ε_0 solves the following variational inequality:

$VI^{\perp}$: find $f \in \Omega^{\perp}$ satisfying $\quad [\nabla_f C(f^*, \varepsilon^*) f + \nabla_{\varepsilon} C(f^*, \varepsilon^*) \varepsilon_0]^T (f' - f) \geq 0 \quad$ for any $f' \in \Omega^{\perp}$.

Now we show that variational inequality $VI^{\perp}$ can also be interpreted as a network equilibrium problem. Let $u(\varepsilon) = (u^1(\varepsilon), u^2(\varepsilon), u^3(\varepsilon), u^4(\varepsilon), u^5(\varepsilon))$ be the corresponding dual solution. In this case, the complementary slackness conditions of the linear program (4.1.1) are

$$u^1 - u^2 - C(f, \varepsilon) = 0$$

$$-\Delta^T u^1 + \Delta^T u^2 - \Gamma^T u^3 + \Gamma^T u^4 + u^5 = 0$$

$$(u^5)^T h = 0$$

$$f = \Delta h$$

$$\Gamma h = d$$

$$h \geq 0, u \geq 0$$

or, equivalently,

$$\Delta^T C(f, \varepsilon) - \Gamma^T v \geq 0$$

$$[\Delta^T C(f, \varepsilon) - \Gamma^T v]^T h = 0$$

$$f = \Delta h$$

$$\Gamma h = d$$

$h \geq 0$

where $v = u^4 - u^3$. Notice v is the minimum travel time vector between the O-D pairs. Let J_1 denote the index set of paths that carry a positive amount of flow at equilibrium, i.e.,

$$J_1 = \{p \mid h_p^* > 0, \sum_{a \in A} \delta_{ap} c_a(f^*, \varepsilon^*) = v_w^*, p \in P_w, w \in W\}.$$

Similarly, define

$$J_2 = \{p \mid h_p^* = 0, \sum_{a \in A} \delta_{ap} c_a(f^*, \varepsilon^*) = v_w^*, p \in P_w, w \in W\}, and$$

$$J_3 = \{p \mid h_p^* = 0, \sum_{a \in A} \delta_{ap} c_a(f^*, \varepsilon^*) > v_w^*, p \in P_w, w \in W\}.$$

Under any small perturbation, the paths in J_1 will continue to have a positive amount of flow and the paths in J_3 will remain at zero flow level. The paths in J_2, however, may change their status for any small perturbation. As indicated in the proof of Theorem 4.1.1, $\Omega^\perp$ can also be written as

$$\Omega^\perp = \{f \mid f = \Delta h, \Gamma h = 0, h_p \text{ UIS } \text{ for } p \in J_1, h_p \geq 0 \text{ for } p \in J_2, h_p = 0 \text{ for } p \in J_3\}.$$

Therefore, the variational inequality $VI^\perp$ may be viewed as a network equilibrium problem with the cost function $\nabla_f C(f^*, \varepsilon^*) f + \nabla_\varepsilon C(f^*, \varepsilon^*) \varepsilon_0$, with zero demands for all O-D pairs, and with lower and upper bounds imposed on the path flows (in particular some path flows must be zero). In Section 5 we provide an example to show how to construct an auxiliary network to compute the directional derivatives.

4.2. Perturbed traffic equilibria with elastic demands

In the previous subsection, we have considered a traffic equilibrium model with a fixed demand pattern. Now we allow the demand of each O-D pair to be a function of the minimum travel time between all O-D pairs. As we will see, the resulting equilibrium model can also be described as a variational inequality of the general form we suggested in Subsection 4.1. But unfortunately, Assumption 2.4 is now too restrictive in this case. Therefore, we need to modify some of our proofs in Section 3.1 in order to obtain the Lipschitz continuity property.

Let $D(\cdot, \cdot) : R^{|W|} \times R^m \to R^{|W|}$ be a perturbed demand function. Then the equilibrium conditions can be described as follows:

$$\sum_{a \in A} \delta_{ap} c_a(f,\varepsilon) \begin{cases} = v_w & if\ h_p > 0 \\ \geq v_w & if\ h_p = 0 \end{cases} \quad \Big\}\ for\ p \in P_w, w \in W$$

$$f = \Delta h$$

$$\Gamma h = D(v, \varepsilon)$$

$$h \geq 0.$$

As indicated by Dafermos and Nagurney [1984], these equilibrium conditions can also be written as a variational inequality with $x = (f, v, d)$, $F(x, \varepsilon) = (C(f, \varepsilon)^T, d - D(v, \varepsilon)^T, -v^T)^T$, and $P = \{(f, v, d) \mid f = \Delta h, d = \Gamma h, h \geq 0\}$. (Notice that in this section d refers to demand and not to a derivative as in the earlier sections). In terms of the general form,

$$A = \begin{bmatrix} I & O & O \\ -I & O & O \\ O & O & I \\ O & O & -I \\ O & O & O \end{bmatrix}, \qquad B = \begin{bmatrix} -\Delta \\ +\Delta \\ -\Gamma \\ +\Gamma \\ +I \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now suppose (f^*, v^*, d^*) is an equilibrium solution for $\varepsilon = \varepsilon^*$. Let h^* be any corresponding equilibrium path flow pattern and $J = \{p \mid h_p^* = 0\}$. Let

$$P^{\perp} = \{(f, v, d) \mid C(f^*, \varepsilon^*)^T f = 0, f = \Delta h, d = \Gamma h, h_p \geq 0 \ \text{ for } p \in J\}.$$

Now assume $C(\cdot, \cdot)$ and $-D(\cdot, \cdot)$ both satisfy Assumptions 2.1-2.4. Note that $F(\cdot, \cdot)$ might not satisfy Assumption 2.4 in this case (since $[F(f, v, d') - F(f, v, d'')]^T[(f, v, d') - (f, v, d'')] = 0$ for all d' and d"). As a result, Lemma 3.1.2. (b) may not be valid. However, we can still show that $S'(\varepsilon)$, the solution set to $VI'(\varepsilon)$, is Lipschitz continuous at (f^*, v^*, d^*). We will prove this fact in Appendix C. So this general model still satisfies properties (i), (ii), and (iii) except the variational inequality $VI^{\perp}$ is now of the following form:

find $(f, v, d) \in P^{\perp}$ satisfying $\quad [\nabla_f C(f^*, \varepsilon^*) f + \nabla_\varepsilon C(f^*, \varepsilon^*) \varepsilon_0]^T (f' - f) + [d - \nabla_v D(v^*, \varepsilon^*) v$

$$- \nabla_\varepsilon D(v^*, \varepsilon^*) \varepsilon_0]^T (v' - v) - v^T (d' - d) \geq 0 \qquad \text{for any } (f', v', d') \in P^{\perp}$$

It is not hard to see that this linear variatonal inequality problem has a unique solution. In fact, suppose (f^1, v^1, d^1) and (f^2, v^2, d^2) are two solutions to $VI^{\perp}$. Then

$$\nabla_f C(f^*, \varepsilon^*)(f^1 - f^2) - \nabla_v D(v^*, \varepsilon^*)(v^1 - v^2) \leq 0.$$

Since $\nabla_f C(f^*, \varepsilon^*)$ and $-\nabla_v D(v^*, \varepsilon^*)$ are positive definite by our assumption, $f^1 = f^2$ and $v^1 = v^2$. Finally, notice that $d = \nabla_v D(v^*, \varepsilon^*) v + \nabla_\varepsilon D(v^*, \varepsilon^*) \varepsilon_0$ for any solution of $VI^\perp$. So $d^1 = d^2$. Again, $P^\perp$ can be written as

$$P^\perp = \{ (f, v, d) \mid f = \Delta h, d = \Gamma h, h_p \text{ UIS for } p \in J_1, h_p \geq 0 \text{ for } p \in J_2, h_p = 0 \text{ for } p \in J_3 \}$$

Therefore, $VI^\perp$ can be interpreted as a network equilibrium problem with linear cost and demand functions and path flows restricted between upper and lower bounds.

5. Example

Consider the network of Figure 5.1 with the perturbed cost function of each link given next to that link. There are two O-D pairs $w_1 = (1, 3)$ and $w_2 = (2, 4)$, with demands $d_1 = 2$ and $d_2 = 1$. The possible paths connecting these two O-D pairs are $P_{w_1} = \{$ 1-3, 1-2-3, 1-4-3, 1-2-4-3 $\}$ and $P_{w_2} = \{$ 2-4 $\}$.

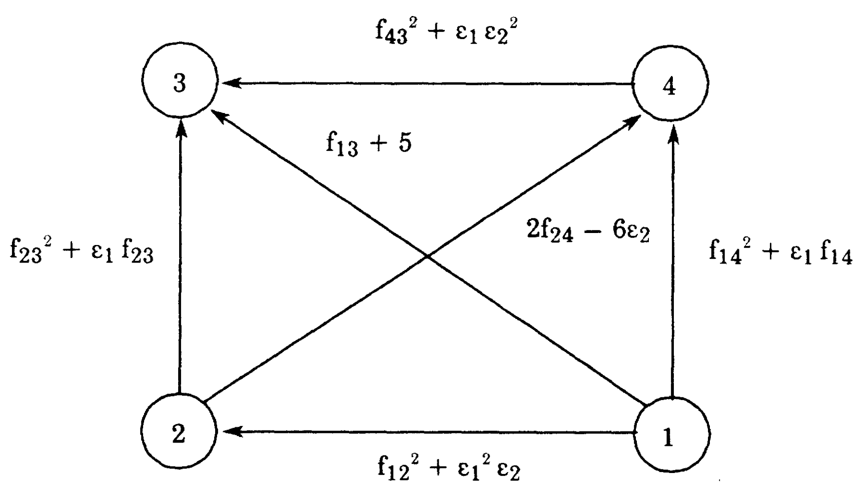

Figure 5.1. Traffic Equilibrium Example

We let $h = (h_1, h_2, h_3, h_4, h_5)$ denote the corresponding path flow vector. It is possible to show that the equilibrium flow pattern at $\varepsilon^* = (2, 0)$ is given by

$$h^* = (h_1^*, h_2^*, h_3^*, h_4^*, h_5^*) = (0, 1, 1, 0, 1), \text{ and}$$

$$f^* = (f_{12}^*, f_{13}^*, f_{14}^*, f_{23}^*, f_{24}^*, f_{43}^*) = (1, 0, 1, 1, 1, 1).$$

We now compute the derivative of the perturbed link flow pattern $f(\varepsilon)$ at $\varepsilon^* = (2, 0)$ along direction $\varepsilon_0 = (2, 1)$. As we mentioned in the previous section, the variational inequality $VI^{\perp}$ can be solved over a network. In this example, $J_1 = \{2, 3, 5\}$, $J_2 = \{4\}$ and $J_3 = \{1\}$. We construct an auxiliary network (See Figure 5.2) with the cost function as $\nabla_f C(f^*, \varepsilon^*) f + \nabla_\varepsilon C(f^*, \varepsilon^*) \varepsilon_0$ and with zero demands for both O-D pair w_1 and w_2. Furthermore, we have the following restrictions on the path flows: $-\infty < h_p < +\infty$ for $p \in J_1$, $0 \le h_p < +\infty$ for $p \in J_2$, and $h_p = 0$ for $p \in J_3$.

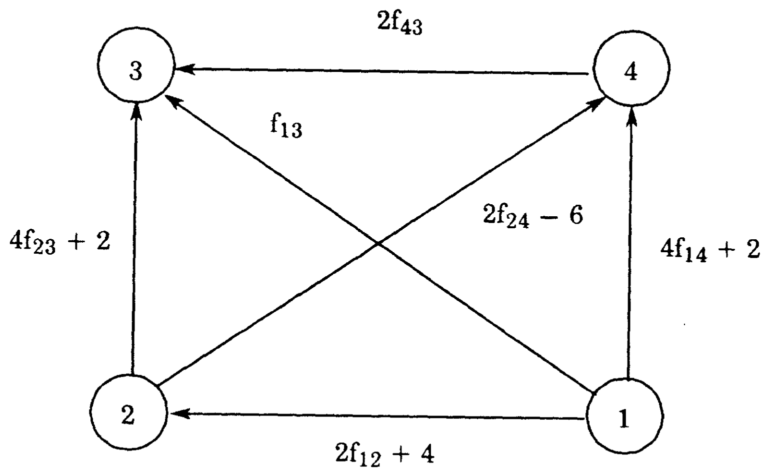

Figure 5.2. Derived Network for Computing Directional Derivative

Solving this linear variational inequality $VI^{\perp}$, we obtain $h = (0, 2/15, -8/15, 6/15, 0)$, and hence $f = \Delta h = (8/15, 0, -8/15, 2/15, 6/15, -2/15)$. Thus in this case, the directional derivative of the perturbed equilibrium link flow pattern is $d(\varepsilon_0) = (8/15, 0, -8/15, 2/15, 6/15, -2/15)$.

6. Conclusions

This study considers the perturbation problem for variational inequalities defined on polyhedral sets. The approach suggested in this paper consists of two major steps — first establishing a Lipschitz continuity property, and then a directional differentiability property of the perturbed local solution set. This particular feature of the method allows application to a number of equilibrium problems including the traffic equilibrium problem and the spatial market equilibrium problem. The analysis of this paper was carried out in a fairly general context — we considered the variation of the local solution set, rather than a unique solution, with respect to small perturbations. Thus we have introduced the notion of differentiability for a point-to-set mapping about a certain point. Even when the local solution set is not directionally differentiable, we attempted to characterize the first-order behavior of the local solution set. In a subsequent paper, we will extend the work of this paper to variational inequalities defined on nonpolyhedral sets.

Appendix A: Proof of Lemma 3.1.1.

Proof. (a) Property (a) follows immediately from the differentiability of $F(\cdot,\cdot)$ at (x^*, ε^*).

(b) Suppose property (b) does not hold. Then there exists a sequence $\{x^n\}$, $x^n \in x^* + P^\perp$ and $x^n \to x^*$ satisfying $[F(x^n, \varepsilon^*) - F(x^*, \varepsilon^*)]^T (x^n - x^*) \le \|x^n - x^*\|^2 / n$. Let z be a limit point of the sequence $\{(x^n - x^*)/\|x^n - x^*\|\}$. Then we have $z \in P^\perp$, $\|z\| = 1$, and $z^T \nabla_x F(x^*, \varepsilon^*) z \le 0$, which contradicts the assumption that $\nabla_x F(x^*, \varepsilon^*)$ is positive definite on $P^\perp$.

(c) If $F(x^*, \varepsilon^*) = 0$, then clearly $P \subseteq x^* + P^\perp$. Property (c) becomes trivial. If $F(x^*, \varepsilon^*) \neq 0$, consider a neighborhood U_1 of x^* of the form $U_1(\delta) = \{x \mid x_i^* - \delta < x < x_i^* + \delta,\ i = 1, \cdots, n\}$, where $\delta > 0$ is to be determined. Note that $H = \{x \mid F(x^*, \varepsilon^*)^T (x - x^*) = 0\}$ is a supporting hyperplane of $P \cap \mathrm{Cl}(U_1(\delta))$ at x^*, and that the solution set of linear program

$$\min\{F(x^*, \varepsilon^*)^T x \mid x \in P \cap \mathrm{Cl}(U_1(\delta))\}$$

is contained in $H \cap P$ (see Figure 3.1.1). In view of the polyhedral structure of $P \cap \mathrm{Cl}(U_1(\delta))$ and the continuity of the function F at (x^*, ε^*), we know (by the upper semicontinuity property of the solution set of linear programs) that there exist a neighborhood $V_1 \subseteq V$ of ε^* and δ small enough so that for any $x^0 \in \mathrm{Cl}(U_1(\delta))$ and $\varepsilon^0 \in V_1$, the solution set of linear program

$$\min\{F(x^0, \varepsilon^0)^T x \mid x \in P \cap \mathrm{Cl}(U_1(\delta))\}$$

is contained in $H \cap P$. Finally, note that $H \cap P \subseteq x^* + P^\perp$, which completes the proof.

□

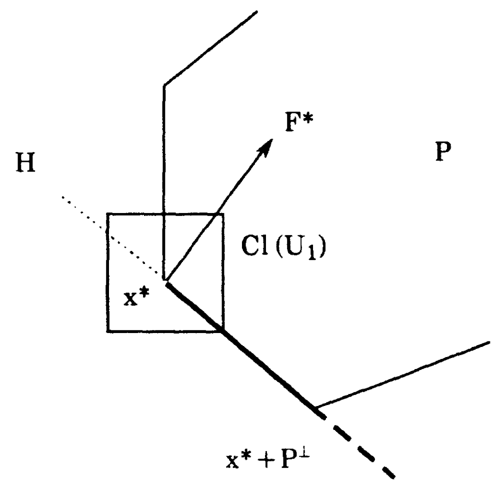

Figure.3.1.1. Local Approximation About x*

Appendix B: Proof of Lemma 3.2.1.

Proof. $x(\varepsilon)$ and $\pi(\varepsilon)$ satisfy the following complementary slackness conditions:

$$A^T \pi(\varepsilon) - F(x(\varepsilon), \varepsilon) = 0$$

$$\sum_{i=1}^{n} a_{ki} x_i(\varepsilon) - b_k \geq 0 \quad \text{if } \pi_k(\varepsilon) = 0$$

$$\sum_{i=1}^{n} a_{ki} x_i(\varepsilon) - b_k = 0 \quad \text{if } \pi_k(\varepsilon) > 0$$

$$\pi(\varepsilon) \geq 0.$$

Let $x'(t_n) = [x(\varepsilon^* + t_n \varepsilon_0) - x(\varepsilon^*)]/t_n \to x^L$

$\pi'(t_n) = [\pi(\varepsilon^* + t_n \varepsilon_0) - \pi(\varepsilon^*)]/t_n \to \pi^L$.

Then for n large enough, we have by subtracting the equations or inequalities in the previous complementarity system for values ε^* and $\varepsilon^* + t_n \varepsilon_0$, and by using the complementary slackness condition,

$$A^T \pi'(t_n) - [F(x(\varepsilon^* + t_n \varepsilon_0), \varepsilon^* + t_n \varepsilon_0) - F(x^*, \varepsilon^*)]/t_n = 0$$

$$\sum_{i=1}^{n} a_{ki} x_i'(t_n) = 0 \quad for\, k \in K_1, k \in K_2\, and\, \pi_k^L > 0$$

$$\sum_{i=1}^{n} a_{ki} x_i'(t_n) \geq 0 \quad for\, k \in K_2\, and\, \pi_k^L = 0$$

$$\pi_k'(t_n) \geq 0 \quad for\, k \in K_2, \quad \pi_k'(t_n) = 0 \quad for\, k \in K_3.$$

Letting $n \rightarrow \infty$, we observe that x^L and π^L satisfy system (3.2.3).

□

Appendix C: S' (ε) is Lipschitz continuous at (f*, v*, d*) in the context of general traffic equilibrium problem.

Proof. For any $x'(\varepsilon) \in S'(\varepsilon)$, since $x'(\varepsilon)$ solves $VI'(\varepsilon)$ and x^* solves $VI'(\varepsilon^*)$, we have

$$-F(x^*, \varepsilon^*)^T (x'(\varepsilon) - x^*) \leq -F(x'(\varepsilon), \varepsilon)^T (x'(\varepsilon) - x^*).$$

By Assumption 2.4, there are $\alpha > 0$, and $\beta > 0$ satisfying

$\alpha \| f'(\varepsilon) - f^* \|^2 + \beta \| v'(\varepsilon) - v^* \|^2$

$\leq [C(f'(\varepsilon), \varepsilon^*) - C(f^*, \varepsilon^*)]^T (f'(\varepsilon) - f^*) + [D(v'(\varepsilon), \varepsilon^*) - D(v^*, \varepsilon^*)]^T (v'(\varepsilon) - v^*)$

$= [F(x'(\varepsilon), \varepsilon^*) - F(x^*, \varepsilon^*)]^T (x'(\varepsilon) - x^*)$

$\leq [F(x'(\varepsilon), \varepsilon^*) - F(x'(\varepsilon), \varepsilon)]^T (x'(\varepsilon) - x^*)$

$= [C(f'(\varepsilon), \varepsilon^*) - C(f'(\varepsilon), \varepsilon)]^T (f'(\varepsilon) - f^*) + [D(v'(\varepsilon), \varepsilon^*) - D(v'(\varepsilon), \varepsilon)]^T (v'(\varepsilon) - v^*)$

$\leq \| C(f'(\varepsilon), \varepsilon^*) - C(f'(\varepsilon), \varepsilon) \| \| f'(\varepsilon) - f^* \| + \| D(v'(\varepsilon), \varepsilon^*) - D(v'(\varepsilon), \varepsilon) \| \| v'(\varepsilon) - v^* \|.$

Notice that $\alpha x^2 + \beta y^2 \leq a x + b y$ implies $\alpha x^2 + \beta y^2 \leq a^2 / \alpha + b^2 / \beta$. So

$\alpha \| f'(\varepsilon) - f^* \|^2 + \beta \| v'(\varepsilon) - v^* \|^2$

$\leq \| C(f'(\varepsilon), \varepsilon^*) - C(f'(\varepsilon), \varepsilon) \|^2 / \alpha + \| D(v'(\varepsilon), \varepsilon^*) - D(v'(\varepsilon), \varepsilon) \|^2 / \beta$

$\leq [L_1^2 / \alpha + L_2^2 / \beta] \| \varepsilon - \varepsilon^* \|^2.$

In the last expression L_1 and L_2 are the convergence constants (see Assumption 2.2) corresponding to $C(f, \varepsilon)$ and $D(v, \varepsilon)$. We notice that $d'(\varepsilon) - D(v'(\varepsilon), \varepsilon) = 0$ is always valid for ε near ε^* (since $v'(\varepsilon)$ is continuous at ε^* and hence the corresponding component v in x = (f, v, d) can be viewed relatively as a free variable for ε near ε^*). Thus by the differentiability assumption, there are constants $M_1, M_2 > 0$ satisfying

$$\| d'(\varepsilon) - d^* \| = \| D(v'(\varepsilon), \varepsilon) - D(v^*, \varepsilon^*) \|$$

$$\leq M_1 \| v'(\varepsilon) - v^* \| + M_2 \| \varepsilon - \varepsilon^* \|$$

$$\leq [(L_1^2 / \alpha\beta + L_2^2 / \beta\beta)^{1/2} M_1 + M_2] \| \varepsilon - \varepsilon^* \|.$$

Thus $S'(\varepsilon)$ is Lipschitz continuous at (f^*, v^*, d^*).

□

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