DEVELOPMENT OF A NEW WALL SHEAR STRESS GAUGE FOR FLUID FLOWS

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ABSTRACT

A new technique has been developed to measure the wall shear stress and its direction in the turbulent boundary layer. In the turbulent boundary layer there is a very thin region, a viscous sublayer, which extends from \( y^+ = 0 \) to 5. This technique involves the measurement of torque upon a very small cylindrical body placed above the wall in the viscous sublayer, so that the device is operating in the creeping flow regime. The method of approach has involved calibration tests on a gauge 8 mm. long by 0.8 mm. in diameter, located in the uniform shearing flow created in a cone-and-plate apparatus. Our theoretical, computational and experimental results show that the torque has a linear relation with the wall shear stress. The gauge response is reversed for reversing the flow. By directivity measurements using this gauge, maximum wall shear stress direction and its magnitude are obtained. Linear response is obtained up to Reynolds number 3.2 which we could get in our apparatus. We have been using a spectral element code, Nekton, for solving the creeping flow equations for the gauge development. There is a very good agreement between the experimental and computational results.

Thesis Supervisor: Patrick Leehey

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NOMENCLATURE OF SYMBOLS

\( d \) Cylinder diameter

\( F \) Drag

\( h \) Distance between wall and cylinder

\( H \) Distance between stationary and moving walls

\( L \) Length scale

\( L_s \) Distance between stationary mirror and screen

\( N \) Number of interpolation points around cylinder

\( p \) Pressure

\( r \) Radial distance

\( \text{Re} \) Reynolds number, \( \frac{U d^2}{\nu} \)

\( T \) Temperature

\( u \) \( x \) component of velocity

\( \vec{u} \) Velocity vector

\( U_\infty \) Free stream velocity

\( U \) Constant shear rate

\( u_d \) Disturbance velocity

\( u_{d,\text{max.}} \) Maximum disturbance velocity

\( u_* \) Friction velocity, \( \sqrt{\frac{\tau_w}{\rho}} \)

\( u^+ \) \( \frac{u}{u_*} \)

\( \nu \) \( y \) component of velocity

\( y \) Distance from wall
$y^+$ Distance from wall in viscous units, $\frac{y u_*}{v}$

$\alpha$ Angle between flow direction and normal direction of gauge

$\beta$ Cone angle

$\theta_r$ Rotation angle of the gauge

$\mu$ Absolute viscosity of fluid

$\nu$ Kinematic viscosity of fluid

$\delta L$ Laser displacement

$\rho$ Fluid density

$\omega$ Angular velocity of cone

$\epsilon$ Aspect ratio, $\frac{h}{d}$

$\epsilon_l$ Aspect ratio, $\frac{L}{d}$

$\sigma$ Variance

$\tau_w$ Wall shear stress

$\vec{\tau}$ Stress vector

$\tau_{ij}$ Stress tensor

$\tau_{rr}$ Normal viscous stress in radial direction

$\tau_{\theta\theta}$ Normal viscous stress in tangential direction

$\tau_{r\theta}$ Viscous shear stress

$\psi$ Stream function

$\xi, \eta$ Bipolar coordinates

$\eta_d$ Ratio of the computational drag to analytical drag

$\eta_t$ Ratio of the computational torque to analytical torque
1. INTRODUCTION:

The measurement of mean and fluctuating shear stresses created on the wall under a turbulent boundary layer flow is very important in analyzing a flow field.

A variety of techniques such as the Stanton tube, the Preston tube, the surface fence, the floating element and the thermal methods have been used for the measurement of wall shear stress in a turbulent boundary layer. Stanton(1920) used a rectangular pitot tube, mounted on a wall in a fully developed laminar flow, and used the difference between the pressure measured with this pitot tube and the static pressure to determine the velocity at the center of the tube, and established that if the center of the Stanton tube was located close enough to the wall, the wall shear stress could be calculated as $\tau_w = \mu \frac{u'(y)}{y}$. This is an indirect measurement technique.

Preston, J.H.(1953) used a round pitot tube on a surface in the fully developed turbulent pipe flow and established a calibration curve for a round pitot tube. The surface fence that consists of a wall obstruction was invented by Konstantinov and Dragnysh(1955). The difference in pressure before and behind this gauge is related to the wall shear stress. The advantages of this gauge over the Stanton tube is that it gives a doubled pressure reading. Head and Rechenberg,(1962) calibrated a surface fence and a Preston tube in a turbulent flow and compared these two gauges. Although there was an agreement in moderately unfavorable pressure gradients, the two gauges indicated different values of the wall shear stress in strongly unfavorable pressure gradients.

Vagt, J.D. and Fernholtz, H.(1973) calibrated the surface fence versus the Preston tube and gave a calibration curve of the surface fence for the flow direction. The floating element technique based on the measurement of skin friction forces, acting on a floating element buried in the wall inside a turbulent boundary layer. Frei, D. and Thomann, H.(1980) used the floating element technique to investigate the error of Preston tubes in adverse pressure gradients. The gaps between the floating element and the surrounding wall filled with a liquid in order to prevent disturbing forces on the element. Petri, S.(1984) developed a 4 mm. by 4 mm. floating element to measure the wall shear stress. Although that gauge was satisfactory in operation, it did not provide the desired frequency response. The main difficulty with this technique is the gap effects.
The gap creates a disturbance field which affects the performance of the gauge. Another method of measuring the wall shear stress is the use of flush mounted hot film probes. The operation principle of these probes is that the fluid at the surface of the probe, which is mounted on a wall inside a turbulent boundary layer, is controlled at a specific temperature, which is different from that in the bulk fluid and the heat transfer rate between the fluid and the probe is measured. Using a calibration curve given between the measured heat transfer rate and the velocity, the wall shear stress can be determined. This technique measures the mean wall shear stress as well as fluctuating wall shear stress in a turbulent boundary layer.

Most of these techniques have limited applicability, and lack analytical and computational fluid studies for the gauges. We have developed a new technique for the measurement of the wall shear stress in a turbulent boundary layer. The measurement principle of this technique is to place a cylindrical body inside the viscous sublayer and measure the torque acting on the body. The gauge operates in the creeping flow regime by its inside the viscous sublayer. The torque acting on the gauge due to the linear shear flow of the viscous sublayer has a linear response to the wall shear stress. This is a direct measurement technique. Since the gauge has a linear response to the wall shear stress, both mean and fluctuating components of the wall shear stress can be measured. Another important feature of the gauge is that the gauge gives the maximum wall shear stress direction and its magnitude. This gauge could also be used to determine the flow direction.
2. ANALYTICAL PART

2.1. Wall shear stress device for viscous sublayer:

Our measurement technique for the wall shear stress is put a cylindrical body very close to a wall inside the viscous sublayer and measure the torque acting on the body due to the shearing flow of the viscous sublayer.

The experimental results show that the viscous sublayer extends from the wall to $y^+ = 5$. In this region the mean velocity profile is linear ($u^+ = y^+$). Although the mean velocity profile within the sublayer is a linear velocity profile, the flow within it is not laminar, but accompanied by considerable irregular fluctuations. The Reynolds number based on the characteristic length which is the distance from the wall to $y^+$ and the velocity at $y^+$, has the following relation

$$\text{Re} = (y^+)^2$$

As the gauge extends to the edge of the viscous sublayer, the Reynolds number becomes 25. The flow field can not be a Stokes flow for this Reynolds number, therefore, the gauge diameter should be such that the gauge stays below $y' = 1$.

2.2. Behavior of Stokes Flows:

The Navier-Stokes equation is

$$\rho \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p + \mu \nabla^2 \vec{u}$$

Consider a flow field with characteristic length $L$, and velocity $U_\infty$. With the proper nondimensional distance $x_i^* = \frac{x_i}{L}$, velocities $u_i^* = \frac{u_i}{U_\infty}$ and pressure $p^* = \frac{p}{U_\infty^2}$, the Navier-Stokes equation becomes
For \( \text{Re} \rightarrow 0 \), this equation becomes the Stokes equation

\[
\nabla^* p^* = \nabla^* 2 \bar{u}^*
\]

All properties of a stokes flow are governed by linear equations for \( p, u_i, \omega_i, \tau_{ij} \). The linear property may be used in adding flow fields to produce new flows. For a symmetrical object, the streamlines are symmetric and all the other properties, \( u_i, p, \omega_i, \tau_{ij} \), are antisymmetric with respect to symmetry axis.

### 2.3. Description of the problem:

Our measurement principle for the wall shear stress is to put a cylindrical body very close to a wall inside the viscous sublayer where the flow can be assumed to be a shearing flow and measure the torque acting on the body. Therefore, our analytical and numerical studies are aimed at solving the creeping flow equations when there is a linear shear flow and a cylindrical gauge which is very close to a wall.

Dimensional analysis gives the following functional relations for the torque and the drag acting on the cylindrical body due to a shearing flow.

\[
T = f (\mu, Ud, d, h)
\]

\[
F = g (\mu, Ud, d, h)
\]

Where \( U, d, \mu, h \) are the constant shear rate, the diameter of the cylinder, absolute viscosity and the distance between the wall and cylinder respectively. Since we consider a Stokes flow, density does not appear as a variable in the above functional relations. Using the Buckingham \( \pi \) theorem,
these functional relations become

\[
\frac{T}{\mu Ud^2} = f_2\left(\frac{h}{d}\right)
\]

\[
\frac{F}{\mu Ud} = g_2\left(\frac{h}{d}\right)
\]

As can be seen, the torque and drag are linearly related to the wall shear stress, \(\tau_w = \mu U\). The above relations are valid for a two dimensional flow field. For a three dimensional flow field, there is one more variable, the length scale of the cylinder, \(L\). For this case, the functional relations become

\[
\frac{T}{\mu Ud^2} = f_3\left(\frac{h}{d}, \frac{L}{d}\right)
\]

\[
\frac{F}{\mu Ud} = g_3\left(\frac{h}{d}, \frac{L}{d}\right)
\]

2.4. 2-D Solution of the Stokes equation for the cylindrical gauge:

Figure (1). Cylindrical gauge
The solution of the Stokes equation, when there is a linear shear flow and a cylindrical gauge which is very close to a wall, was solved by Davis and O’Neill(1977) by using a stream function formulation. Inflow velocity components were taken as \((U_y, 0, 0)\). The equation of continuity is

\[ \nabla \cdot \mathbf{u} = 0 \]

Using the stream function, \(\psi\), velocities are given by

\[ \mathbf{u} = \left( U \frac{\partial \psi}{\partial y}, -U \frac{\partial \psi}{\partial x}, 0 \right) \]

The boundary conditions are

on the plane, \(\psi = \frac{\partial \psi}{\partial y} = 0\)

on the cylinder, \(\psi = M\), \(\frac{\partial \psi}{\partial n} = 0\)

The constant \(M\) depends on the gap ratio, \(\frac{h}{d}\), and can be determined from the flux of fluid through the gap. The boundary condition at infinity is \(\psi \sim \frac{1}{2} y^2 (y \to \infty)\). The equations of motion for Stokes flow are

\[ \nabla p = \mu \nabla^2 \mathbf{u} \]

Using the stream function, the equations of motion become a biharmonic equation for \(\psi\)
The Reynolds number for the flow is \( \frac{Ud^2}{4v} \). This Reynolds number must be small for the Stokes equation to hold. Davis and O’Neill obtained the solution of this biharmonic equation in terms of bipolar coordinates. Their results show that the torque and drag force due to a linear shear flow are linearly related to the wall shear stress. Their solutions give the torque and drag force on the body per unit length as

\[
T = \frac{\pi}{2} \tau_w d^2
\]

\[
F = 2 \pi \tau_w d \left( 1 + \frac{2}{3} \varepsilon - \frac{4}{45} \varepsilon^2 + \cdots \right)
\]

where \( \varepsilon = \frac{h}{d} \), \( d \) is the cylinder diameter and \( h \) is the gap between the wall and cylinder. The drag equation agrees in terms of functional dependency with the result, found in section (2.3) using a dimensional analysis. An interesting result is that the torque acting on the body is independent of the gap. The pressure and shear stress are antisymmetric with respect to symmetry axis. As can be seen from the following figure, \( y \) components of the pressure and shear stress terms at \( \theta \) and \( -\theta \) cancel each other, therefore, the integration of the shear stress and pressure on the cylinder does not give a lift force.
According to analytical results of Davis and O’Neill when the gap is approximately 0.685 times the cylinder radius or less, the flow separates from the boundaries. Flow visualization for this problem was done by Taneda(1979). In his experiment, the gap was 0.57 times the cylinder radius and Reynolds number was 0.011. In his flow visualization picture, shown in figure (11), two closed vortices due to the flow separation can be seen. Taneda’s flow visualization result agrees with Davis and O’Neill’s result, since the gap in Taneda’s experiment was smaller than the critical value, given by Davis and O’Neill.

2.5. Maximum Disturbance Velocity:

We measure the wall shear stress using a cylindrical gauge. It is necessary to know the disturbance velocity due to the cylindrical gauge to determine the proper size of the cylinder gauge for the wall shear stress measurements in the turbulent boundary layer. Another reason for investigating the disturbance velocity is that we use a cone-and-plate apparatus for testing the gauge, and it consists of a stationary wall as well as a moving upper wall which creates a blockage effect. The blockage effect due to the upper wall is important since the presence of the moving upper wall increases the velocity gradient on the cylinder, therefore, increasing the torque acting on the body. Decreasing the distance between the walls increases the torque. The disturbance velocity, \( u_d \), is defined as the difference between the free stream velocity and the velocity in the presence of the cylindrical body at the same location. When there is no body in the domain, the streamlines are straight lines. The streamlines take symmetrical curved shapes in the presence of the cylindrical body and the maximum disturbance velocity, \( u_{d_{\text{max}}} \), occurs along the symmetry axis because of the presence of only x component of the velocity. Finite span decreases this velocity, hence the two dimensional calculation yields a conservative estimate of blockage. The maximum disturbance velocity is obtained using Davis and O’Neill’s stream function formulation. Derivation of the maximum disturbance velocity is shown in Appendix (A) and it is plotted in figure (2). It can be seen from figure (2) that the maximum disturbance velocity decays almost exponentially in the y direction, and it almost becomes zero when the distance from the wall is
about 12 times the diameter plus the gap. The cone-and-plate experiments have been done for
four different values of \( \frac{H}{d + h} \), (2.355, 3.548, 4.754, 7.167). As can be seen in the maximum
disturbance velocity plot, the ratio of \( \frac{u_{d\text{ max.}}}{u} \) changes from 0.06 to 0.003 in our experiments.
These results show that there is a blockage effect in the experiment. The cone-and-plate experi-
ments should be done for \( \frac{H}{d + h} = 12 \) in order not to have a blockage effect. This experiment
will be done in the near future.
Figure (2). Maximum disturbance velocity

\[ \frac{u_{d \text{ max}}}{u} = \frac{\text{Maximum disturbance velocity}}{\text{Free stream velocity}} \]
3. EXPERIMENTAL SET-UP AND RESULTS :

3.1. Cone-and-plate apparatus and flow behavior in the apparatus :

The cone-and-plate apparatus used for the calibration of the cylindrical gauge is shown in figure (9). This apparatus was developed by Prof. C.F. Dewey, Jr. of M.I.T. The reason for using this apparatus is that it creates a linear shear flow and gives a constant shear rate everywhere inside the apparatus. The cone-and-plate apparatus consists of a shallow rotating cone and a stationary circular plate. The diameter of the apparatus used in the experiment is 206 mm. Two different cones (30° and 60°) are used in the experiment to determine the blockage effect. This apparatus is filled with glycerol or glycerol-water mixtures and driven by an electrical motor.

For small Reynolds numbers (Re < 1), flow streamlines inside the apparatus are concentric circles and cone surface velocity, u(r), increases linearly with radius. The wall shear stress is

\[ \tau_w = \mu \frac{\partial u}{\partial y} \]

\[ \tau_w = \mu \frac{\omega r}{H} = \mu \frac{\omega}{\tan \beta} \]

For small cone angle, \( \beta \), the wall shear stress is

\[ \tau_w = \mu \frac{\omega}{\beta} \]

where \( \beta \) is the cone angle, \( \mu \) is the absolute viscosity, \( \omega \) is the angular velocity of the cone, and \( H \) is the gap height. The wall shear stress is constant everywhere inside the cone for small Reynolds numbers.

The flow behavior inside the cone-and-plate apparatus were investigated by H.P Sdougos, S.R. Bussolari, and C.F. Dewey, Jr. (1982). They investigated the flow behavior inside the cone-and-plate apparatus using flow visualization, hot film heat-transfer probes and measurements of
the torque required to rotate the cone against the retardation of the viscous fluid. They also presented theoretical results to these experiments. According to their findings, flow regimes are well characterized by the single dimensionless parameter:

$$R^* = \frac{r^2 \omega \beta^2}{12 \nu}$$

where \( r \) is the radial distance from the axis of the cone. The parameter \( R^* \) is analogous to Reynolds number. They found a good agreement between the theoretical results of the wall shear stress values for \( R^* > 0.5 \) and observed turbulence due to secondary flow for \( R^* \geq 4 \). Their results did not attempt to describe the flow near the outer rim of the cone, where edge effects and the boundary conditions are important. The dimensionless parameter \( R^* \) is less than 0.02 in our cone-and-plate experiments for determining the blockage effect.

Fewell, M.E. and Hellums, J.D.(1977) investigated the secondary flow in cone-and-plate viscometers by numerical integration of the equations of motion for steady incompressible flow of Newtonian fluids. They assumed a spherical shape for the outer rim of the cone, and determined the effects of the Reynolds number and cone angle in secondary flow. They defined the Reynolds number as \( \frac{R_0^2 \omega}{\nu} \), where \( R_0 \) is the outer radius of the cone. According to their results, there is no secondary flow effect in our experiments since the Reynolds number \( \frac{R_0^2 \omega}{\nu} \) is 39.5 in the experiments.

3.2. Construction of the gauge and measurement principle:

The cylindrical gauge is shown in figure (4). The diameter of the gauge is 0.8 mm. and the length of the gauge is 8 mm.. The construction technique of this gauge was developed by Peter M. Wagner, research engineer from Technische Universtat Berlin. The gauge is made of platinum and one of its ends polished with special polishing materials up to the centerline for an axial distance of 1.2 mm. in order to have a mirror surface for reflection of the laser light. A plexiglass
plate 50.8 mm. in diameter and 3.17 mm. in thickness is used for the mounting surface of the gauge. Four copper wires with a diameter of 0.4 mm. and a height of 0.6 mm. are mounted on the plexiglass as supports. Two platinum wires having a diameter of 25 micrometers are soldered to the supports after pretensioning. The gauge is also soldered at the center to the pretensioned platinum wires at both ends. The stationary plate of the apparatus has a diameter of 206 mm. and has a circular hole with a diameter of 50.8 mm. at a position 63 mm. from the axis of the cone. The gauge on the plexiglass plate is put in the circular hole of the stationary plate of the apparatus. Glycerol and glycerol-water mixtures are used as working fluids in the apparatus in order to keep the Reynolds number small.

The measurement principle is basically shown in figure (3). A stationary helium-neon laser sends a laser beam to a stationary mirror outside the apparatus. The reflected laser beam goes to the mirror surface of the gauge from the stationary mirror, and is reflected back to the stationary mirror, and which then goes to the screen which is at a distance of 1 meter from the stationary mirror. When the cone does not rotate, there is a reference spot on the screen. When it rotates, it creates a linear shear flow and due to that shearing flow the gauge rotates and the mirror surface of the gauge reflects the laser beam back with an angle. The reflected laser beam gives a displaced spot on the screen. We measure the displacement of the laser spot on the screen and measure the rotational speed of the cone at the same time using a digital counter. By measuring the rotational speed, we know the wall shear stress since \( \tau_w = \mu \frac{\omega}{\beta} \). The relation between the rotation angle of the gauge and the laser displacement on the screen is

\[ \delta L = 2 \theta_T L_s \]

where \( \delta L \) is the laser displacement, \( \theta_T \) is the rotation angle of the gauge, and \( L_s \) is the distance between the stationary mirror and the screen. Basically, by measuring the rotational speed of the cone and the laser displacement, we get a calibration curve for wall shear stress versus the rotation angle of the gauge.
Figure (3). The cone-and-plate apparatus

Figure (4). Geometry of the shear stress gauge
3.3. Experimental Results:

As explained in section (3.1), the wall shear stress has a constant value over the plate surface of the cone-and-plate apparatus for small Reynolds numbers. In order to observe the blockage effect of the gauge, the same gauge is used in two different positions from the rotational axis of the cone while two different cones (3° and 6°) are used in the experiment. A linear response is obtained in each case.

The calibration curve for the 3° cone and \( \frac{H}{d} = 3.26 \) is shown in figure (12). The absolute viscosity of glycerol used in this experiment is 1275 centipoise at the measurement temperature, 20°C. The average slope, the ratio of the laser displacement on the screen to the rotational speed of the cone, is 31.68 [mm.secs] and the ratio of the variance to the average slope is 2.5%. The average slope in the measurement can be written as

\[
\text{Average slope} = \frac{\text{laser displacement}}{\text{rotational speed}} = \frac{\delta L}{\omega}
\]

By putting the values \( \delta L \) and \( \omega \) into above relation, we get the following relations

\[
\text{Average slope} = \frac{2 \mu L_s}{\beta \theta_T} \frac{\theta_T}{\tau_w}
\]

or

\[
\frac{\tau_w}{\theta_T} = \frac{2 \mu L_s}{\beta} \frac{1}{\text{Average slope}}
\]

Using the above relation, we find the ratio of the wall shear stress to the rotation angle of the gauge, \( \frac{\tau_w}{\theta_T} \), as 26.8 [N/m².deg.].
The calibration curve for the $3^0$ cone and $\frac{H}{d} = 4.91$ is shown in figure (13). The viscosity of glycerol is also 1275 centipoise. The average slope, the ratio of the variance to the average slope, and the value of $\frac{\tau_w}{\theta_T}$ are 30.56 [mm. secs], 11 %, and 27.8 [N/m$^2$/deg.] respectively.

The calibration curve for the $6^0$ cone and $\frac{H}{d} = 6.58$ is shown in figure (14). The viscosity of glycerol is 745 centipoise in this experiment. The average slope is 4.70 [mm. secs] and the ratio of the variance to the slope is 4.7 %. The value of $\frac{\tau_w}{\theta_T}$ is 52.78 [N/m$^2$/deg.].

The calibration curve for the same cone angle, $6^0$, and the same viscosity, but the different gap, $\frac{H}{d} = 9.92$, is shown in figure (15). The average slope in this experiment is 4.28 [mm. secs], and the ratio of the variance to the slope is 4.7 %, and the value of $\frac{\tau_w}{\theta_T}$ is 58.01 [N/m$^2$/deg.].

Maximum Reynolds number ($Re = \frac{U d^2}{v}$) is 0.023 in the above experiments.

The rotational stiffness of the gauge could not be measured, therefore, a constant rotational stiffness is assumed. This is a good assumption since the maximum rotation of the gauge in our experiments is around 2 [degree], and the response of the gauge is linear. Since we do not have the stiffness of the gauge, we cannot compare the experimental torque results with the theoretical torque results.

As can be seen from the experimental values of $\frac{\tau_w}{\theta_T}$ for various gap heights, shown in figure (16), while the gap height, H, is decreasing, the rotation angle of the gauge increases for constant shear stress values since decreasing the gap height increases the velocity gradient acting on the gauge, therefore, increasing the torque which is linearly related to the rotation angle of the gauge through the rotational stiffness of the gauge ($T = k_T \theta_T$, where $k_T$ is the rotational stiffness of the gauge, and $\theta_T$ is the rotation angle of the gauge).
We will do additional experiments in the $30^\circ$ cone in various $H/d$ positions to obtain an empirical curve between $\frac{t_w}{\theta_T}$ and $\frac{H}{d}$.

In each experiment, the measurements are taken both for positive and negative rotational speeds of the cone. As can be seen in figures (12), (13), (14), and (15), the gauge response is reversed for reversing the flow direction. Therefore, the gauge gives the magnitude and direction of the wall shear stress. That is a very important feature of the gauge since most of the other techniques do not give the direction of the shear stress.

In order to see the direction sensitivity of the gauge, the angle between the flow direction and the normal to the symmetry axis of the gauge, $\alpha$, is changed from $0^\circ$ to $60^\circ$ with $10^\circ$ increments in $30^\circ$ cone and at the position of constant $\frac{H}{d} = 4.08$. The directivity measurements, shown in figure (20), are obtained. A linear response is obtained for $\alpha = 0^\circ$. Increasing the angle, $\alpha$, gives an increasing nonlinear response. The results become uncorrelated after $\alpha = 40^\circ$. If the flow field were two dimensional, what we would expect is that only velocity component normal to the gauge axis would generate a torque. In this case, the ratio of the wall shear stress at $\alpha = 0^\circ$ to the shear stress at $\alpha$ becomes $\cos(\alpha)$, but the flow field can not be two dimensional for our aspect ratio, $e_1 = 10$. Increasing the angle, $\alpha$, increases the third dimension effect in the flow field and gives more than a cosine effect. This can be seen in figure (20). In this figure, a peak appears at $\alpha = 50^\circ$ with a very large variance. In some sense it is an "unstable point" since small changes from that angle give a very large difference in the response. On the other hand our objective is to find out an unknown flow direction from this directivity measurement; therefore, that peak is not important because of its "unstable" behavior. The important result is that by changing the angle, $\alpha$, it is possible to find an unknown flow direction and maximum wall shear stress direction from the response of the gauge.

In order to see up to what Reynolds number the flow field might be assumed a Stokes flow, glycerol-water mixtures with dynamic viscosities ranging from 815 centipoise (100% glycerol)
to 32 centipoise (60% glycerol, 40% water in volume) are used. The maximum Reynolds number, which we could obtain in our cone-and-plate apparatus, based on the diameter of the gauge and shearing velocity at the mid section of the gauge is 3.2. Up to that Reynolds number, a linear response is obtained. These results are shown in figures (21), (22), and (23). This result indicates that the flow field is a Stokes flow for the Reynolds number, 3.2, and allows us to use that Reynolds number to determine the actual size of the gauge for wind tunnel and turbulent oil channel measurements.
Figure (5). Calibration curve for $\beta = 3^0$, $\mu = 1275$ cp.
4. COMPUTATIONAL PART:

4.1. Introduction to the spectral element code, NEKTON:

We use the spectral element code, NEKTON, developed by Prof. A.T. Patera at M.I.T., for computing the Stokes solution for the cylindrical gauge. NEKTON is a computer code for the simulation of steady and unsteady incompressible fluid flow with forced and natural convection heat transfer. The code has three parts: Prenek, Nekton and Postnek. Prenek is an interactive program in which all the necessary information for the flow problem such as geometrical, physical, and numerical parameters can be given. The Nekton part of the code performs the numerical integration of the Navier-Stokes and energy equations for the flow problem which is specified in Prenek. Postnek is an interactive graphic package, in which the results of a Nekton simulation can be analyzed.

The spectral element technique is a high order finite element technique. In the spectral element, the computational domain is broken up into macro elements as in the finite element technique and the velocity and pressure terms in each element are represented by high order Lagrangian interpolants. In each element, the velocity and pressure terms are expanded in terms of \((N-1)\)th order polynomial Lagrangian interpolants through Chebyshev collocation points. Inserting the assumed forms of the dependent variables \((\mathbf{u}, p, T)\) in the governing equations, and using weighted residual techniques, discrete equations are generated. The solution for the dependent variables, velocities, pressure, and temperature, are obtained at the collocation points of the mesh. Convergence to the exact solution can be obtained either by increasing the number of elements or by increasing the order of the interpolants.

We used the Stokes version of Nekton to compute the cylindrical gauge problem. The discretizations of steady and unsteady Stokes flows and their solution procedure are given by Ronquist, E.M. and Patera, A.T. (1988). The Stokes version of Nekton is run for steady Stokes cases. In order to find the blockage effect of the gauge and compare the analytical solution of the torque and drag acting on the cylinder with computational results, the code is run for constant
\( \epsilon = \frac{h}{d} \) and different values of \( \frac{H}{d + h} \) for different cases and the velocities and pressure computed in collocation points are used in the approach, explained in the appendix (B), to calculate the stress distribution around the cylinder, and the torque and drag are computed by integrating the shear stress and pressure terms on the cylinder.

4.2. Nondimensionalization of the problem:

We used the Stokes version of NEKTON to solve the steady Stokes equation for the cylindrical gauge. The Stokes equations are properly nondimensionalized for the problem in order to generalize the results.

The Stokes equation is

\[
0 = - \nabla p + \mu \nabla^2 \mathbf{u}
\]

By using the proper nondimensional distance \( x_i^* = \frac{x_i}{d} \), velocities \( u_i^* = \frac{u_i}{U/d} \), and pressure \( p^* = \frac{p}{\mu U} \), the nondimensional form of the Stokes equation becomes

\[
0 = - \nabla^* p^* + \mu \nabla^* u^2
\]

With this nondimensionalization, the nondimensional stress and pressure terms are

\[
p^* = \frac{p}{\mu U}, \quad \tau^* = \frac{\tau}{\mu U}
\]

The torque and drag equations in terms of nondimensional quantities becomes

\[
T = \mu U \left[ \frac{d}{2} \right]^2 \int_{0}^{2\pi} \left[ \tau_{\theta}^* \right]_{r=d/2} d\theta
\]

\[
F = \mu U \left( \frac{d}{2} \right) \int_{0}^{2\pi} \left[ (\tau_{\theta}^*)_r \right]_{r=d/2} \cos\theta + p^* \sin\theta d\theta
\]
Figure (6). Computational domain

Figure (6) shows the boundary conditions for the problem. The inflow and outflow are taken as $u = Uy$, $v = 0$. Wall boundary conditions ($u = v = 0$) are specified on the cylinder and on the lower wall. On the upper wall the boundary conditions are $u = UH$, $v = 0$.

4.3. Computational results:

The computational domain, close to the cylinder, is broken up into circular elements in order to get the collocation points data in the radial direction. By using this approach, we could easily convert the data, given in Cartesian coordinates by the code, to cylindrical coordinates, and compute the shear stress on the cylinder by using the approach, given in appendix (B). The other parts of the domain are broken up into rectangular elements. The size of the elements are decreased while approaching the cylinder in order to increase the accuracy of the results. In the
computational calculation, the ratio of the gap, the distance between the cylinder and the lower wall, to the cylinder diameter, $\varepsilon$, is taken as a constant, 0.384. The value of $\varepsilon$ is chosen higher than the critical value, given by Davis and O’Neill for symmetrical flow separation. The code is run for constant $\varepsilon = 0.384$ and seven different values of $\frac{H}{d + h}$, (2.355, 3.58, 4.754, 4.98, 7.167, 8.94, 12.77), while two different orders of Lagrangian interpolants (5 th and 7 th order) are used. In each case, the shear stress on the cylinder, the torque and the drag are computed with the approach, given in appendix (B).

The velocity field plot for $\frac{H}{d + h} = 3.8$ is given in figure (25). In this figure, the vectors show the direction and magnitude of the velocities in the collocation points.

The spectral element mesh for $\varepsilon = 0.384$ and $\frac{H}{d + h} = 2.355$ is shown in figure (26). The shear stress distribution and pressure distribution on the cylinder are shown in figure (28) and (29), respectively. From these figures, the antisymmetric behavior of the shear stress and pressure can be seen. The maximum shear stress which appears at $\theta = 0^0$ is 9.68 times the wall shear stress, $\tau_w$, and the minimum shear stress is $-2.70\tau_w$ at $\theta = 180^0$. The maximum pressure on the cylinder is 7.886 $\tau_w$.

The spectral element mesh for $\frac{H}{d + h} = 8.94$ and 7 th order Lagrangian interpolants is shown in figure (37). Figures (38) and (39) show the shear stress and pressure distributions on the cylinder. The maximum shear stress is 4.99 $\tau_w$ and the minimum shear stress is $-0.78\tau_w$. The maximum pressure is 3.42 $\tau_w$ at $\theta = 57^0$.

As can be seen from table (1), the maximum wall shear, appears at $\theta = 0^0$, decreases while the distance between the lower and upper walls, $H$, increases. The reason for this is that increasing $H$ decreases the velocity gradient on the cylinder in order to maintain the constant flow rate. The shear stress starts decreasing from $\theta = 0^0$ and becomes zero at $\theta = 90^0$. After $\theta = 90^0$, the shear stress changes its direction, and starts increasing, and approaches another extreme value at $\theta = 180^0$. The reason for this increase is that the geometry between the lower wall and cylinder
from $\theta = 90^0$ through $180^0$ acts as a convergent channel, therefore, the velocity gradient has to increase while $\theta$ is increasing in order to maintain the constant flow rate. Parabolic velocity profiles are obtained in the region between the lower wall and cylinder. Another interesting point which should be mentioned is that the maximum pressure on the cylinder appears at $\theta = 57^0$.

The torque parameter, $\eta_t$, the ratio of the computational torque to analytical torque, is $1.941$ for $\frac{H}{d + h} = 2.355$ and it is $1.044$ for $\frac{H}{d + h} = 12.77$. The computational torque and drag results for each case are given in table (1) and plotted in figure (7). As can be seen from figure (7), the computational torque and drag results approach exponentially to the analytical results while the distance between the upper and lower walls, $H$, is increasing. This result shows the correctness of the computational results since the analytical results are obtained for $H \to \infty$.

We were not able to measure the rotational stiffness of the gauge, therefore, we assumed a constant stiffness for our measurement range of the rotation angle of the gauge

\[ \theta_T = 2 \text{[ deg.]} \]. The calibration curves, obtained in the cone-and-plate experiments, include blockage effects because of small gap heights. An attempt has been made to correct these experimental results. The assumptions of constant stiffness and two dimensional flow field are used in this correction. The torque parameter $\eta_t$, given in table (1), is used to convert the calibration curve, obtained in the cone-and-plate measurements, to the actual calibration curve to be used in the boundary layer measurements where $H$ could be assumed as infinite. Basically, $\frac{\tau_w}{\theta_T}$, obtained in the cone-and-plate experiment, is multiplied by $\eta_t$ to obtain the actual $\frac{\tau_w}{\theta_T}$ for the boundary layer measurements. The converted values of $\frac{\tau_w}{\theta_T} \eta_t$, based on the assumptions of a constant stiffness and two dimensional flow field, should be the same both for the $3^0$ and $6^0$ cone experiments. The converted results of $\frac{\tau_w}{\theta_T} \eta_t$ can be seen in table (2). This attempt works well for the $6^0$ cone results, but it does not work for the $3^0$ cone results. The $3^0$ cone results differ 32% from the $6^0$ cone results. The gap height, $H$, in the $3^0$ cone is smaller than its in the $6^0$ cone,
therefore, the third dimension effect in the flow field becomes more important in the $3^0$ cone. Disturbances due to the gauge cannot travel in the vertical direction because of small gap heights. Therefore, the disturbances have to travel along the gauge, and that makes the flow field three dimensional. We need to do additional experiments for various gap heights in the $3^0$ cone in order to check the consistency of the $3^0$ results and obtain an empirical curve between $\frac{\tau_w}{\theta_T}$ and $\frac{H}{d}$.

These results also show the importance of the third dimension in the flow field, therefore, we will also solve the creeping flow equations for the gauge by using the matched asymptotic expansions.
\[ \eta_t = \frac{\text{Computational Torque}}{\text{Analytical Torque} \ (H \to \infty)} \rightarrow x \]

\[ \eta_d = \frac{\text{Computational Drag}}{\text{Analytical Drag} \ (H \to \infty)} \rightarrow \circ \]

Figure (7). Computational torque and drag
5. CONCLUSION:

1. A new technique has been developed for the measurement of the wall shear stress in a turbulent boundary layer. The measurement principle of this technique is to place a cylindrical body inside the viscous sublayer and measure the torque acting on the body due to the shearing flow of the viscous sublayer.

2. The torque and drag force are linearly related to the wall shear stress.

3. The gauge diameter should be such that the gauge stays below $y^+ = 1$ in the wall shear stress measurements.

4. The gauge 8 mm. long by 0.8 mm. in diameter is calibrated and a linear response is observed.

5. By directivity measurements using this gauge, the maximum wall shear stress direction and its magnitude are obtained. That is a very important feature of the gauge since most of the other techniques do not give the direction of the shear stress.

6. A linear response is obtained up to Reynolds number 3.2 which we could get in our cone-and-plate apparatus. This result allows us to use that Reynolds number in order to determine the size of the gauge for wind tunnel and turbulent oil channel experiments.

7. The computational torque and drag results match with the analytical results.

8. By using the computational torque results, the calibration curves, obtained in the cone-and-plate experiments, are converted to the actual calibration curves for the boundary layer measurements. This approach works well for the $6^\circ$ cone results, but it does not work for the $3^\circ$ cone results because of the third dimension effect in the flow field.

9. Additional experiments are necessary in the $3^\circ$ cone to check the consistency of the results and obtain an empirical curve between $\frac{\tau_w}{\theta_T}$ and $\frac{H}{d}$. 

10. Three dimensional solution of the creeping flow equations for the gauge is necessary.
REFERENCES


APPENDIX (A):

DERIVATION OF MAXIMUM DISTURBANCE VELOCITY:

The solution of the biharmonic equation for a cylinder close to a wall was given given by Davis and O’Neill (1977) in terms of bipolar coordinates. Bipolar coordinates can be obtained by considering the complex function for obtaining the potential for two opposite line sources a distance 2c apart:

\[ W = \ln \left( \frac{c + z}{c - z} \right) = 2 \tanh^{-1} \left( \frac{z}{c} \right) \]  \hspace{1cm} (A.1)

Using \( W = \eta + i \xi \) and \( z = x + iy = c \tanh \left( \frac{W}{2} \right) \), the Cartesian coordinates in terms of bipolar coordinates are obtained as

\[ x = \frac{c \sinh \eta}{\cosh \xi - \cos \eta}, \quad y = \frac{c \sinh \xi}{\cosh \xi - \cos \eta} \]  \hspace{1cm} (A.2)

with \( c = \frac{d}{2} \sinh \alpha \). The plane is given by \( \xi = 0 \) and the cylinder is given by \( \xi = \alpha \). The solution of the biharmonic equation which satisfies the wall boundary conditions on the plane and on the cylinder and free streaming boundary condition at infinity were given by

\[ \psi = \frac{1}{2} y^2 - \chi + M \phi \]  \hspace{1cm} (A.3)

where \( \chi \) and \( \phi \)
\[ \chi = \frac{1}{2} \left( \frac{\xi}{a} \right)^2 (\cosh \xi - \cos \eta)^{-1} \sum_{n=0}^{\infty} \chi_n(\xi) \cos n \eta \] \hfill (A.4)

\[ \phi = (\cosh \xi - \cos \eta)^{-1} \left[ \phi_0(\xi) + \phi_1(\xi) \cos \eta \right] \hfill (A.5) \]

where

\[ \chi_0(\xi) = A_0 \xi \sinh \xi + B_0 \left( \xi \cosh \xi - \sinh \xi \right) \] \hfill (A.6a)

\[ \chi_1(\xi) = A_1 (\cosh 2 \xi - 1) + B_1 (\sinh 2 \xi - 2 \xi) \] \hfill (A.6b)

\[ \chi_n(\xi) = A_n \left[ \cosh (n+1) \xi - \cosh (n-1) \xi \right] + B_n \left[ (n-1) \sinh (n+1) \xi \right. - \left. (n+1) \sinh (n-1) \xi \right] \quad (n \geq 2) \] \hfill (A.6c)

\[ \phi_0(\xi) = a_0 \xi \sinh \xi + b_0 (\xi \cosh \xi - \sinh \xi) \] \hfill (A.6d)

\[ \phi_1(\xi) = a_1 (\cosh 2 \xi - 1) + b_1 (\sinh 2 \xi - 2 \xi) \] \hfill (A.6e)

the coefficients in the above equations are
\[ a_0 = -\frac{\sinh^2 \alpha}{\alpha^2 - \sinh^2 \alpha}, \quad b_0 = \frac{\alpha + \sinh \alpha \cosh \alpha}{\alpha^2 - \sinh^2 \alpha} \]  
(A.7a)

\[ a_1 = -\frac{0.5 \tanh \alpha}{\alpha - \tanh \alpha}, \quad b_1 = \frac{0.5}{\alpha - \tanh \alpha} \]  
(A.7b)

\[ A_0 = \frac{\alpha - \cosh \alpha \sinh \alpha}{\alpha^2 - \sinh^2 \alpha}, \quad B_0 = \frac{\sinh^2 \alpha}{\alpha^2 - \sinh^2 \alpha} \]  
(A.7c)

\[ A_1 = \frac{\alpha e^{-2\alpha} - e^{-\alpha} \sinh \alpha + \sinh^2 \alpha}{\sinh 2\alpha (\alpha - \tanh \alpha)}, \quad B_1 = -\frac{0.5 \tanh \alpha}{\alpha - \tanh \alpha} \]  
(A.7d)

\[ A_n = \frac{n(\alpha - \coth \alpha) \sinh^2 \alpha + e^{-n\alpha} \sinh \alpha}{\sinh^2 n \alpha - n^2 \sinh^2 \alpha}, \quad B_n = \frac{-n \sinh^2 \alpha}{\sinh^2 n \alpha - n^2 \sinh^2 \alpha} \quad (n \geq 2) \]  
(A.7e)

The x component of the velocity is

\[ u = \frac{\partial y}{\partial y} = U (y - \frac{\partial x}{\partial y} + M \frac{\partial \phi}{\partial y}) \]  
(A.8)

where

\[ \frac{\partial x}{\partial y} = \frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial x}{\partial \eta} \frac{\partial \eta}{\partial y} \]  
(A.9a)
Partial derivatives of Cartesian coordinates with respect to bipolar coordinates are

\[
\frac{\partial x}{\partial \xi} = -\frac{c \sin \eta \sinh \xi}{(\cosh \xi - \cos \eta)^2} \tag{A.10a}
\]

\[
\frac{\partial x}{\partial \eta} = \frac{c (\cos \eta \cosh \xi - 1)}{(\cosh \xi - \cos \eta)^2} \tag{A.10b}
\]

\[
\frac{\partial y}{\partial \xi} = \frac{c (-\cosh \xi \cos \eta + 1)}{(\cosh \xi - \cos \eta)^2} \tag{A.10c}
\]

\[
\frac{\partial y}{\partial \eta} = \frac{c (-\sin \eta \sinh \xi)}{(\cosh \eta - \cos \eta)^2} \tag{A.10d}
\]

As can be seen from the above relations,

\[
\frac{\partial y}{\partial \xi} = -\frac{\partial x}{\partial \eta}, \quad \frac{\partial x}{\partial \xi} = \frac{\partial y}{\partial \eta} \tag{A.11}
\]

The Jacobian is

\[
J = \frac{\partial (x, y)}{\partial (\xi, \eta)} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta} = \left( \frac{\partial y}{\partial \eta} \right)^2 + \left( \frac{\partial y}{\partial \xi} \right)^2 \tag{A.12}
\]
By putting the derivatives in to the Jacobian and after some reduction,

\[
J = \frac{c^2}{(\cosh\xi - \cos\eta)^2}
\]  

(A.13)

Partial derivatives of bipolar coordinates with respect to Cartesian coordinates are

\[
\frac{\partial \xi}{\partial x} = J^{-1} \frac{\partial y}{\partial \eta}, \quad \frac{\partial \eta}{\partial x} = -J^{-1} \frac{\partial y}{\partial \xi}
\]  

(A.14a)

\[
\frac{\partial \xi}{\partial y} = -J^{-1} \frac{\partial x}{\partial \eta}, \quad \frac{\partial \eta}{\partial y} = J^{-1} \frac{\partial x}{\partial \xi}
\]  

(A.14b)

By putting equation (A.13) into equation (A.14),

\[
\frac{\partial \eta}{\partial x} = -\frac{\partial \xi}{\partial y} = \frac{1}{c} \left( \cosh\xi \cos\eta - 1 \right)
\]  

(A.15a)

\[
\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} = \frac{1}{c} \left( -\sin\eta \sinh\xi \right)
\]  

(A.15b)

The disturbance velocity, \( u_d \), is the difference between the free stream velocity and the velocity in the presence of the cylindrical body. The maximum disturbance velocity occurs along the symmetry axis because of the presence of only \( x \) component of the velocity. In other words, the maximum displacement of each streamline due to the body occurs on the symmetry axis.
Bipolar coordinate $\eta$ is zero on the $y$ axis.

$$\frac{\partial \chi}{\partial \eta} = 0, \quad \frac{\partial \phi}{\partial \eta} = 0 \quad \text{at} \quad \eta = 0 \quad (A.16)$$

Using equation (A.16) in the velocity equation (A.8), we obtain the velocity on the symmetry axis as

$$u_0 = U \left( y - \frac{\partial \chi}{\partial \zeta} \frac{\partial \zeta}{\partial y} + M \frac{\partial \phi}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \quad \text{at} \quad \eta = 0 \quad (A.17)$$

Maximum disturbance velocity is

$$u_{d \ max.} = u_0 - U y \quad (A.18)$$

In terms of stream function,

$$u_{d \ max.} = U \left( -\frac{\partial \chi}{\partial \zeta} + M \frac{\partial \phi}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} \quad (A.19)$$

where

$$\frac{\partial \chi}{\partial \zeta} = -\frac{1}{2} c^2 \left( \cosh \zeta - 1 \right)^{-2} \sinh \zeta \sum_{n=0}^{\infty} \gamma_n (\xi) + \frac{1}{2} c^2 \left( \cosh \zeta - 1 \right)^{-1} \sum_{n=0}^{\infty} \gamma_n (\xi) \quad (A.20a)$$
\[ \frac{\partial \phi}{\partial \xi} = -(\cosh \xi - 1)^{-2} \sinh \xi \left[ \phi_0(\xi) + \phi_1(\xi) \right] + (\cosh \xi - 1)^{-1} \left[ \phi_0'(\xi) + \phi_1'(\xi) \right] \] (A.20b)

\[ \frac{\partial \xi}{\partial y} = \frac{1}{c} (1 - \cosh \xi) \] (A.20c)

\[ M = \frac{1}{4} d^2 \cosh^2 \alpha \] (A.20d)

\[ \chi_0'(\xi) = A_0 \left( \sinh \xi - \xi \cosh \xi \right) + B_0 \xi \sinh \xi \] (A.20e)

\[ \chi_n' = A_n \left[ (n + 1) \sinh(n + 1)\xi - (n - 1)\sinh(n - 1)\xi \right] \] (A.20f)

\[ + B_n (n^2 - 1) \left[ \cosh(n + 1)\xi - \cosh(n - 1)\xi \right] \quad (n \geq 2) \]

\[ \phi_0'(\xi) = a_0 \left( \sinh \xi + \xi \cosh \xi \right) + b_0 \xi \sinh \xi \] (A.20g)

\[ \phi_1'(\xi) = a_1 \sinh 2\xi + b_1 2 \left( \cosh 2\xi - 1 \right) \] (A.20h)

The other functions \( \chi_0(\xi) \), \( \chi_1(\xi) \), \( \chi_n(\xi) \), \( \phi_0(\xi) \), \( \phi_1(\xi) \) and the coefficients in these functions are given in equations (A.6) through (A.7). The maximum disturbance velocity, \( u_{d \text{ max}} \), is computed and plotted in figure (2).
APPENDIX (B):

COMPUTATION OF TORQUE AND DRAG:

In this section computational approach for torque and drag is given. The velocities, computed with NEKTON in Gaussian collocation points, is converted to cylindrical coordinates and then the closest three interpolation point velocities around the cylinder in radial directions are used in second order polynomials to find their variations in radial directions on the cylinder. Shear stress distribution on the cylinder is obtained by calculating the derivatives of the velocities and the torque and drag are calculated by integrating the shear stress and pressure on the cylinder.

For a Newtonian fluid, the viscous stress vector at the surface of a body must lie on the surface. Therefore, at the surface all the normal viscous stresses vanish. The only nonzero stress, \( \tau_{r\theta} \), on the cylinder is

\[
\left. \tau_{r\theta} \right|_{r=d/2} = \mu \left[ \frac{\partial u_\theta}{\partial r} \right]_{r=d/2} \tag{B.1}
\]

The torque and drag acting on the cylinder are

\[
T = \int_0^{2\pi} \left[ \left. \tau_{r\theta} \right|_{r=d/2} \right] \left[ \frac{d}{2} \right]^2 d\theta \tag{B.2}
\]

\[
F = \int_0^{2\pi} \left[ \left. \tau_{r\theta} \right|_{r=d/2} \cos\theta + \rho \sin\theta \right] \frac{d}{2} d\theta \tag{B.3}
\]
The velocities in cylindrical coordinates in terms of Cartesian coordinates are

\[ u_\theta (i,j) = u(i,j) \cos \theta_i + v(i,j) \sin \theta_i \]  \hspace{1cm} (B.4)

\[ u_r (i,j) = -u(i,j) \sin \theta_i + v(i,j) \cos \theta_i \]  \hspace{1cm} (B.5)

Taking a second order polynomial for the radial velocity, \( u_{\theta_i} \), as

\[ u_{\theta_i} = C_1 r^2 + C_2 r + C_3 \]  \hspace{1cm} (B.6)
and using the values of \( u_\theta (i,0) , u_\theta (i,1) , u_\theta (i,2) \) to find the coefficients, \( C_1 , C_2 , C_3 \), and after some manipulation, we get

\[
\frac{\partial u_{\theta_i}}{\partial r} \bigg|_{r=d/2} = \frac{1}{D_1} \left[ dr_1^2 u_\theta (i,2) - (dr_1 + dr_2)^2 u_\theta (i,1) \right]
\]

(B.7)

where

\[
D_1 = -dr_1 dr_2 (dr_1 + dr_2)
\]

(B.8)

Shear stress on the cylinder in dimensional form is

\[
\tau_{r\theta_i} = \mu \left[ \frac{\partial u_{\theta_i}}{\partial r} \right] \bigg|_{r=d/2}
\]

(B.9)

The shear stress equation (B.9) is evaluated by using equation (B.7) and (B.8).

The torque and drag equations in discrete form are

\[
T = \sum_{n=0}^{N} \left[ \frac{\tau_{r\theta_i} + \tau_{r\theta_{i+1}}}{2} \left[ \frac{d}{2} \right]^2 \left[ \theta_{i+1} - \theta_{i} \right] \right]
\]

(B.10)

\[
F = \sum_{n=0}^{N} \left[ \frac{\tau_{r\theta_i} + \tau_{r\theta_{i+1}}}{2} \cos \left[ \frac{\theta_{i+1} + \theta_{i}}{2} \right] + \frac{p_{i+1} + p_{i}}{2} \sin \left[ \frac{\theta_{i+1} + \theta_{i}}{2} \right] \right] \frac{d}{2} \left[ \theta_{i+1} - \theta_{i} \right]
\]

(B.11)
Figure (9). The cone-and-plate apparatus
Figure (10). The shear stress gauge and experimental set-up
Figure (11). Shear flow over a cylinder near a wall
$\frac{H}{d} = 3.26$

Figure (12). Calibration curve for $\frac{H}{d} = 3.26$, $\beta = 3^0$
Figure (13). Calibration curve for $H/d = 4.91$, $\beta = 3^0$
Rotational Speed, \[ \left( \frac{\tau_w \beta}{\mu} \right) \] [1/sec.]

Figure (14). Calibration curve for \( \frac{H}{d} = 6.58 \), \( \beta = 6^0 \)
Figure (15). Calibration curve for $\frac{H}{d} = 9.92$, $\beta = 6^0$
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Figure (18). Direction sensitivity for $\alpha = 20^0$.
Figure (19). Direction sensitivity for $\alpha = 40^0$

$\text{Laser Displacement, } \left[ 2\theta_T L_s \right] \text{ [nm]}$

$\text{Rotational Speed, } \left[ \frac{\tau_w \beta}{\mu} \right] \text{ [1/sec.]}$

$\alpha = 40^0$
measurement range of constant shear rate

\(-30 \leq U \leq 30 \ [1/\text{sec.}]\)

\[\begin{align*}
\mid : & \ 2\sigma \text{ range} \\
\times : & \ \text{mean value}
\end{align*}\]

\[
\frac{\tau_w|_\alpha}{(\tau_w|_{\alpha=0})_{\text{mean}}} = \frac{\text{wall shear stress at } \alpha}{\text{mean wall shear stress at } \alpha=0^0}
\]

Figure (20). Directivity curve
Figure (21). Calibration curve for $Re_{max} = 0.13$.
Figure (22). Calibration curve for $\text{Re}_{\text{max}} = 0.47$

Rotational Speed, $\left[ \frac{\tau_w \beta}{\mu} \right] \ [1 / \text{sec.} ]$

Laser Displacement, $[\theta_r, L_s] \ [\text{mm.}]$
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Figure (27). Spectral element mesh for $\frac{H}{d + h} = 4.754$
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Figure (35). Pressure distribution on the cylinder for $\frac{H}{d+h} = 7.167$
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Figure (37). Shear stress distribution on the cylinder for \( \frac{H}{d + h} = 8.94 \)
Figure (38). Pressure distribution on the cylinder for $\frac{H}{d + h} = 8.94$
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Figure (40). Shear stress distribution on the cylinder for $\frac{H}{d + h} = 12.77$
Figure (41). Pressure distribution on the cylinder for $\frac{H}{d + h} = 12.77$
### TABLE (1). Computational Results

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<tr>
<th>( \frac{H}{d + h} )</th>
<th>2.355</th>
<th>3.580</th>
<th>4.754</th>
<th>4.980</th>
<th>7.167</th>
<th>8.940</th>
<th>12.770</th>
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<tbody>
<tr>
<td>( \eta_l )</td>
<td>1.941</td>
<td>1.505</td>
<td>1.259</td>
<td>1.238</td>
<td>1.143</td>
<td>1.095</td>
<td>1.044</td>
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<tr>
<td>( \eta_d )</td>
<td>2.651</td>
<td>1.673</td>
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<td>1.293</td>
<td>1.162</td>
<td>1.106</td>
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<tr>
<td>( \frac{\tau_{\text{max.}}}{\tau_w} )</td>
<td>9.684</td>
<td>6.686</td>
<td>5.758</td>
<td>5.602</td>
<td>5.197</td>
<td>4.985</td>
<td>4.761</td>
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<tr>
<td>( \frac{\tau_{\text{min.}}}{\tau_w} )</td>
<td>-2.698</td>
<td>-1.291</td>
<td>-1.041</td>
<td>-0.730</td>
<td>-0.847</td>
<td>-0.778</td>
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<td>( \frac{P_{\text{max.}}}{\tau_w} )</td>
<td>7.886</td>
<td>5.029</td>
<td>4.022</td>
<td>3.880</td>
<td>3.577</td>
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### TABLE (2). Experimental Results

<table>
<thead>
<tr>
<th>( \frac{H}{d} )</th>
<th>3.26</th>
<th>4.91</th>
<th>6.58</th>
<th>9.92</th>
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<tbody>
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<td>( \frac{H}{d + h} )</td>
<td>2.355</td>
<td>3.548</td>
<td>4.754</td>
<td>7.167</td>
</tr>
<tr>
<td>( \beta \ [\text{deg.}] )</td>
<td>3</td>
<td>3</td>
<td>6</td>
<td>6</td>
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<td>( \frac{\tau_w}{\theta_T} )</td>
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<td>27.8</td>
<td>52.78</td>
<td>58.01</td>
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<tr>
<td>( \eta_l )</td>
<td>1.941</td>
<td>1.60</td>
<td>1.259</td>
<td>1.143</td>
</tr>
<tr>
<td>( \frac{\tau_w}{\theta_T} \eta_l )</td>
<td>52.02</td>
<td>44.08</td>
<td>66.45</td>
<td>66.31</td>
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