PROPAGATORS OF ATMOSPHERIC MOTIONS

by

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ABSTRACT

The propagation of linear wave motions in inviscid, stratified, ideal gas atmospheres is described by obtaining the relevant propagators (or Green's functions for the initial value problem). The transient acoustic oscillations, buoyancy oscillations, and gravity waves for an unbounded non-rotating atmosphere are derived.

Introduction of the hydrostatic assumption is found to eliminate the acoustic and buoyancy oscillations and modify the gravity wave. Time independent potential vorticity motions result for an atmosphere in constant rotation, but these also lose their energy by radiation when the influence of the earth's variable vorticity is taken into account by the " β " approximation. A "filtering" method of synthesizing propagation equations for elementary propagators from their contour integral representations is given.

The excitation of the Lamb boundary wave from a point heat source is analyzed. Rossby wave motions and gravity wave motions for an unbounded planar atmosphere excited by several different kinds of switch-on forcing are obtained. The quantitative details are obtained by steepest descent integrations, but the "group velocity" concept is adequate for a qualitative description of the resulting motions.

A theoretical analysis of forced hydrostatic atmospheric wave motions on a rotating sphere is given. A conservation of energy equation is obtained, several related spectral theorems are established, and the integration of the equation for forced tidal motions on a sphere by expansion in Legendre polynomials is discussed.

An example of the motion of a convectively unstable atmosphere is given to illustrate instabilities that grow asymptotically as exp(ct), $\exp(ct^{1/2})$, and $\exp(ct^{1/3})$, Re $c > 0$.

Thesis supervisor: Victor P. Starr
Title: Professor of Meteorology

"if you will have a tree bear more fruit than it used to do, it is not anything you can do to the boughs, but it is the stirring of the earth and the putting of new mould about the roots that must work it."

quoted from F. Bacon by M. Stone

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I. INTRODUCTION

A. On Atmospheric Wave Propagation

A fundamental goal in the study of the dynamics of continuous media is to describe the evolution of initial data with time, given various possible forms of externally imposed forcing. This study is concerned with the small amplitude motions of ideal gas atmospheres in simple planar and spherical geometries. We shall obtain Green's functions for the small amplitude motions of such atmospheres. These functions, which we designate propagators, following the usage in the physical literature, provide a dynamical description of motion over a time period sufficiently brief that the nonlinear effects can be approximated as time invariant spatial functions. The study of propagators leads to a systematic classification of the various types of atmospheric motions and provides guidance in the more detailed analysis of specific dynamical phenomena in planetary and stellar atmospheres.

When a stable continuous medium is displaced, it experiences an acceleration in the direction of its original equilibrium, waves are excited which then transmit energy throughout the medium. The equations governing such macroscopic phenomena are usually nonlinear. Because nonlinear motions are in general quite difficult to analyze directly in complicated physical systems such as atmospheres, it is convenient to consider nonlinear effects as one type of forcing function which produces a response in a linearized system of equations. This permits a simpler analysis of the coupling between various scales of motion. The concept has been used by Lighthill (1952) and Unno-Kato (1962) in discussion of the generation of acoustic motions by turbulence; by various authors including Kibel (1955), and Dobrischman (1964) in analysing the approach to geostrophic balance of large scale atmospheric motions, and by Saltzman (1965) in the discussion of forced mean planetary scale motions, as well as by many other authors in other contexts. We shall likewise take this viewpoint so that the atmospheric wave phenomena considered will be treated as linear phenomena with nonlinear terms in the equations of motion considered to be one kind of forcing function. In mathematical terms, we are concerned with the reduction of nonlinear differential equations to nonlinear integral equations. This is frequently the first task that must be done in an abstract mathematical study of a system of nonlinear differential equations.

Observational studies of wave phenomena in the terrestrial and the solar atmosphere have led to an increased understanding of the various manifestations of transient motions that occur in compressible ideal gas atmospheres. Unfortunately, little observational information on the motions that occur in other planetary atmospheres is yet available. Such studies would assist astronomers and meteorologists in distinguishing between the accidental and the essential features of the individual systems. This is achieved to a certain extent by the studies of laboratory generated fluid motions, but usually, different boundary conditions lead to very different mathematical problems in the analysis of such motions and make direct comparison somewhat difficult.

Many of the observed atmospheric wave phenomena are excited by energy inputs that are localized in time and space. Energy sources are frequently a part of the nonlinear internal dynamics of the system. Recent astronomical studies have shown the presence of motions in the solar atmosphere ranging from those associated with the small granulations and having periods of a few minutes, up to large scale motions with periods of many solar days and with spatial extent not much smaller than the radius of the sun itself. Many physical effects including radiative transfer, variable gas constants, and magnetic effects must be incorporated into a complete dynamical description of stellar motions. (These are not directly considered in this study). The smaller scale motions of the solar atmosphere are excited by a zone of convection. The large scale surface motions are thought to occur as a result of the baroclinic release of available potential energy associated with a north-south horizontal solar temperature gradient. Whatever the exact mechanism by which these large scale motions are generated, it is clear that their subsequent propagation will be intimately related to the solar rotation and will be dependent on the confining effect of the approximately spherical geometry of the sun. One should also mention the extensive observational and theoretical study of variable stars, which are found to oscillate radially with periods of a few hours to several months.

The smallest scale motions in the earth's atmosphere are the acoustic waves, and the detailed study of such motions is in itself an extensive discipline of the applied sciences. The higher frequency components of these motions are rapidly damped by viscosity and so it is the "low frequency" acoustic waves which are observed geophysically, such as those excited by auroral disturbances and detected at the ground. Acoustic waves are frequently observed in the form of shocks or pulses. This is accounted for theoretically by the lack of dispersion of energy in different wavelengths in the linear theory plus the tendency of the nonlinear effects to intensify existing pressure differences. At sufficiently low "frequencies" the motion is highly modified by gravity and resulting motion is known as an acoustic-gravity wave.

Acoustic-gravity waves may be excited in all highly turbulent regions such as thunderstorms, boundary layer turbulence, clear air turbulence arising as a result of shearing instability of the jet stream, air flow over irregular topography, or irregular surface heating. Study of the elementary propagators of these motions permits conclusions to be drawn on the common aspects of all such motions and simplifies the detailed analysis of each individual problem.

When the direction of propagation of a wave motion in a stratified atmosphere is nearly horizontal so that the sine and the tangent of the angle of propagation, measured from a surface of constant gravity, are approximately equal, one is permitted a convenient theoretical approximation known as the long wave, or hydrostatic, approximation. Oscillatory motions in this case have frequencies small compared to the parcel frequency. Figure 1-1, taken from Mahoney (1966), depicts the vertical structure of the horizontal velocity of a gravity wave imposed on a current of much longer temporal duration. Wind data for such studies is obtained from the radar tracking of falling spheres. Complicated surface phenomena such as squall lines and rapidly moving cold fronts are related to ideal wave phenomena in this approximation, both as sources of wave energy and as manifestations of essentially hydrostatic degrees of freedom of the atmosphere.

Many atmospheric motion phenomena are of sufficiently great horizontal extent that the sphericity of the earth becomes important in analysis. For example, it has been found observationally and theoretically that essentially instantaneous local introduction of thermal energy of order of magnitude of 10^{24} ergs or greater, produces an acoustic gravity wave, which propagates radially along the ground and travels completely around the earth several times. The first realization of such phenomena which was subjected to scientific investigation was the explosion of the Krakatoa volcano in 1883. See the discussion of Taylor (1929/30). Recent man-made "aeroclysms" have provided a repeat performance, but those parties directly responsible have agreed to discontinue such studies because of the resulting adverse effects on the health of the planetary inhabitants. Figure 1-2, which is reproduced from Wexler and Hass (1962), gives the pressure trace from atmospheric pulses originating in Siberia in 1961, and Whipple's composite of the great Siberian meteor of 1908.

When the time scale of motions approaches and exceeds the terrestrial day, one finds observationally and theoretically a wealth of new motion phenomena, not directly present in nonrotating systems. It is found that all atmospheric gravity waves which are excited have frequencies greater than some average Coriolis frequency. Thus transient gravity waves will have a time period less than a day, except possibly in the immediate vicinity of the equator, where zonal oscillations of periods up to several weeks may occur. All atmospheric motions outside the tropics with time periods of several days or more can be considered a vorticity mode motion. The conservative quantity associated with this mode is sometimes called potential vorticity. Such motions are distinguished by an approximate state of balance between the pressure field and the wind field, which is known as geostrophy. These motions are a prime concern of dynamic meteorologists, since it is this mode of motion which releases the available potential energy and gives rise to cyclonic storms.

As a result of the variation of the earth's vorticity, potential vorticity motions in a resting atmosphere propagate as waves. When these waves are analyzed into spherical harmonic components, the phase propagates to the west. The phase speed is very slow for the smaller scale potential vorticity waves, so that the distorting effects of horizontal and vertical wind shears are extremely important in the prediction of the actual evolution of such motions, and it is necessary in practice to perform the requisite computations by the use of high speed computers. One may distinguish between the smaller scale potential vorticity waves and the transient and steady "planetary scale motions", where the dynamics should be formulated on a spherical earth. Figure 1-3, taken from Teweles (1963), shows a stationary harmonic planetary wave as revealed by observation.

Finally, we mention the study of atmospheric "tidal" oscillations, a branch of atmospheric dynamics with perhaps the most interesting, and certainly the oldest, history of theoretical and observational study. These are hydrostatic gravity-acoustic waves, occuring on a rotating sphere. The prevailing opinion over the last century as to the form of the forcing of these waves has wavered between thermal and solar gravitational sources. The much greater magnitude of thermal forcing has recently been firmly established, but the exact form of the forcing is not yet certain. Latest computations of Butler and Small (1963) favor heating in the atmospheric ozone layer. Figure 1-4, taken from a study of Avery and Haurwitz (1964) shows the observed surface amplitude of the semidiurnal tide over the United States.

B. Historical Notes

The first systematic study of the dynamics of an atmosphere on a rotating sphere was that of Laplace. He assumed that atmospheric dynamics could be reduced to the dynamics of a homogenous ocean with a free surface. The first initial value problem in geophysics to be studied, was that for a point disturbance exciting a water wave. The methods of solution, as given by Cauchy and by Poisson, were rather complicated and the treatment of other wave motions in a similar fashion was thus discouraged. Most other nineteenth century wave studies considered time periodic motion. There the motions were mathematically simpler and at the same time experimentally accessible.

Within this framework the basic foundations for the analysis of waves in stratified geophysical media were laid by nineteenth century physicists, especially Green and Stokes. Hough (1898) considerably advanced the theory of the dynamics of homogenous oceans with free surfaces. Various meteorologists generalized Laplace's model of the atmosphere as a homogenous incompressible fluid in order to consider the release of potential energy in the theory of cyclones. Lamb (1908), (1910) first gave a formulation of the motions of stratified atmosphere in a constant gravitational field. Eddington (1919) and later astrophysicists formulated generalizations appropriate for study of self-gravitating gaseous spheres. See the Handbuch der Physik article of Ledoux and Walraven for detailed review of this subject. Taylor (1936) and Pekeris (1937) formulated the Lamb theory of atmospheric dynamics for a spherical rotating earth with the aid of the Laplace-Hough tidal theory. Rossby (1939) gave simple approximate dynamic models for Laplace's oscillations of the second class, and greatly stimulated the development of dynamic meteorology by realizing the applicability of these models to weather forecasting. The question of the approach of an initial line disturbance on a geostrophic ocean to geostrophic equilibrium was raised by Rossby (1938), studied by Cahn (1945), and by Bolin (1955) for the motions of a stratified incompressible fluid between two boundaries. Obukov (1949) gave a mathematical analysis of the initial value problem for a localized disturbance on a homogenous ocean, and Kibel (1955) generalized the analysis of Obukov to a stratified fluid. Monin (1958) simplified and extended this analysis, while Veronis (1958) gave results for the initial value problem with the earth's variable vorticity considered in the β- plane approximation. The propagation of a pulse in a non-rotating nonhydrostatic atmosphere was first studied by Pekeris (1948) and has been considered by many later writers, the latest being Van Hulsteyn (1965).

Eliassen (1949) noticed that the use of pressure as a new independent variable in place of geometrical height led to useful simplification in the formulation of the equations of atmospheric motion in the hydrostatic approximation. Since the pressure is proportional to mass, the effects of compressibility are eliminated except in the boundary conditions. A similar but more limited formulation has been achieved by Weekes-Wilkes (1947) in the theory of atmospheric tides by using standard mathematical substitutions.

Theories have been developed by Dorodnitsyn, Lyra, Queney, and later writers, for the study of the motion forced by air flows over irregular topography or heat sources. Flows over individual hills and ridges are essentially nonhydrostatic phenomena, unaffected by the earth's rotation, while for sufficiently large scale motions, the hydrostatic approximation and sometimes also the geostrophic approximation are made to facilitate analysis. On the other hand, the earth's rotation can no longer be neglected for these forced long waves. See Corby (1954) and Krishnamurti (1964), for reviews and further reference to the existing theory and observations for small scale topographic waves, and see Rao (1965) for a brief review of geostrophic planetary waves forced by topography and heating.

The importance of describing atmospheric dynamics as the evolution of given initial conditions has been emphasized by Case (1962) and Pedlosky (1964).

C. The Purpose of This Study

In this thesis we shall examine various linear models of atmospheric motion, formally considering omitted homogenous terms in the dynamic equations to be included as part of the externally imposed forcing. There are some observed atmospheric motions where the omitted terms will be of smaller magnitude than the retained linear terms and our results can be directly applied to the description of these phenomena. When this is not the case, our analytic results will not be quantitatively correct, but will nevertheless be of physical interest.

In theoretical study of individual atmospheric phenomena, it is frequently desirable to discard many dynamic terms in order to isolate those aspects that are of greatest importance in quantitatively determining the observed motion. In order rationally to determine approximations to be applied, it is necessary to understand what is being omitted. Some omissions can be expected to result only in numerical errors whose magnitude may be evaluated by estimating the magnitude of the terms so omitted. Other omissions change the basic physics described by the equations. Such omissions can be completely understood only by consideration of the relationship of solutions of the approximated equations to solutions of the more correct equations. A fundamental physical law of macrophysics is the principle of causality which states that input into the atmosphere at a given time can only produce a response at some later time. It is highly undesirable to formulate dynamical models in which this condition is violated, nor does it make sense to allow inputs to enter a system from regions exterior to the system under consideration, when such external sources are not explictly specified. The flow of information in a physical system which is governed by partial differential equations is intimately related to time differentiated terms in the equations and to specified boundary conditions. We may wish to obtain some kind of approximate solution to a well posed system by obtaining solutions to a more approximate system of equations. These approximate equations may not by themselves describe unambiguously some approximate dynamics and it is then necessary to refer back to the well posed dynamical description in order to determine what additional conditions should be used to insure a unique approximation to the well posed system.

The primary objective of this thesis is to provide an analytic description of the role of time differentiated components in determining solutions for the equations of atmospheric dynamics. These results may be used for guidance in the rational selection of approximate equations for the detailed study of individual motion phenomena.

Because the principle of superposition applies when a linear model is used, it is possible to arrive at a satisfactory understanding of all possible motions by consideration of a few selected model problems. The evolution of given initial conditions can be described by the elementary solutions to simple impulses. To describe the excitation of forced motions, it is convenient to study "switch on" problems, where the source assumed is suddenly switched on at $t = 0$. By superposition the same resulting forced motion must occur when a steady source slightly changes its amplitude. Hence, regardless of initial conditions, the final motion resulting from a steady source must be equivalent to that predicted from switch-on initial forcing. This analysis then gives a satisfactory derivation of forced wave motions, when the assumption of steady forcing results in a model equation without time differentiated terms to indicate the proper direction of energy flow.

D. Outline of Thesis Content

In chapter II we present a graphical description of the important wave propagation parameters for the first one hundred kilometers of the earth's atmosphere. This material is based on numerical data provided to the author by Dr. R. E. Newell.

In chapter III, is provided a formulation for the dynamics of stratified atmospheres. Chapter IV is devoted to a detailed analysis of the elementary propagators of an isothermal nonrotating atmosphere in the Boussinesq approximation. Many of the mathematical techniques to be used throughout the remainder of the thesis are introduced in this section. The elementary propagators studied are the "gravity wave" and the "buoyancy oscillation" propagator. These two propagators coalesce at points of observation vertically above the source.

The general problem of the propagation of an impulsive disturbance in a nonrotating isothermal atmosphere is first discussed in Chapter V. The initial disturbance propagates spherically with the speed of sound. The "wave tail" consists of an acoustic oscillation, a buoyancy oscillation, and a propagating gravity wave. Next we consider the propagation of a pulse in a hydrostatic stratified rotating atmosphere. A gravity wave propagates outward behind a cylindrical front, leaving behind a residual potential vorticity mode motion. The introduction of the " β -effect" via a simple model leads to several new modes, the most interesting of which is an unstable inertial oscillation. The energy in the potential vorticity mode now propagates outward as Rossby waves. A brief discussion is given of the "filtering" process for synthesizing equations where solutions will be an elementary propagator, given the contour integral representation of the propagator.

In chapter VI, we analyze some simple examples of Lamb waves propagating from point sources. The Lamb waves themselves are nondispersive (in an isothermal atmosphere) but these waves are found to excite concomitantly, buoyancy oscillations in a nonhydrostatic nonrotating atmosphere, and inertial oscillations in a rotating hydrostatic atmosphere.

In chapter VII, we introduce some Fourier integral techniques which will be used in the following two chapters, and it is shown that one may approximate details of wave sources, or internal wave dissipation by the use of a multiple stationary phase computation.

Chapter VIII is devoted to several examples of the excitation of Rossby waves in a stratified atmosphere, and chapter IX provides several examples of internal gravity wave excitation.

In X is given the fundamentals of normal modes expansions of the hydrostatic atmospheric wave equations on a spherical earth.

This subject, initiated by Laplace, is known as tidal theory.

The material of the previous chapters is summarized in XI and suggestions for further development of the theory of atmospheric wave propagation phenomena is given. An example of atmospheric instability is given.

The reader will find a somewhat more detailed statement as to the content at the beginning of each chapter.

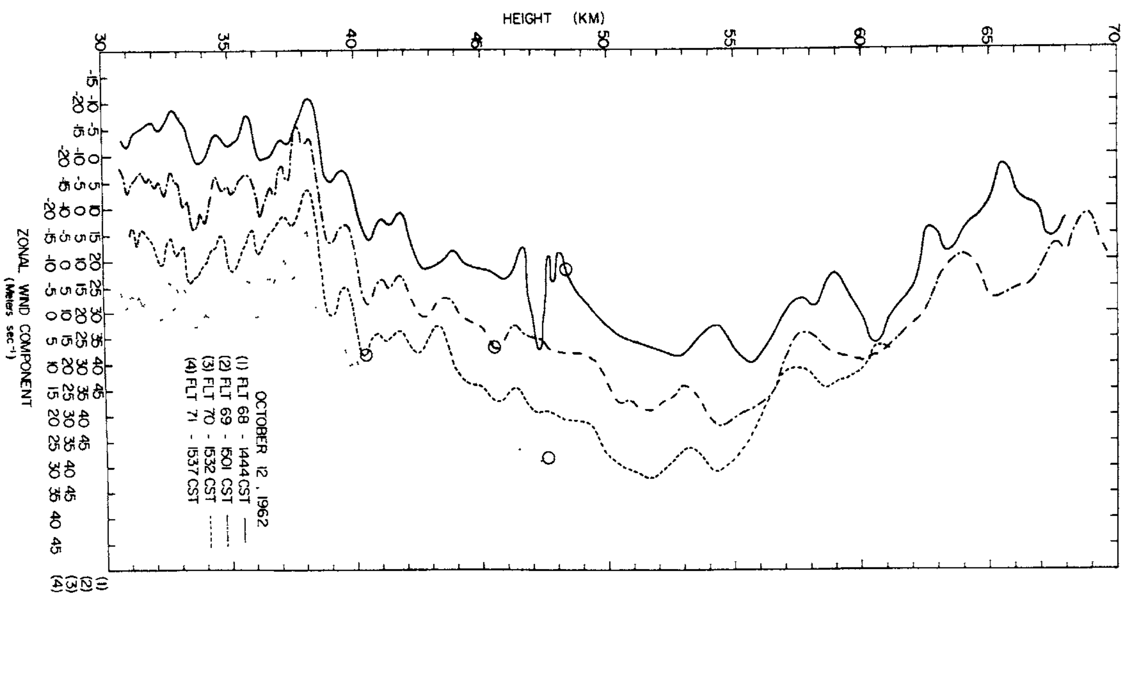


Figure 1-1. Vertical propagating gravity waves. Taken from Mahoney (1966).

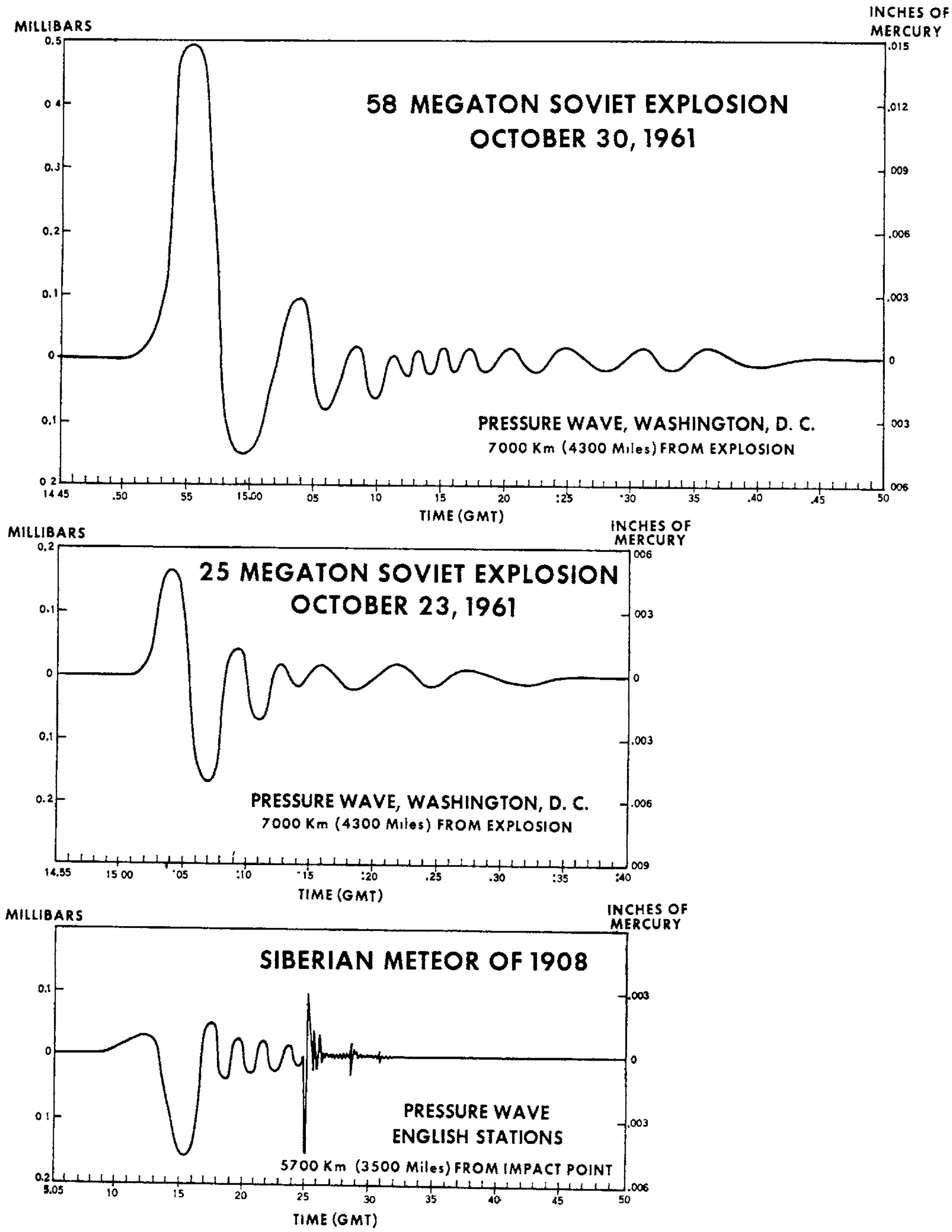


Figure 1-2. Pressure waves excited by explosions. Taken from Wexler and Hass (1962).

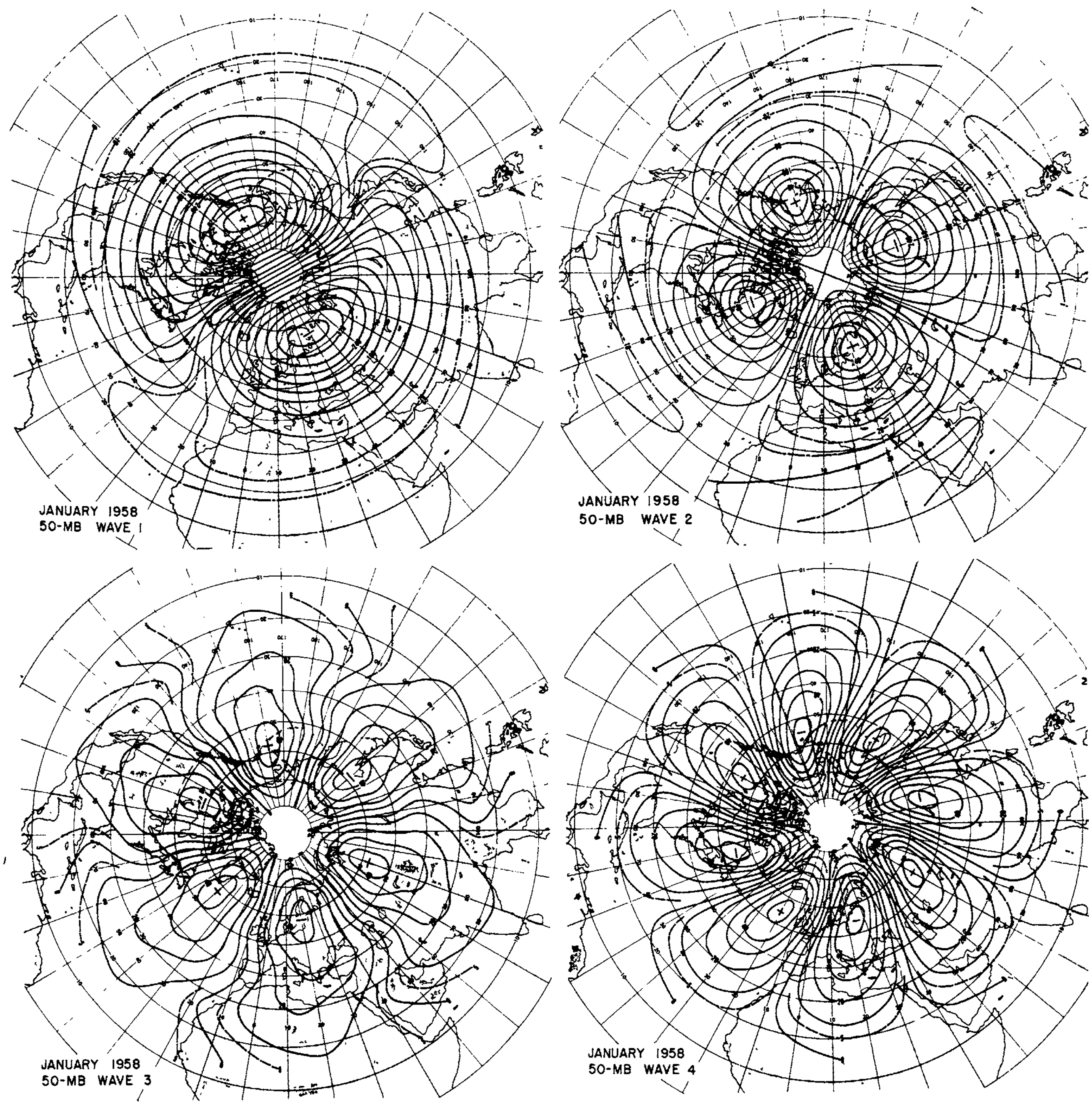


Figure 1-3. "Standing Planetary Waves", (geopotential height). Taken from Teweles (1963).

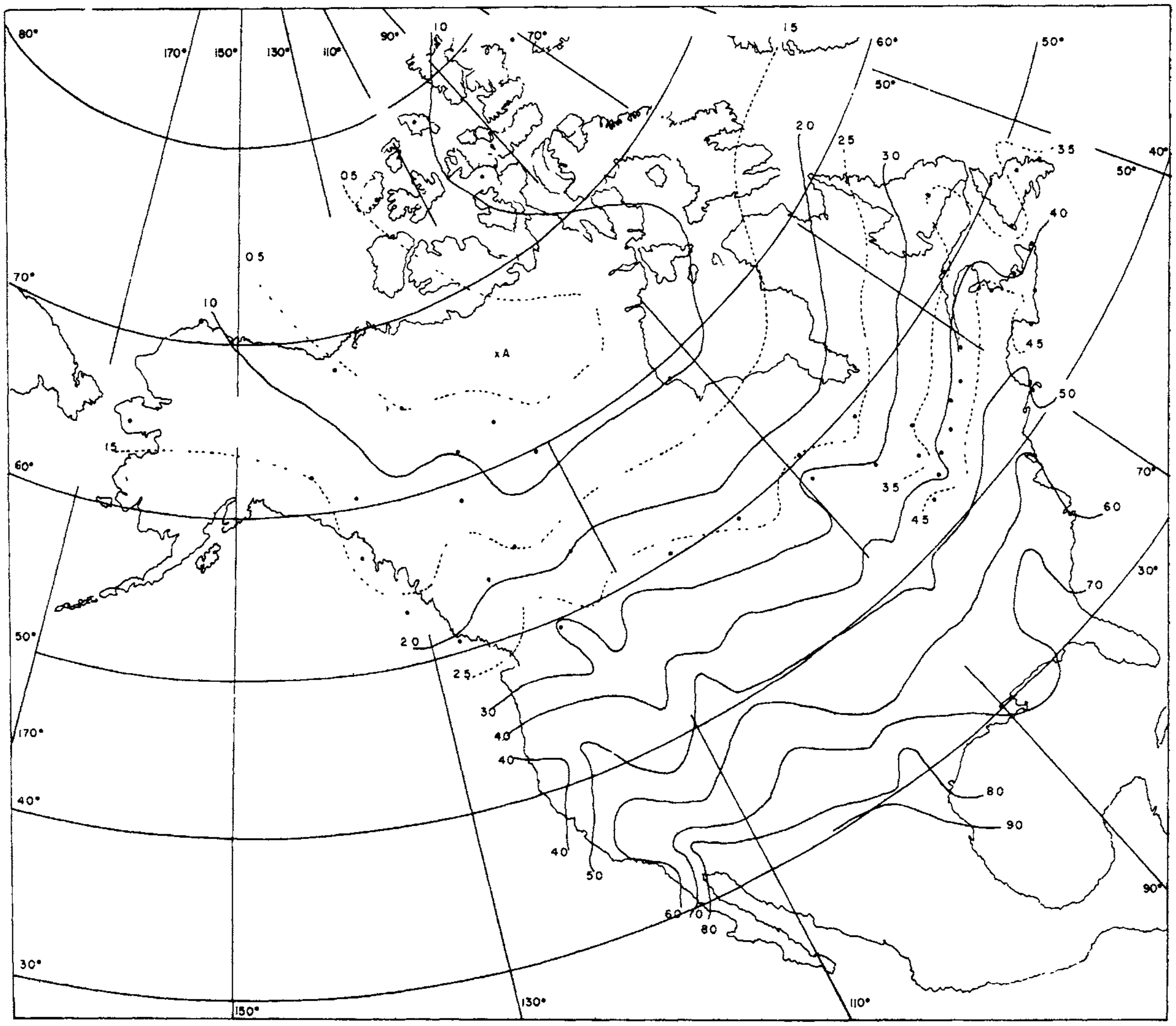


Figure 1-4. Amplitude of the "S_2" semidiurnal pressure wave over North America (units are 10^{-1} mb). Taken from Avery and Haurwitz (1964).

II. CLIMATOLOGY OF WAVE PROPAGATION PARAMETERS

This chapter is intended to serve as an introduction to some of the more important parameters which occur in the theory of atmospheric wave propagation. To facilitate the discussion, we have prepared cross sections and graphs of some of these parameters as they occur in the earth's atmosphere. Data for the first hundred kilometers was provided to the author by Dr. R. E. Newell.

This graphical data is primarily intended for qualitative use, and is of less than the highest accuracy achievable by use of present climatological data sources. This is especially true in the troposphere and lower stratosphere where many years of global radiosonde data permit a much more sophisticated description of atmospheric statistics than is possible by using only a few cross sections.

The atmosphere between 30 and 100 kilometers is studied primarily by meteorological rockets, falling spheres, and meteor winds. Dr. Newell has processed much of the available raw data for this region in order to define latitudinal averages of the winds and temperatures as well as the variances and covariances of these parameters.

The mean temperature data, averaged over the six summer months, and the six winter months, has been used to compute quantities which are important in the theory of atmospheric wave motions. These quantities include the "scale height", the "buoyancy frequency", and the "planetary stability". The scale heights, buoyancy frequency, and planetary stability, computed independently from the numerical temperature data, are given as meridional cross sections, and as hemispheric average profiles computed from these cross sections. Small discrepencies between the various cross sections presented may be ascribed to random computational error inherent in finite difference computations, and in the subjective analysis of the cross sections. All the larger scale features are in general agreement. The results are summarized by means of hemispheric average vertical profiles, in Figs. 2-3, 2-6 and 2-9. We use the notation $(\overline{\quad})$, to denote hemispheric average of a quantity.

The scale height (RT/g) shown in Figs. 2-1, 2-2 and 2-3, is merely the temperature, measured in more convenient units. The most important features are the rapid decrease from 0 to 15 km; the minimum at 15 to 20 km, especially over the equator; the increase from 15 to 50 km, with a maximum found over the summer pole at 50 km; the decrease to 90 km, with minimum at 90-95 km, especially strong over the summer pole. Beyond this level, the scale height increases monotonically to very large values in the thermosphere.

The speed of sound $c = (\gamma g H)^{1/2}$ is important as the speed of signal transmission in the atmosphere. Its inverse, c^{-1}, may be considered proportional to an index of refraction which determines the path taken of acoustic pulses. (Acoustic pulses obey the laws of geometrical optics). Acoustic motions are refracted downward in regions of increasing c (20-50 km and above 95 km). It is likely that viscous dissipation strongly attenuates acoustic motions above 100 km. In this region, c monotonically increases to very large values.

The buoyancy frequency squared, $N^2 = g/H \left(\kappa + \frac{\partial H}{\partial z}\right)$ is given in Figs. 2-4, 2-5 and 2-6. Low values are found in the troposphere, a sharp increase to high values in the stratosphere, a gradual decrease to a minimum at 60 km, increase to high N^2 in the 90-120 km region, and a monotonic decrease above this region. The sharp increase of N^2 from the troposphere to stratosphere, may be considered the tropopause, considerably smoothed out as a result of taking time and latitudinal averages. At a given time, and a given location, the increase of N^2, is yet much sharper, and may be considered a discontinuity in N^2, on the scale which our cross sections are drawn.

The planetary stability, S, is defined as

$$S = N^2 H^2 / 4\Omega_e^2 r_e^2$$

and shown in Figs. 2-7, 2-8 and 2-9. The large scale features of S correspond to those on the N^2 and H cross sections. Small discrepencies may be considered random errors. It is not possible to obtain exactly the hemispheric average of S, from the product of N^2 and H^2 due to latitudinal variations in S, N and H. That is:

$$\bar{S} \neq \overline{N^2}\,\overline{H^2}/4\Omega_e^2 r_e^2$$

Furthermore,

$$\bar{S} \neq \frac{g}{\bar{T}}\left(\frac{g}{C_p} + \frac{d\bar{T}}{dz}\right)\bar{H}^2$$

The most important qualitative features of S is the large maximum at 40 km and the broad minimum in the 60 to 90 km region.

The stability parameter S (or other quantities proportional to it) is important for the theory of hydrostatic atmospheric wave propagation, including atmospheric tides. That is, it often happens that discussion of the vertical dependence of the motions may be separated from the horizontal dependence by separation of variables, with the result that it becomes necessary to solve a Sturm-Liouville equation of the form:

$$\left[\frac{\partial^2}{\partial z^2} - q(z) + \lambda\, S(z)\right]\psi = -f(z)$$

where the coordinate z is the log of the ambient pressure, $q(z) \simeq 1/4$, ψ is a dependent variable describing the motion, $f(z)$ is some external forcing and λ is a separation of variables constant.

The quantity $\lambda S(z) - g(z)$ then determines the oscillation of vertically propagating hydrostatic wave motions. Referring to Fig. 2-9 , we may think the region from 10 to 60 km as a "resonant cavity" for wave energy, with the ground and the region from 60-90 km acting as "barriers". Since S increases monotonically from 90 km, the upper barrier will always be leaky, and permit some upward wave propagation into the ionosphere. These considerations are modified in the presence of wind systems, and especially when applied to discussions of very low frequency wave propagation. See Charney and Drazin (1961).

In Fig. 2-10, is given the time for viscous decay by the factor e^{-1} of a harmonic wave of wavelength λ . This gives an order of magnitude estimate of the decay time of a motion where the shortest "distance scale" is λ . For most atmospheric motions, this "shortest scale" will be a characteristic vertical dimension of the motion. A simple derivation of the formula used may be obtained by applying the one dimensional diffusion operator $\left(\frac{\partial}{\partial t} - \nu(z) \frac{\partial^2}{\partial z^2} \right)$ to a wave of the form $f(t)\, e^{2\pi i z/\lambda}$, obtaining the solution

$$f \sim e^{-\nu(z)(4\pi^2/\lambda^2)t}$$

(Here $\nu(z)$ is kinematic viscosity). Viscous decay of more general motions occurs due to similar diffusion operators which are in the equations of motion.

Because of the much greater variability of the wind fields compared to the temperature, it is more difficult to describe simply the wind structure which is important for wave propagation problems. "Mean winds" vary extensively according to geographical location and the time over which the wind is averaged. For example an annually averaged wind might be an exceedingly poor approximation to the mean wind "seen" by a transient gravity wave with lifetime less than an hour. (Some approximation to the wind prevailing during the lifetime of the gravity wave is what would be required). Rather than give a detailed discussion of many possible atmospheric flows which will significantly affect results from wave propagation, we have limited our presentation to a simple model profile, Fig. 2-11, which is characteristic of the winds observed in middle latitudes in winter. The profile is characterized by a wind maximum at 15 km, the jet of the upper troposphere, a minimum at 30 km in the upper stratosphere, and the mesospheric jet, peaking at 60 km. These jets may be observed to be as much as double the amplitude shown, in given synoptic situations. The tropospheric jet contributes to the trapping of the energy of small scale orographic waves, while the mesospheric jet provides an important trapping mechanism for low frequency planetary scale wave motions.

Fig. 2-12 gives the average derivative of the scale height. Many of the less important dynamic terms due to variable stratification can be represented in terms of this parameter.

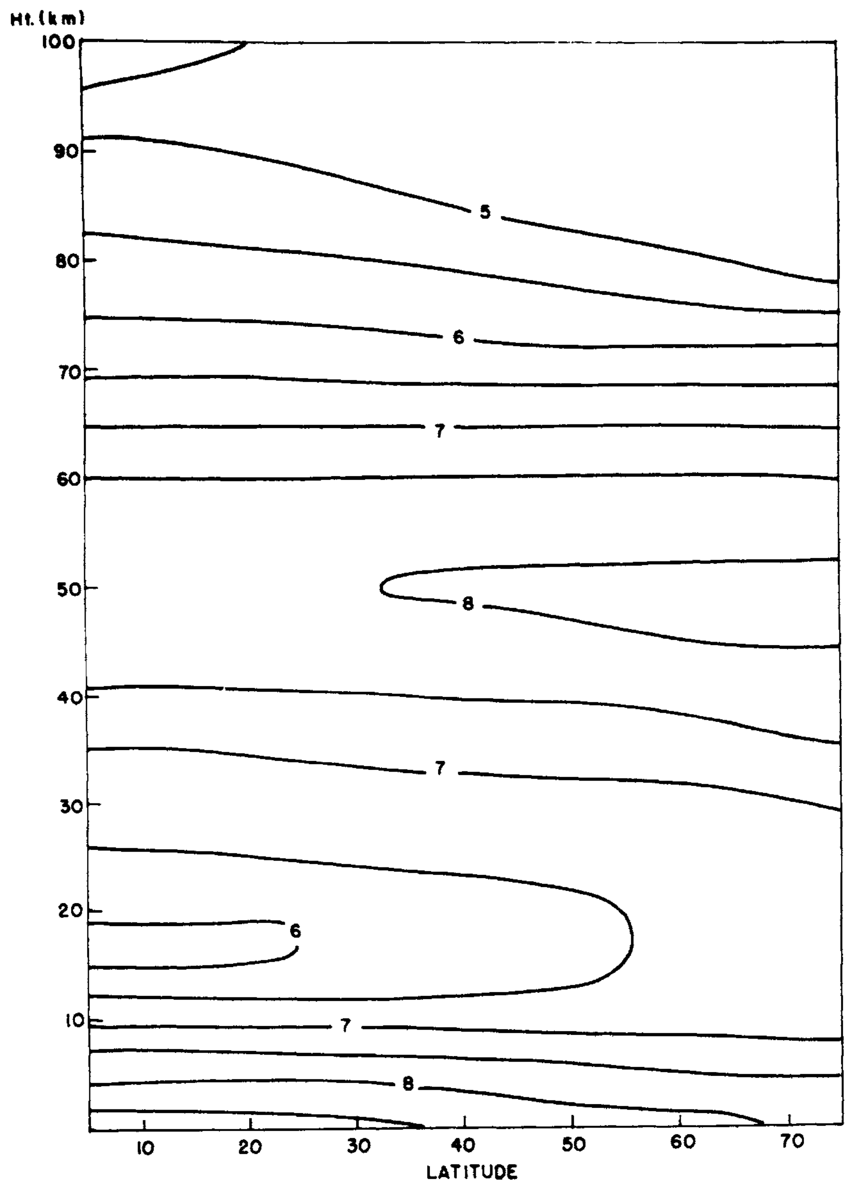


Figure 2-1. "Summer" atmospheric scale height in km.

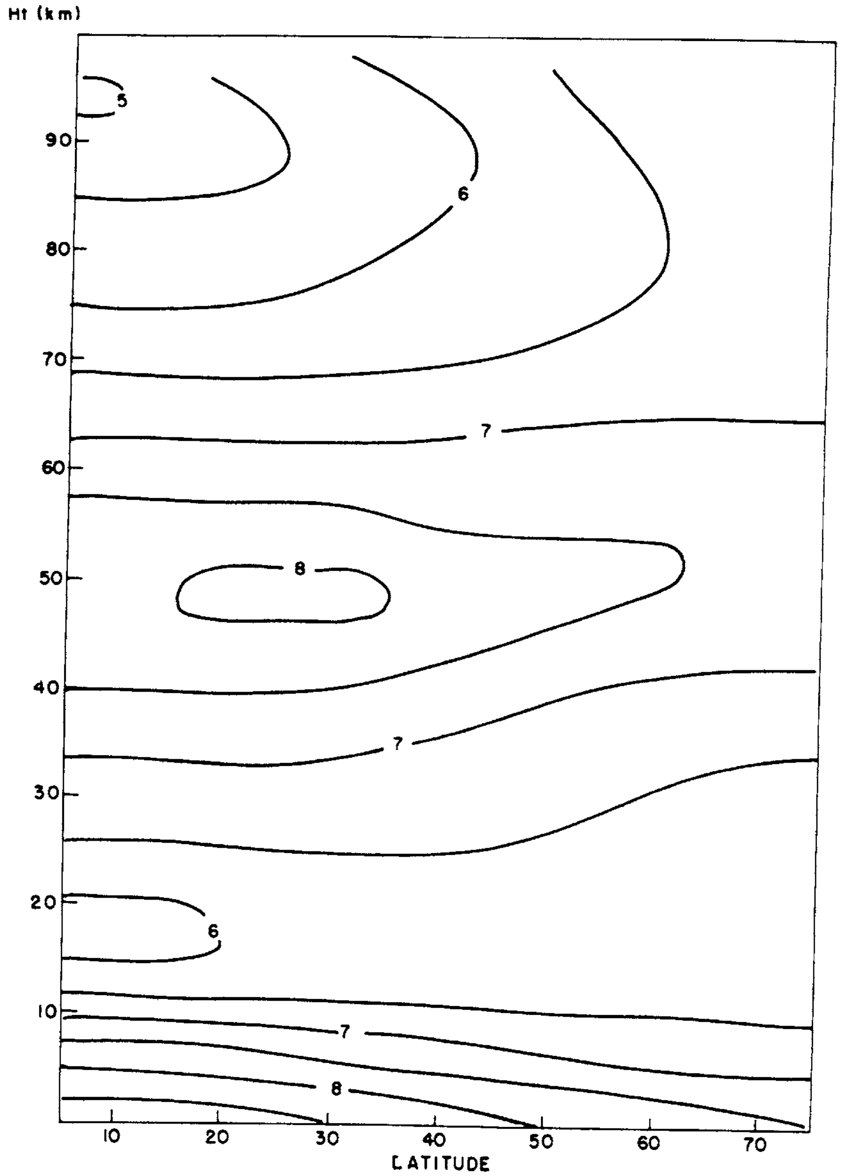


Figure 2-2. "Winter" atmospheric scale height in km.

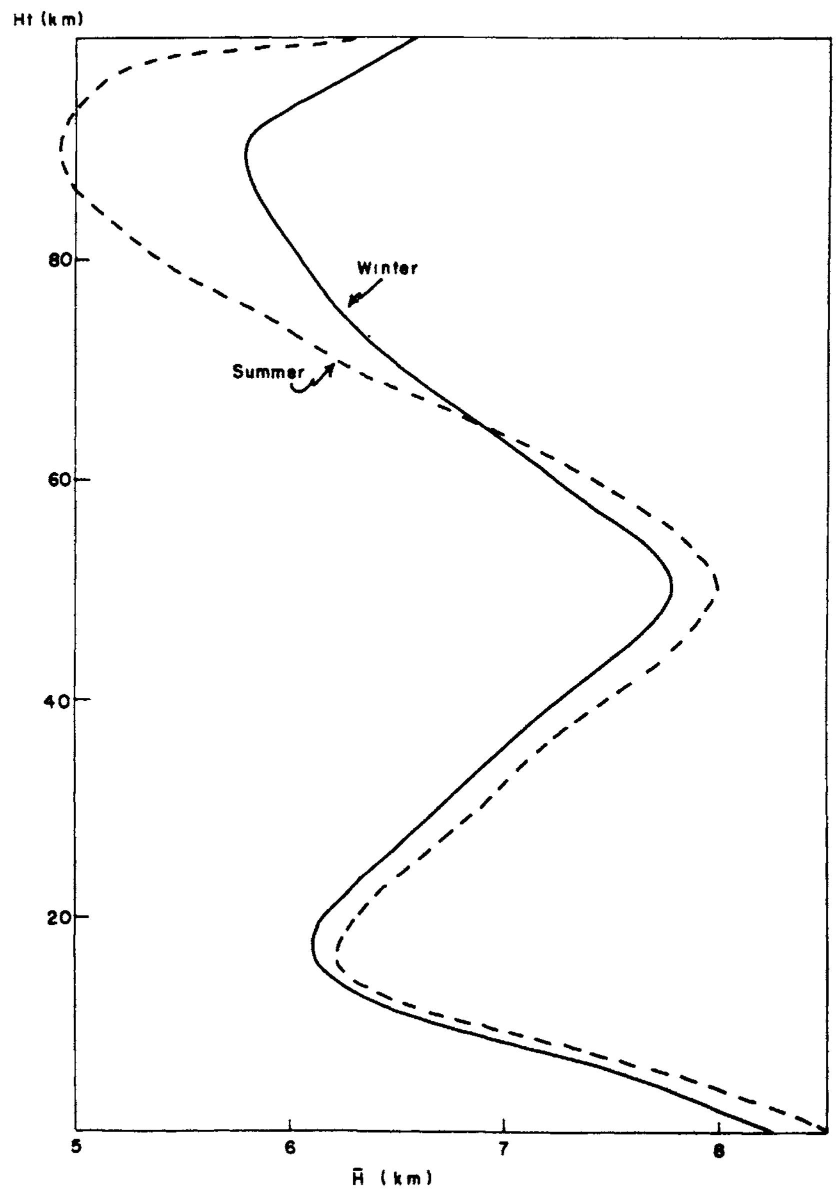


Figure 2-3. Hemispheric average scale height.

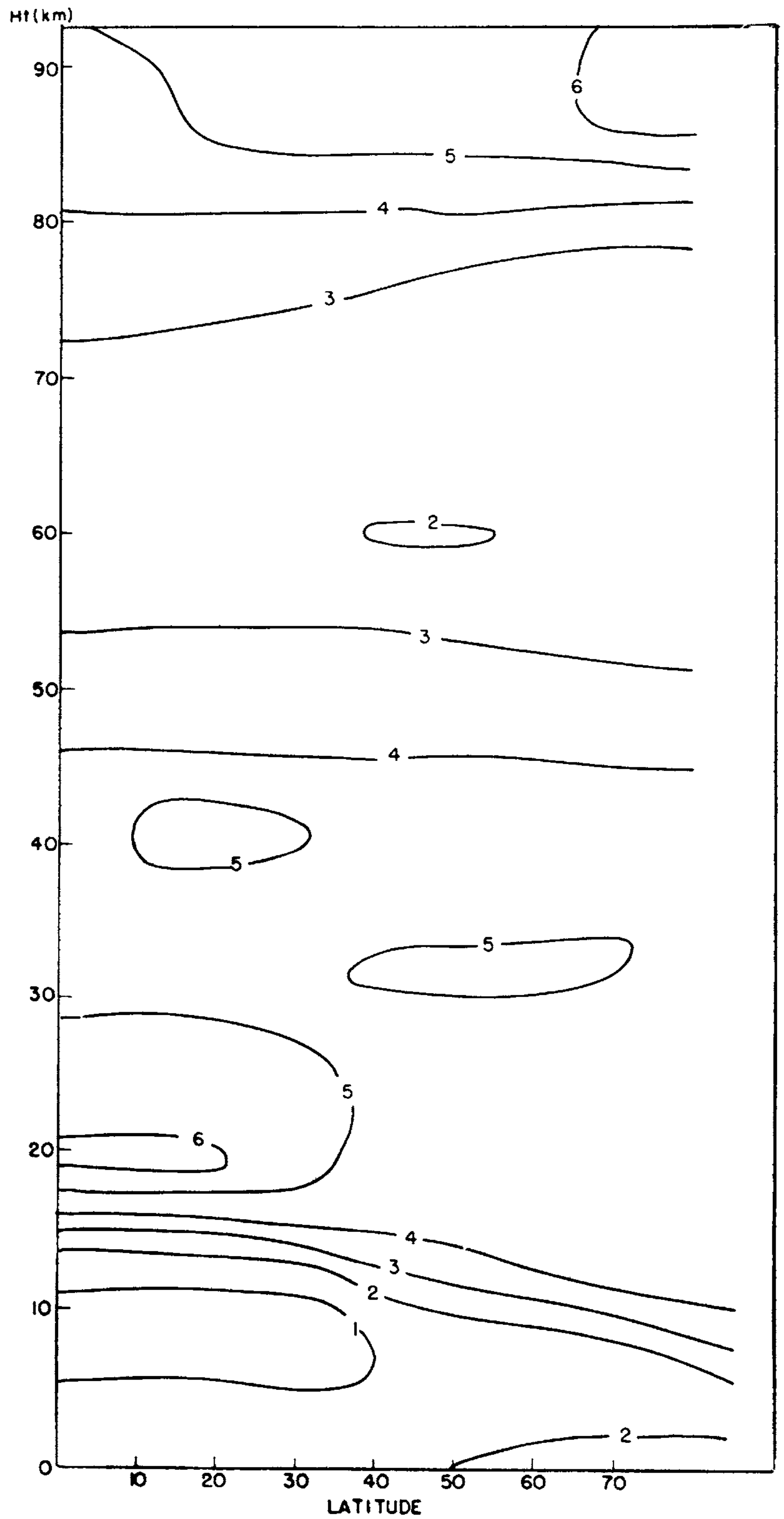


Figure 2-4. "Summer" N^2 in units of 10^{-4} sec^{-2}.

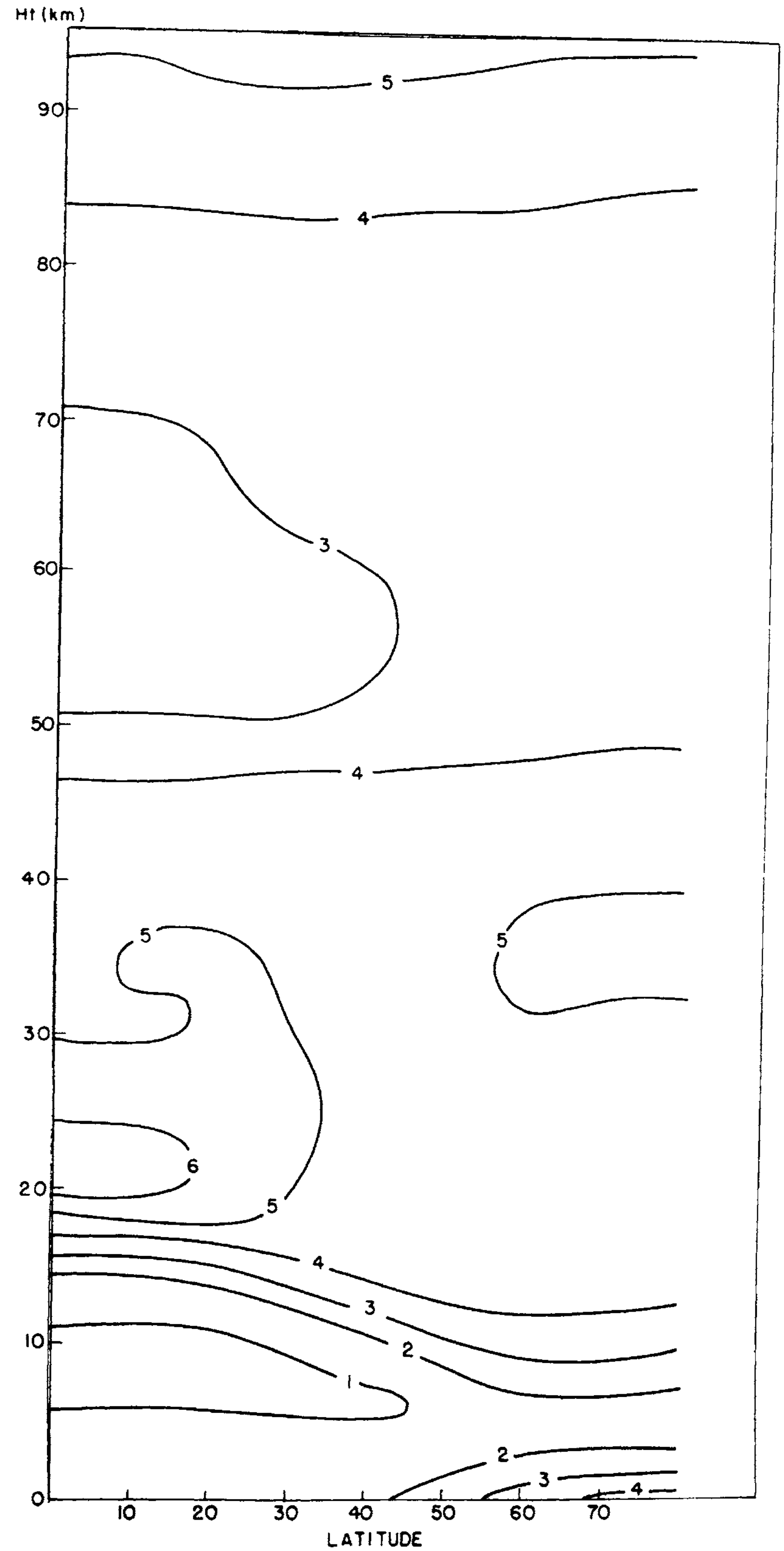


Figure 2-5. "Winter" N^2 in units of 10^{-4} sec^{-2}.

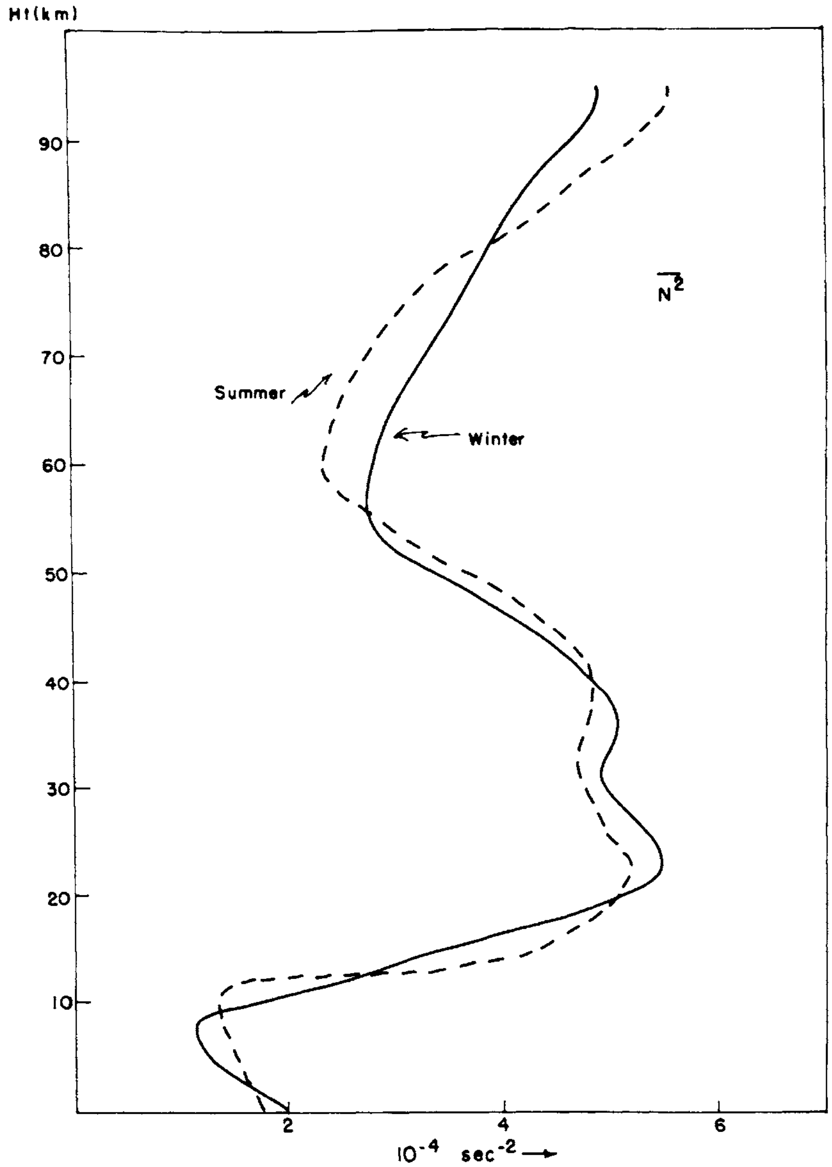


Figure 2-6. Hemispheric average N^2.

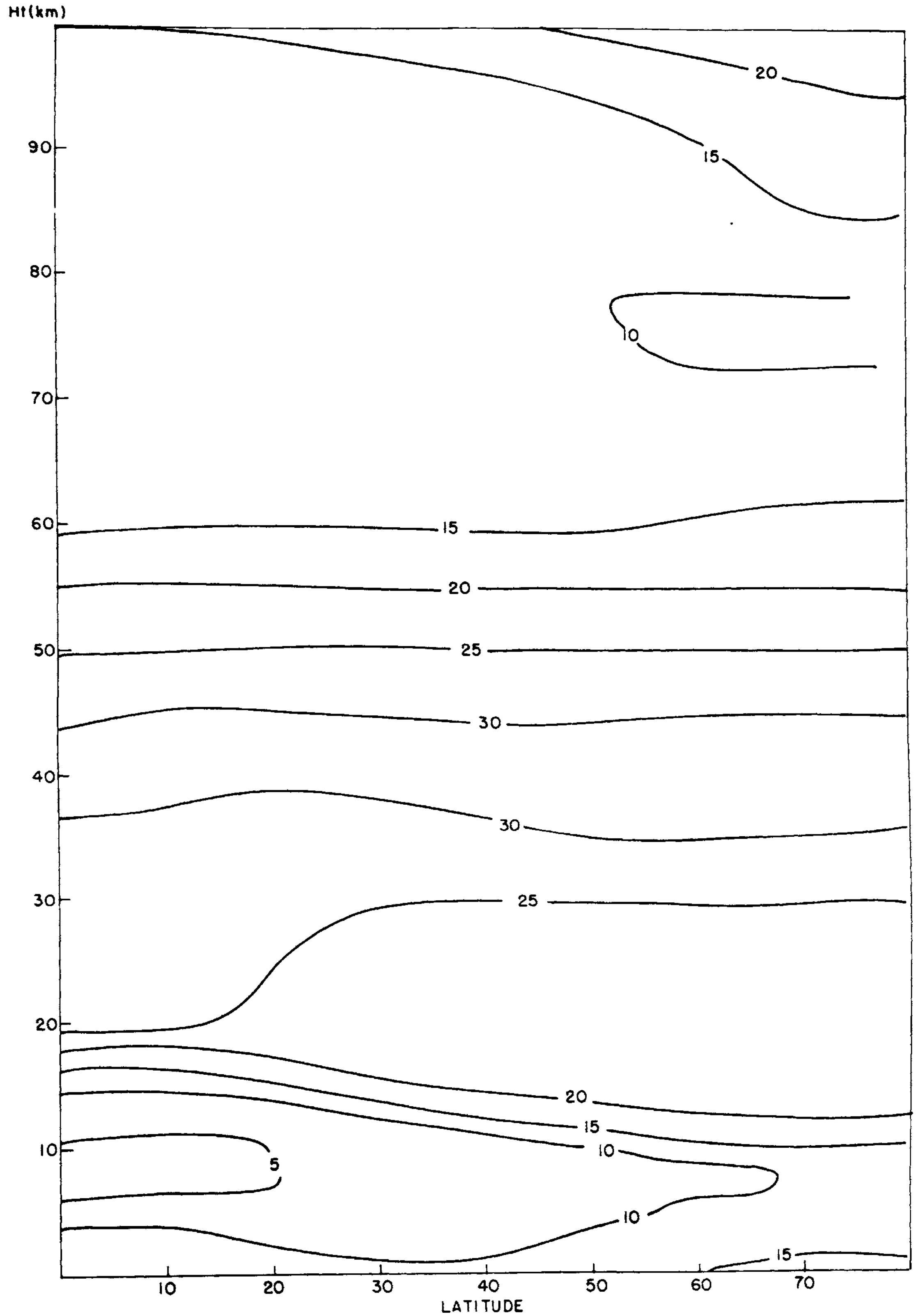


Figure 2-7. "Summer" planetary stability, S x 10^3, (nondimensional units).

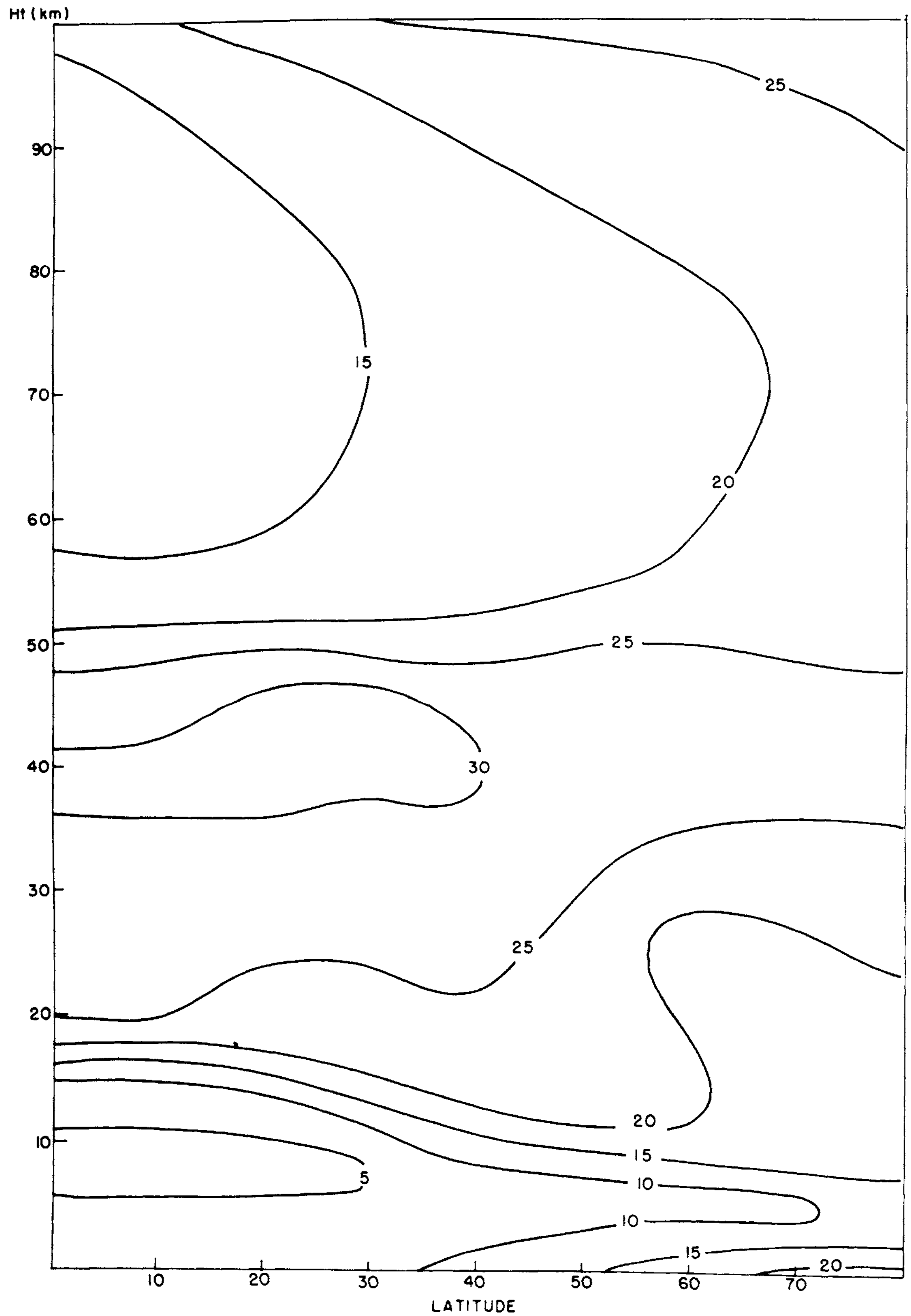


Figure 2-8. "Winter" planetary stability, S x 10^3, (nondimensional units).

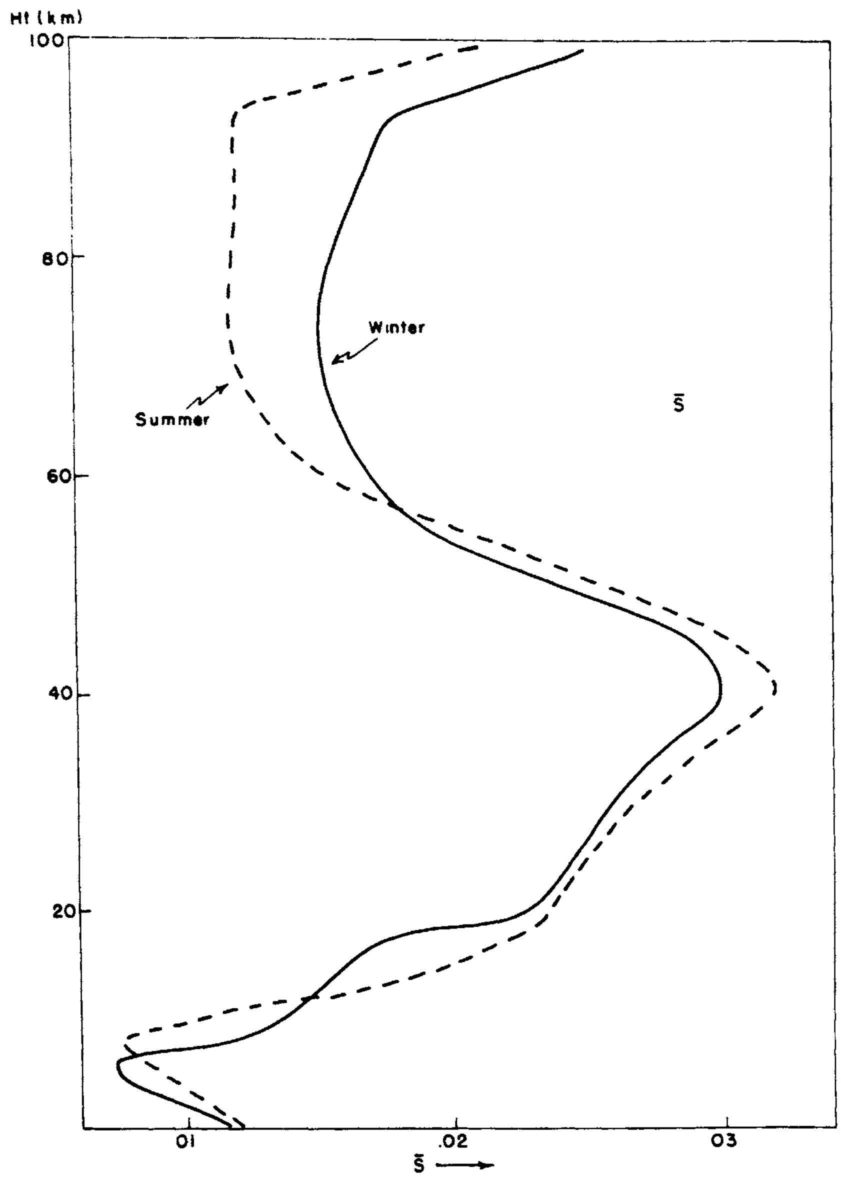


Figure 2-9. Hemispheric averages S (nondimensional units).

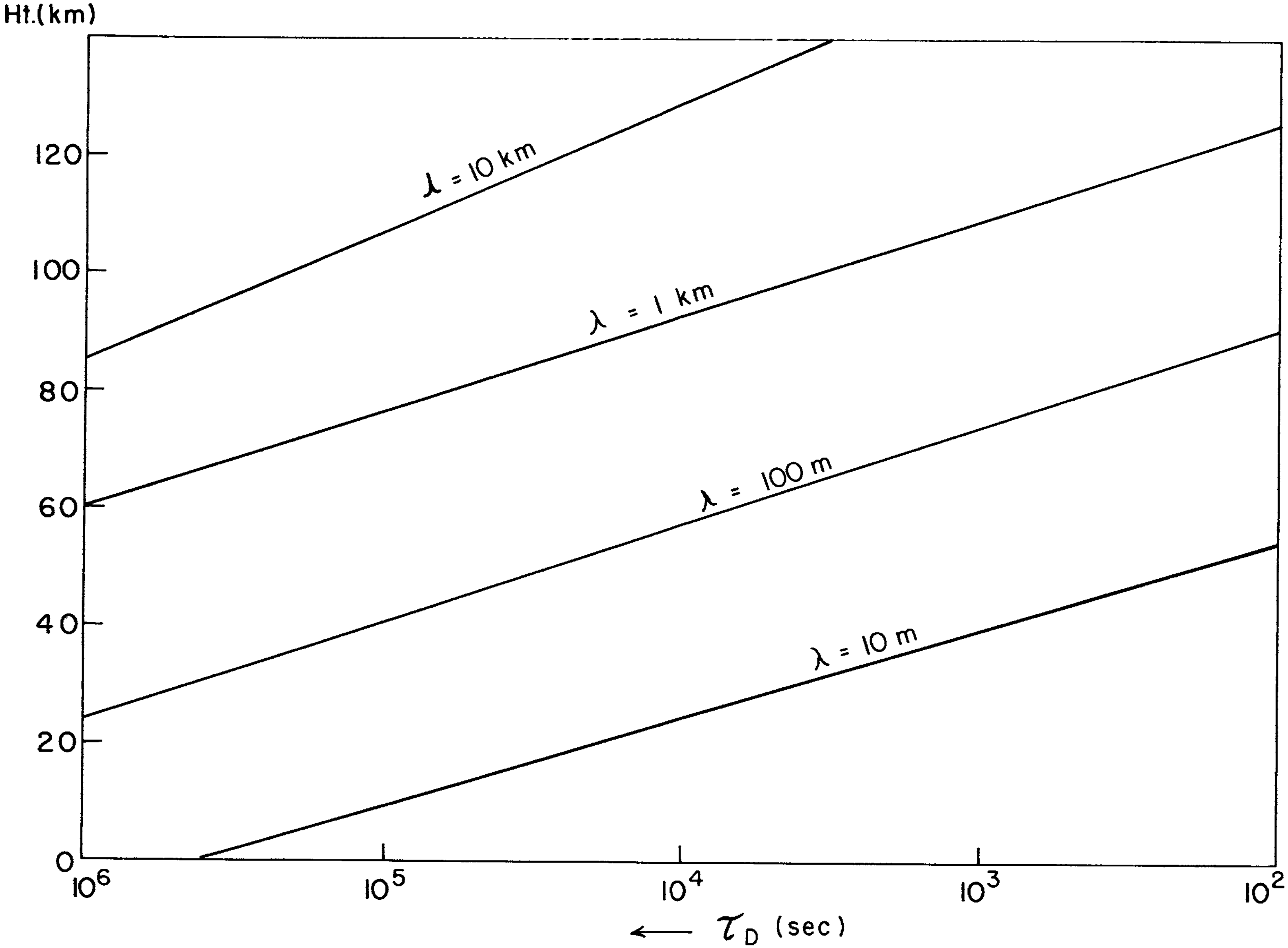


Figure 2-10. Viscous "damping" time for a sinusoidal wave, $e^{i\lambda z}$, where $\tau_D = \frac{\lambda^2}{4\pi^2 \nu(z)}$

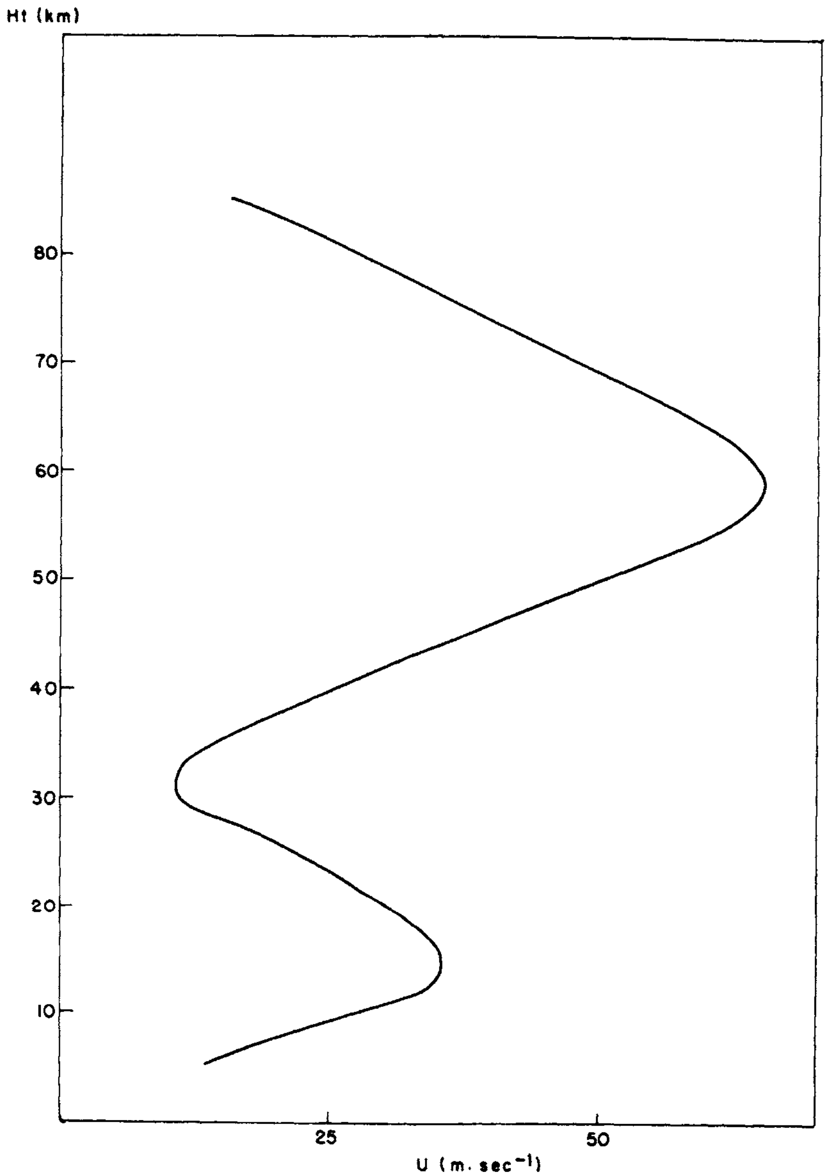


Figure 2-11. Model " U " profile (middle latitude winter).

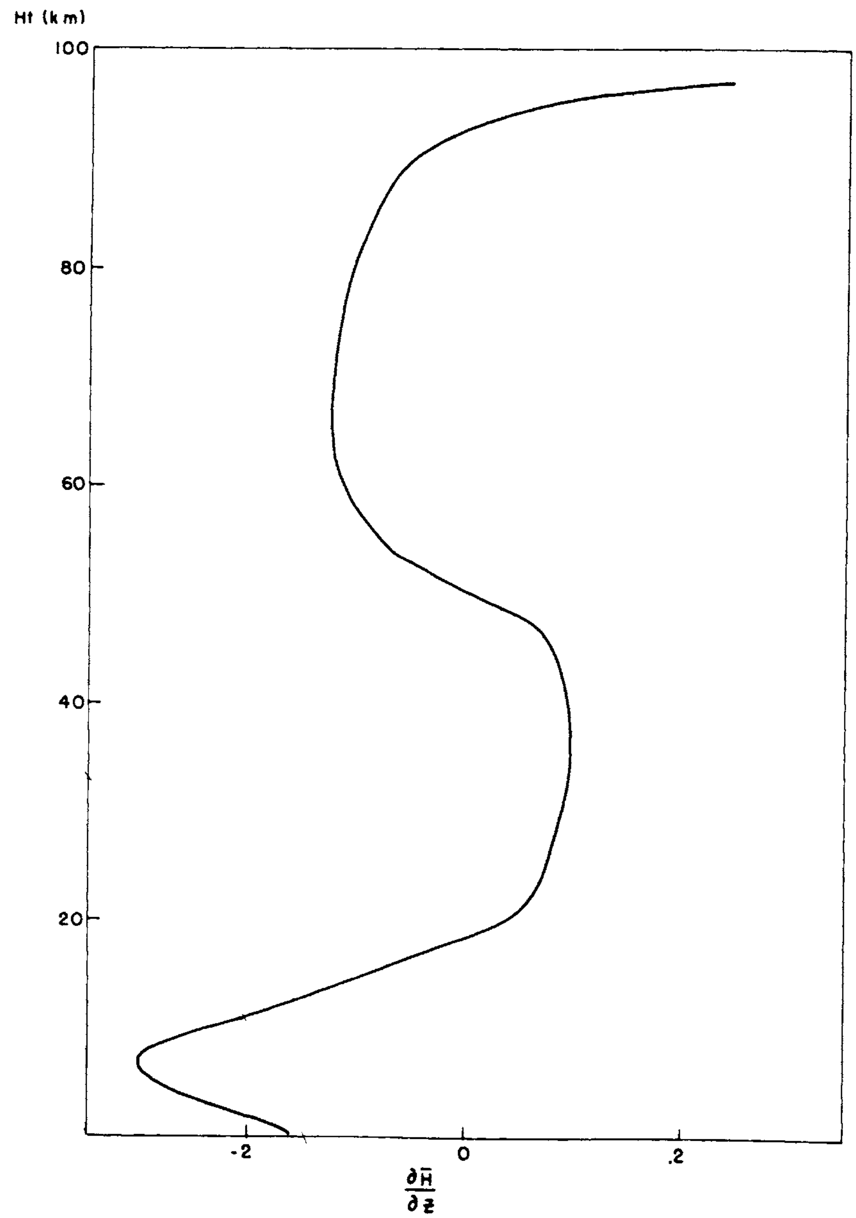


Figure 2-12. Derivative of scale height.

III. FORMULATION

A. On the Mathematical Formulation

The more common mathematical notation of modern dynamic meteorology is defined in the list in Appendix I. Less familiar symbols are also defined in the text as they are introduced. The general approach to be used in formulating linearized equations from the nonlinear equations which describe atmospheric motions will be illustrated below for the equations of inviscid atmospheric dynamics, applicable to nonrotating Cartesian coordinate systems. These equations are

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla_3)\vec{u} + 1/\rho\, \nabla_3 P - g\hat{k} = \vec{F} \tag{3.1}$$

$$\frac{\partial \theta}{\partial t} + (\vec{u} \cdot \nabla_3)\theta = Q/C_p \tag{3.2}$$

$$\frac{\partial \rho}{\partial t} + \nabla_3 \cdot \rho\vec{u} = \mathcal{S} \tag{3.3}$$

$$P = \frac{R^\gamma}{P_{00}^{\gamma-1}}(\rho\theta)^\gamma \tag{3.4}$$

We rearrange the above system of differential equations so that a tractable linear operator remains on the left hand side of the system and the remaining terms of the nonlinear system together with the forcing terms are given on the right hand side. Thus the total forcing is given by external forcing plus internal forcing due to stresses which are similar to Reynold's stresses. In order to effect this rearrangement, we shall choose some reference wind $\vec{U}$, a reference potential temperature Θ , a reference pressure Π , and a reference density ρ_o . These are chosen, consistent with the requirement of mathematical tractability for the left hand side, to correspond to some climatology of the atmospheric field variables. The total fluid velocities, pressure, temperature, and potential temperature become respectively

$$\begin{aligned} (u, v, w) &= (u' + U,\ v' + V,\ w' + W) \\ \theta &= \Theta + \theta' \\ p &= \Pi + p' \\ \rho &= \rho_o + \rho' \end{aligned} \tag{3.5}$$

The following constraints are imposed on the selection of the reference variables.

(1) They are chosen to be time independent.

(2) The reference pressure is related hydrostatically to the reference density. That is

$$\frac{d}{dz}\Pi = -\rho_o g \tag{3.6}$$

(3) The reference pressure, density and potential temperature obey the ideal gas law.

$$\Pi = \frac{R^{\gamma}}{P_{oo}^{\gamma-1}}(\rho_o\ \Theta)^{\gamma}$$

The previously given system of equations then becomes

$$\frac{\partial \vec{u}'}{\partial t} + (\vec{U} \cdot \nabla_3)\vec{u}' + (\vec{u}' \cdot \nabla_3)\vec{U} + 1/\rho_0 \nabla_3 p' + (g/\rho_0)\rho'\hat{k} = \vec{F}_{tot.} \tag{3.7}$$

$$\frac{\partial \theta'}{\partial t} + (\vec{U} \cdot \nabla_3)\theta' + (\vec{u}' \cdot \nabla_3)\Theta = \frac{Q_{tot.}}{C_p} \tag{3.8}$$

$$\frac{\partial \rho'}{\partial t} + \nabla_3 \cdot \rho_0 \vec{u}' + \nabla_3 \cdot \rho'\vec{U} = \mathcal{A}_{tot.} \tag{3.9}$$

$$\frac{1}{\gamma}\frac{p'}{\Pi} - \frac{\rho'}{\rho_0} - \frac{\theta'}{\Theta} = \epsilon \tag{3.10}$$

The source terms are now dependent on the dependent variables of the problem. Given proper initial conditions, we may solve linear systems such as that given above. It is assumed in this discussion that boundary conditions have been specified, and have been linearized in the same manner. The "solution" obtained will be a function of the initial conditions and also the dependent variables. For suffiently small amplitude initial conditions, one may assume for a first approximation that the dependent variables are zero and by iteration obtain improved approximations. The present work is primarily concerned with the inversion of the linear operators, and we do not consider further the nonlinear equations which are derived by this inversion. We assume that all the systems of equations used in this thesis are derived in the manner given above. The terms of the complete nonlinear dynamic equations which are assumed part of the forcing functions will not be given explicitly. These may be found in standard textbooks on atmospheric dynamics. For the remainder of this thesis, we shall omit the primes on perturbation quantities and the subscript, "tot", on forcing functions which has been used in the notation above.

The boundary conditions are chosen from physical considerations. One should consider the boundary conditions as important in determining the atmospheric dynamics as the system of differential equations. One or more of the horizontal space variables must be taken to extend to infinity if a spherical geometry is to be approximated by a planar geometry for describing motions with distance scales small compared to the radius of the earth. For sufficiently small time after an initial disturbance is started, solutions will be independent of the boundary conditions assumed, while for large enough times the solution may be very dependent on the type of boundary condition. If this is so, and furthermore, if it is not possible to specify the boundary condition very accurately, then the accuracy of results obtained will decay in time.

The most appropriate boundary condition that may be used when a boundary extends to infinity is a specification as to how the region of integration is to be continued to infinity. Since in practice, a solution is only required for a finite domain, the specification of the medium beyond this domain is only important insofar as it affects the solution in the finite domain. One may use the term internal domain to refer to the region in which a solution is required. When the differential equations have constant coefficients, we assume that the coefficients remain constant out to infinity. For variable coefficient problems, the boundary conditions are given as the asymptotic behavior of the coefficients outside the internal domain. This asymptotic behavior is to be selected to reproduce the effect of the external domain on the internal domain, and not necessarily to reproduce faithfully the actual variability of the coefficients outside this domain. For instance, if an acoustic wave problem is considered in which the external region is known to dissipate through the action of viscosity all acoustic wave motions which leave the internal domain, then we would choose an external domain transparent to the acoustic waves. This may be achieved by assuming that the speed of sound monotonically decreases in the exterior region. It is unimportant whether this condition is contrary to an actual monotonic increase of the speed of sound in the exterior region or even whether it leads to a negative sound speed at great distances. All that is necessary is that the solution of the inviscid equations with the assumed sound speed is correct within the internal domain. The assumption of monotonic decrease of the sound speed in the external domain is made since it leads to wave solutions where practically all the wave energy leaving the internal region will radiate to infinity. More generally, one may choose the asymptotic behaviour of coefficients so that wave energy will be radiated outward whenever most of the energy leaving the internal domain is dissipated rather than being down-reflected. In most cases, the rate of energy dissipation can only be roughly estimated, and furthermore will be dependent on the form of the disturbances excited. Thus upper boundary conditions obtained by the above considerations will in some cases give only a crude approximation to the actual wave behaviour which would result from inclusion of dissipation. These effects of viscous dissipation become important for large scale atmospheric motions above one hundred kilometers, and furthermore the actual winds and temperatures above this region are very poorly known. Hence any motions in the lower atmosphere which are highly dependent on conditions above one hundred kilometers are at present basically unpredictable.

B. The Equations for a Nonrotating Resting Atmosphere.

The linearized equations for the small amplitude motions of a resting atmosphere in a nonrotating frame are :

$$\frac{\partial \vec{u}}{\partial t} + 1/\rho_0 \nabla_3 p + g\hat{k}\, \rho/\rho_0 = \vec{F} \qquad (3.1$$

$$\frac{\partial \theta}{\partial t} + w \frac{\partial \Theta}{\partial z} = Q/c_p \qquad (3.1$$

$$\frac{\partial \rho}{\partial t} + w \frac{\partial \rho_0}{\partial z} + \rho_0 \nabla_3 \cdot \vec{u} = \mathcal{S} \qquad (3.1$$

$$p - c^2 \left(\rho + \rho_0 \theta / \Theta \right) = 0 \tag{3.14}$$

where

$$c^2 = \frac{\gamma \Pi}{\rho_0} \tag{3.15}$$

These follow from system (3.7) - (3.10), when the reference wind is taken to be zero. An equivalent system was first discussed by Lamb (1910). It is convenient to introduce the "Brunt Väisälä" or buoyancy frequency, N , defined by

$$N = \left(g/\Theta \ \frac{\partial \Theta}{\partial z} \right)^{1/2} \tag{3.16}$$

To simplify the reduction of this system to a single P. D. E., we assume the forcing functions are zero. They may be reintroduced as needed.

The reduction is as follows:

1). Take $\partial/\partial t \times \hat{k} \cdot$ (3.11), eliminate ρ by (3.14), and (3.12), to obtain

$$\left(\frac{\partial^2}{\partial t^2} + N^2 \right) w + 1/\rho_0 \left(\frac{\partial}{\partial z} + g/c^2 \right) \frac{\partial p}{\partial t} = 0 \tag{3.17}$$

2) Take ($\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j}$) $\cdot$ (3.11), and $\frac{\partial}{\partial t} \times$ (3.13), and eliminate $\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$ from these equations. Eliminate ρ by (3.14) and by (3.12), use $\frac{\partial \rho_0}{\partial z} + (\rho_0/\Theta) \frac{\partial \Theta}{\partial z} = -\rho_0 g/c^2$ to get

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right) p + \rho_0 \frac{\partial}{\partial t} \left(\frac{\partial}{\partial z} - g/c^2 \right) w = 0 \tag{3.18}$$

The variability of coefficients due to vertical decrease of density may be largely eliminated by introduction of the new variables

$$\left.\begin{aligned} \hat{w} &= \rho_0^{1/2}\, w \\ \hat{p} &= \rho_0^{-1/2}\, p \end{aligned}\right\} \qquad (3.19)$$

It is useful to use for further reduction , the scale height H,

$$H = \Pi / \rho_0 g \qquad (3.20)$$

and Eckart's parameter Γ defined by

$$\Gamma = \left(\frac{g}{c^2} + \frac{1}{2\rho_0} \frac{\partial \rho_0}{\partial z} \right) = \frac{1}{H} \left(\frac{1}{\gamma} - \frac{1}{2} \right) - \frac{1}{2H} \frac{\partial H}{\partial z} \qquad (3.21)$$

Also useful is the identity

$$N^2/c^2 = \frac{1}{\gamma H^2} \left(\kappa + \frac{\partial H}{\partial z} \right) \qquad (3.22)$$

We assume the approximation

$$\frac{\partial}{\partial z}\, p/c^2 \simeq \frac{1}{c^2} \frac{\partial p}{\partial z} \qquad (3.23)$$

to apply to the first term in (3. 18) whence p may be eliminated from (3. 17) -(3. 18) in terms of $\hat{w}$ defined by (3. 19). This gives the scalar hyperbolic P. D. E.

$$\left[\frac{\partial^2}{\partial t^2} \left(\Delta_3 - \frac{1 + \epsilon(z)}{4H^2} \right) + N^2 \Delta_2 - 1/c^2 \frac{\partial^4}{\partial t^4} \right] \hat{w} = 0 \qquad (3.24)$$

where $\epsilon(z)$ is given by

$$\epsilon(z) = \left[4H^2\left(\frac{N^2}{C^2} + \Gamma^2 - \frac{\partial \Gamma}{\partial z}\right) - 1\right] = \left[\frac{4}{\gamma}\frac{\partial H}{\partial z} - \left(\frac{\partial H}{\partial z}\right)^2 + \frac{2}{H}\frac{\partial^2 H}{\partial z^2}\right] \tag{3.25}$$

Comments: a) The approximation (3. 22) is not suitable for discussion of the one dimensional vertical oscillations of a variably stratified atmosphere, but another easy reduction is available in this case, c.f., Lamb, p. 541. b) The parameter $\epsilon(z)$ will be determined primarily by the first term $4/\gamma \frac{\partial H}{\partial z}$, which in the earth's atmosphere ranges from approximately -2 in the troposphere to +1 in the stratosphere, and vanishes for an isothermal atmosphere. If we were to carry out similar reductions for other dependent variables, we would obtain different expressions for ϵ. The variable $\rho_0^{1/2}\ \nabla_3 \cdot \vec{u}$ may be shown to satisfy (3. 24) with $\epsilon(z) = 0$. (c.f., Moore and Speigel, 1964).
c) The inviscid equation (3. 24) should be augmented by viscous and thermal diffusion terms for discussion of motions above the first one hundred kilometers of the earth's atmosphere. If we use the simplifying assumption that the viscous and thermal diffusion coefficients are equal and vary slowly relative to the scale of motion then it may be shown that the motions of a viscous atmosphere are obtained from (3. 24) by substituting in (3. 24) for the time derivative

$$\frac{\partial}{\partial t} \rightarrow \left(\frac{\partial}{\partial t} - \nu(z)\,\Delta_3\right) \tag{3.26}$$

The inviscid equations are satisfactory for descriptions of the atmospheric wave motions in the first one hundred kilometers of the earth's atmosphere. d) Since 100 km $\ll r_0$, we may assume with negligible error that the Laplacian operator in spherical coordinates, may be given by

$$\Delta_3 \simeq \left(\frac{\partial^2}{\partial z^2} + \frac{1}{r_0^2}\Delta_2\right) \tag{3.27}$$

where Δ_2 is the Laplacian on the surface of a unit sphere. We may hence use (3.24) for describing motions in a spherical nonrotating earth, provided the Laplacians are interpreted as above. e) The derivation of (3.24) may similarly be carried out for a coordinate system in constant rotation.

The result is that (3.24) is replaced by

$$\left[\left(\frac{\partial^2}{\partial t^2} + f_0^2\right)\left(\frac{\partial^2}{\partial z^2} - \left(\frac{1+\epsilon(z)}{4H^2} + \frac{N^2}{c^2}\right)\right) + \left(\frac{\partial^2}{\partial t^2} + N^2\right)\left(\Delta_2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\right]\hat{w} = 0 \tag{3.28}$$

Some useful mathematical approximations to (3.24) are obtained by assuming the following limits:

a) $H \to \infty$, we obtain

$$\left[\frac{\partial^2}{\partial t^2}\Delta_3 + N^2\Delta_2 - \frac{1}{c^2}\frac{\partial^4}{\partial t^4}\right]\hat{w} = 0 \tag{3.29}$$

b) $c \to \infty$

$$\left[\frac{\partial^2}{\partial t^2}\left(\Delta_3 - \frac{1+\epsilon(z)}{4H^2}\right) + N^2\Delta_2\right]\hat{w} = 0 \tag{3.30}$$

c) $c, H \to \infty$

$$\left[\frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial z^2} + \left(N^2 + \frac{\partial^2}{\partial t^2}\right) \Delta_2 \right] \hat{w} = 0 \tag{3.31}$$

d) $c, N \rightarrow \infty$

$$\left[\frac{\partial^2}{\partial t^2} \left(\frac{\partial^2}{\partial z^2} - \frac{1 + \epsilon(z)}{4H^2} \right) + N^2 \Delta_2 \right] \hat{w} = 0 \tag{3.32}$$

The physical justification for use of these more approximate equations, is that for certain types of sources and for sufficiently large intervals of time after the application of the source, the motions excited often can be described by a more approximate equation than (3. 24).

It may be shown that in a) certain "acoustic" oscillations with the frequency

$$\omega_A = \frac{c}{2H} \left(1 + \epsilon(z) \right)^{1/2} \tag{3.33}$$

are no longer present, so we should refer to (3. 29) as the "acoustic oscillation filtered" model. This equation still is hyperbolic, and allows "acoustic motions" and "gravity wave" motions. In the approximate "anelastic" model b), acoustic motions have been suppressed and disturbances propagate with infinite speed. The model c), is usually referred to as the equation for a Boussinesq incompressible fluid. The model d) is referred to as the "hydrostatic model" for atmospheric motions, or the "long wave" approximation, and due to

Lamb.

C. Formulation of Hydrostatic Atmospheric Dynamics.

The horizontal equations of motion, the thermodynamic and continuity equations and the equation of state are written

$$\frac{\partial \vec{c}}{\partial t} + f\,\hat{k} \times \vec{c} + \nabla g h = \vec{F}_h \tag{3.34}$$

$$(\rho_0 \Theta)^{-1} \frac{\partial \theta}{\partial t} - \left(\frac{N H}{\Pi}\right)^2 \omega = (\rho_0\, C_p\, \Theta)^{-1} \varphi \tag{3.35}$$

$$\nabla \cdot \vec{c} = \frac{1}{\Pi} \frac{\partial \omega}{\partial \hat{z}} \tag{3.36}$$

$$\frac{\theta}{\Theta} = \frac{1}{H} \frac{\partial h}{\partial z} \tag{3.37}$$

The vertical coordinate used here is $\hat{z} = \log \Pi^{-1}$, the log of the reference pressure. See Eliassen (1949). We use $f = 2\,\Omega_e$ sin θ for the Coriolis parameter, $\vec{c} = u\hat{\imath} + v\hat{\jmath}$ is the horizontal velocity field. The vertical motion is described by ω,

$$\omega = \frac{Dp}{Dt} \simeq \frac{\partial p}{\partial t} - \rho_0 g w \tag{3.38}$$

Also h is perturbation geopotential height, θ is perturbation, and Θ mean potential temperature, H is scale height. The "heights" h and H are related hydrostatically to the perturbation pressure p and mean pressure Π , by

$$h = p/\rho_0 g$$
$$H = \Pi/\rho_0 g \tag{3.39}$$

The vector operators, ∇ , $\nabla\cdot$, written in terms of latitude θ and longitude λ are given in Appendix I.

It is often convenient to replace the equations of motion (3.34) with the vorticity equation and divergence equation, which may be written

$$\partial/\partial t\ \Delta\psi + \frac{2\Omega_e}{r_e}\frac{\partial\psi}{\partial\lambda} + \nabla\cdot f\nabla\phi = \hat{k}\cdot\nabla\times\vec{F} \tag{3.40}$$

$$\partial/\partial t\ \Delta\phi + \frac{2\Omega_e}{r_e}\frac{\partial\phi}{\partial\lambda} - \nabla\cdot f\nabla\psi + g\Delta h = \nabla\cdot\vec{F} \tag{3.41}$$

where $\Delta = \nabla\cdot\nabla$ is the Laplacian on the surface of the sphere. This system was first obtained by Love (1913).

We have substituted

$$\vec{c} = \nabla\phi + \hat{k}\times\nabla\psi \tag{3.42}$$

Assuming boundary conditions have been specified to make Laplacian invertible, ϕ and ψ may be obtained from $\vec{c}$ by

$$\Delta\psi = \hat{k}\cdot\nabla\times\vec{c}$$
$$\Delta\phi = \nabla\cdot\vec{c} \tag{3.43}$$

The first of these expressions gives the vorticity written in terms of ψ or $\vec{c}$, and the second, the horizontal divergence either in terms of Φ or $\vec{c}$.

We may eliminate θ and ω in (3.35) - (3.37), to relate h and the divergence as

$$\Delta \Phi = \frac{1}{\Pi}\frac{\partial \omega}{\partial \hat{z}} = \frac{1}{\Pi}\frac{\partial}{\partial \hat{z}}\left(\frac{\Pi}{(NH)^2}\left[\frac{\partial^2 gh}{\partial \hat{z}\partial t} - \frac{gHQ}{C_p \Theta}\right]\right) \tag{3.44}$$

If we eliminate Φ in (3.40), we obtain

$$\left(\frac{\partial}{\partial t}\Delta + \frac{2\Omega_e}{r_e}\frac{\partial}{\partial \lambda}\right)\psi + \frac{\partial}{\partial t}\nabla \cdot f\nabla \Delta^{-1}\left(\frac{1}{\Pi}\frac{\partial}{\partial \hat{z}}\frac{\Pi}{(NH)^2}\frac{\partial}{\partial \hat{z}}\right)gh = s \tag{3.45}$$

where

$$s = (\hat{k} \cdot \nabla \times \vec{F}) + \nabla \cdot f \nabla \frac{1}{\Pi}\frac{\partial}{\partial \hat{z}}\left(\Pi \frac{g\Delta^{-1}Q}{N^2 H\, C_p \Theta}\right) \tag{3.46}$$

The equation (3.45) is known as the inhomogenous potential vorticity equation. The potential vorticity q , is defined as

$$q = \Delta_2 \psi + \left[\nabla \cdot f \nabla \Delta^{-1}\left(\frac{1}{\Pi}\frac{\partial}{\partial \hat{z}}\frac{\Pi}{(NH)^2}\frac{\partial}{\partial \hat{z}}\right)\right]gh \tag{3.47}$$

so that (3.45) may be written

$$\frac{\partial q}{\partial t} + \frac{2\Omega_e}{r_e}\frac{\partial \psi}{\partial \lambda} = s \tag{3.48}$$

We likewise eliminate Φ in (3.41) to obtain

$$\left[\frac{\partial^2}{\partial t^2} 1_B \left(\frac{1}{\Pi}\frac{\partial}{\partial \hat{z}}\frac{\Pi}{(NH)^2}\frac{\partial}{\partial \hat{z}}\right)gh + \Delta gh - \nabla \cdot f \nabla \psi\right] = r \tag{3.49}$$

where we define $\mathbb{1}_\beta$ as the operator

$$\mathbb{1}_\beta = 1 + \frac{2\Omega_e}{r_e}\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial t}\,\Delta\right)^{-1} \tag{3.50}$$

and r the source, is

$$r = \nabla\cdot\vec{F} + \frac{\partial}{\partial t}\mathbb{1}_\beta\,\frac{1}{\Pi}\frac{\partial}{\partial\hat{z}}\left(\frac{\Pi\, g H}{(NH)^2}\,\frac{Q}{c_p\,\Theta}\right) \tag{3.51}$$

To further carry out the elimination, we define the "Coriolis operators", F_1 and F_2 by

$$F_1 = \nabla\cdot f\,\nabla(\,\Delta^{-1}\mathbb{1}_\beta) \qquad\qquad F_2 = (\nabla\cdot f\,\nabla)\,\Delta^{-1} \tag{3.52}$$

We eliminate ψ in (3.49) and (3.45) to obtain

$$\left[\left(\frac{\partial^2}{\partial t^2}\mathbb{1}_\beta + F_1F_2\right)\frac{1}{\Pi}\frac{\partial}{\partial\hat{z}}\frac{\Pi}{(NH)^2}\frac{\partial}{\partial\hat{z}} + \Delta\right] gh = r + \nabla\cdot f\,\nabla\left(\frac{\partial}{\partial t}\Delta\mathbb{1}_\beta\right)^{-1} s \tag{3.53}$$

We shall call (3.53) "Laplace's tidal equation" since a similar equation was first discussed by Laplace in regard to the theory of tides in the ocean and atmosphere.

The essential physical approximation involved in the above derivation are:

a) the motions are hydrostatic, e.g., $\frac{\partial^2}{\partial t^2} << N^2$

b) the atmosphere is a thin shell relative to the radius of the earth.

c) viscous and thermal diffusion are neglected

d) the earth is approximately spherical

The nonlinear terms omitted in the above equations may be considered as part of the forcing functions r and s . See Lorenz (1960) for the nonlinear terms appropriate to the above system.

Assumption a) may be relaxed (c. f., Eckart's book) provided we do not include Coriolis terms in the vertical equation of motion. Assumptions b) and c) are consistent. That is, the inviscid equations are only applicable in the first 100 km of the earth's atmosphere and the vertical scale of this system is small compared to the earth's radius. In order to remove assumption b), it is necessary to carry out a more careful analysis in using geometric vertical height, as done for instance by Yanowitch (1963). One resulting modification is that in the continuity equation a term like $\frac{\partial^2}{\partial z^2}$ is replaced with a term like $\frac{\partial^2}{\partial z^2} + \frac{2}{r_e + z}\frac{\partial}{\partial z}$, the second term resulting from divergence of the radial coordinate lines. Such terms as the latter are of some importance for long distance propagation of radio waves on a spherical earth, even though they may be quite small, since they lead to a certain amount of downward refraction of the waves. For atmospheric motions, the variable temperature and winds have a much greater refractive effect on wave propagation, and the neglect of radial coordinate line divergence may be neglected on this account. (This approximation is sometimes referred to by radio engineers as the flat earth approximation).

In order to introduce viscous and thermal diffusion effects into the above system, one may assume the coefficients of viscous and thermal diffusion are the same, and that their spatial variability may be neglected in reducing the equations. Then one merely substitutes whereever there is a time derivative $\frac{\partial}{\partial t}$, the term $\left(\frac{\partial}{\partial t} - \nu\Delta_3\right)$ where $\nu = \nu(x,y,t)$ is the diffusion coefficient. More exact treatment greatly complicates the system.

Assumption d), the neglect of the earth's eccentricity, is generally accepted without question. For an analysis of this question, see Hough (1897).

It is sometimes convenient for discussion of motions over distance scales small compared to the radius of the earth, to approximate the Coriolis operators, F_1 and F_2 by a mean scalar Coriolis parameter f_0 . That is

$$F_1 \simeq F_2 \simeq f_0 \tag{3.54}$$

and likewise to approximate the operator 1_θ by unity.

$$1_\theta \simeq 1 \tag{3.55}$$

or

$$\left(\frac{\partial}{\partial t}\Delta + \frac{2\Omega_e}{r_e}\frac{\partial}{\partial \lambda}\right) \simeq \frac{\partial}{\partial t}\Delta$$

whence Laplace's tidal equation reduces to

$$\partial/\partial t\left[\left(\frac{\partial^2}{\partial t^2}+f_o^2\right)\frac{1}{\pi}\frac{\partial}{\partial \hat{z}}\frac{\pi}{(NH)^2}\frac{\partial}{\partial \hat{z}}+\Delta\right]gh = \frac{\partial r}{\partial t}+f_o s \tag{3.56}$$

which is known as the equation for internal gravity waves.

Alternately, we may assume the low frequency approximations,

$$\begin{aligned} F_1 &\simeq f_o 1_\theta \\ F_2 &\simeq f_o \\ \frac{\partial^2}{\partial t^2} &\ll f_o^2 \end{aligned} \tag{3.57}$$

whence the Laplace tidal equation reduces to

$$\partial/\partial t\left[f_o^2\left(\pi^{-1}\frac{\partial}{\partial \hat{z}}\frac{\pi}{(NH)^2}\frac{\partial}{\partial \hat{z}}\right)gh+\Delta gh\right]+\frac{2\Omega e}{re}\frac{\partial h}{\partial \lambda} = \frac{\partial r}{\partial t}+\nabla\cdot f\nabla(1_\theta\Delta)^{-1}s \tag{3.58}$$

which is known as the equation for atmospheric Rossby waves. This low frequency tidal equation may alternately be derived by approximating potential vorticity q given by (3.47) by

$$q \simeq \Delta\psi + f_o^2\frac{1}{\pi}\frac{\partial}{\partial \hat{z}}\frac{\pi}{(NH)^2}\frac{\partial}{\partial \hat{z}}\frac{gh}{f_o} \tag{3.59}$$

and approximating (3.49) by the geostrophic relation

$$\Delta gh - f_o\Delta\psi \simeq 0 \tag{3.60}$$

and eliminating either h or ψ in (3.48).

These approximations apply for motions with frequency small compared to the rotational frequency of the earth, and for the motion scale small compared to the earth's radius. For yet smaller distance scales of the motion, a planar approximation to the sphere may be appropriately applied. That is, we may introduce a local Cartesian system centered at λ_0 , θ_0 by

$$x = r_e (\cos \theta_0) \quad (\lambda - \lambda_0)$$
$$y = r_e \, (\theta - \theta_0) \tag{3.61}$$

We shall use the usual notation

$$\beta = \frac{2\Omega_e}{r_e} \cos \theta_0$$

so that for the planar approximation

$$\frac{2\Omega_e}{r_e} \partial/\partial\lambda \doteq \beta \; \partial/\partial x \tag{3.62}$$

We shall for the purpose of discussion, assume one further approximate equation obtained by combining the low frequency Rossby wave equation (3.58) with the high frequency, gravity wave equation. This may be written in Caresian coordinates as

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial t^2} + f_0^2\right)\left(\frac{1}{\Pi}\frac{\partial}{\partial \hat{z}}\frac{\Pi}{(NH)^2}\frac{\partial}{\partial \hat{z}}\right) + \Delta\right] h + \beta\frac{\partial h}{\partial x} = 0 \tag{3.63}$$

Variations of this equation have been used in the literature in an effort to extend the " β -plane" model to "non-geostrophic" motions. One of the conclusions obtained from analysis of this equation is that it is not suitable for the integration of initial value problems since it predicts unstable "inertial oscillation".

Since boundary conditions are frequently expressed in terms of vertical motion w it is useful to relate the geopotential h to w Eliminating ω from (3.38), (3,39) and (3.44) gives

$$w = \frac{\partial h}{\partial t} + \frac{g}{N^2 H}\left[-\frac{\partial^2 h}{\partial \hat{z}\,\partial t} + \frac{H\,Q}{C_p\,\Theta}\right] \tag{3.64}$$

Further details concerning sources, boundary conditions, and more approximate equations will be introduced when required in the following chapters.

IV. SOME PROPAGATORS-MATHEMATICAL METHODOLOGY FOR INVERSION OF DIFFERENTIAL OPERATORS

The purpose of this chapter is two-fold. It is intended to present in detail the simplest theoretical model for combined gravity waves and buoyancy oscillations, and at the same time to present systematically some of the mathematical techniques to be used in the remainder of this thesis. The theoretical model we study is the "incompressible" system given by (3. 31) which may be written

$$\left[\frac{\partial^2}{\partial t^2}\Delta_3 + N^2\Delta_2\right]\hat{w} = -F_T(x,y,z,t) \tag{4. 1}$$

where here $F_T(x,y,z,t)$ is an arbitrary forcing function. We assume N to be constant. The most highly differentiated terms in (4. 1) give the motions of an "ideal" fluid while the lower order term $N^2\Delta_2$ gives a correction for stratification.

We shall first study the motion due to the simplest possible localized source. That is we replace F_T with $\delta(x)\delta(y)\delta(z)\delta(t)$

$$F_T = \delta(x)\delta(y)\delta(z)\delta(t) \tag{4. 2}$$

An understanding of this motion is helpful in the more difficult task of studying the motions due to actual physical sources, $\vec{F}$, Q , and $\mathcal{A}$. Such studies are necessary for complete understanding of a given kind of wave motion, but for understanding the gross features of the many different possibilities for atmospheric wave motions, it is desirable to discuss first solutions for the most elementary sources.

We define the propagator of (4.1) to be the solution $W(x, y, z, t)$ for the source (4.2). That is

$$\left[\frac{\partial^2}{\partial t^2}\Delta_3 + N^2\Delta_2\right] W = -\delta(x)\,\delta(y)\,\delta(z)\,\delta(t) \tag{4.3}$$

It is possible without loss of generality to take W and all of its derivatives to be identically zero for $t<0$ since by using proper delta function sources at $t=0$, one may obtain the same results as if the initial conditions were nonzero.

The time delta function may be written in terms of a Laplace contour integral as

$$\delta(t) = \frac{1}{2\pi i}\,\partial/\partial t \int_{-i\infty+\epsilon}^{i\infty+\epsilon} e^{\sigma t}\,\frac{d\sigma}{\sigma} \tag{4.4}$$

where the variable of integration is taken to be a complex variable, and the path of integration is taken to run to the right of the imaginary axis. This permits a natural association (isomorphism) of differential operators in time, acting on W, with the complex variable σ.

It is seen from (4.4) that a function of the operator $\partial/\partial t$, $F(\partial/\partial t)$ operating on $\delta(t)$ may be written

$$F(\partial/\partial t)\,\delta(t) = (\partial/\partial t)^n\,\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{F(\sigma)}{\sigma^n}\,e^{\sigma t}\,d\sigma \tag{4.5}$$

where the term $(\partial/\partial t)^n$ outside the integral and the term σ^{-n} in the integrand are seen to cancel each other formally. The integer n is to be selected large enough to insure existence of the integral. It is assumed that

a) The contour is taken to the right of all singularities of $F(\sigma)/\sigma^n$

b) n is chosen large enough so that

$$\lim_{\substack{\sigma \to \infty \\ Re\,\sigma \geq 0}} \left| F(\sigma)/\sigma^n \right| \to 0$$

uniformly with respect to $-\pi/2 \leq \arg \sigma \leq \pi/2$.

The condition a) insures that the integral of $F(\sigma)/\sigma^n$ taken over a large semicircle around the origin in the right half plane will go to zero as the radius of the semicircle is taken to infinity. The fact that the semicircle integral vanishes under these conditions is known as Jordan's lemma. From a) and b) it follows that

$$F(\partial/\partial t)\,\delta(t) = 0$$

for $t < 0$

which is the condition of causality. No response can occur before the source $\delta(t)$ is applied.

For ease in writing we shall introduce the convention that σ may be used both for the operator $\partial/\partial t$ occurring outside contour integrals as well as for the complex variable σ which occurs under the integral sign. In this notation (4.5) is written

$$F(\sigma)\,\delta(t) = \sigma^n \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{F(\sigma)}{\sigma^n}\, e^{\sigma t}\, d\sigma \tag{4.6}$$

The direct manipulation of equations involving an operator such as $\partial/\partial t = \sigma$ and evaluation of the subsequent expressions by contour integration is known as the direct operational calculus.

Using the above notation we write (4.1) as

$$\left(\sigma^2\Delta_3 + N^2\Delta_2\right)W = -\delta(x)\,\delta(y)\,\delta(z)\,\delta(t) \tag{4.7}$$

Assuming $\sigma \neq 0$, we define a new variable

$$Z = \left(N^2+\sigma^2\right)^{1/2}\sigma^{-1} z \tag{4.8}$$

and the delta function

$$\delta(Z) = \sigma\left(N^2+\sigma^2\right)^{-1/2}\delta(z) \tag{4.9}$$

where $(N^2+\sigma^2)^{1/2}$, taken as a complex function of σ, is defined so that the branch cuts are those of Fig. 4-1. Equation (4.6), using (4.7) and (4.8) becomes

$$\left(\frac{\partial^2}{\partial Z^2} + \Delta_2\right)W = -\left(N^2+\sigma^2\right)^{-1/2}\sigma^{-1}\,\delta(x)\,\delta(y)\,\delta(Z)\,\delta(t) \tag{4.10}$$

The operator on the left is a Laplacian, which is inverted to give

$$W = \frac{\left(N^2+\sigma^2\right)^{-1/2}\sigma^{-1}}{4\pi\left(x^2+y^2+Z^2\right)^{1/2}}\,\delta(t) = \frac{1}{4\pi}\left(\sigma^2R^2+N^2z^2\right)^{-1/2}\left(N^2+\sigma^2\right)^{-1/2}\delta(t) \tag{4.11}$$

where

$$R = (x^2 + y^2 + z^2)^{1/2}$$

The contour integral used to evaluate the above fractional operator is

$$W = \frac{1}{4\pi R} \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{e^{\sigma t} d\sigma}{(\sigma^2+N^2)^{1/2}(\sigma^2+N^2 z^2/R^2)^{1/2}} \tag{4.12}$$

where the path of integration is taken to lie to the right of the imaginary σ- axis.

Let $\quad \tilde{W}(\sigma) = (4\pi R)^{-1}(\sigma^2 + N^2)^{-1/2}(\sigma^2 + N^2 z^2/R^2)^{-1/2}$

be the Laplace transform of W. The contour in (4.12) may be deformed so as to enclose the singularities of $\tilde{W}(\sigma)$ as shown in Fig. 4-1 provided the integral around a large semicircle in the left half of the Riemann sheet under consideration becomes vanishingly small as its radius is taken to infinity. The examination of contour integrals around large semicircles will frequently be necessary but will not be explicitly discussed further in this thesis, since the requisite manipulations are not of physical interest. The reader may assume, unless it is stated otherwise, that such integrals have been verified to be vanishingly small.

An integral such as (4.12) may be evaluated directly by numerical integration, so there is no need to reduce it to other complicated functional relationships, such as in this case, a convolution integral of two Bessel functions. However, in a study such as this one, which is intended to lay bare the essential features of the dynamics, it is desirable to approximate complicated solutions such as (4.12) by means of functions sufficiently simple that their numerical graph can be visualized without the need for detailed figures. These simple, approximate functions are usually applicable only over a certain range of parameters. Laplace contour integrals such as (4.12) may be approximated by polynomial functions for small enough time, and also by other simple functions for large enough time. The graph for intermediate time may then be visualized by mentally interpolating between the readily visualized expressions for small and large time.

An expression useful for small time may be obtained from (4.12) by expanding $\tilde{W}(\sigma)$ in an inverse power series of σ. This gives

$$W = \frac{1}{4\pi R}\int_{-i\infty}^{i\infty} e^{\sigma t}\frac{d\sigma}{\sigma^2}\sum_{k=0}^{\infty} N^{2k}\frac{\alpha_k}{\sigma^{2k}} = \frac{H(t)}{4\pi N R}\sum_{k=0}^{\infty}\frac{\alpha_k (Nt)^{2k+1}}{(2k+1)!} \tag{4.13}$$

where

$$\alpha_k = \sum_{j=0}^{k}\binom{-1/2}{j}\binom{-1/2}{k-j}\left(\frac{z}{R}\right)^{2(k-j)}$$

The series is seen to converge for all finite t, and hence W is an entire function of time. Keeping only the first two terms of this power series solution, we write

$$W = \frac{H(t)}{4\pi R}\left[t - \frac{1}{12}N^2(1 + z^2/R^2)\,t^3 + O(N^4 t^5)\right] \tag{4.14}$$

which is the desired simple expression for small time. The first term may be intepreted as giving the motion of an ideal (inviscid incompressible unstratified) fluid for a special kind of point source, and the higher order term as giving corrections for stratification.

Expressions valid for large time are somewhat more difficult to obtain but are often of much greater physical interest, since they describe the ultimate evolution of the dynamical system. For large time, we consider the contour integrals around the singularities of $\overline{W}(\sigma)$, which are obtained by deformation of the original contour integral. In Appendix III, A. we establish the complete asymptotic solutions of the branch line contour integrals which occur in this chapter. The numerical values of the singularities of $\overline{W}(\sigma)$ in the problem presently being considered are branch points which occur at

$$\left.\begin{array}{l} \sigma = \pm i N \\ \sigma = \pm i N^{2}/R \end{array}\right\} \tag{4.15}$$

The term "buoyancy oscillation" is used to refer to an oscillatory motion with oscillation frequency N . The two contour integrals around the branch points $\pm i N$ describe such a motion as $t \to \infty$, and hence we refer to the sum of these contour integrals as the buoyancy oscillation propagator.

$$W_B = \frac{1}{2\pi i} \int_{\Gamma_B} e^{\sigma t} \bar{W}(\sigma)\, d\sigma \tag{4.16}$$

In textbooks the term "gravity wave" is used to describe plane wave solutions of homogeneous equations corresponding to (4.1). For such a solution to satisfy the homogeneous equation, the "dispersion relation"

$$\omega^2 = \frac{N^2(k_x^2 + k_y^2)}{(k_x^2 + k_y^2 + k_z^2)} \tag{4.17}$$

must be satisfied. (The restoring force of gravity is expressed through the buoyancy frequency N). This expression may be written

$$\omega = N \sin \theta \tag{4.18}$$

where

$$\theta = \sin^{-1}\left[\frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + k_z^2}\right]$$

is the angle which the direction of propagation of the sinusoidal wave motion makes with respect to the horizontal. The term z/R in (4.15) is likewise the sine of the angle of an observation point with respect to a source lying in the plane $z = 0$. Furthermore the contour integrals around branch points at $\sigma = \pm i N z/R$ give, as $t \to \infty$, an oscillation with frequency, $N z/R$. Hence we call the sum of the contour integrals around the branch point singularities at $\pm i N z/R$ the gravity wave propagator.

$$W_G = \frac{1}{2\pi i}\int_{\Gamma_G} e^{\sigma t}\,\overline{W}(\sigma)\,d\sigma \tag{4.19}$$

The propagator W is thus decomposed into "elementary" propagators associated with the different singularities of $\overline{W}(\sigma)$ and we write

$$W = W_B + W_G \tag{4.20}$$

where W_B refers to the buoyancy oscillation propagator and W_G refers to the gravity wave propagator.

The integral of $\overline{W}(\sigma) = (4\pi R)^{-1}(\sigma^2 + N^2)^{-1/2}(\sigma^2 + N^2 z^2/R^2)^{-1/2}$ around the two contours Γ_B, given in Fig. 4-1, is evaluated by use of the asymptotic solution given in Appendix III, A. of this chapter. The contribution from the integral taken around $\sigma = -iN$ is the complex conjugate of the contribution from the integral taken around $\sigma = iN$. We find

$$W_B = (4\pi N)^{-1}(x^2+y^2)^{-1/2}(2\pi Nt)^{-1/2}\,2\,\mathrm{Re}\; e^{i(Nt - \frac{3\pi}{4})}\Big[1 + \sum_{1}^{\infty}\alpha_K \frac{\Gamma(K+\frac{1}{2})}{\Gamma(\frac{1}{2})}(Nt)^{-K}\Big] \tag{4.21}$$

where

$$\alpha_K = \sum_{j=0}^{K}\left(\frac{i}{2}\right)^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})} f_{K-j} \tag{4.22}$$

and f_{K-j} is defined by the generating function

$$\left(1+\frac{2i\lambda}{1-z^2/R^2}-\frac{\lambda^2}{1-z^2/R^2}\right)^{-1/2}=\sum_{\ell=0}^{\infty} f_\ell\,\lambda^\ell \tag{4.23}$$

We thus find a simple expression for the buoyancy propagator for large time

$$W_B=\frac{1}{4\pi N(x^2+y^2)^{1/2}}\left(\frac{2}{N\pi t}\right)^{1/2}\left[\cos\left(Nt-\frac{3\pi}{4}\right)+\epsilon_B\right] \tag{4.24}$$

where

$$\epsilon_B=O\left[\left(1-z^2/R^2\right)^{-1}(Nt)^{-1}\right] \tag{4.25}$$

The gravity wave propagator is likewise found by integrating $\overline{W}(\sigma)$ around the two contours Γ_G .

$$W_G=\frac{1}{4\pi N(x^2+y^2)^{1/2}}\left(\frac{2}{N\pi(z/R)t}\right)^{1/2}\mathrm{Re}\;e^{iN(z/R)t-i\pi/4}\left[1+\sum_{1}^{\infty}\alpha_\kappa\frac{\Gamma(\kappa+\frac{1}{2})}{\Gamma(\frac{1}{2})}\left(\frac{Nzt}{R}\right)^{-\kappa}\right] \tag{4.26}$$

where α_κ is given above, but the $f_{\kappa-j}$ used are generated by

$$\left(1+\frac{2i\lambda R/z}{(R^2/z^2-1)}+\frac{\lambda^2}{R^2/z^2-1}\right)^{-1/2}=\sum_{\ell=0}^{\infty} f_\ell\,\lambda^\ell \tag{4.27}$$

We hence find the simple expression

$$W_G=\frac{1}{4\pi N(x^2+y^2)^{1/2}}\left(\frac{2}{N\pi\,z/R\,t}\right)^{1/2}\left[\cos\left(\frac{Nz}{R}t-\pi/4\right)+\epsilon_g\right] \tag{4.28}$$

where

$$\epsilon_g = O\left(1 - R^2/z^2\right)^{-1}\left(\frac{Nzt}{R}\right)^{-1} \tag{4.29}$$

The parameters N and $\frac{Nz}{R}$ are characteristic frequencies at which the atmosphere oscillates. Note that for $\left(1 - z^2/R^2\right) \ll 1$, the time that must elapse before (4.24) and (4.28) are applicable is much greater than if $\left(1 - z^2/R^2\right) \approx 1$. When the above strong inequality holds, the gravity and buoyancy propagators will oscillate together for many periods before there is effective separation between these two different kinds of waves. To handle this situation, we obtain an approximate solution useful for the time interval $\left(\frac{Nz}{R}\right)^{-1} < t < \left(\frac{Nz}{R}\right)^{-1}\left(1 - z^2/R^2\right)^{-1}$

Consider the integral

$$W = \frac{1}{4\pi R}\,\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{d\sigma\, e^{\sigma t}}{\left(\sigma^2 + N^2\right)^{1/2}\left(\sigma^2 + N^2(1-\epsilon)\right)^{1/2}} \tag{4.30}$$

where $\epsilon = (x^2+y^2)/R^2 \ll 1$. We find an expansion:

$$W = \frac{1}{4\pi N R}\, Re\; e^{iNt - i\pi/2}\left[1 + \sum_{k=1}^{\infty} \beta_k \epsilon^k\right]$$

where

$$\beta_1 = \frac{1}{4}\left(\frac{Nt}{i} + \frac{1}{2}\right)$$

and in general

$$\beta_k = \frac{\Gamma(k + 1/2)}{\Gamma(1/2)\,(k!)^2} \sum_{j=0}^{k} \frac{(j+k)!}{(k-j)!\, j!}\left(\frac{Nt}{2i}\right)^{k-j} \tag{4.31}$$

To a first approximation

$$W = \frac{1}{4\pi N R} \sin(Nt) \tag{4.32}$$

provided

$$\left(\frac{Nz}{R}\right)^{-1} \ll t \ll \left(\frac{Nz}{R}\right)^{-1}\left(1 - z^2/R^2\right)^{-1}$$

The present results are summarized as follows:

a) For $t < 0$, the atmosphere is at rest.

b) A delta function source is applied at $t = 0$.

At this instant a vertical motion is excited throughout the atmosphere. At first it increases from zero as a linear function of time and decays in space away from the source like R^{-1}. As time increases, the growth of the wave becomes less than linear.

c) For some time which is the order of N^{-1} the vertical motion is oscillatory in time, with a frequency N and an amplitude that decays in time. There is an amplitude decay in space like $(x^2+y^2)^{-1/2}$ This motion with frequency N is called a buoyancy oscillation.

d) At some later time of order $(Nz/R)^{-1}$, another motion with frequency (Nz/R) is also observed. This motion is called a gravity wave. One may consider the motion for times $t < N^{-1}$ to be a consequence of the gravity wave and buoyancy oscillation being nearly completely superimposed and hence cancelling each other by destructive interference. For times greater than $(Nz/R)^{-1}$ these two wave components, as defined by the singularities of $\widetilde{W}(\sigma)$, have dispersed into separate motions. For very large time the buoyancy oscillation and the gravity wave amplitudes decay like $t^{-1/2}$. This decay is a consequence of the energy flow out of any finite region surrounding the source. Such decay is commonly found in wave problems where the domain of solution is open to infinity.

In order to obtain a feeling for the accuracy of the description of atmospheric motions afforded by the asymptotic solutions of this study, it is instructive to examine a simple example where the "exact" solution is readily computed from mathematical tables. For this purpose, we use the gravity wave propagator obtained from (3.41) with the assumption $(N^2+\sigma^2)^{1/2} \simeq N^2$. This is equivalent to making the hydrostatic approximation so we shall call this propagator "the hydrostatic gravity wave propagator". This propagator has the contour integral representation

$$W_G = \frac{1}{4\pi NR} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{e^{\sigma t}\, d\sigma}{(\sigma^2 + N^2 z^2/R^2)^{1/2}} \tag{4.33}$$

The small time representation to second order is

$$4\pi NR \; W_G = 1 - \frac{N^2 z^2}{R^2}\frac{t^2}{4} + O\left(\frac{Nzt}{R}\right)^4 \tag{4.34}$$

The large time representation to lowest order is

$$4\pi N R W_G = \left(\frac{2}{\pi N(z/R) t}\right)^{1/2} \cos\left(\frac{Nz}{R} t - \pi/4\right) \tag{4.35}$$

The exact solution is

$$4\pi N R W_G = J_0\left(\frac{Nz}{R} t\right) \tag{4.36}$$

where J_0 is a zeroth order Bessel function.

In Fig. 4-2, we have plotted (4.34) and 4.35) and denoted points of the exact solution (4.36) in the region of worst fit by x. We have only plotted the first half oscillation of the solution. The asymptotic solution (3.64) gives results within (1%) for $Nzt/R > 4$. The conclusion reached is that W_G is approximated by (4.35) with error less than 5% for $\frac{Nzt}{R} > .8$, while W_G is approximated by (4.34) by error less than 5% when $\frac{Nzt}{R} < 1.2$.

If the transition from the large time solution to the small time solution were smoothly drawn in by hand, the resulting graph would describe the entire solution with only a few percent error.

The point to be made here is that in an inherently inaccurate science such as that of atmospheric motions, the approximate solutions such as obtained in this study provide as accurate a description of the dynamics as is warranted, given the defects in the physical model. Approximate solutions are frequently much more useful than exact solutions since they may be more easily manipulated and are easier to "understand". In the remainder of this thesis we shall determine only the lowest order approximations to wave propagation problems.

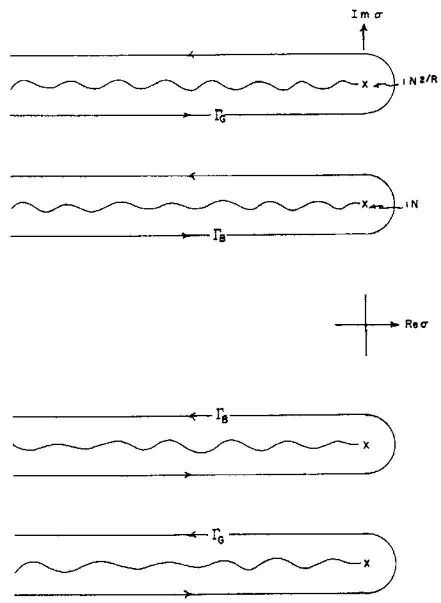


Figure 4-1. Branch line contours for integration of gravity waves and buoyancy oscillations.

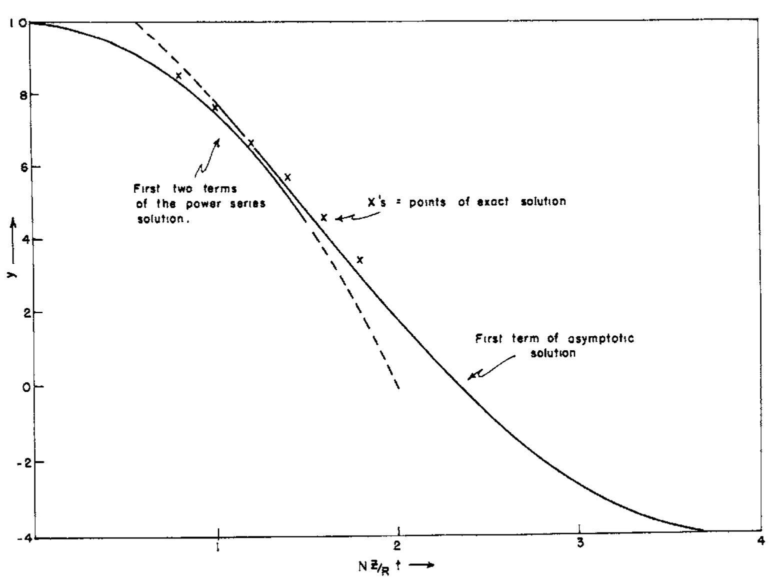


Figure 4-2. Matching of the small time power series solution to the large time asymptotic solution for the hydrostatic gravity wave propagator.

V. PROPAGATORS OF A STRATIFIED COMPRESSIBLE ATMOSPHERE

A. Propagators for a Nonhydrostatic Atmosphere

In this chapter we consider equation (3.24). We shall confine our attention to solutions for elementary point sources (the propagators) and do not consider the analysis for actual physical sources, which may be studied once the propagators are well understood.

An important difference between the solutions of (5.1) and those of the incompressible model (4.1) is that the solutions of (5.1) are identically zero outside of a sphere with radius $R = ct$. The set of points on the sphere $R = ct$ is known as the acoustic front. This sphere is a characteristic surface of the wave equation, and propagators defined on this sphere will have discontinuous derivatives of some order, or may themselves be discontinuous or have delta function singularities. Discontinuous or singular functions may be used in two different ways for describing the state of the atmosphere in the theory of atmospheric motions.

(1) They may be used as idealizations of functions which are continuous but change their magnitude over a very small distance as measured on some distance scale. For instance, the temperature across a "cold front" on a hemispheric weather map is often idealized to be a discontinuous function.

(2) They may be used to represent the response of the atmosphere to physically unrealistic, but mathematically convenient, elementary sources. This is how they occur in this thesis. One integrates over the solution for the elementary source to get the solution to the physical source. In many cases usage (2) reduces to (1). That is, at sufficient distance from a source, the atmospheric response may be asymptotically equal to the response to some elementary source.

We now consider the inhomogenous acoustic gravity wave equation

$$\left[\frac{\partial^2}{\partial t^2}\Delta_3 + N^2\Delta_2 - \left(\omega_A^2 + \frac{\partial^2}{\partial t^2}\right)\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]W = -\delta(\vec{R})\,\delta(t) \tag{5.1}$$

where we define the acoustic oscillation frequency ω_A by

$$\omega_A^2 = c^2/4H^2\,\left(1+\epsilon(z)\right) \tag{5.2}$$

For purposes of mathematical simplification, we shall assume that the parameters ω_A, N, c are independent of z, and that

$$\omega_A \gg N \gg Nz/R \tag{5.3}$$

The formal solution to (5.1) may then be written as

$$W = \frac{1}{4\pi R}(\sigma^2+N^2)^{-1/2}\left(\sigma^2+N^2z^2/R^2\right)^{-1/2} e^{-P(\sigma,\vec{R})}\,\delta(t) \tag{5.4}$$

where $P(\sigma,\vec{R})$ is defined as

$$P(\sigma, \vec{R}) = R/c\left[(\sigma^2+\omega_A^2)^{1/2}\frac{(\sigma^2+N^2 z^2/R^2)^{1/2}}{(\sigma^2+N^2)^{1/2}}\right] \tag{5.5}$$

The fractional differential operator in (5.4) differs from that for the incompressible atmosphere solution (4.11), by the factor $e^{-P(\sigma,\vec{R})}$, which may be considered to give a finite propagation speed for waves in a compressible stratified atmosphere.

In order to evaluate the propagator (5.4), we represent it by the contour integral

$$W = \frac{1}{4\pi R}\,\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\sigma\,\frac{e^{\sigma t - P(\sigma,\vec{R})}}{(\sigma^2+N^2)^{1/2}(\sigma^2+N^2 z^2/R^2)^{1/2}} \tag{5.6}$$

For $t < R/c$ we may close the contour in the $Re\ \sigma > 0$ plane, and hence by Cauchy's theorem obtain a null result. For $(t - R/c) \ll 1$, it is convenient to evaluate (5.6) by expanding $P(\sigma, \vec{R})$ in a power series in σ^{-1}, and then to also expand the integrand of (5.6) in a series in σ^{-1}. That is

$$\frac{e^{\sigma t - P(\sigma,\vec{R})}}{(\sigma^2+N^2)^{1/2}(\sigma^2+N^2 z^2/R^2)^{1/2}} = \frac{1}{\sigma^2}\,e^{\sigma t - R/c\,\sigma}\left(1 - \tfrac{1}{2}\frac{R}{c}\frac{\omega_A^2}{\sigma} + O(\sigma^{-2})\right) \tag{5.7}$$

Substitution of (5. 7) in (5. 6) and integration gives

$$4\pi R W = H(t - R/c)\left[(t - R/c) - \frac{R}{c}\frac{\omega_A^2}{4}(t - R/c)^2 + O(t - R/c)^3\right] \tag{5.8}$$

The integration is carried out by the method of steepest descent. The integrand has saddle points where $\frac{d}{d\sigma}\left(\sigma t - P(\sigma, \vec{R})\right) = 0$

That is

$$t = \sigma P(\sigma, \vec{R})\left[\frac{1}{\sigma^2 + \omega_A^2} + \frac{1}{\sigma^2 + N^2 z^2/R^2} - \frac{1}{N^2 + \sigma^2}\right] \tag{5.9}$$

which for sufficiently large time has the eight roots

$$\sigma = \sigma_A \simeq \frac{\pm i\,\omega_A}{(1 - R^2/c^2t^2)^{1/2}}, \quad t > R/c \tag{5.10}$$

$$\sigma = \sigma_G \simeq \frac{\pm i\, N z/R}{\left(1 - \left(\frac{\omega_A}{N}\right)^2\left(\frac{R}{ct}\right)^2\right)^{1/2}} \tag{5.11}$$

$$\sigma = \sigma_{B\pm} \simeq \pm i N\left(1 + \left(\frac{\omega_A}{2N}\frac{R}{ct}\right)^{2/3} e^{\pm\frac{2}{3}i\pi}\right) \tag{5.12}$$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} t >> \frac{\omega_A R}{N c}$$

In the vicinity of σ_A, $\sigma \simeq \sigma_A$, $P(\sigma, \vec{R})$ may be approximated by

$$P(\sigma, \vec{R}) \simeq R/c\,(\sigma^2 + \omega_A^2)^{1/2} \tag{5.13}$$

In the vicinity of σ_G, $\sigma \simeq \sigma_G$, $P(\sigma, \vec{R})$ may be approximated by

$$P(\sigma, \vec{R}) \simeq \frac{R}{c}\frac{\omega_A}{N}\left(\sigma^2 + N^2 z^2/R^2\right)^{1/2} \tag{5.14}$$

while in the vicinity of σ_{B+}, $\sigma \simeq \sigma_{B+}$, $P(\sigma,\vec{R})$ may be approximated by

$$P(\sigma, \vec{R}) \simeq \frac{R}{c}\frac{\omega_A (N/2)^{1/2}}{(\sigma - iN)^{1/2}} e^{-i\pi/4} \tag{5.15}$$

For large time the integral (5. 6) may be approximated by evaluation in the neighborhood of saddle points, and hence

$$W \simeq W_A + W_B + W_G \tag{5.16}$$

where

$$W_A = \frac{1}{4\pi R \sigma_A^2} \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\sigma\, e^{\sigma t - (\sigma^2 + \omega_A^2)^{1/2} R/c} \tag{5.17}$$

$$W_G = \frac{1}{4\pi R N} \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\sigma\, \frac{e^{\sigma t - a(\sigma^2 + N^2 z^2/R^2)^{1/2}}}{(\sigma^2 + N^2 z^2/R^2)^{1/2}} \tag{5.18}$$

$$W_B = \frac{(2/N)^{1/2}}{4\pi N R} \,\mathrm{Re}\, \frac{e^{-3\pi i/4}}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\sigma\, \frac{e^{\sigma t - a\left[(\sigma - iN)^{-1/2}\left(\frac{N^3}{2}\right)^{1/2} e^{-i\pi/4}\right]}}{(\sigma - iN)^{1/2}} \tag{5.19}$$

where we use

$$a = \frac{R}{c} \frac{\omega_A}{N} \tag{5.20}$$

and we make use of the assumption (5.3) that the characteristic frequencies are widely separated in order to simplify the integrals.

It is assumed that t is sufficiently large that the value of these integrals can be determined asymptotically by integration in the neighborhood of the saddle points, and so the sum of W_A, W_B and W_G differs from the value of (5.6) only by quantities that are asymptotically zero.

The functions W_A, W_B and W_G may be evaluated exactly using Appendix II, 3 and 7. We find

$$W_A = - H(t - R/c) \frac{1}{4\pi R \, \sigma_A^2} \frac{\partial}{\partial (R/c)} J_0 \, \omega_A \left(t^2 - \left(\frac{R}{c}\right)^2 \right)^{1/2} \tag{5.21}$$

$$W_G = H(t-a) \frac{1}{4\pi R N} J_0 \left[\left(\frac{N z}{R} \right) (t^2 - a^2)^{1/2} \right] \tag{5.22}$$

$$W_P = H(t-a) \frac{1}{4\pi R N} \left(\frac{2}{N} \right)^{1/2} \frac{(1+\epsilon)}{(3\pi t)^{1/2}} \cos \left[N \left(t + \frac{3}{2} \left(t a^2 \right)^{1/3} \right) - \frac{3\pi}{4} \right] \tag{5.23}$$

where we use Appendix III, C, to evaluate the integral for W_P asymptotically.

The asymptotic error ϵ is

$$\epsilon = O\left(N^{-1}(a^2 t)^{-1/3}\right)$$

The term W_A is designated the "acoustic wave" propagator. Note in particular than an expansion of W_A in powers of ($t - R/c$) matches the first two terms of W given by (5. 8).

The term W_B and W_G are designated the buoyancy and gravity wave propagators respectively. A precise estimate of the solution in the neighborhood of $t \doteq a$ is quite complicated, but it is adequate for descriptive purposes to assume (5. 22) and (5. 23) give the motion up to this point. The actual transition, somewhere in the neighborhood of $t \doteq a$, to asymptotically negligible buoyancy and gravity wave oscillations will occur smoothly, so that when (5. 22) and (5. 23) are differentiated the derivative of $H(t-a)$ should be neglected.

The motion described by (5. 16) consists of: a spherical acoustic front traveling outward with speed c, beyond which the atmosphere is at rest; a tail of acoustic oscillations of frequency which decreases to ω_A with increasing time; which is joined somewhere back of the front by buoyancy oscillations whose frequency asymptotes to N as $t \to \infty$; and gravity wave motions whose frequency approaches Nz/R , $t \to \infty$.

B. Propagators for a Rotating Hydrostatic Atmosphere

In this section we examine (3.56) for perturbations on a resting planar atmosphere in the absence of boundaries. We shall assume N^2H^2, and $\tilde{\epsilon}(\hat{z})$ in (3.55) are constants (not necessarily corresponding to isothermal atmosphere values). It is convenient to use the "geostrophically scaled" Cartesian coordinates, $\vec{\tilde{R}} = (x, y, \frac{NH}{f_0}\hat{z})$ That is, we define the stretched height coordinate z

$$\frac{NH}{f_0}\hat{z} = z \tag{5.24}$$

We shall use the notation $\tilde{R} = \left[x^2 + y^2 + \frac{N^2H^2}{f_0^2}\hat{z}^2\right]^{1/2}$ for radial distance in these coordinates, $\rho = (x^2+y^2)^{1/2}$ for horizontal radial distance, and $C = \left(\frac{1+\tilde{\epsilon}}{4N^2H^2}\right)^{-1/2}$ for the velocity of internal gravity waves. Assume also that time is scaled so that $f_0 = 2\Omega e \sin\theta_0 = 1$ The propagator for a rotating hydrostatic atmosphere is defined to be the solution to

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial t^2}+1\right)\left(\frac{\partial^2}{\partial z^2} - 1/C^2\right) + \Delta\right]\hat{h} = -\delta(\vec{\tilde{R}})\,\delta(t) \tag{5.25}$$

where $\delta(\vec{\tilde{R}}) = \delta(x)\,\delta(y)\,\delta(z)$. In the remainder of this chapter, we omit the " ^ " on $\hat{h} = \frac{\Pi^{-1/2}h}{NH}$. The spatial operator is inverted to obtain the operational solution

$$h = \frac{e^{-\frac{1}{C}\left[\sigma^2\rho^2 + \tilde{R}^2\right]^{1/2}}}{4\pi\left(\sigma^2\rho^2 + \tilde{R}^2\right)^{1/2}}\,\sigma^{-1}\delta(t) \tag{5.26}$$

The operator may be expanded in powers of σ^{-1} to obtain a power series evaluation

$$h = (4\pi\rho)^{-1} H(t - \rho/C)\left[(t-\rho/C) - \frac{1}{4}\frac{\tilde{R}^2}{\rho^2}\left(\frac{\rho}{C}(t-\rho/C) + \frac{1}{3}\left(t-\frac{\rho}{C}\right)^2\right) + \cdots\right] \tag{5.27}$$

which is useful for

$$(t - \rho/c) << 1$$

According to the hydrostatic model, the atmosphere is at rest beyond the surface ($t - \rho/C$), a cylinder with vertical axis through the source. Signals propagate vertically with infinite speed. This infinite speed of vertical signal transmission is a consequence of the hydrostatic relationship. The more correct nonhydrostatic model of the previous section shows that actually the gravity waves propagate behind the sphere $(\rho^2 + z^2)^{1/2} \simeq \frac{N}{\omega_A} ct \simeq Ct$, and that the amplitude of all derivatives smoothly asymptotes to zero, rather than discontinuously as in the hydrostatic model.

In order to evaluate the integrand for points of observation far behind the wave front, we express the solution (5.26) as a contour integral

$$h = \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \bar{h}(\sigma)\, e^{\sigma t}\, d\sigma \tag{5.28}$$

with

$$\bar{h}(\sigma) = (4\pi)^{-1}[\sigma^2\rho^2 + \tilde{R}^2]^{-1}\sigma^{-1}\, e^{-1/c\,[\sigma^2\rho^2 + \tilde{R}^2]}$$

The contour is deformed into steepest descent contours about the branch points of $\bar{h}(\sigma)$ at $\sigma = \sigma_g = \pm i\,\tilde{R}/\rho$, and a loop around the pole at $\sigma = 0$. The potential vorticity propagator, h_v, is defined to be the integral of $\frac{1}{2\pi i}\,\bar{h}(\sigma)\,e^{\sigma t}$ taken around the pole at $\sigma = 0$ and excluding the branch points. The gravity wave propagator h_g is defined to be $\frac{1}{2\pi i}\partial/\partial t$ of the integral of $\bar{h}(\sigma)\,e^{\sigma t}$ taken around the branch points at $\sigma = \pm i\,\tilde{R}/\rho$ and excluding the pole at the origin. Hence

$$h(x, y, z, t) = h_v(x, y, z, t) + \int_t^{\infty} h_g(x, y, z, t')\, dt' \tag{5.29}$$

where

$$h_v = \frac{1}{2\pi i}\int_{(\sigma = 0)} \bar{h}(\sigma)\, e^{\sigma t}\, d\sigma \tag{5.30}$$

and

$$h_g = \frac{1}{2\pi i}\int_{(\sigma = \pm i\,\tilde{R}/\rho)} \sigma\,\bar{h}(\sigma)\, e^{\sigma t}\, d\sigma \tag{5.31}$$

The above definition of h_g is such that $h_g \simeq W_G$ given by (5.22) when rotation is negligible and the hydrostatic approximation is valid. The conditions under which the hydrostatic and nonrotating atmosphere assumptions are tenable may then be investigated by comparing

h_g and W_G .

The integral for h_v is evaluated by Cauchy's residue theorem:

$$h_v = \frac{e^{-\tilde{R}/c}}{4\pi\tilde{R}} \quad , \quad t > R/C \tag{5.32}$$

The integral for h_g is evaluated as

$$h_g = H(t - \rho/c)\frac{1}{4\pi\rho} J_0\left[\tilde{R}/\rho\left(t^2 - \rho^2/c^2\right)^{1/2}\right] \tag{5.33}$$

For more complicated sources, the simplest "exact" solution possible is the Laplace integral solution, and approximate description of the ensuing motion by series solutions asymptotically valid for small and large time is appropriate. In Fig. 5-1, is shown $J_0(\tau^2-1)^{1/2}$. This figure illustrates the initial front and the oscillatory tail of the gravity wave propagator.

If we compare h_g with the earlier obtained W_G, it is seen that the primary difference in the motion far behind the wave front in the two cases is the difference of the oscillation frequencies: the present propagator has the dimensional frequency

$$\frac{\sigma_g}{\pm i} = \left(\frac{N^2H^2\hat{z}^2 + f_0^2\hat{\rho}^2}{x^2+y^2}\right)^{1/2}\left(1 - \frac{\rho^2}{c^2t^2}\right)^{1/2} \tag{5.34}$$

while the oscillation frequency of (5.22) is

$$\sigma_G/\pm i = \frac{Nz}{R}\left(1 - \omega_A/_N\left(\frac{R}{t}\right)^2\right)^{1/2} \tag{5.35}$$

For comparing these two frequencies, we use time parameters appropriate to the earth's atmosphere. These are

$$f_0 \sim 10^{-4} \ sec^{-1}$$

$$N \sim 10^{-2} \ sec^{-1} \tag{5.36}$$

This leads to the conclusions for the earth's atmosphere that

a) The effect of rotation is negligible at points of observation such that $\tan\Theta \simeq \Theta >> 10^{-4}$ where Θ is the declination of the point of observation with respect to the source.

b) The hydrostatic approximation is valid for points of observation such that $\sin\Theta \simeq \Theta << 1$

The present criteria are a consequence of the directional dispersion of a locally generated gravity wave motion. The atmosphere acts as a prism, such that the more horizontal the point of observation with respect to a local source, the lower the wave frequency. An equivalent statement of the above criteria is that the hydrostatic approximation is applicable to motions with oscillation frequencies very small compared to N, and the earth's rotation is negligible for motions with frequencies large compared to f_0, the Coriolis frequency.

There is yet another class of motion possible on a rotating plane, which are known as inertial motions. These are motions with an asymptotic frequency for large time which is the Coriolis frequency, f_0. One type of inertial motion arises as a limiting case of the gravity mode of motion. As seen from the frequency relationship (5.34), a gravity wave which propagates in a horizontal direction is an inertial oscillation with frequency f_0. Such motions commonly occur as a result of the horizontal guiding of a gravity wave, either as a Lamb wave, or as a trapped internal gravity wave guided by boundaries or thermal inhomogeneities. It is also possible to excite inertial oscillations which are independent of the gravity wave mode. We find such a motion as one of the modes of oscillation present in $h_{\beta\nu}$, the propagator for the β-plane equations. The physical significance of this oscillation is somewhat obscure, but it is of mathematical importance for the equation considered, since it is found to represent a motion that grows in time; that is, an instability.

In summary, for arbitrary impulsive point sources for the equation (5.25) one would expect to observe a motion consisting of

a) Gravity waves which decay algebraically for large time as their energy is lost to infinity.

b) A potential vorticity motion which is independent of time. In the presence of a mean flow, the motion will be approximately carried with the fluid particles. In a resting atmosphere, this mode loses no energy to infinity.

c) Possibly some inertial oscillations, which will decay in time (in the present case), as a result of wave radiation to infinity. (This decay follows from conservation of energy of solutions).

C. The β -Plane Propagators and a Method of Filtering

We now consider motions on a β -plane. The β -plane propagator for hydrostatic atmospheric motions is defined to be the solution to

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial t^2}+1\right)\left(\frac{\partial^2}{\partial z^2}-1/C^2\right)+\Delta\right]h+\beta\frac{\partial h}{\partial x}=-\delta(\vec{R})\delta(t) \qquad (5.37)$$

where we use the notation introduced in B. Time is scaled so that $f_0 = 1$. Inversion of the l. h. s. gives

$$h(x,y,z,t) = \bar{h}(\sigma)\ \delta(t) \qquad (5.38)$$

where we define $\bar{h}(\sigma)$ by

$$\bar{h}(\sigma) = (4\pi\sigma)^{-1}[\sigma^2\rho^2+\tilde{R}^2]^{-1/2}\exp\left[-\frac{\beta x}{2\sigma}+\left(C^{-2}+\frac{\beta^2/4}{\sigma^2(\sigma^2+1)}\right)^{1/2}(\sigma^2\rho^2+\tilde{R}^2)^{1/2}\right] \qquad (5.39)$$

We shall evaluate (5.38) for large time by means of contour integration

$$h(x,y,z,t) = \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon}\bar{h}(\sigma)\,e^{\sigma t}\,d\sigma \qquad (5.40)$$

The principle part of $\bar{h}(\sigma)$ as $\sigma\to\infty$ is

$$\bar{h}(\sigma) \sim (4\pi\rho)^{-1} e^{-\sigma\rho/c} (1 + O\sigma^{-1}) \tag{5.41}$$

Thus as in B. the motion is confined within the cylindrical hydrostatic gravity wave front

$$t = \rho/c \tag{5.42}$$

The solution may be expanded in powers of ($t - \rho/C$) behind this front. We omit this computation. To evaluate (5.40) asymptotically for large time, we must determine the saddle points of the exponent of $\bar{h}(\sigma)$. As $t \to \infty$, these s.p. approach the singularities of $\bar{h}(\sigma)$ which are

a) branch points at

$$\sigma = \pm i \omega_g = \pm i \tilde{R}/\rho \tag{5.43}$$

b) branch points at the four roots of

$$4\sigma^2(\sigma^2 + 1) + C^2\beta^2 = 0 \tag{5.44}$$

For values of β, N, H, $f_0 = 1$, appropriate to the earth's atmosphere, these roots are widely separated so that the following approximate formulae may be used for these b.p.

$$\sigma = \pm i \omega_{c1} = \pm i \left(1 - \frac{\beta^2 C^2}{8}\right)$$

$$\sigma = \pm i \omega_{c2} = \pm i \left[\frac{C\beta}{2}\left(1 + \frac{\beta^2 C^2}{8}\right)\right]$$

c) essential singularities at

$$\sigma = \pm i \tag{5.45}$$

d) an essential singularity at

$$\sigma = 0 \tag{5.46}$$

We deform the path of integration into the $Re\,\sigma < 0$ plane, and hence into contours encircling these nine singularities of $\bar{h}(\sigma)$, and thus decompose $h(x, y, \hat{z}, t)$ into the elementary propagators

$$h = \int_t^{\infty} h_g(t')\,dt' + h_{c_1} + h_{c_2} + h_I + h_{\beta v} \tag{5.47}$$

Assuming $e^{i\beta x/2\omega_g} \simeq 1$, the gravity wave propagator, h_g, will again be given approximately by (5.33).

The propagators h_{c_1} and h_{c_2}, not previously obtained, are tentatively labeled compression propagators. They appear to result from "coupling" of atmospheric compressibility with the inertial wave mode and potential vorticity wave respectively. In the limit as $\sigma \to i\omega_{c_1}$

$$\bar{h}(\sigma) \simeq b_1\, e^{-a_1(\sigma^2 + \omega_{c_1}^2)^{1/2}} \left(1 + O(\sigma - i\omega_{c_1})\right) \tag{5.48}$$

where

$$a_1 = \left| (\omega_{c_1}{}^2 - \omega_{c_2}{}^2)^{1/2} \left(\frac{\tilde{R}^2}{\omega_{c_1}{}^2} - \rho^2 \right)^{1/2} \frac{1}{C\,(\omega_{c_1}{}^2 - 1)^{1/2}} \right| \tag{5.49}$$

$$b_1 = (4\pi\, \omega_{c_1})^{-1} \left(\tilde{R}^2 - \omega_{c_1}^2 \rho^2 \right)^{-1/2} e^{+\left(\frac{i\beta x}{2\omega_{c_1}} - \frac{i\pi}{2} \right)}$$

As $\sigma \to i\,\omega_{c2}$

$$\bar{h}(\sigma) \sim b_2\, e^{-a_2 (\sigma^2 + \omega_{c2})^{1/2}} \left(1 + O(\sigma - i\omega_{c_1}) \right) \tag{5.50}$$

where the parameters a_2, b_2 are obtained from (5.49) by exchanging the subscripts 1 and 2. The large time behavior of h_{c_1} and h_{c2} can thus be computed by application of Appendix III, B. We omit this computation and merely quote the result that as $t \to \infty$:

$$h_{c1} \sim (\omega_{c_1} t)^{-3/2}$$
$$h_{c2} \sim (\omega_{c_2} t)^{-3/2} \tag{5.51}$$

so that h_{c_1} and h_{c2} will be asymptotically negligible relative to the other propagators in (5.47) for sufficiently large time.

The inertial mode propagator h_I is determined by the behavior of $\bar{h}(\sigma)$ in the vicinity of $\sigma = \pm i$. As $\sigma \to i$, we have

$$\bar{h}(\sigma) \sim b\; e^{-a(\sigma - i)^{-1/2}} e^{-i\pi/4} \left(1 + O(\sigma - i) \right) \tag{5.52}$$

where

$$a \simeq e^{-i\pi/2}\, \beta z / 2^{3/2}$$

$$b \simeq (4\pi z i)^{-1}\, e^{i\beta x/2} \tag{5.53}$$

Referring to Appendix III, C for the approximate evaluation of h_I by a saddle point integration, we find

$$h_I \simeq h_{I_+} \simeq 2\,\mathrm{Re}\left(\frac{a}{2t}\right)^{1/3} \frac{b}{(3\pi t)^{1/2}} \exp\left[i\left(t+\frac{\pi}{8}\right) + e^{2/9\,\pi i}\,\frac{3}{2}\left(\frac{\beta^2 z^2}{4}t\right)^{1/3}\right] \tag{5.54}$$

and hence

$$h_I \sim e^{\kappa t^{1/3}} \times \text{ algebraic terms in } t \tag{5.55}$$

where $\kappa = \left(\cos\frac{2}{9}\pi\right)\frac{3}{2}\left(\frac{\beta z}{2}\right)^{2/3}$

A rough estimate from parameters appropriate to the earth's atmosphere gives the e-folding growth and decay time for $h_{I_\pm}$ to be an order of 10-100 days. Since the β-plane equation does not have any simple energy integrals, it is not possible to discuss this instability of the inertial oscillation in terms of the release of some kind of potential energy. This instability does not correspond to any actual physical process, but appears to be a numerical instability of the approximation scheme, and hence an undesirable feature of equation (5.37).

The last propagator which occurs in the β -plane problem considered is $h_{\beta v}$, the potential vorticity propagator. The asymptotic behavior of this propagator may be determined by expanding $\tilde{h}(\sigma)$ in a power series in σ and integrating the series term by term with the kernel function $\frac{1}{2\pi i} e^{\sigma t}$. Keeping only the principle part of $\tilde{h}(\sigma)$ in expanding it in a series in σ , we find

$$\tilde{h}(\sigma) \simeq \frac{1}{4\pi\tilde{R}\sigma} e^{-\beta/2\sigma(x+\tilde{R})}\left(1+O(\sigma)\right) \tag{5.56}$$

The asymptotic solution for the propagator is

$$h_{\beta v} = \frac{(1+O(t^{-\frac{1}{2}}))}{4\pi\tilde{R}} J_0\left[2\beta t(x+\tilde{R})\right]^{1/2} \tag{5.57}$$

The wave oscillation frequency is seen from (5.57) to be given by

$$\frac{d}{dt}\left[2\beta t(x+R)\right]^{1/2} = \left[(x+\tilde{R})\,\beta/t\right]^{1/2} \tag{5.58}$$

At fixed time, lines of constant phase will be located on paraboloids of revolution in the coordinate system ($x, y, \frac{NH}{f_0}\hat{z}$). The focii are located at the origin.

That part of the initial disturbance which goes onto the potential vorticity propagator contains all wave lengths. Along the positive axis at fixed x , the wave number of the disturbance is seen to be

$$\frac{d}{dx}\left[2(\beta x t)^{1/2}\right] = (\beta t / x)^{1/2}$$

The very long wave length components are first past the point x . The asymptotic formula is not valid along the negative x axis, and there is no wave propagation in this direction.

It is of interest to compare the β -plane potential vorticity propagator with the potential vorticity propagator (5. 37) in a constantly rotating coordinate system. It is seen that the introduction of a gradient of potential vorticity in the β -plane approximation transforms a time invariant potential vorticity propagator into a wavelike propagator, which decays in time as a result of radiation of energy to infinity.

A question that arises is whether this elementary propagator can be studied independently of the other modes of motion. These other modes may be of no physical interest. This will be so, if it is possible to find a propagator equation which has only a single elementary propagator, namely $h_{\beta v}$. It is not possible to obtain such an equation which gives $h_{\beta v}$ exactly, but it is possible to obtain an equation with a propagator which is asymptotically equal. To achieve this result, we define a propagator h_R by retaining only the principle part of $\bar{h}(\sigma)$ in expanding it about $\sigma = 0$. That is we define

$$h_R = (4\pi \tilde{R})^{-1} e^{-\frac{\beta}{2\sigma}(x + \tilde{R})} \delta(t) \tag{5.59}$$

Apply the operator

$$\sigma \left[\Delta_3 - \beta^2/2\sigma^2 \right] e^{\beta x/2\sigma}$$

to both sides of (5. 59) to obtain the propagator equation

$$\left[\sigma \Delta_3 + \beta \frac{\partial}{\partial x} \right] h_R = - \delta (\vec{\tilde{R}}) \delta(t) \tag{5. 60}$$

It is clear from the manner in which equation (5. 60) was synthesized that it has the same asymptotic behavior as the potential vorticity propagator h_V . The contour integration evaluation of (5. 59) may be carried out exactly to obtain the first term of (5. 56).

$$h_R = (4\pi \tilde{R})^{-1} J_0 \left[2 (x + \tilde{R}) \beta t \right]^{1/2} \tag{5. 61}$$

See Fig. 5-2 and Fig. 5-3. We shall refer to h_R as the "Rossby wave propagator" and refer to (5. 60) as the Rossby wave equation. We shall return to the general theory of this equation in Chapter VIII.

It is possible to derive further equations which have solutions asymptotically equal to elementary propagators, by exactly the same procedure of expanding the Laplace transform of the propagator about its principle part, in the vicinity of other singularities. We call this process the "filtering" of the propagator to obtain an equation for the elementary propagator. One more example of this filtering process will suffice.

Let us obtain from the nonhydrostatic acoustic wave, gravity wave, and buoyancy oscillation propagators, defined by (5.17), (5.18) and (5.19), the corresponding propagation equations. After a little algebra, one finds the following equations for W_A, W_G, and W_B.

$$\frac{\partial^2}{\partial t^2}\left[\Delta_3 - \left(\omega_A^2 + \frac{\partial^2}{\partial t^2}\right)\right] W_A = -\delta(\vec{R})\,\delta(t) \tag{5.62}$$

$$\left[\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial z^2} + \left(\frac{\partial^2}{\partial t^2} + N^2\right)\left(\Delta_2 - \left(\frac{\omega_A}{Nc}\right)^2\frac{\partial^2}{\partial t^2}\right)\right] W_G = -\left(1 + \frac{1}{N^2}\frac{\partial^2}{\partial t^2}\right)^{1/2}\delta(\vec{R})\,\delta(t) \tag{5.63}$$

$$\left[\left(\frac{\partial^2}{\partial t^2} + N^2\right)\Delta_3 - \left(\frac{N\omega_A}{c}\right)^2\right] W_B = -\left(1 + \frac{1}{N^2}\frac{\partial^2}{\partial t^2}\right)^{1/2}\delta(\vec{R})\,\delta(t) \tag{5.64}$$

The above equations for W_A, W_G, and W_B have solutions which are asymptotically equivalent to the elementary propagators defined by (5.17) - (5.19) or by (5.21) - (5.23).

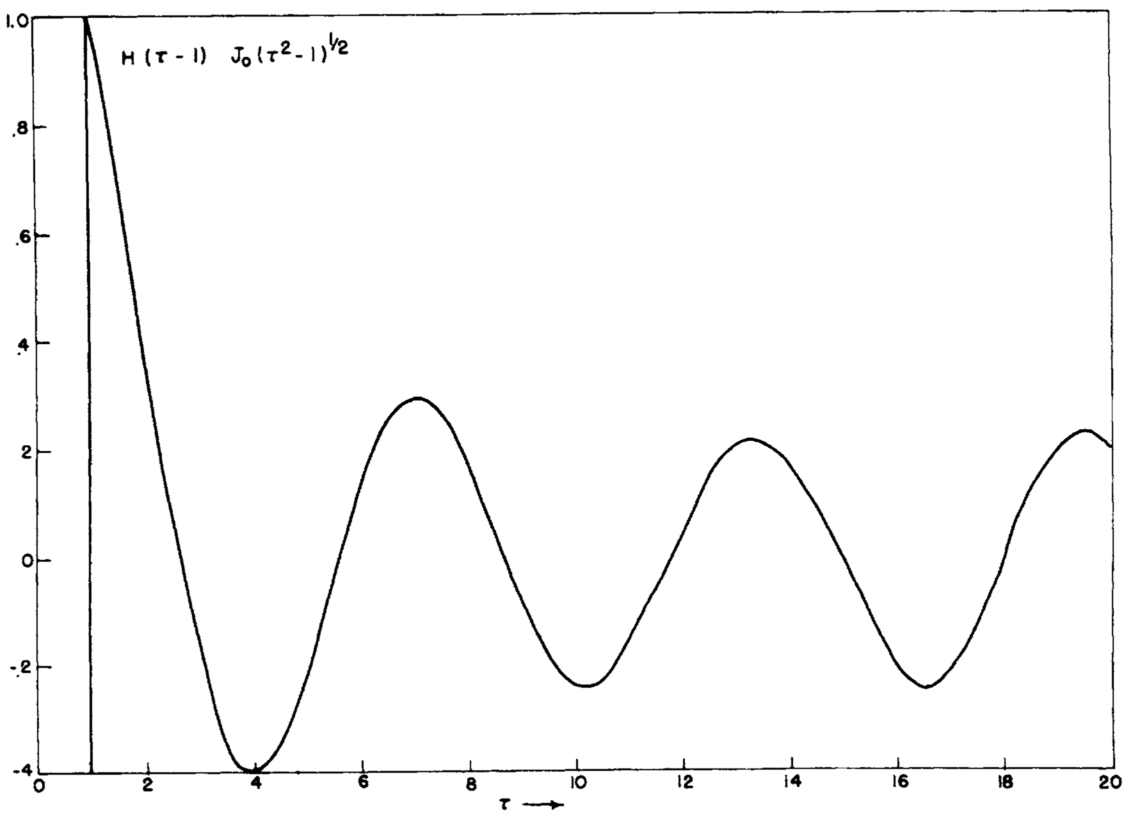


Figure 5-1. $4\pi\rho x$ hydrostatic gravity wave propagator, when we let $\tau = Nzt/\rho$, and we assume $Nz/C = 1$

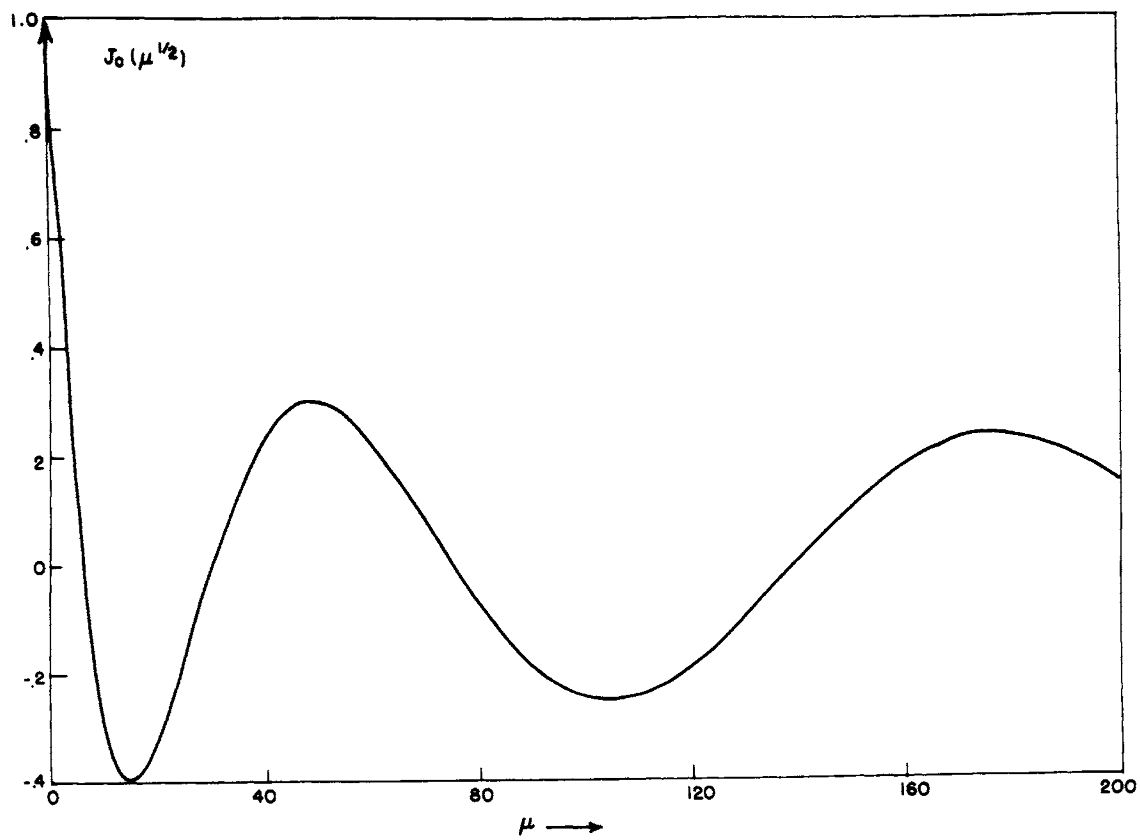


Figure 5-2. $4\pi\tilde{R}$ x Rossby wave propagator. $\mu = \beta t\left(x + \left(y^2 + \frac{N^2}{f_0^2} z^2\right)^{1/2}\right)$.

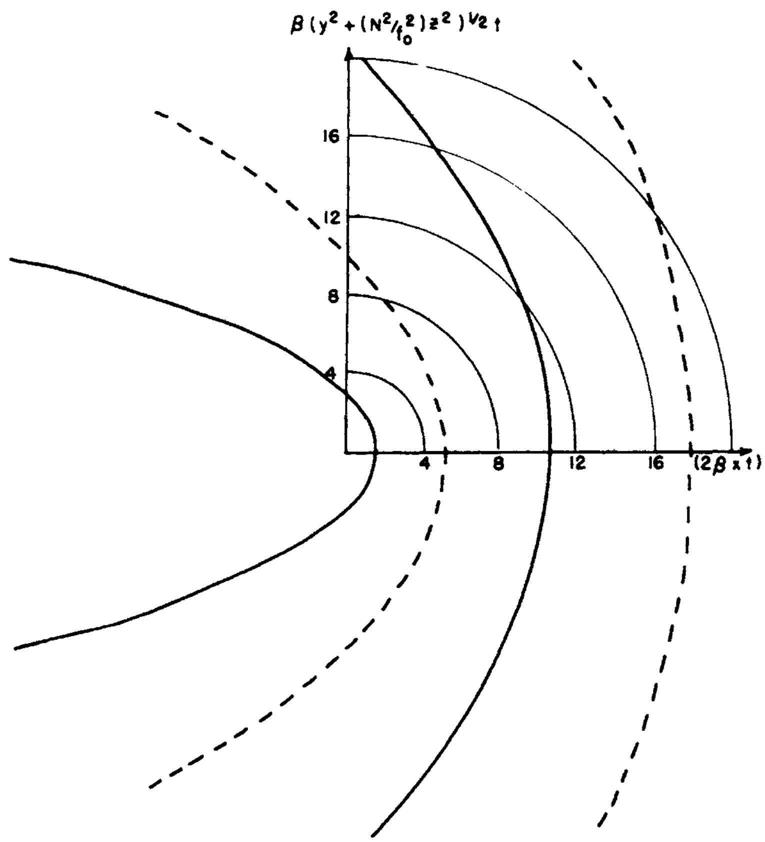


Figure 5-3. The trough-ridge pattern associated with the three dimensional Rossby wave propagator (——— = trough line, – – – – – = ridge line (units are in tens).

VI. THE LAMB BOUNDARY WAVE IN A STRATIFIED COMPRESSIBLE ATMOSPHERE

A. Introduction.

If we eliminate w in (3.17), (3.18) in terms of $\hat{p}$ defined by (3.19), we find that $\hat{p}$ satisfies a hyperbolic P. D. E. similar to (3.24). This is

$$\left[\frac{\partial^2}{\partial t^2}\Delta_3 - \frac{1+\epsilon^*(z)}{4H^2} + N^2\Delta_2 - 1/c^2\frac{\partial^4}{\partial t^4}\right]\hat{p} = -f(x,y,z,t) \tag{6.1}$$

where

$$\epsilon^*(z) = \epsilon(z) + 8H^2\frac{\partial\Gamma}{\partial z}$$

We have assumed

$$w\,\frac{\partial N^2}{\partial z}\Big/\left(N^2+\frac{\partial^2}{\partial t^2}\right)w \quad << 1$$

and may be neglected. The form of $f(x,y,z,t)$ depends on the energy sources assumed to be present. If the only energy source is addition of heat, Q, then $f(x,y,z,t)$ is

$$f(x,y,z,t) = \left[g\left(\frac{\partial}{\partial z}-\Gamma\right)+\left(N^2+\frac{\partial^2}{\partial t^2}\right)\right]\frac{\rho_0^{1/2}}{C_p\,\Theta}\frac{\partial Q}{\partial t} \tag{6.2}$$

where Γ is defined by (3.21). From (3.17), the boundary condition $w=0$ may be written

$$\left(\frac{\partial}{\partial z}+\Gamma\right)\hat{p} = 0 \qquad , \quad z=0 \tag{6.3}$$

It was first recognized by Lamb that for the isothermal atmosphere approximation, (6.1) has a homogenous solution

$$\hat{p} = e^{-\Gamma z} P_{(x,y)} \tag{6.4}$$

where $P_{(x,y)}$, the pressure amplitude at the lower boundary satisfies the equation for horizontally traveling acoustic waves.

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) P_{(x,y)} = 0$$

For this isothermal atmosphere mode, the vertical motion is identically zero, and $\hat{p}$ decays exponentially away from the lower boundary. We shall follow the text of Eckart in referring to this mode as a Lamb wave. In recent years there has been a renewed interest in two classical problems of atmospheric dynamics in which the Lamb wave is fundamental to the theory. These are:

a) The propagation of "pulse" around the globe from a point of origin. The high frequency nonhydrostatic components of this motion have been of interest for the discussion of the long distance propagation of energy from a nuclear blast, recorded as pressure fluctuations by microbarographs. The low frequency, hydrostatic components (on a rotating earth) are of interest for the discussion of the manner in which wind systems attain a state of "geostrophic balance".

b) The theory of atmospheric tides. The Lamb wave modes of oscillation on a spherical rotating earth have been extensively discussed as possibly the largest amplitude component of the pressure fluctuations in atmospheric "tides" observed at the earth's surface. In particular, present evidence indicates, that the semidiurnal traveling pressure wave, corresponds to a Lamb wave forced by thermal heating in the stratosphere. In tidal theory, the Lamb wave is usually designated simply as the lowest mode, or more explicitly as the mode with " h_n ", "the equivalent height", being approximately 10 km.

In order to synthesize the solution of the inhomogenous system (6.1) and (6.2), one might use a "spectral expansion". That is, we seek solutions of the homogeneous system depending on the parameter λ

$$\left(\frac{1}{N^2} \frac{\partial^2}{\partial t^2} \left[\left(\frac{\partial^2}{\partial z^2} - \frac{1+\epsilon^*}{4H^2} \right) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] + \lambda \right) P_\lambda (z, t) = 0 \qquad (6.5)$$

$$\frac{\partial P_\lambda}{\partial z} + \Gamma P_\lambda = 0 \quad , \; z = 0$$

such that an arbitrary function of z and t may be expressed as an integral over λ (the spectrum of P_λ). Then in particular, we might express $\hat{p}$, and the forcing $f(x, y, z, t)$ as integrals over the λ . That is

$$f(x,y,z,t) = \int_\lambda d\,\rho_\lambda(z,t)\;F(x,y,\lambda)$$

$$\hat{p}(x,y,z,t) = \int_\lambda d\,\rho_\lambda(z,t)\;p(x,y,\lambda)$$

and hence reduce the solution of (6. 1) to evaluation of the much simpler inhomogenous problem

$$[\Delta_2 - \lambda]\;p(x,y) = -F(x,y) \tag{6. 6}$$

In general, the spectral expansion will be of some mathematical difficulty to establish, but in the simple case that the coefficients in(6. 1) are for an isothermal atmosphere and hence independent of z , it is quite straight-forward, to separate out the z dependence by expansion in the spectrum of the singular Sturm Liouville system

$$\left[\frac{d^2}{dz^2} - \frac{1}{4H^2} + K^2\right]\rho(K,z) = 0 \tag{6. 7}$$

$$\left(\frac{d}{dz} + \Gamma\right)\rho(K,z) = 0 \quad , \; z = 0$$

The expansion is obtained " by integrating around the singularities of the Green's function. " (See chapter 10, B). The result may then be expressed as

$$\delta(z-z') = 2\Gamma e^{-\Gamma(z+z')} + \frac{2}{\pi}\int_0^\infty \cos\kappa(z-z')\,d\kappa + \frac{2}{\pi}\,Re\int_0^\infty \frac{\kappa+i\Gamma}{\kappa-i\Gamma}\,e^{i\kappa(z+z')}\,d\kappa \tag{6.8}$$

To derive more complicated expansions for physical problems, it is often necessary to proceed heuristically by contour integral techniques, but here theory is rigorous. See for instance Titchmarsh, Vol. I.

The physical intepretation of (6. 8) is that we may express any function of z , as a Lamb wave and two Fourier integrals. The first integral is the Fourier expansion for an unbounded medium and may be associated with motion that arises in the absence of boundaries, while the second term may be considered to represent image motion due to the lower boundary.

We shall make use of the expansion (6. 8) to discuss in the remainder of this chapter, propagation of Lamb waves in an isothermal atmosphere. In the next section, we continue the discussion of Lamb waves given by the nonhydrostatic model, while in the last section, we shall consider Lamb waves given by the rotating hydrostatic model for atmospheric motions.

B. The Propagation of a Lamb Wave in a Nonhydrostatic Isothermal Atmosphere.

We consider the equation

$$\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial z^2} - \Gamma^2\right) + \left(N^2 + \frac{\partial^2}{\partial t^2}\right)\left(\Delta_2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\right]\hat{p} = -f(x,y,z,t) \tag{6.9}$$

with a point heat source . (See (6.2)).

$$f(x,y,z,t) = \left[g\left(\frac{\partial}{\partial z} - \Gamma\right) + \left(N^2 + \frac{\partial^2}{\partial t^2}\right)\right]\frac{\partial}{\partial t}\,\delta(\vec{R})\,\delta(t) \tag{6.10}$$

and the boundary condition

$$\left(\frac{\partial}{\partial z} + \Gamma\right)\hat{p} = 0 \tag{6.11}$$

We shall assume the isothermal atmosphere approximation so that

$$\Gamma^2 = \frac{1}{4H^2} - N^2/c^2$$

For simplicity we shall first develop the theory of (6.9) for unbounded planar coordinates and then sketch the extension to spherical coordinates.

We use the eigenfunction expansion (6.8) to separate out the dependence in the z variable. This gives

$$\hat{p} = p_L + p_D + p_I \tag{6.12}$$

where P_L is the Lamb wave propagator for a point heat source; P_D , and P_I are referred to as direct source and image propagators, respectively.

Here P_L , P_D and P_I are given by

$$P_L = 2\Gamma e^{-\Gamma(z+z')} P_0(x,y) \tag{6.13}$$

$$P_D = \frac{2}{\pi} Re \int_0^\infty P_K(x,y,K)\, e^{iK(z-z')} dK \tag{6.14}$$

$$P_I = \frac{2}{\pi} Re \int_0^\infty P_K(x,y,K)\left(\frac{K+i\Gamma}{K-i\Gamma}\right) e^{iK(z+z')} dK \tag{6.15}$$

and P_0 , and P_K satisfy the P. D. E.

$$\left(\frac{\partial^2}{\partial t^2} + N^2\right)\left(\Delta_2 - \frac{1}{C^2}\frac{\partial^2}{\partial t^2}\right) P_0 = -\left(-2\Gamma g + \left(N^2 + \frac{\partial^2}{\partial t^2}\right)\right)\frac{\partial}{\partial t}\, \delta(x)\delta(y)\delta(t) \tag{6.16}$$

$$\left[\left(\frac{\partial^2}{\partial t^2} + N^2\right)\left(\Delta_2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) - (\Gamma^2 + K^2)\frac{\partial^2}{\partial t^2}\right] P_K = -\left(g(iK-\Gamma) + \left(N^2 + \frac{\partial^2}{\partial t^2}\right)\right)\frac{\partial}{\partial t}\,\delta(x)\delta(y)\delta(t) \tag{6.17}$$

We are not here particularly interested in the analysis for P_D and P_I , so we merely here give the solution for P_L , and mention that P_I can be expressed as a line image source of P_D . We find

$$P_D = \left[g\left(\frac{\partial}{\partial z} - \Gamma\right) + (N^2+\sigma^2)\right] \frac{e^{-\frac{P(\sigma,\vec{R})}{\sigma}}}{4\pi R(\sigma^2+N^2)^{1/2}(\sigma^2+N^2 z^2/R^2)^{1/2}} \delta(t) \qquad (6.18)$$

which is essentially the result (5.06), manipulated to give the gravity waves and acoustic waves excited by a point heat source in an unbounded atmosphere. For further approximate reductions, we refer the reader to the discussion following (5.06).

We now proceed to a more detailed discussion of the Lamb wave propagation. Inverting the spatial operator in (6.16) gives

$$P_0 = \left[1 - \frac{2\Gamma g}{\sigma^2+N^2}\right] \sigma \frac{1}{2\pi} K_0(\sigma \rho/c) \; \delta(t) \qquad (6.19)$$

The first term gives

$$\frac{1}{2\pi}\left[\sigma \; K_0(\sigma \rho/c)\right] \delta(t) = \frac{1}{2\pi} \frac{\partial}{\partial t}\left(\frac{H(t-\rho/c)}{(t^2-\rho^2/c^2)^{1/2}}\right) \qquad (6.20)$$

The second term may be evaluated exactly as a contour integral or convolution integral using

$$\left[\frac{\sigma}{\sigma^2+N^2} \; K_0(\sigma \rho/c)\right] \delta(t) = \int_{\rho/c}^{t} d\tau \frac{\cos N(t-\tau)}{(\tau^2-(\rho/c)^2)^{1/2}} \qquad (6.21)$$

$$= \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \sigma \, e^{\sigma t} \frac{K_0(\sigma \rho/c)}{(\sigma^2+N^2)} \, d\sigma$$

An approximate description for large ρ/c is given by

$$\left[\frac{\sigma}{\sigma^2+N^2}\, K_0\!\left(\frac{\sigma\rho}{c}\right)\right]\delta(t) \doteq H(t-\rho/c)\left(\frac{c}{2\rho}\right)^{1/2}\left[2(t-\rho/c)^{1/2} - \frac{8}{15}N^2(t-\rho/c)^{5/2} + \epsilon\right] \quad (6.22)$$

$$\epsilon = O\left(N^2(t-\rho/c)^{7/2} + \frac{c}{\rho}(t-\rho/c)^{1/2}\right)$$

$$\left[\frac{\sigma}{\sigma^2+N^2}\, K_0\!\left(\frac{\sigma\rho}{c}\right)\right]\delta(t) = \left(\frac{c\,\pi}{2\rho N}\right)^{1/2}\left[\cos\left(N(t-\rho/c) - \frac{\pi}{4}\right) + O(t-\rho/c)^{-n}\right] \quad (6.23)$$

Here (6.23) is suitable for evaluation near the front $(t-\rho/c)$, and (6.23) suitable for $N(t-\rho/c) \gtrsim 1$.

Now an actually excited Lamb wave motion will be greatly complicated by the details of the physical source. Any reasonable source will smooth out the singularity of (6.20) at $t = \rho/c$, and lead to a damping of the oscillations described by (6.21) and (6.23). The motion then predicted is :

a) A sharp, essentially nondispersive pulse traveling horizontally with the speed of sound.

b) A long decaying tail of buoyancy oscillations, with a period of 5-10 minutes, (depending on what part of the troposphere or lower stratosphere determines the buoyancy frequency). Comparison with Fig. 1-2, (from Wexler and Hass) suggests that the above theory may account for the initial observed pulse and low frequency tail of the observed pressure wave, but that without elaborate hypothesis as to source, we can not expect to explain the higher frequency intermediate portion of the trace, or the frequently observed very high frequency waves. Rather extensive numerical computations of the acoustic-gravity wave normal modes have been presented by Pfeffer and Zarichny, (1962), (1963) and Press and Harkrider, (1962) and Harkrider (1964). These studies have been relatively successful in explaining the observed pressure wave in terms of the nonisothermal atmosphere Lamb wave and the first few internal gravity wave and acoustic wave modes. These internal modes may account for discrepencies between the present isothermal atmosphere theory and the observed wave. As a contribution to further understanding of this phenomenon, we wish to emphasize that the term $\left(\frac{\partial^2}{\partial t^2} + N^2\right)^{-1}$ in the solution operator will convert an otherwise nondispersive wave motion into oscillations with the buoyancy frequency. This point appears to have been anticipated by Pekeris (1948), but has been somewhat neglected in more recent studies, which have emphasized those aspects of the motion which are related to dispersive wave propagation theory.

In order to extend the above analysis to a sphere, it is convenient to use a two dimensional spherical coordinate system with the source located at the "north pole". We take θ to measure colatitude - that is $r_e\theta$ is the great circle distance from the source. The Lamb wave propagation equation (6.16) then is equivalent to

$$\left[\frac{1}{\sin\theta\, r_e^2}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]P_o = -\ \delta(t)\frac{\delta(\theta)}{2\pi\sin\theta} \qquad (6.24)$$

We have here simplified somewhat the forcing so that it is a simple unit impulse. It is simple to obtain the Green's function for a unit heat impulse, by analogy to the analysis following (6.16), once the result of (6.24) is known. The inversion of the spatial dependence of (6.24) (c.f., for instance Friedlander, p. 170) gives

$$P_o = \left(\cosh\left[\pi\left(\left(\frac{r_e\sigma}{c}\right)^2-\tfrac{1}{4}\right)^{1/2}\right]\right)^{-1}\frac{1}{4}F\left(\tfrac{1}{2}+i\left(\left(\frac{r_e\sigma}{c}\right)^2-\tfrac{1}{4}\right)^{1/2}, \tfrac{1}{2}-i\left(\left(\frac{r_e\sigma}{c}\right)^2-\tfrac{1}{4}\right)^{1/2}, 1; \frac{1+\cos\theta}{2}\right)\delta(t) \qquad (6.25)$$

One finds a uniformly valid approximation for large time is given by

$$P_o \simeq \frac{(\pi/2\sigma)^{1/2}}{2\pi\left(r_e/c\ \sin\theta\right)^{1/2}}\left[e^{-\sigma r_e\theta/c} + e^{-\sigma r_e/c(\pi-\theta)}\right]\sum_{j=0}^{\infty} e^{-2\pi j r_e\sigma/c}\,\delta(t) \qquad (6.26)$$

$$\simeq \frac{1}{2\pi}\left(\frac{c}{2r_e\sin\theta}\right)^{1/2}\sum_{j=0}^{\infty}\left[\frac{H(\tau_j - r_e\theta)}{(\tau_j - r_e\theta)^{1/2}} + \frac{H(\tau_j - r_e(\pi-\theta))}{(\tau_j - r_e(\pi-\theta))^{1/2}}\right] \qquad (6.27)$$

where we have assumed in (6.25) that

$$\left(\frac{r_e^2\sigma^2}{c^2} - \tfrac{1}{4}\right)^{1/2} \simeq \frac{r_e\sigma}{c}$$

and where

$$\tau_j = t - 2nj\, r_e \tag{6.28}$$

(A result similar to (6.26) has been found by D. B. Van Hulsteyn, (1965),

The first term in the brackets of (6.21) gives the wave directly from the source plus the contribution for the waves that have circled the earth j times ($j = 1, 2, \cdots$), refocused at the poles, and propagated out again, while the second term in brackets gives the wave that has propagated around the globe j times ($j = 1, 2, \cdots$), refocused at the antipodal point and propagated out from there. (Because of dissipation, the number of actually observed waves is limited to the first few terms in this series).

The solution to (6.24) for an unbounded plane approximation is (c. f., (6.20)).

$$P_o = \frac{1}{2\pi}\left[K_o\left(\sigma r_e \theta/c\right)\right] \delta(t) = \left(\frac{c}{2 r_e \theta}\right)^{1/2} \frac{1}{2\pi} \frac{H(t - r_e\theta/c)}{(t - r_e\theta/c)^{1/2}} \tag{6.29}$$

This is the first term of (6.27) multiplied by $(\sin\theta/\theta)^{1/2}$. For small θ so that $(\sin\theta/\theta) \doteq 1$, the unbounded planar model gives the correct direct source wave. However for $\left(\frac{\theta}{\sin\theta}\right)^{1/2} > 1$, the planar model overestimates the geometrical attenuation by this factor. After the wave passes the equator of the coordinate system, the planar model continues to predict attenuation of a spreading wave, while the spherical solution correctly predicts that the wave amplitude begins again to increase as a result of propagation inward to the antipodal point.

C. The Propagation of a Lamb Wave in a Rotating Hydrostatic Atmosphere

We discuss the equation

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial t^2}+f_0^2\right)\left(\frac{\partial^2}{\partial \hat{z}^2}-\frac{1}{4H^2}\right)+N^2\Delta_2\right]\Pi^{1/2}h = f(x,y,\hat{z},t) \tag{6.30}$$

where $z = H\hat{z}$

and

$$f(x,y,\hat{z},t) = \left(\frac{\partial^2}{\partial t^2}+f_0^2\right)\Pi^{-1/2}\frac{\partial}{\partial \hat{z}}\left(\frac{\Pi}{NH}\right)^2\frac{Q}{\rho_0 C_p \Theta} \tag{6.31}$$

Eq. (6.30) is the "internal gravity wave" equation (3.56), where we assume that the atmosphere is isothermal and that the only forcing is a heat input Q. The boundary condition assumed is $w=0$, at $z=0$, or from (3.64).

$$\Pi^{1/2}\left(\frac{\partial}{\partial z}-\frac{\kappa}{H}\right)h = \left(\frac{\partial}{\partial \hat{z}}+\Gamma\right)\Pi^{1/2}h = 0 \tag{6.32}$$

We shall take for $f(x,y,\hat{z},t)$

$$f(x,y,\hat{z},t) = \left(\frac{\partial^2}{\partial t^2}+f_0^2\right)\frac{\partial}{\partial \hat{z}}\,\delta(\vec{R})\,\delta(t) \tag{6.33}$$

which according to (6.31), is a unit impulse of heat. The projection of this unit heat pulse forcing onto the Lamb wave mode is given by (6.8) as f_L, where

$$f_L(x,y,z,t) = 2\Gamma \int_0^\infty dz' \, f(x,y,z',t)\, e^{-\Gamma(z+z')} = -2\Gamma^2\left(\frac{\partial^2}{\partial t^2}+f_0^2\right)\delta(x)\delta(y)\delta(t) \quad (6.34)$$

Let the projection of the perturbation geopotential height onto the Lamb wave mode be written as

$$h_L = 2\Gamma \int_0^\infty dz' \, e^{-\Gamma(z+z')} \pi^{1/2} h(z-z') \quad (6.35)$$

Then h_L satisfies the P. D. E.

$$\frac{\partial}{\partial t}\left[\Delta_2 - 1/c^2\left(\frac{\partial^2}{\partial t^2}+f_0^2\right)\right] h_L = \frac{f_L}{N^2} \quad (6.36)$$

with f_L defined by (6.34) and c is the speed of sound

$$c = N\left(\frac{1}{4H^2} - \Gamma^2\right)^{-1/2} = (\gamma g H)^{1/2} \quad (6.37)$$

The Lamb wave is then obtained by inversion of the spatial dependence of (6.36) as

$$h_L = \frac{-\Gamma^2}{\pi N^2}(\sigma^2+f_0^2)\,\sigma^{-1} K_0\left[(\sigma^2+f_0^2)^{1/2}\rho/c\right] \delta(t) \quad (6.38)$$

which has the exact evaluation

$$h_L = \frac{-\Gamma^2}{\pi N^2}\left[\left(\frac{\pi}{2}\right)^{1/2}\sigma^{1/2}\delta(t-\rho/c) + H(t-\rho/c)\left(\frac{\partial^2}{\partial t^2}+f_0^2\right)\int_{\rho/c}^{t}\frac{\cos f_0(\tau^2-(\rho/c)^2)^{1/2}}{(\tau^2-(\rho/c)^2)^{1/2}}d\tau\right] \quad (6.39)$$

For $0 < t - \rho/c < 1$ one may use

$$h_L \simeq \frac{-1}{\pi} \frac{\Gamma^2}{N^2} \frac{\partial}{\partial t} \; H(t-\rho/c) \left[\frac{\cos f_0 (t^2 - (\rho/c)^2)^{1/2}}{(t^2 - (\rho/c)^2)^{1/2}} \right] \tag{6.40}$$

while for $(t - \rho/c) > 1$, we may use

$$h_L \simeq \frac{2\Gamma^2}{\pi N^2} \left(\frac{1}{t^2}\right) \cos(f_0 t) \quad + h_{Lp}$$

where h_{Lp} is given by

$$h_{Lp} = \frac{-\Gamma^2}{\pi N^2} f_0^2 \; K_0 (f_0 \rho/c) \tag{6.41}$$

The term h_{Lp}, which does not decay with time, is interpreted as a geostrophically balanced Lamb wave.

Thus in summary: The Lamb wave motion excited by a unit impulse of heat consists of

a) An outward horizontally propagating inertial-gravity wave, consisting of a discontinuous front $t = \rho/c$, and a long train of decaying inertial oscillations.

b) A residual time independent motion, associated with geostrophically balanced motions.

Variations of the above problem are common in the meteorological literature. Earliest analytic work is that of Cahn (1945) for one dimensional shallow water waves (or a rotating ocean), and Obukhov (1949) for two dimensional shallow water waves.

If the variability of the earth's vorticity is introduced through the " β -effect", it is found that the residual geostrophic motions also radiate to infinity. See Dobrischman (1964). Assuming a constant Coriolis parameter, the planar approximation results may be extended to the direct source solution on a sphere, by weighting the solution by a geometrical factor $\left(\frac{\theta}{\sin\theta}\right)^{1/2}$ to account for the focusing of the spherical coordinate system, (θ = angular great circle distance from source). Further wave terms representing the contribution from waves that have refocused one or more times at the source point and at the antipodal point may be added, if dissipation is sufficiently small to permit propagation for great distances. The quantitative generalization of (6.40) to inviscid propagation on a sphere is

$$h_L \simeq \left(\frac{c}{2 r_e \sin\theta}\right)^{1/2} \frac{-P^2}{\pi N^2} \frac{\partial}{\partial t}\left[\sum_{j=0}^{\infty} H(\tau_j - r_e\theta)\, F_{j+} + H(\tau_j - r_e(\pi-\theta))\, F_{j-}\right] \quad (6.42)$$

where

$$F_{j+} = \frac{\cos\left[f_o\left(\tau_j^2 - \left(\frac{r_e\theta}{c}\right)^2\right)^{1/2}\right]}{(\tau_j - r_e\theta)^{1/2}}$$

$$F_{j-} = \frac{\cos\left[f_o\left(\tau_j^2 - \left(r_e\left(\frac{\pi-\theta}{c}\right)\right)^2\right]}{(\tau_j - r_e\theta)^{1/2}} \quad (6.43)$$

and τ_j is given by (6.28).

Presumably a more realistic model, including the variability of the Coriolis force would give results qualitatively similar to (6.42).

VII. APPLICATION OF FOURIER INTEGRAL METHODS TO ATMOSPHERIC WAVE MOTIONS

A. Alternate Methods of Analysis for Atmospheric Wave Motions - The Method of Multiple Stationary Phase

In the preceeding chapters, we have presented solutions for various types of atmospheric wave motions excited by point sources. While the scale of motions ranged from acoustic motions to Rossby waves, it was possible to discuss all these motions by similar mathematical techniques. This uniformity in procedure was possible because the spatial dependence of the solutions could be determined by the inversion of second order constant coefficient elliptic operators, and thus exact solutions could be obtained as certain operational expressions .

The purpose of this chapter is to introduce some convenient Fourier integral techniques which may be used as alternate methods for solution of the problems previously discussed. We can gain in this manner further insight into the physical content of the theory and obtain some modifications that are not possible to derive by means of the techniques previously employed.

In this section we describe, in the context of a simple example, the various alternate procedures which may be employed for solution of the preceeding problems. Our primary result will be an approximate method for the evaluation of multiple Fourier integrals, known as the method of multiple stationary phase.

We discuss here the two dimensional hydrostatic gravity wave equation for a point implusive source. It is to be remembered that these results are somewhat artificial since the ultimate rate of decay of the motion will depend on the spatial dimensions of the source. The equation considered is

$$\left(\frac{\partial^4}{\partial t^2 \partial z^2} + N^2 \frac{\partial^2}{\partial x^2}\right)\hat{w} = -\frac{\partial}{\partial x}\delta(x)\delta(z)\delta(t) \tag{7.1}$$

The previously employed operational technique gives the solution to this problem as

$$\hat{w} = -\frac{\sigma x}{2\pi N}\left(\sigma^2 x^2 + N^2 z^2\right)^{-1}\delta(t) \tag{7.2}$$

Evaluation of this expression by contour integration gives

$$\hat{w} = -\frac{1}{2\pi N x}\cos\left(N z t/x\right) \tag{7.3}$$

Alternatively we may use a Fourier integral approach based on the spectral expansion

$$\delta(x)\delta(y) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} e^{ikx+i\gamma z}\,dk\,d\gamma \tag{7.4}$$

We hence seek solutions of the form

$$\hat{w}(x,z,t) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} W(\kappa,\gamma,t)\, e^{i\kappa x + i\gamma z}\, d\kappa\, d\gamma \tag{7.5}$$

which will satisfy (7.1) provided W satisfies the ordinary differential equation

$$\left(\gamma^2 \frac{\partial^2}{\partial t^2} + \kappa^2 N^2\right) W = i\kappa\, \delta(t) \tag{7.6}$$

which has the solution

$$W = \frac{i}{N\gamma} \sin\left[(\kappa N/\gamma)\, t\right] \tag{7.7}$$

Three obvious integration procedures for the evaluation of (7.5) are a) integrate over κ, then γ, b) integrate over γ, then κ, c) carry out both integrationssimultaneously. These proceed as follows:

a) After integrating in κ, (7.5) reduces to

$$\hat{w} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\delta(x+Nt/\gamma) - \delta(x-Nt/\gamma)\right]\frac{e^{i\gamma z}\, d\gamma}{2N\gamma} \tag{7.8}$$

The γ integration then gives (7.3).

b) After integrating in γ, (7.5), reduces to

$$\hat{w} = \frac{1}{4\pi N}\int_{-\infty}^{\infty} J_0\left[2(N\kappa z t)^{1/2}\right] e^{i\kappa x}\, d\kappa \tag{7.9}$$

The integration in κ may then be carried out exactly by introducing an integral representation for the Bessel function, and by carrying out the integration under the integral sign. This reduces the integral to (7.8), and is hence equivalent to a). We may also evaluate (7.9) approximately by the method of stationary phase. That is, we take

$$J_0\, 2(N\kappa z t)^{1/2} \simeq \frac{1}{\pi^{1/2}(N\kappa z t)^{1/4}} \cos\left[2(N\kappa z t)^{1/2} - \pi/4\right] \tag{7.10}$$

Points of stationary phase occur when $\kappa^{1/2} = \mp (N z t)^{1/2}/x$.
Omitting the details of this computation, we find

$$\hat{w} = \frac{1}{2\pi N x} \cos\left(\frac{N z t}{x}\right)\left(1 + O\left[\left(1+\frac{x}{z}\right) N t\right]^{-1}\right) \tag{7.11}$$

The lowest order term is actually the exact solution previously obtained.

c) The κ and γ integrations are now carried out jointly by the method of multiple stationary phase. Since this is frequently the most powerful method for the analysis of more complicated atmospheric wave propagation problems, we shall describe the method, abstractly, in some detail.

The integral under consideration is of the form

$$w(x,z,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\kappa,\gamma)\, d\kappa\, d\gamma\; e^{i\kappa x + i\gamma z \pm i\omega(\kappa,\gamma)t} \tag{7.12}$$

where it is assumed that x , z , and t are large parameters.

Rigorous derivations of the multiple stationary phase technique are available in the mathematical literature. See for instance, Jones and Kline (1958). We give here a simple derivation of the theory which shows the computational procedure used. We shall assume that $f(\kappa,\gamma)$ has no singularities on the real κ and γ axes, and that the curvature in κ and γ space of the phase surface

$$\phi(\kappa,\gamma) = \kappa x + \gamma z \pm \omega(\kappa,\gamma)\, t \tag{7.13}$$

does not vanish.

The phase is stationary at the points in (κ , γ) space and (x , z , t) where

$$\left.\begin{aligned} \frac{x}{t} &= \frac{\partial \omega}{\partial \kappa} \\ \frac{z}{t} &= \frac{\partial \omega}{\partial \gamma} \end{aligned}\right\} \tag{7.14}$$

This forms a nonlinear system of equations in κ and γ , which we assume may be inverted to obtain the wavenumbers of stationary phase

$$\begin{aligned} \kappa_{sp} &= \kappa_{sp}(x, z, t) \\ \gamma_{sp} &= \gamma_{sp}(x, z, t) \end{aligned} \tag{7.15}$$

For simplicity, assume there is only one (κ_{sp} , γ_{sp}) point of stationary phase in (κ , γ) space. We evaluate the partial second derivatives of ϕ (κ , γ) at the point of stationary phase. Let these be:

$$\left.\begin{aligned} \frac{1}{2}\left(\frac{\partial^2 \omega}{\partial \kappa^2}\right)_{\substack{\kappa_{sp}\\ \gamma_{sp}}} &= a \\ \frac{1}{2}\left(\frac{\partial^2 \omega}{\partial \kappa \partial \gamma}\right)_{\substack{\kappa_{sp}\\ \gamma_{sp}}} &= b \\ \frac{1}{2}\left(\frac{\partial^2 \omega}{\partial \gamma^2}\right)_{\substack{\kappa_{sp}\\ \gamma_{sp}}} &= c \end{aligned}\right\} \tag{7.16}$$

We expand the phase Φ in a Taylor's series about the point of stationary phase.

$$\Phi(\kappa,\gamma) = \Phi(\kappa_{sp},\gamma_{sp}) + a\mu^2 + 2b\mu\eta + c\eta^2 + R(\mu,\eta) \tag{7.17}$$

where we use

$$\mu = \kappa - \kappa_{sp}$$
$$\eta = \gamma - \gamma_{sp}$$

and $R(\mu,\eta)$ are all the cubic or higher order term of the Taylor's series expansion of Φ .

We likewise expand $f(\kappa,\gamma)$ in a Taylor's series.

$$f(\kappa,\gamma) = f(\kappa_{sp},\gamma_{sp}) + \mu\left(\frac{\partial f}{\partial \mu}\right)_{s.p.} + \eta\left(\frac{\partial f}{\partial \gamma}\right)_{s.p.} + \cdots \tag{7.18}$$

and furthermore expand $e^{i\Phi(\kappa,\gamma)}$ in a power series in $R(\mu,\eta)$,

$$e^{i\Phi(\kappa,\gamma)} = \left(e^{i\Phi(\kappa_{sp},\gamma_{sp}) + a\mu^2 + 2b\mu\eta + c\eta^2}\right)\sum_{n=0}^{\infty}\frac{R^n(\mu,\eta)}{n!} \tag{7.19}$$

The product of $f(\kappa,\gamma)/f(\kappa_{sp},\gamma_{sp})$ and this sum may then be expressed as a power series

$$\frac{f(\kappa,\gamma)}{f(\kappa_{s0},\gamma_{s0})}\sum_{0}^{\infty}\frac{R^{n}(\mu,\eta)}{n!} = 1+\sum_{\kappa=1}^{\infty}\sum_{j=0}^{\kappa} g_{\kappa j}\,\lambda^{j}\mu^{\kappa-j} \tag{7.20}$$

The first few terms of the series will usually be easy to obtain.

The quadratic form $a\mu^2+2b\mu\eta+c\eta^2$ may be transformed to normal form

$$a\mu^2+2b\mu\eta+c\eta^2 = \lambda_1 K_1^2+\lambda_2 K_2^2 \tag{7.21}$$

where (K_1 , K_2) are normal coordinates and λ_1, λ_2 are the eigen-values of the matrix, M

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{7.22}$$

We introduce the polar coordinates K , θ by the substitution,

$$\begin{aligned} K_1 &= \lambda_1^{-1/2} K\cos\theta \\ K_2 &= \lambda_2^{-1/2} K\sin\theta \end{aligned} \tag{7.23}$$

and expand (7.20) as

$$\sum_{\kappa=1}^{\infty}\sum_{j=0}^{\kappa} g_{\kappa j}\,\lambda^{j}\mu^{\kappa-j} = \sum_{\kappa=1}^{\infty}\sum_{j=0}^{\kappa} K^{\kappa}\,G_{\kappa j}\left(\frac{\cos\theta}{\lambda_1^{1/2}}\right)^{j}\left(\frac{\sin\theta}{\lambda_2^{1/2}}\right)^{\kappa-j} \tag{7.24}$$

Assuming the analysis (7.15) - (7.24) has been carried out, the integration of (7.12) reduces to

$$\hat{w} = \frac{e^{i\phi(\kappa_{sp},\gamma_{sp})}}{4\pi^2(\lambda_1\lambda_2)^{1/2}} f(\kappa_{sp},\gamma_{sp}) \int_0^\infty e^{iK^2} K\,dK \int_0^{2\pi} \left(1+\sum_{n=1}^{\infty}\sum_{j=0}^{n} K^n G_{nj}\left(\frac{\cos\theta}{\lambda_1^{1/2}}\right)^j\left(\frac{\sin\theta}{\lambda_2^{1/2}}\right)^{n-j}\right) d\theta \quad (7.25)$$

The integration may now be explicitly carried out, with the result

$$\hat{w} = \frac{f(\kappa_{sp},\gamma_{sp})}{4\pi(\lambda_1\lambda_2)^{1/2}} e^{i\phi(\kappa_{sp},\gamma_{sp}) - i\pi/4} (1+\varepsilon) \quad (7.26)$$

where

$$\varepsilon = \sum_{n=1}^{\infty} e^{-ni\pi/4}\, \Gamma\left(\frac{n}{2}+1\right) \sum_{j=1}^{n} G_{nj} I_{nj} \lambda_1^{-j/2} \lambda_2^{-(n-j)/2}$$

$$I_{nj} = \frac{1}{2\pi}\int_0^{2\pi} (\cos\theta)^j (\sin\theta)^{n-j}\, d\theta$$

This is an asymptotic power series in $\lambda_1^{-1/2}$, and $\lambda_2^{-1/2}$.

The following modifications of the above analysis will frequently be necessary.

1) When more than one point of multiple stationary phase is present in (κ , γ) space, the sum of the contributions from all stationary phase points must be included.

2) When a , b , and c all vanish, it is necessary to retain in the exponential the third order terms in the expansion of $\phi(\kappa,\gamma)$

Generally the parameters λ_1 and λ_2 will increase as $t \to \infty$ so that the first term in the sum will be adequate to describe the large time evaluation of the system.

Returning to the integration of (7.5), we see from (7.7) that the denominator has a pole at $\gamma = 0$. However, because of the rapid oscillations of the numerator, the integrand is integrable at this point, as may be seen by mapping the point to infinity.

The points of multiple stationary phase of the integrand are given by

$$\left.\begin{aligned} x/t &= \mp N/\gamma \\ z/t &= \pm N\kappa/\gamma^2 \end{aligned}\right\} \tag{7.27}$$

which may be solved for κ and γ to obtain

$$\left.\begin{aligned} \kappa_{\pm sp} &= \pm Nt z/|x|^2 \\ \gamma_{\pm sp} &= \mp \frac{Nt}{x} \end{aligned}\right\} \tag{7.28}$$

Expansion of the phase about these points gives

$$\left.\begin{aligned} \Phi = \frac{Nzt}{|x|} &+ \frac{x^2}{Nt}(\kappa - \kappa_{sp})(\gamma - \gamma_{sp}) \\ &+ \frac{x\,z}{Nt}(\gamma - \gamma_{sp})^2 + R(\kappa^i \gamma^{3-i}) \end{aligned}\right\} \tag{7.29}$$

where $R(\kappa^j \gamma^{3-j})$ is the remainder involving cubic or higher terms.

The integral (7.5) is hence approximated as

$$\hat{w} = Re \frac{e^{iNzt/|x|}}{4\pi^2 N} \int_{-\infty}^{\infty} d\kappa \int_{-\infty}^{\infty} \frac{d\gamma}{2\gamma}(1+\epsilon)\, e^{i\left(\frac{x^2}{Nt}\right)\left[(\kappa-\kappa_{sp})(\gamma-\gamma_{sp}) + \frac{1}{2}\frac{z}{|x|}(\kappa-\kappa_{sp})^2\right]} \quad (7.30)$$

where

$$\epsilon = O\left(\frac{\kappa-\kappa_{sp}}{\kappa_{sp}}\right) + O\left(\frac{\gamma-\gamma_{sp}}{\gamma_{sp}}\right)$$

Application of the multiple stationary phase computation then gives

$$\hat{w} = -\frac{1}{2\pi N x} \cos\left(\frac{Nzt}{x}\right) \left(1 + O\left[\left(1+\frac{x}{|z|}\right) Nt\right]^{-1/2}\right) \quad (7.31)$$

It is known from the above analysis that

$$\hat{w}(x,z,t) = -(2\pi N x)^{-1}\, Re\, e^{iNzt/x} \quad (7.32)$$

and also that

$$\hat{w}(x,z,t) = -(4\pi^2 N)^{-1} \iint_{-\infty}^{\infty} \frac{d\kappa\, d\gamma}{\gamma}\, e^{i\kappa x + i\gamma z}\, Re\, e^{i\left(\frac{N\kappa t}{\gamma} - \pi/2\right)} \quad (7.33)$$

has (7.32) as an asymptotic stationary phase approximation. This permits a rapid asymptotic evaluation of the stationary phase contribution to the Fourier integral

$$\phi(x,z,t) = -(4\pi^2 N)^{-1} \iint_{-\infty}^{\infty} \frac{F(k,\gamma)}{\gamma} e^{ikx + i\gamma z} \, Re \, e^{i\left(\frac{Nkt}{\gamma} - \frac{\pi}{2}\right)} dk \, d\gamma \tag{7.34}$$

where we assume that $F(k,\gamma)$ has no singularities that contribute to (7.29), nor to a first approximation does it alter the phase. Then it will follow from application of the stationary phase method that

$$\phi(x,z,t) = (4\pi N x)^{-1} \left(F(k_{+sp}, \gamma_{+sp}) \, e^{iNzt/x} + F(k_{-sp}, \gamma_{-sp}) \, e^{-iNzt/x} \right) \tag{7.35}$$

Knowing that the exact solution (7.32) is asymptotically equal to the stationary phase approximation to (7.33), we can obtain (7.35) by inspection, provided we can obtain the points of stationary phase ($k_{\pm sp}$, $\gamma_{\pm sp}$), here given by (7.28).

More generally, let $\Psi(k,\gamma,t)$ be a Fourier transform of a function $\psi(x,y,z,t)$ to be evaluated, and let $\Phi(x,y,z,t)$ be the Fourier transform of a known comparison function $\phi(x,z,t)$. Assuming that the asymptotic integration of Ψ and Φ is determined by single points of stationary phase which are approximately equal, then $\psi(x,z,t)$ is given approximately by

$$\psi(x,z,t) = \phi(x,z,t) \left(\frac{\Psi(k_{sp}, \gamma_{sp}, t)}{\Phi(k_{sp}, \gamma_{sp}, t)} \right) \tag{7.36}$$

where k_{sp} , γ_{sp} is the stationary wavenumber for the Fourier integral representations of ψ and ϕ .

This procedure will be applied to some further atmospheric wave propagation problems in the next section.

B. Approximate Evaluation of Atmospheric Wave Propagation Problems by the Method of Stationary Phase in Conjunction with Comparison Functions.

In this section we discuss some more applications of the stationary phase technique to the solution of atmospheric wave propagation problems when a comparison problem has a known exact solution. Two possibilities may be distinguished: a) The solution for a simpler source is known exactly. b) The solution for a simpler equation is known exactly. Problem a) may be referred to as a source perturbation problem, and b) as an operator perturbation problem.

We shall first illustrate the operator perturbation procedure by reference to the problem of determining viscous corrections to inviscid atmospheric wave propagation in an unbounded medium

Assuming for simplicity that the coefficient of viscosity, ν , may be equated to the coefficient of conductivity, and that these coefficients may be considered locally constant, we may introduce dissipation into the inviscid equations by making the substitution in the wave propagation equation

$$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} - \nu \Delta_3 \tag{7.37}$$

wherever the operator $\partial/\partial t$ appears.

Assume we have found Fourier plane wave solutions for inviscid motions of the form e^{Φ_i} where

$$\Phi_i = i\,\vec{K}\cdot\vec{R} + i\,\omega(\vec{k})\,t \tag{7.38}$$

Then the addition of viscosity will be equivalent to replacing Φ_i by Φ_v where Φ_v is given by

$$\Phi_v = \Phi_i - \nu|\vec{k}|^2 t \tag{7.39}$$

with $|\vec{k}|^2 = k_x^2 + k_y^2 + k_z^2$. We shall assume that ν is a small parameter.

Let $w_i(x,y,z,t)$ be a known inviscid solution for a localized source, which has the Fourier integral representation

$$w_i(x,y,z,t) = \iiint_{-\infty}^{\infty} d\vec{k}\; f(\vec{k},t)\, e^{\Phi_i(\vec{k})} \tag{7.40}$$

and let $\vec{k}_{sp}$ be the only wave number which makes $\Phi_i(\vec{k})$ stationary:

Then in the limit as $\nu \to 0$, the solution to the viscous problem, $w_v(x,y,z,t)$ will merely be

$$w_v(x,y,z,t) = e^{-\nu|\vec{k}_{sp}|^2 t}\, w_i(x,y,z,t) \tag{7.41}$$

It is necessary to modify this procedure if there exist two or more points of stationary phase, $\vec{k}_1$, $\vec{k}_2$, such that $|\vec{k}_1|^2 \neq |\vec{k}_2|^2$.

The derivation of (7.41) follows from noting that the Fourier integral representation of w_v is known from (7.39) to be

$$w_v(\vec{R},t) = \iiint_{-\infty}^{\infty} d\vec{k}\; f(\vec{k},t)\; e^{\phi_v(\vec{k})} \tag{7.42}$$

and that approximate evaluation of the integrand (7.42) for small v is determined by the same stationary phase points as in (7.40). That is

$$\nabla_{\vec{k}}\, \phi_v = O(v) \simeq 0$$

for $\vec{k}$ evaluated at $\vec{k}_{sp}$.

As an example, consider the viscous gravity wave equation

$$\left[\left(\frac{\partial}{\partial t} - v\,\Delta_3\right)^2 \Delta_3 + f_o^2 \frac{\partial^2}{\partial z^2} + N^2 \Delta_2\right] w_v = 0 \tag{7.43}$$

The phase to the inviscid problem is given by

$$\phi_{Gi} = i\left(\vec{k}\cdot\vec{R} \pm \frac{t}{|\vec{k}|}\left[f_o^2 k_z^2 + N^2(k_x^2 + k_y^2)\right]^{1/2}\right) \tag{7.44}$$

where $\vec{k} = (k_x, k_y, k_z)$ is the vector wave number for plane gravity waves. The points of stationary phase are given by

$$\nabla_{\vec{k}}\, \phi_{Gi} = 0 \tag{7.45}$$

This provides three equations for k_x , k_y , k_z which may be solved to obtain

$$\begin{pmatrix} k_{x\,sp} \\ k_{y\,sp} \\ k_{z\,sp} \end{pmatrix} = \begin{pmatrix} \frac{x t}{x^2+y^2} \\ \frac{y t}{x^2+y^2} \\ -t/z \end{pmatrix} \psi(x,y,z) \tag{7.46}$$

where

$$\psi(x,y,z) = \left(\frac{N^2 z^2 + f_0^2 (x^2+y^2)}{|\vec{R}|^2} \right)^{1/2} \tag{7.47}$$

Let the solution to a given inviscid problem for motions propagating from a local disturbance, be known as $w_i(x,y,z,t)$. Then the solution to the viscous problem, (7.43), for the same I. C., may be approximated by

$$w_v = w_i \, e^{-\nu t^3 \left[\frac{N^2}{x^2+y^2} + \frac{f_0^2}{z^2} \right]} \tag{7.48}$$

From this result, the decay rate for viscous damping will increase like t^2 for increasing time and will be most rapid in the neighborhood of the source. This result is only accurate for $w_v \simeq w_i$, but will give qualitative information concerning the damping over a wider range of t and $\vec{R}$.

As another useful example, we consider the viscous Rossby wave equation

$$\left[\left(\frac{\partial}{\partial t} - \nu \Delta_3 \right) \Delta_3 + \beta \frac{\partial}{\partial x} \right] \psi_v = 0 \tag{7.49}$$

The phase for inviscid Rossby waves may be written

$$\Phi_{Ri} = i \left[\vec{k} \cdot \vec{R} + \frac{k_x \beta t}{|\vec{k}|^2} \right] \tag{7.50}$$

Then the stationary phase condition, $\nabla_{\vec{k}} \Phi_{Ri} = 0$, inverted for wave-number gives

$$\vec{k}_{\substack{sp \\ \pm}} = \begin{pmatrix} k_{xsp} \\ k_{ysp} \\ k_{zsp} \end{pmatrix} = \pm \frac{\left(\frac{1}{2} \beta t \right)^{1/2}}{(x+R)^{1/2} R} \begin{pmatrix} x+R \\ y \\ z \end{pmatrix} \tag{7.51}$$

Assuming the same initial disturbance, then solutions the the viscous Rossby wave equation, ψ_v , are related to the inviscid solutions, ψ_i , by (7.41), which using (7.51) gives

$$\psi_v = \psi_i \, e^{-\beta \nu t^2/R} \tag{7.52}$$

again provided $e^{-\beta \nu t^2/R} \simeq 1$.

As an example of the source perturbation procedure, assume the Rossby wave equation

$$\left(\frac{\partial}{\partial t} \Delta_3 + \beta \frac{\partial}{\partial x} \right) \psi = - f(\vec{R}) \, \delta(t) \tag{7.53}$$

We know that for $f(\vec{R}) = \delta(\vec{R})$ (7. 53) may be inverted to obtain the propagator

$$\psi = \Psi_R = \frac{1}{4\pi R} J_0 \left(2\beta t (x+R)\right)^{1/2} \tag{7.54}$$

Let $F(\vec{k})$ be the F. T. of $f(\vec{R})$. (The F. T. of $\delta(\vec{R})$ is 1).

Then (7. 53) may be solved as

$$\psi = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} \frac{F(\vec{k})}{|\vec{k}|^2} d\vec{k} \; e^{i\vec{k}\cdot\vec{R} + i\frac{k_x \beta t}{|\vec{k}|^2}} \tag{7.55}$$

Then by comparison with Ψ_R given by (7. 54), we see that the stationary phase contribution of the integral (7. 55) is approximately

$$\psi_{s.p.} \simeq \frac{1}{4\pi R} \frac{1}{2} \left[F(k_{sp+}) H_0^{(1)} \left(2\beta t (x+R)\right)^{1/2} + F(k_{sp-}) H_0^{(2)} \left(2\beta t (x+R)\right)^{1/2} \right] \tag{7.56}$$

where $k_{sp\pm}$ is given by (7. 51).

We shall find these procedures useful in the next two chapters.

VIII. ROSSBY WAVES EXCITED BY TIME DEPENDENT DISTURBANCES

A. Preliminary Remarks

The first class of problems analyzed in this chapter concerns the Rossby wave motions excited by various point sources in a three dimensional unbounded atmosphere. In section B, we discuss the motion which ensues after switch-on of the sources proportional to $H(t)\cos(\omega_0 t)$, and $H(t)\,J_0(\omega_0 t)$. These are simple models for undamped and algebraically damped oscillatory sources respectively. The sudden switch-on excites transient forerunner motions of all frequencies, and a forced wave motion of frequency ω_0. For comparison, we examine in C, the motion excited by the source $e^{-(t/\tau)^2}\cos(\omega_0 t)$, an oscillatory source smoothly switched-on and off by a Gaussian modulation factor. In this case, only a forced wave packet motion is excited, which for not too large time, propagates outward without change of shape.

In D, we discuss the Rossby waves excited by switching-on a traveling disturbance. When the source is taken to travel in the positive x-direction, the only wavelike motions excited are switch-on transients; the forced motion excited attenuates exponentially away from the source. In contrast, a source assumed traveling in the negative x-direction excites wavelike forced motions as well.

One of the major defects of horizontally unbounded models for atmospheric motions is that wave energy propagating from a source around the earth one or more times is not included in evaluating the excited motion. At present it is not possible to estimate the importance of this energy relative to the wave energy arriving directly from the source, but it is possible to describe qualitatively the propagation of these secondary waves. We consider, in E.,Rossby's one dimensional wave equation for propagation in a domain that is periodic in x. The motion for an impulse, and x unbounded, was given by Rossby (1945). Charney and Eliassen (1949) previously discussed the solution to various problems for a periodic domain by summing numerically the Fourier series representation of the solutions.

Many of the results of this chapter may be described in terms of the asymptotic phase for the Rossby wave motions excited by a point impulse. The phase η is given by

$$\eta = (2\beta t)^{1/2}\left(x + \left(x^2 + y^2 + N^2/f_0^2\, z^2\right)^{1/2}\right)^{1/2} + C \tag{8.1}$$

which may be considered a surface defined over the space x, y, z, t. A surface of constant phase in time-space is the four dimensional manifold obtained by the intersection of the $\eta(x, y, z, t)$ surface with the plane $\eta = \eta_0$. The projection of these constant phase surfaces onto the R , t plane is a family of hyperbolas, while the projection onto the x , $\left(y^2 + N^2/f_0^2\, z^2 \right)^{1/2}$ plane is a family of parabolas.

One may consider each point of the phase surface to be approximated by its tangent plane, and hence to be considered locally a plane wave, the local wave number, frequency, being the same as the normal vector to η . That is

$$\begin{pmatrix} k_x \\ k_y \\ k_z \\ \omega \end{pmatrix} = \begin{pmatrix} \partial\eta/\partial x \\ \partial\eta/\partial y \\ \partial\eta/\partial z \\ \partial\eta/\partial t \end{pmatrix} \tag{8.2}$$

we find from differentiating (8.1) and using $\tilde{R} = (x^2 + y^2 + N^2/f_0^2\, z^2)^{1/2}$

$$\begin{pmatrix} k_x \\ k_y \\ k_z \\ \omega \end{pmatrix} = \pm \frac{(\beta t/2)^{1/2}}{\tilde{R}\,(x+\tilde{R})^{1/2}} \begin{pmatrix} x + \tilde{R} \\ y \\ z \\ 1/t\, \tilde{R}\,(x+\tilde{R})^{1/2} \end{pmatrix} \tag{8.3}$$

which is the same as the stationary phase wave number given by (7.51).

The results obtained in the next section may be described as follows. As soon as a point source is switched on, at a given point, very high frequency Rossby wave motions of very small amplitude are excited. As time increases the frequency decreases proportional to $t^{-1/2}$. We use the usual term <u>forerunners</u> to describe the transient Rossby wave motions of all frequencies that are so excited by the switch-on of the source. Eventually, at a given point of observation, the frequency decreases to ω_0 , the forcing frequency. At this time there is a resonance-like increase of the wave amplitude to the value which is forced by the time dependent source. This forced motion propagates outward with the same time dependence as the source, but delayed by a given phase factor.

We shall coin the term the Brillouin front to describe the surface within which the forced wave motion is confined. The transition from only forerunner motion ahead of the front to forerunner and forced motion behind the front occurs over a transition region of finite but ever decreasing thickness. (L. Brillouin, in the theory of electromagnetic wave propagation in a dielectric medium, first described such a transition zone). The velocity with which the front progresses outward is known as the signal velocity. This velocity may be identified with the local group velocity for motion with frequency ω_0

The Gaussian wave packet of section C. propagates outward, centered on the same Brillouin front, and attenuated by the Gaussian factor fore and aft.

The traveling disturbance problem described in section D. shows that when forced wavelike motions are excited, they are confined within another Brillouin front. The front in this case, for a westward traveling disturbance, is a sphere intersecting the x-axis at the location of the source and at a distance $2Ut$ downstream (to the right of the source). The signal velocity of this front corresponds to the group velocity for motions with a scalar wavenumber equal to $(\beta/U)^{1/2}$.

The one dimensional motions of E. are similar to those described above except that the Brillouin fronts propagate only in the positive x-direction and travel around the system to give secondary wave motions.

In F., we discuss the upward propagation of internal Rossby waves for horizontally sinusoidal sources. The primary intent here is to determine the initial excitation of disturbances. This section incidentally provides a rigorous verification of the internal wave upper boundary condition selected by Charney and Drazin (1961), in a discussion of the steady forced Rossby wave motions in a variably stratified atmosphere.

In G. is given a discussion of a modified form of Rossby waves which are frequently postulated in physical applications. When a "divergence" or "compressibility" term is added to the Rossby wave equation, the group velocity has a maximum, so that two modes, a Rossby wave and a compression wave, are present for points sufficiently near the source, but in the neighborhood of the (x, t) region propagating outward with the maximum group velocity, the two modes coalesce into an Airy front.

In Figures 8-1 - 8-5, are sketched some of the features of the various Rossby wave solutions.

B. On Rossby Waves Excited by Oscillating Sources

Consider the Rossby wave equation

$$\left(\frac{\partial}{\partial t}\Delta_3 + \beta\frac{\partial}{\partial x}\right)\psi = -f(t)\,\delta(\vec{R}) \tag{8.4}$$

where the vertical coordinate has been assumed to be stretched so that $N/f_0 = 1$. The motion forced by a source beginning at $t = 0$ is given by

$$\psi = \frac{1}{4\pi R}\int_0^t f(t-\tau)\, J_0\left(2\beta\tau(x+R)\right)^{1/2} d\tau \tag{8.5}$$

We obtain explicit approximate solutions for

i) $$f(t) = H(t)\cos(\omega_0 t) \tag{8.6}$$

ii) $$f(t) = H(t)\, J_0(\omega_0 t) \tag{8.7}$$

For this purpose we use the contour integral representation of (8.5), written

$$\psi = \frac{1}{4\pi R} \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} F(\sigma)\, e^{\sigma t - \frac{\beta}{2\sigma}(x+R)} \frac{d\sigma}{\sigma} \tag{8.8}$$

$$F(\sigma) = \int_0^\infty e^{-\sigma t} f(t)\, dt \tag{8.9}$$

Sources (8. 5) and (8. 6) give

$$F_1(\sigma) = \int_0^\infty e^{-\sigma t} \cos(\omega_0 t)\, dt = \frac{\sigma}{\sigma^2 + \omega_0^2} \tag{8.10}$$

$$F_2(\sigma) = \int_0^\infty e^{-\sigma t} J_0(\omega_0 t)\, dt = \frac{1}{(\sigma^2 + \omega_0^2)^{1/2}} \tag{8.11}$$

The path of integration for (8. 8) is deformed into one of steepest descents. Saddle points are located at

$$\sigma = \sigma_{sp} = \pm i \left(\frac{\beta}{2t}(x+R)\right)^{1/2} \tag{8.12}$$

When

$$|\sigma_{sp}| < \omega_0 \tag{8.13}$$

the original contour must be deformed past the singularity at ω_0 , and the loop integral about ω_0 added to the s. p. integral. This evaluation thus gives

$$\psi_{1,2} = H\left(t - \frac{\beta(x+R)}{2\omega_0^2}\right)\psi_{F\,1,2} + \psi_{s.p.\,1,2} \tag{8.14}$$

where we use

$$\psi_{1,2\,F} = \frac{1}{4\pi R}\,Re\,\frac{1}{2\pi i}\int_{\sigma=-\infty}^{(\sigma=i\omega_0)}\frac{d\sigma}{\sigma}\,\exp\left[\sigma t - \frac{\beta}{2\sigma}(x+R)\right]F_{1,2}(\sigma) \tag{8.15}$$

$$\psi_{1,2\,s.p.} = \frac{1}{4\pi R}\,\frac{1}{2\pi i}\int_{\sigma=-\infty}^{(s.D.)}\frac{d\sigma}{\sigma}\,\exp\left[\sigma t - \frac{\beta}{2\sigma}(x+R)\right]F_{1,2}(\sigma) \tag{8.16}$$

See Appendices III, D. and IV, D. for further details of the evaluation.

The forced motion, ψ_{1F} , is given by

$$\psi_{1F} = \frac{1}{4\pi\omega_0 R}\sin\left(\omega_0 t + \frac{\beta}{2\omega_0}(x+R)\right) \tag{8.17}$$

while ψ_{2F} is a branch line integral evaluated approximately as

$$\psi_{2F} = \left(1 + O(\omega_0 t)^{-1}\right)\frac{1}{4\pi\omega_0 R}\left(\frac{2}{\pi\omega_0 t}\right)^{1/2}\cos\left[\omega_0 t + \frac{\beta(x+R)}{2\omega_0} - \frac{3\pi}{4}\right] \tag{8.18}$$

The forerunner motion $\psi_{1,2}$ s.p. is evaluated approximately as

$$\psi_{1,2\,s.p.} \simeq \frac{1}{4\pi R}\,Re\,F_{1,2}(\sigma_{s.p.})\,H_0^{(1)}\left(2\beta t(x+R)\right)^{1/2} \tag{8.19}$$

The results for ψ_1, s.p. and ψ_2, s.p. may be written

$$\psi_{1,\,s.p.} \simeq \frac{1}{4\pi R}\left(\omega_0^2 - \frac{\beta}{2t}(x+R)\right)^{-1}\frac{\partial}{\partial t}\,J_0\left[2\beta t(x+R)\right]^{1/2} \tag{8.20}$$

$$\psi_{2,\,s.p.} \simeq \begin{cases} \frac{1}{4\pi R}\left(\omega_0^2 - \frac{\beta}{2t}(x+R)\right)^{-1/2} J_0\left(2\beta t(x+R)\right)^{1/2} ; & |\sigma_{sp}| < \omega_0 \\ \frac{1}{4\pi R}\left(1 - \frac{2\omega_0^2 t}{\beta(x+R)}\right)^{-1/2}\left(\frac{2t}{\beta(x+R)}\right)\frac{\partial}{\partial t} J_0\left(2\beta t(x+R)\right)^{1/2} ; & |\sigma_{sp}| > \omega_0 \end{cases} \tag{8.21}$$

These asymptotic approximations are valid for t such that

$$\left[2\beta t(x+R)\right]^{1/2}\left(1 - \frac{2\omega_0^2 t}{\beta(x+R)}\right) \gtrsim 1 \tag{8.22}$$

$$J_0\left(2\beta t(x+R)\right)^{1/2} \simeq \frac{\left(\frac{2}{\pi}\right)^{1/2}\cos\left[2\beta t(x+R) - \pi/4\right]}{\left(2\beta t(x+R)\right)^{1/4}} \tag{8.23}$$

In particular the analysis fails in the neighborhood of the Brillouin front

$$t = \frac{\beta(x+R)}{2\omega_0^2} \tag{8.24}$$

where there is a smooth transition from forerunner motions to forced plus forerunner motions. (For (8.24), the saddle point and singularity at ω_0 coalesce). The details for ω_0 a pole are discussed in Appendix IV, D. One finds in this case that ψ, in the neighborhood of t given by (8.24), is

$$\psi_1 \simeq \frac{1}{4\pi\omega_0 R} \operatorname{Re}\left[\exp\left(i\omega_0 t + i\frac{\beta(x+R)}{2\omega_0} - \frac{i\pi}{2}\right)\frac{1}{2}\operatorname{erfc}\left(e^{i\pi/4}\left(\frac{2t^3}{\beta(x+R)}\right)^{1/4}\left[\left(\frac{\beta(x+R)}{2t}\right)^{1/2} - \omega_0\right]\right)\right] \tag{8.25}$$

c.f. (8.12) for σ s.p.

It may be shown that ψ_1 given by (8.25) reduces to the transient motion ψ_1 s.p. as $\left(\frac{\sigma_{sp}}{\omega_0}-1\right)(\omega_0 t)^{1/2} \to \infty$, and to $\psi_{1,F} + \psi_1$, s.p. as $\left(\frac{\sigma_{sp}}{\omega_0}-1\right)(\omega_0 t)^{1/2} \to -\infty$. On the Brillouin front ψ_1 reduces to $\frac{1}{2}\psi_{1F}$, one half the final forced motion.

The predicted Rossby wave motion at a point distant from the source for the $\cos\omega_0 t$ forcing may now be summarized as follows. Soon after the source is switched on (but long after the acoustic front has arrived) forerunner Rossby wave motions of high frequency will begin to arrive. The ratio of their amplitude to the final forced motion will be approximately proportional to

$$A = (A_1)(A_2) = \left(\frac{2\omega_0^2 t}{\beta(x+R)}\right)^{1/2}\left(2\beta t(x+R)\right)^{-1/4}$$

where we assume $A_1 \ll 1$, $A_2 \ll 1$. As t increases, the factor A_1 will approach 1, and there will occur a rapid increase of amplitude, such that for $A_1 = 1$, ψ will have attained one half the amplitude of the steady forced motion given by (8.17), and for $A_1 \simeq 1+\Delta$ where Δ is some small increment, the motion may be represented by

$$\psi = \frac{1+O(\epsilon)}{4\pi\omega_0 R}\sin\left(\omega_0 t + \frac{\beta}{2\omega_0}(x+R)\right)$$

where $\epsilon = O(t^{-3/4})$

represents the error made if the transient forerunner motions are neglected.

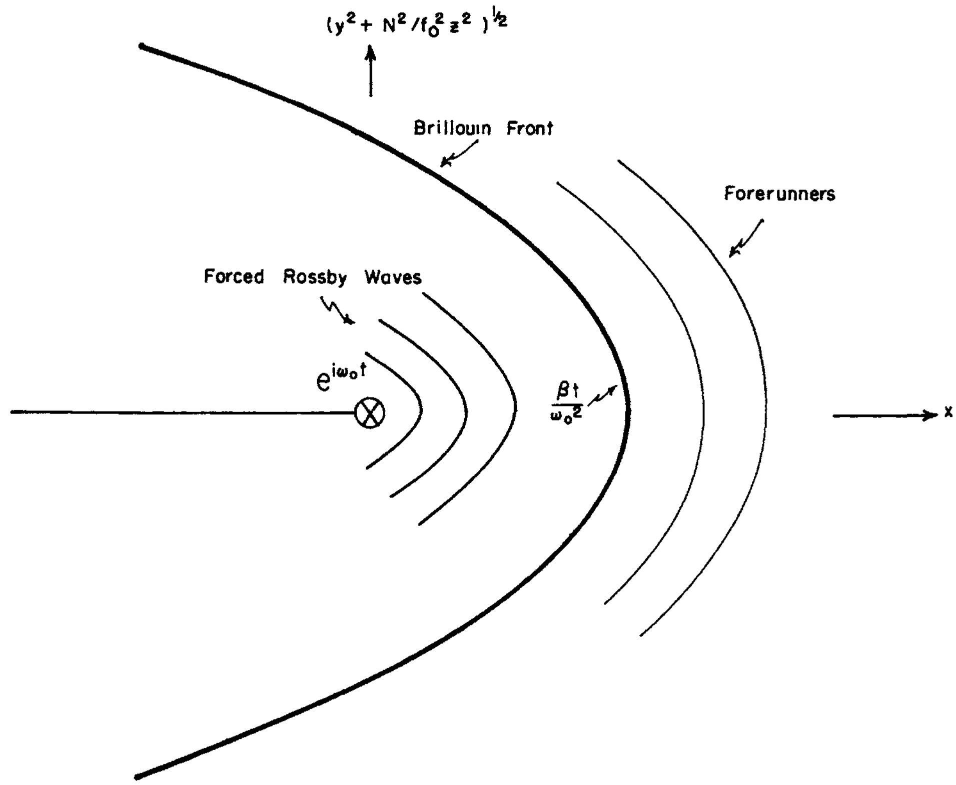


Figure 8-1. Sketch of 3-D Rossby waves excited by a switch-on oscillating point disturbance.

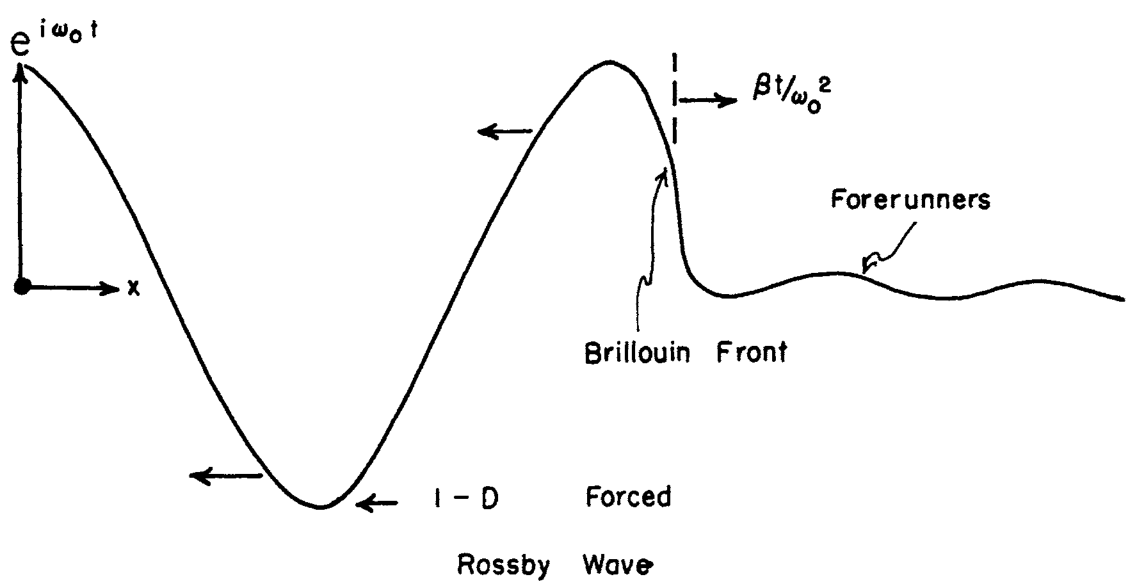


Figure 8-2. Sketch of 1-D Rossby waves excited by a switch-on oscillating point disturbance.

In contrast, for the $H(t)J_0(\omega_0 t)$ source the forerunner motions decay as $t \to \infty$ like $t^{-1/4}$ (c.f., (8.21)) while the forced motions decay like $t^{-1/2}$. The forced motion (c.f., (8.18)) are confined within the Brillouin front (8.24), but upon their arrival, there is only a small increase of Rossby wave amplitude over that due to the forerunners. The difference of the wave amplitude directly ahead and behind the front decays like $t^{-1/4}$ as $t \to \infty$. It is easy to formulate other Bessel function-like sources, where the forced wave motion will be dominant behind the front. This is the case for the source

$$f(t) = \frac{\partial}{\partial t}\left[H(t)\; J_0(\omega_0 t)\right]$$

where it may be shown that again the forced wave motion decays like $t^{-1/2}$, but now the transient motion decays like $t^{-3/4}$. The reason for this is the seemingly paradoxical result that the more singular in time is the switch on, the more rapid the forerunner motions decay as $t \to \infty$. This is true for any wave motion defined asymptotically by a steepest descent integration over frequency such that the saddle point approaches the origin as $t \to \infty$. This is presumably a consequence of the fact that for more singular sources, higher frequency, and hence more rapidly radiating motions are then excited.

C. Rossby Wave Packet Excited by an Oscillating Gaussian Modulated Source.

Consider the Rossby wave equation (8.4) with $f(t)$ the source

$$f(t) = e^{-(t/\tau)^2} \cos \omega_0 t \tag{8.26}$$

where we assume $\omega_0 \gg \tau^{-1}$. That is, the motion is smoothly switched on by a Gaussian modulation factor. The solution may be written as a Fourier integral

$$\psi = \operatorname{Re} \frac{1}{4\pi R} \int_{-\infty}^{\infty} \frac{d\omega}{i\omega} e^{i\omega t} \left(\frac{\tau}{2\pi^{1/2}}\right) e^{-(\omega-\omega_0)^2 \tau^2/4} e^{i\frac{\beta}{2}(x+R)\frac{1}{\omega}} \tag{8.27}$$

For evaluation of (8.27), we expand the phase in a power series in ($\omega - \omega_0$), and retain terms to second order.

$$\exp i\left[\omega t + \frac{\beta}{2}(x+R)\frac{1}{\omega}\right] = \exp i\left[\omega_0 t + \frac{\beta}{2}(x+R)\frac{1}{\omega_0}\right] F(\omega-\omega_0) \tag{8.28}$$

where

$$F(\omega) = \exp\left[i(\omega-\omega_0)\left[t - \frac{\beta(x+R)}{2\omega_0^2}\right] + i\frac{1}{8\omega_0^3}\beta(x+R)(\omega-\omega_0)^2 + \dots\right] \tag{8.29}$$

Substitution of (8.28) into (8.27) and computation of the integral gives

$$\psi = \frac{1+\epsilon}{4\pi\omega_0 R} \sin\left(\omega_0 t + \frac{\beta}{\omega_0^2}(x+R)\right) e^{-\left(\frac{t}{\tau} - \frac{\beta(x+R)}{2\omega_0^2\tau}\right)^2} \tag{8.30}$$

where the error term ϵ, defined as

$$\epsilon = O\left(\frac{\beta(x+R)}{2\tau^2\omega_0^3}\right) \tag{8.31}$$

is assumed much less than one.

The result shows that the Gaussian time-modulated disturbance excites a Gaussian spatially modulated wave packet centered on the outward propagating Brillouin front, $t = \frac{\beta(x+R)}{2\omega_0^2}$. The phases move through the wave packet towards the source. Note the $\frac{1}{4\pi R}$ geometrical attenuation, and the phase variation is the same as for the switch on source motion, given by (8.17).

For the above evaluation to be approximately correct, it is sufficient that the saddle points of the integrand (8.27) near the real axis be approximately at $\omega = \omega_0$. But these are located at the roots of

$$\omega_{sp} = \omega_0 + \frac{2}{\tau^2}\left(t - \frac{\beta}{2}\frac{x+R}{\omega_{sp}^2}\right) \tag{8.32}$$

We assume

$$\omega_{sp} \simeq \omega_0$$

which implies

$$\left.\begin{array}{ll} \dfrac{\beta(x+R)}{2\tau^2\omega_0^3} & \ll 1 \\[2ex] \tau^{-2}\left(\dfrac{t}{\omega_0}\right) & \ll 1 \end{array}\right\} \tag{8.33}$$

The first of these conditions is the same as required in (8.31), $\epsilon \ll 1$, but the second gives the added restriction that the t be sufficiently small. As $t \to \infty$, the motion described by (8.27) spreads out into a motion similar to that excited by a point impulse. The evaluation of (8.27), as $t \to \infty$, gives

$$\lim_{t\to\infty} \psi = \frac{\tau}{2\pi^{1/2}} e^{-\omega_0^2\tau^2/4} \frac{1}{4\pi R} J_0\left[2\beta t(x+R)\right]^{1/2} \tag{8.34}$$

D. Three Dimensional Rossby Waves Excited by a Traveling Point Disturbance.

Consider the equation

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\Delta_3 + \beta\frac{\partial}{\partial x}\right]\psi = -UH(t)\frac{\partial}{\partial x}\delta(x)\,\delta(y)\,\delta(z) \tag{8.35}$$

where a coordinate system in translation with the source has been selected. Multiple Fourier transform gives the solution

$$\psi = \frac{U}{8\pi^3}\iiint_{-\infty}^{\infty} \frac{\left[1 - e^{i\left(\frac{\beta k_x}{|k|^2} - Uk_x\right)t}\right]}{(k^2U-\beta)} e^{i\vec{k}\cdot\vec{R}}\, d\vec{k} \tag{8.36}$$

where $\vec{k} = (k_x, k_y, k_z)$ is the wave number vector. Introducing the spherical wave number coordinates,

$$\left.\begin{aligned} k &= (k_x^2 + k_y^2 + k_z^2)^{1/2} \\ \theta &= \tan^{-1}\left[(k_y^2 + k_z^2)^{1/2}/k_x\right] \end{aligned}\right\} \tag{8.37}$$

and using the fact that (8.36) is even in k, we write the integral as

$$\psi = \frac{U}{8\pi^2}\int_0^{\pi}\sin\theta\, d\theta\int_{-\infty}^{\infty}k^2 dk\left[1 - e^{i\left(\frac{\beta}{k} - Uk\right)t\sin\theta}\right]\frac{e^{ikR\cos(\theta-\varphi)}}{(k^2U - \beta)} \tag{8.38}$$

where φ is the azimuthal angle

$$\varphi = \tan^{-1}\left[(y^2+z^2)^{1/2}/x\right] \tag{8.39}$$

This representation is more convenient for steepest descents evaluation, but the details of the procedure are still rather complicated and hence omitted. The contribution of stationary phase points of (8.36) may be evaluated directly by comparison with the Fourier representation of the Rossby wave propagator

$$\frac{1}{4\pi R}\, J_0\left[2\beta t((x-Ut) + R)\right]^{1/2} = \frac{1}{8\pi^3}\iiint_{-\infty}^{\infty}\frac{d\vec{k}}{k^2}\exp i\left[k_x(x-Ut)+k_y y+k_z z + \frac{k_x\beta t}{k^2}\right] \tag{8.40}$$

We distinguish two cases. For $U < 0$, one finds that

$$\psi = \frac{e^{-\left(\frac{\beta}{U}\right)^{1/2} R}}{4\pi R} + \psi_{sp} \tag{8.41}$$

where

$$\psi_{sp} = \frac{-1}{8\pi^3} \iiint_{s.p.} \frac{e^{i\left(\frac{\beta k_x}{k^2} - U k_x\right)}}{k^2 U - \beta} e^{i \vec{k}\cdot\vec{R}} \, d\vec{k} \tag{8.42}$$

the integration being over the neighborhood of those points where the phase of the integrand is stationary.

The term ψ s.p. may be evaluated asymptotically by using (8.40). The result is

$$\psi_{sp} = \frac{1}{\hat{R}/Ut - 1} \frac{(1+\epsilon)}{4\pi \hat{R}} J_0 \left[2\beta t \left((x-Ut) + \hat{R}\right)\right]^{1/2} \tag{8.43}$$

where

$$\hat{R} = \left((x-Ut)^2 + y^2 + z^2\right)^{1/2} \tag{8.44}$$

$$\epsilon = O\left(\left[2\beta t \left((x-Ut) + \hat{R}\right)\right]^{-1/2} \left(1 - \frac{\hat{R}}{Ut}\right)^{-1}\right) \tag{8.45}$$

For $U > 0$, one finds that

$$\psi = \frac{1}{2\pi R} H(t - \hat{R}/U) \cos\left(\frac{\beta}{U}\right)^{1/2} R + \psi_{sp} \tag{8.46}$$

with ψ_{sp} again given by (8.42) and (8.43).

In summary: for either $U \gtrless 0$, a switch-on transient forerunner motion ψ_{sp} is excited which is advected along with the flow. For $U < 0$ the forced disturbance propagates with infinite speed. Its spatial evanescence is exponential. On the other hand, for $U > 0$, the forced motion has a finite speed of propagation and the forced waves are only found within the sphere

$$\hat{R} = Ut \tag{8.47}$$

which intersects the x-axis at the source point and $2Ut$ downstream from the source.

When (8.38) is evaluated by contour integration, the pole of the integrand of (8.36)

$$\kappa = (\beta/U)^{1/2} \tag{8.48}$$

coincides with the κ saddle point when the Brillouin front (8.47) passes the point of observation. This can be described physically as the arrival time for forced wave energy, which propagates with the group velocity of wavelength of the motion forced by the traveling source.

The motion excited after the source is switched-on may be described as follows. The scalar wave length, $2\pi\left(\frac{R}{\beta t}\right)^{1/2}$, decreases

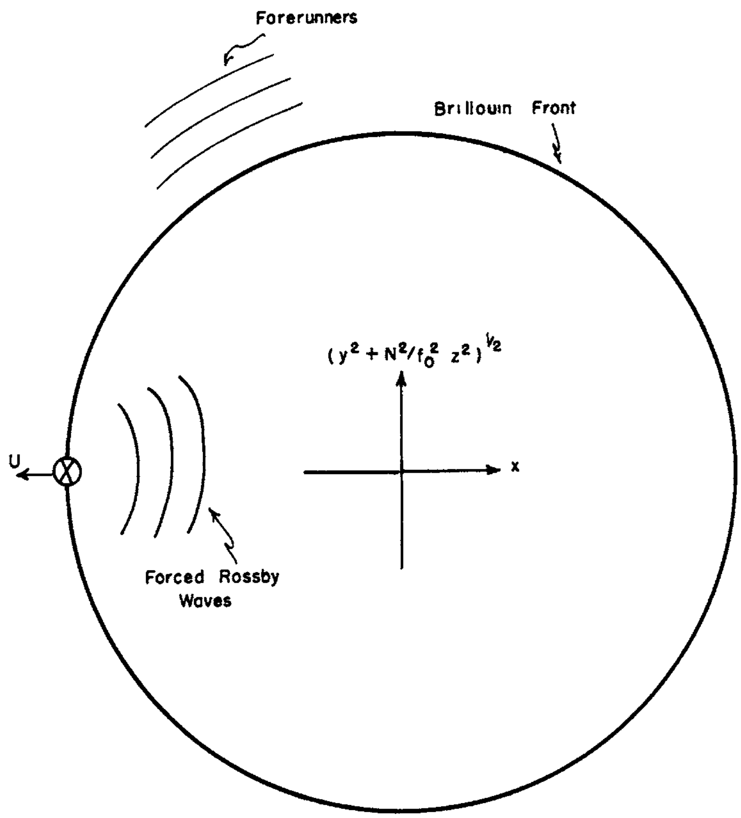


Figure 8-3. Sketch of 3-D Rossby waves excited by a switch-on travelling point disturbance.

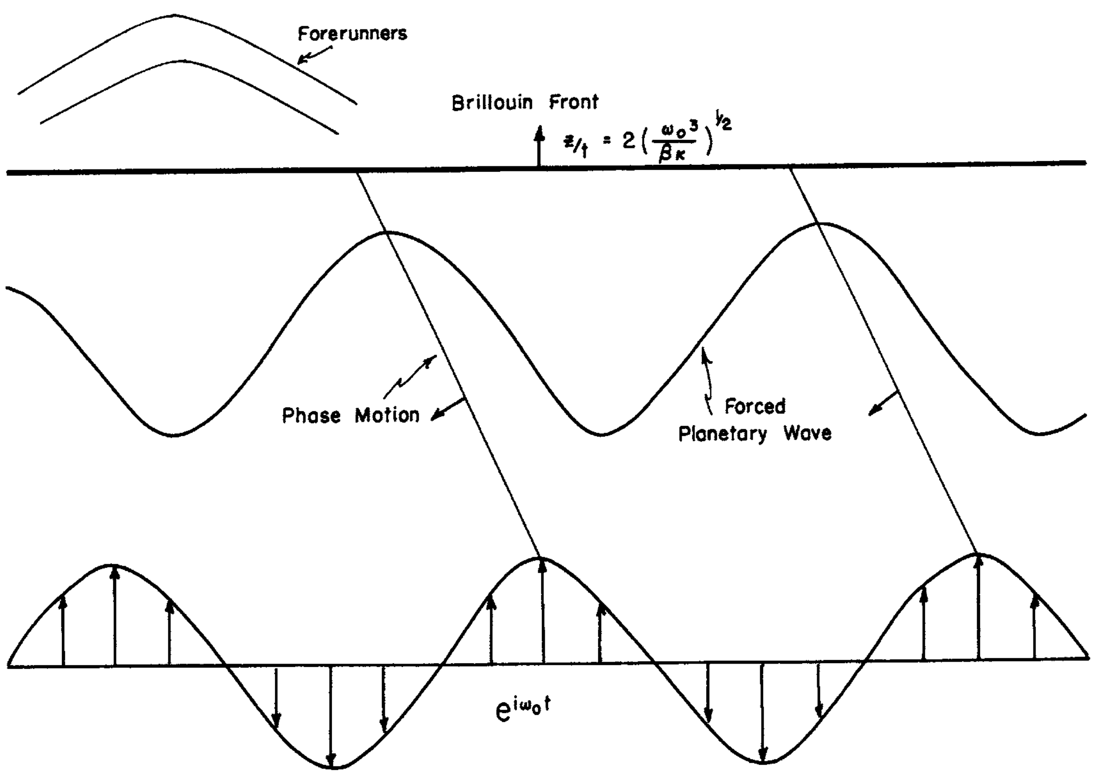


Figure 8-4. Sketch of vertically propagating planetary waves excited by a switch-on oscillating disturbance.

with increasing time so that at any point in the $x > 0$ half space, it finally attains the resonance value $2\pi(U/\beta)^{1/2}$. At this instance when the Brillouin front passes the point of observation, the wave amplitude rises sharply and for all latter time the steady forced wave motion is observed.

The asymptotic evaluation of the forerunners as given by (8.43) is not valid in the neighborhood of the Brillouin front when there is actually a smooth transition region from transient motions ahead of the front to transient plus forced motions behind the front. We omit here the analysis for this transition zone.

E. One Dimensional Rossby Waves in a Periodic Domain

Rossby's one dimensional model equation for the transverse velocity, v, may be written

$$\frac{\partial^2 v}{\partial x \partial t} + \beta v = - f(x,t) \tag{8.49}$$

where we assume the periodic boundary condition

$$v(0, t) = v(l, t) \tag{8.50}$$

and the elementary vorticity sources

a) $$f(x,t) = \delta(x)\,\delta(t) \tag{8.51}$$

b) $$f(x,t) = H(t) \cos \omega_0 t \tag{8.52}$$

c) $$f(x,t) = H(t)\, \delta(x + Ut) \tag{8.53}$$

The solutions will be decomposed into a component of motion excited directly by the source, v_D , and the remaining motion, v_I , which may be considered to have been excited by an infinite sum of image sources. That is

$$v = v_D + v_I \tag{8.54}$$

One then finds for the source **a)** , $f(x,t) = \delta(x)\,\delta(t)$

$$v_D = -H(x)\, e^{-\frac{\beta x}{\sigma}} \sigma^{-1}\, \delta(t) \tag{8.55}$$

$$v_I = -e^{-\frac{\beta(x+\ell)}{\sigma}} \left[1 - e^{-\beta\ell/\sigma}\right]^{-1} \sigma^{-1}\, \delta(t) \tag{8.56}$$

We use the Dirichlet series expansion

$$\left[1 - e^{-\beta\ell/\sigma}\right]^{-1} = \sum_{j=0}^{\infty} e^{-\left(\frac{\beta\ell}{\sigma}\right) j} \tag{8.57}$$

so that using Appendix II, 4,

$$v_D = -H(x)\, J_0\left[2(\beta x t)^{1/2}\right] \tag{8.58}$$

$$v_I = -\sum_{j=1}^{\infty} J_0\left[2(\beta(x+j\ell))^{1/2}\right] \tag{8.59}$$

Here v_D is Rossby's (1945) one dimensional propagator for the primary wave. The j' th term of (8.59) may be considered to result from an image source located at $x=-j\ell$. Such image sources occur on the nonphysical strips, $x<0$, to the left of the source. The Rossby waves only propagate energy in the positive x -direction. For the more realistic I-D model discussed in G. the wave propagation speed is finite and only a finite number of terms will occur in the sum (8.59).

For the oscillatory source b), $f(x,t) = -\delta(x)H(t)\cos\omega_0 t$ the solution is again written $v = v_D + v_I$ where now

$$v_D = -H(x)\,(\sigma^2+\omega_0^2)^{-1} e^{-\frac{\beta x}{\sigma}}\,\delta(t) \tag{8.60}$$

$$v_I = -\left(\sigma^2+\omega_0^2\right)^{-1} \frac{e^{-\frac{\beta(x+\ell)}{\sigma}}}{\left[1-e^{-\beta\ell/\sigma}\right]}\,\delta(t) \tag{8.61}$$

The denominator of (8.61) may again be expanded by (8.57). Using Appendices III, D and IV, D, one finds

$$\left.\begin{aligned} v_D &= H(x)\left(v_{F_0} + v_{T_0}\right) \\ v_I &= \sum_{j=1}^{\infty}\left(v_{F_j} + v_{T_j}\right) \end{aligned}\right\} \tag{8.62}$$

where the forced wave components v_{F_j} are given by

$$v_{F_j} = -H\left(\frac{\omega_0^2 t}{\beta} - (x+j\ell)\right)\frac{1}{\omega_0}\sin\left(\omega_0 t + \frac{\beta(x+j\ell)}{\omega_0}\right) \tag{8.63}$$

and the transient wave components v_{T_j} are

$$v_{T_j} = \frac{(1+\epsilon)\left(\frac{\beta(x+j\ell)}{t^3}\right)^{1/4}\sin\left[2\left(\beta t\,(x+j\ell)\right)^{1/2} - \pi/4\right]}{\left[\omega_0^2 - \frac{\beta(x+j\ell)}{t}\right]} \tag{8.64}$$

with asymptotic error

$$\epsilon = O\left(2\beta t\,(x+j\ell)\right)^{-1/2}\left(1 - \frac{x+j\ell}{\omega_0^2\beta t}\right)^{-1} \tag{8.65}$$

The j'th forced wave, v_{F_j}, is of zero amplitude at a given point x until the arrival of the Brillouin front at $t = \beta(x+j\ell)/\omega_0^2$ so that the wave originally excited at $t=0$ has propagated around the system j times and arrived at x. The direct source wave arrives at $t = \frac{\beta x}{\omega_0^2}$, the once around the system wave at $t = \frac{\beta(x+\ell)}{\omega_0^2}$ and so on.

Since there is no attenuation in the model considered, the wave that has circled the system any number of times has the same amplitude as the wave arriving directly from the source. When the j'th Brillouin front passes a point of observation, the asymptotic error of the j'th switch-on forerunner wave becomes infinite. The actual smooth transition in this region from transient to transient plus forced motion may be described by an error function, as given in Appendix IV, D.

Now let us consider the Rossby wave equation (8.47) for the traveling source, $f(x,t) = H(t)\,\delta(x+Ut)$. In this case the solution may be obtained by a Fourier series expansion as

$$v = \frac{2}{\ell} \sum_{n=-\infty}^{\infty} \frac{e^{i n\pi x/\ell}\left[e^{i\frac{n\pi}{\ell}Ut} - e^{i\beta t/\frac{n\pi}{\ell}}\right]}{\left[U\left(\frac{n\pi}{\ell}\right)^2-\beta\right]} \tag{8.66}$$

Taking $k = \frac{n\pi}{\ell}$, and using the "approximation"

$$\frac{2}{\ell}\sum_{n=-\infty}^{\infty}(\quad) \approx \frac{1}{2\pi}\int_{-\infty}^{\infty}(\quad)\,dk \tag{8.67}$$

the solution reduces to that for an unbounded domain. A more satisfactory comparison between the periodic domain and unbounded domain problems can be obtained by summing the Fourier series by contour integration. That is, (8.66), may be written

$$v = \frac{1}{2\pi}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{dk\, e^{ikx}}{(Uk^2-\beta)}\, \frac{\left[e^{ikUt} - e^{i\beta t/k}\right]}{\left(1-e^{-ik\ell}\right)} \tag{8.68}$$

where the path of integration is taken slightly below the real axis. The integrand is not singular at $k^2 = (\beta/U)$, and so the Fourier series (8.66) may be recovered by completion of the contour by a semicircle in the $\operatorname{Im} k > 0$ plane, and by evaluating the residue at the zeros of $(1 - e^{-ik\ell})$.

The denominator may now be expanded in a similar fashion to (8.57) with the result that v may be written

$$v = H(x)\ v_0 + \sum_{j=1}^{\infty} v_j$$

where

$$v_j = \frac{1}{2\pi}\int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{dk\ e^{ik(x+j\ell)}}{(Uk^2 - \beta)}\left[e^{ikUt} - e^{i\beta t/k}\right] \tag{8.69}$$

Further evaluation proceeds by steepest descent integration. See Appendix V. It is found that $v_j = v_{F_j} + v_{T_j}$ where taking $x_j = x + j\ell$ one finds

$$v_{T_j} = (1+\epsilon_j)\left(\frac{1}{\pi}\right)^{1/2}\frac{t^{1/4}}{(\beta x)^{3/4}}\left(\frac{Ut}{x_j} - 1\right)^{-1}\cos\left[2(\beta x_j t)^{1/2} - \frac{3\pi}{4}\right] \tag{8.70}$$

with

$$\epsilon_j = O\ (\beta x_j t)^{-1/2}\left(\frac{x_j}{Ut} - 1\right)^{-1}$$

while for $U < 0$

$$v_{F_j} = \frac{1}{2(\beta U)^{1/2}}\ e^{-\left(\frac{\beta}{U}\right)^{1/2}(x_j + Ut)}$$

and for $U > 0$

$$v_{F_j} = \frac{1}{(\beta U)^{1/2}}\ H(t + x_j/U)\ H(t - x_j/U)\ \sin\left(\frac{\beta}{U}\right)^{1/2}(x_j + Ut) \tag{8.71}$$

When $x_j = Ut$, the jth Brillouin front passes the point of observation, and the ϵ_j asymptotic error estimate for v_{T_j} blows up. The actual smooth transition from the j 'th transient forerunner motions ahead of $x_j = Ut$, to forerunner plus forced motions behind the front, may be described by an error function formula:

$$v_j \simeq \frac{1}{2}(\beta U)^{-1/2} \, Re\left(\left[\exp\left(i \left(\frac{\beta}{U}\right)^{1/2} (x_j + Ut) - i\pi/2 \right) \right] \left[\operatorname{erfc}\left(e^{i\pi/4} \left(\left(\frac{x_j}{Ut}\right)^{1/2} - 1\right) (\beta U)^{1/4} t^{1/2} \right) \right] - \frac{1}{2\pi^{1/2}} \frac{1}{(\beta U)^{1/4} t^{1/2}} e^{2i(\beta x t)^{1/2} \pm 3i\pi/4} \right) \tag{8.72}$$

for $x_j \approx Ut$

Summarizing: For $U < 0$ (an eastward traveling source) there is excited a steady forced motion, which decays exponentially away from the source, as well as a transient forerunner wave. Both waves vanish to the left of the initial point of excitation but propagate to the right with infinite speed around the system any number of times. On the other hand, for $U > 0$ (a westward traveling source) the steady forced motion is comprised of waves confined between the source at $x = -Ut$ and image sources at $x_j = -Ut$ and the Brillouin front at $x_j = Ut$ To the right of the front is found only the transient forerunner waves. The transition region is described by (8.72).

F. Vertically Propagating Planetary Waves

When topographic or thermal sources excite atmospheric wave energy over distance scales comparable to the radius of the earth, it might be anticipated that only the first few terms of some normal mode expansion will provide satisfactory resolution for the description of the horizontal variability of the motion. In the next chapter, we shall give further details as to the proper normal coordinates for a rotating spherical earth. Here we heuristically simulate a normal mode expansion by assuming a source proportional to $e^{i\kappa x}$ where κ is the "wave number" or the disturbance. Assuming also that the Rossby waves are of the form

$$\psi = \bar{\Psi} e^{i\kappa x}$$

the Rossby wave equation may be written

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \kappa^2\right)\bar{\Psi} + \beta i\kappa \bar{\Psi} = -f(y, z, t) \qquad (8.73)$$

where again $N/f_0 = 1$.

First consider the elementary point source problem,

$$f(y, z, t) = \delta(y)\,\delta(z)\,\delta(t).$$

Then (8.73) may be inverted to obtain

$$\bar{\Psi} = \frac{1}{2\pi\sigma} K_0\left[\left(\kappa^2 - \left(\frac{\beta i\kappa}{\sigma}\right)\right)^{1/2}(y^2+z^2)^{1/2}\right]\delta(t) \qquad (8.74)$$

which for small σ may be approximated by

$$\bar{\Psi} = \left(\frac{\pi}{2a}\right)^{1/2}\frac{1}{2\pi}\sigma^{-3/4}\left(1 + O\left(\frac{a}{\sigma^{1/2}}\right)\right)\exp\left[-ae^{-i\pi/4}\sigma^{-1/2} + i\frac{\pi}{8}\right] \qquad (8.75)$$

with

$$a = (\beta\kappa)^{1/2}(y^2+z^2)^{1/2} \tag{8.76}$$

It follows by Appendix III, C, that as $t \to \infty$,

$$\bar{\varphi} = (1+\epsilon)\left(\frac{2t}{a}\right)^{1/6} \frac{1/2\pi}{(6at)^{1/2}}\, e^{i\frac{3}{2}(2a^2 t)^{1/3}} \tag{8.77}$$

where the asymptotic error ϵ is

$$\epsilon = O\left(\beta\kappa t\,(y^2+z^2)^{1/2}\right)^{-1/3} \tag{8.78}$$

Variations of this model may be used to describe the radiation decay of a planetary heat pulse originally concentrated along a given latitude circle, and with a given longitudinal dependence.

Next we shall consider the forced source problem:

$$f(y,z,t) = \frac{\partial}{\partial t}\delta(y)\,\delta(z)\, e^{i\omega_0 t} H(t) \tag{8.79}$$

which may be considered to model either a pulsating line source or a line source traveling in the x direction. The inverse of (8.73) for the forcing given by (8.79) is written as $\bar{\Psi}_2$ to distinguish it from (8.74). One finds

$$\bar{\Psi}_2 = (\sigma - i\omega_0)^{-1}\ \frac{1}{2\pi} K_0\left[\left(\kappa^2 - \frac{\beta i\kappa}{\sigma}\right)^{1/2}(y^2+z^2)^{1/2}\right]\delta(t) \tag{8.80}$$

which we approximate by

$$\bar{\varphi}_2 = (\sigma - i\omega_0)^{-1}\left(1 + O\left(\frac{a}{\sigma^{1/2}}\right)\right)\left(\frac{\pi}{2a}\right)^{1/2}\sigma^{1/4}\frac{1}{2\pi}\exp\left[-\frac{a e^{-i\pi/4}}{\sigma^{1/2}} + i\pi/8\right] \tag{8.81}$$

This expression may be evaluated with the aid of Appendix IV, C, as

$$\bar{\varphi}_2 = \bar{\varphi}_F + \bar{\varphi}_T \tag{8.82}$$

where the transient forerunner motions $\bar{\varphi}_T$, are given by

$$\bar{\varphi}_T = \left(1 - \omega_0\left(\frac{2t}{a}\right)^{2/3}\right)^{-1}(1+\epsilon)\,\bar{\varphi}\,e^{-i\pi/2} \tag{8.83}$$

provided we take $\bar{\varphi}$ given by (8.77), and take ϵ as

$$\epsilon = O\left(\frac{1}{\omega_0}\left(\frac{a}{2t}\right)^{2/3} - 1\right)^{-1} \tag{8.84}$$

The forced motion $\bar{\varphi}_F$ is given approximately by

$$\bar{\varphi}_F = H\left(t - \frac{(y^2+z^2)^{1/2}}{c_g}\right)\left(\frac{\pi}{2a}\right)^{1/2}\frac{\omega_0^{1/4}}{2\pi}(1+\epsilon)\exp i\left[\omega_0 t + \frac{a}{\omega_0^{1/2}} + \frac{\pi}{4}\right]$$

where the group velocity c_g is

$$c_g = 2\left(\frac{\omega_0^3}{\beta\kappa}\right)^{1/2} \tag{8.85}$$

and the asymptotic error ϵ is given by (8.78). The Brillouin front where the error term ϵ given by (8.84) blows up is described by

$$(y^2+z^2)^{1/2} = c_g t \tag{8.86}$$

In the neighborhood of this front, $\bar{\Psi}_2$ may be approximated with the use of Appendix IV, C, as

$$\bar{\Psi}_2 \simeq \left(\frac{\pi}{2a}\right)^{1/2} \frac{\omega_0^{1/4}}{2\pi} e^{i\left(\omega_0 t + \frac{a}{\omega_0^{1/2}} - \pi/4\right)} \frac{1}{2}\, \mathrm{erfc}\left[e^{i\pi/4}\left(\frac{2t}{a}\right)^{1/3}\left|\frac{3t}{4}\right|^{1/2}\left(\left(\frac{a}{2t}\right)^{2/3} - \omega_0\right)\right] \tag{8.87}$$

Another phenomenon of some interest is the planetary wave motion excited by a switch on source traveling in the vertical direction. For instance, consider the switch-on model problem

$$\left(\frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2} + \beta i k\right)\phi = -H(t)\,\delta(z + Ut) \tag{8.88}$$

The solution to this problem may be represented by the Fourier integral

$$\phi = \frac{1}{2\pi U i}\int_{-\infty}^{\infty}\left[e^{i\gamma U t} - e^{-i\beta k t/\gamma^2}\right]\frac{e^{i\gamma z}\,d\gamma}{\left(\gamma^3 + \frac{\beta k}{U}\right)} \tag{8.89}$$

The Brillouin front for this problem is given by the parallel planes

$$z/t = \pm \tfrac{1}{2} U \tag{8.90}$$

The steepest descents countour integration for evaluation of integrals like (8.89) is discussed in Appendix V. We reproduce here only the final forced motion which may be written

$$\phi \simeq H\left(t - \frac{2|z|}{U}\right)\frac{e^{-i\left(\frac{\beta k}{U}\right)^{1/3}(z + Ut)}}{3\left((\beta k)^2 U\right)^{1/3}} \tag{8.91}$$

G. Modified Rossby Waves

The Rossby wave equation, modified so that the maximum speed of "energy propagation" is finite, may be written

$$\frac{\partial}{\partial t}\left(\Delta_3 - a^2\right)\psi + \beta\frac{\partial\psi}{\partial x} = -f(\vec{R},t) \tag{8.92}$$

The inverse of a is known as the Rossby radius of deformation when it occurs in the theory of Rossby wave propagation on an ocean of finite depth. This term also occurs in the equations for Rossby wave propagation in the atmosphere either as a result of atmospheric compressibility, or as a consequence of the reduction of the degree of the Rossby wave equation by separation of variables.

The solutions to (8.92) contain not only the Rossby wave mode, but another mode as well, which for lack of a better name, we labeled in chapter V, the c_2 compression mode. Here we shall only discuss the 1-D propagation of (8.92) so that $f(\vec{R},t) = f(x,t)$. Let us then consider the problem

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} - a^2\right)\psi + \beta\frac{\partial\psi}{\partial x} = -\delta(x)\,\delta(t) \tag{8.93}$$

with a Fourier integral solution which may be written

$$\psi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k^2+a^2}\,e^{ikx + i\frac{\beta k t}{k^2+a^2}} \tag{8.94}$$

The integral may be evaluated for large time by a saddle point integration as discussed in Appendix VI. Assuming $\frac{\beta t}{x} >> a^2$, one obtains for $x > 0$

$$\psi = H(x)\,\psi_R + \psi_c \tag{8.95}$$

where

$$\psi_R = -(1+\epsilon_R)\left(\frac{x}{\beta t}\right)^{1/4}(\pi\beta t)^{-1/2}\cos\left[2(\beta x t)^{1/2} + \pi/4\right] \tag{8.96}$$

with

$$\epsilon_R = O\left(\frac{a^2 x}{\beta t}\right) + O\,(\beta x t)^{-1/2}$$

and

$$\psi_c = -(1+\epsilon)\left(\frac{2}{\pi a \beta t}\right)^{1/2}\cos\left[\left(1 + \frac{4a^2 x}{\beta t}\right)\left(a x + \frac{\beta t}{2a} - \frac{\pi}{4}\right)\right] \tag{8.97}$$

where

$$\epsilon = O\left(\frac{a^2 x}{\beta t}\right) + O\left(\frac{a}{\beta t}\right)^{1/2}$$

While the Rossby wave mode ψ_R is found to propagate only to the east of the source, the compression mode ψ_c is found on both sides of the source.

At great distances from the source such that $\beta t/x = O(a^2)$, and $x > 0$, the saddle points giving the Rossby wave coalesce with the saddle points giving the compression mode, and the solution is given approximately by

$$\psi \simeq \alpha\, Ai\left[\left(\frac{32a^2}{3\beta t}\right)^{1/3}\left(x - \frac{\beta t}{8a^2}\right)\right]\cos\left[\sqrt{3}\,a\left(x - \frac{\beta t}{4a^2}\right)\right] \qquad (8.98)$$

where

$$\alpha = \frac{1}{4a^2}\left(\frac{32\,a^4}{3\,\beta t}\right)^{1/3}$$

The important point to note is that the Airy function modulation factor decays exponentially beyond the "Airy front" traveling with speed

$$\frac{x}{t} = \frac{\beta}{8a^2} \qquad (8.99)$$

which may be considered the propagation speed of the modified Rossby wave.

One may analyze the existence of modified Rossby wave motions due to oscillating or traveling sources, as in the previous sections. Again transient forerunner motions of all frequencies will be excited, and the steady forced motion of each mode will travel out behind a Brillouin front characterizing that mode. However, no disturbance will propagate to the east at a speed greatly exceeding that given by (8.99). Phillips (1965) has reported experimental laboratory observations of Rossby waves excited by an oscillating paddle. Not only are the Rossby waves observed to the east of the disturbance, but the analogue of the compression mode is found westward of the paddle as well.

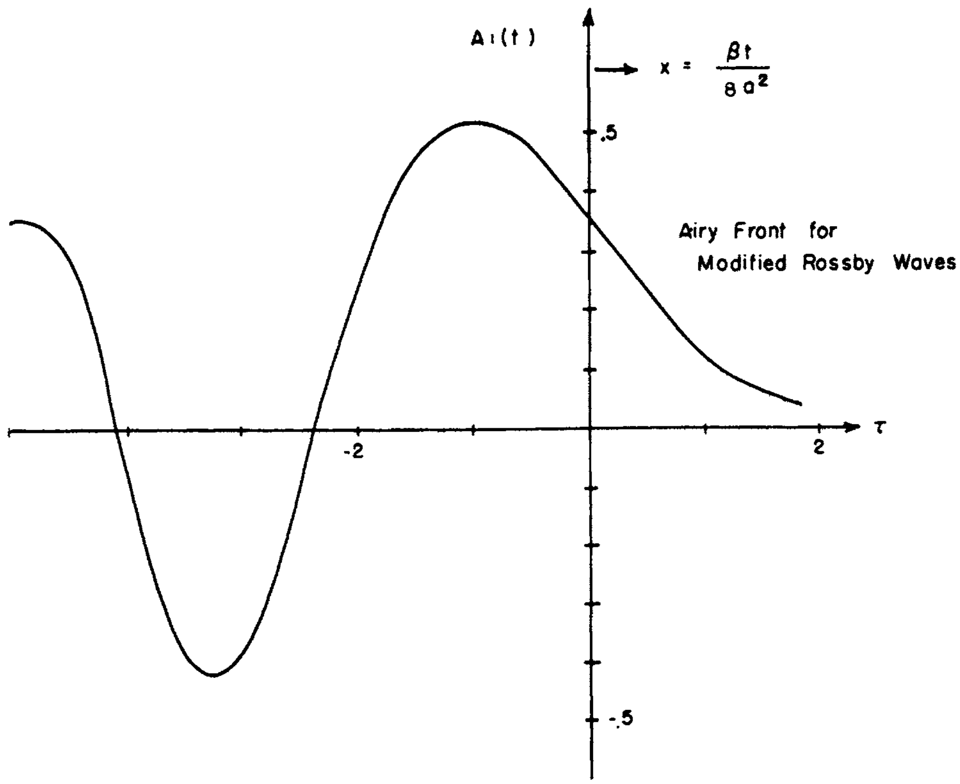


Figure 8-5. Sketch of the Airy front modulation for modified Rossby waves. $\tau = \left(\frac{32a^2}{3\beta t}\right)^{1/3}\left(x - \frac{\beta t}{8a^2}\right)$. See (8.98).

IX. ON GRAVITY WAVES EXCITED BY TIME DEPENDENT DISTURBANCES

A. Some Preliminary Comments

In this chapter are considered some elementary problems for the dynamics of stratified nonrotating atmospheres. We shall first discuss solutions for the hydrostatic model equation

$$\left(\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial z^2} + N^2\frac{\partial^2}{\partial x^2}\right) w = 0. \tag{9.1}$$

which is suitable for the description of the 2-D vertical motion in low frequency gravity wave motions, at some distance behind the switch-on acoustic front.

One may ask for the motion described by (9.1) when the lower boundary oscillates. This gives the boundary condition at $z = 0$.

$$w\big]_{z=0} = W_0(x)\, e^{i\omega_0 t} \tag{9.2}$$

whence assuming w to be given by

$$w = \bar{w}(x,z)\, e^{i\omega_0 t} \tag{9.3}$$

reduces (9.1) to

$$\left(-\omega_0^2\frac{\partial^2}{\partial z^2} + N^2\frac{\partial^2}{\partial x^2}\right)\bar{w}(x,z) = 0 \tag{9.4}$$

$$\bar{w}\big]_{z=0} = W_0(x)$$

Similarly, if a boundary condition

$$w]_{z=0} = w_0(x+Ut)$$

is assumed, and if we assume w is given by

$$w(x,z,t) = \tilde{w}(x+Ut, z) \tag{9.5}$$

then (9.1) is reduced to

$$\left(U^2 \frac{\partial^4}{\partial x^2 \partial z^2} + N^2 \frac{\partial^2}{\partial x^2}\right) w = 0 \tag{9.6}$$

Solutions to (9.4) and (9.6) can not be determined without reference to the more complete equation (9.1). That is, the physical problem specifies only a single boundary condition at $z=0$, but the reduced equations (9.4) and (9.6) are hyperbolic P. D. E., which require two conditions at $z=0$ in order to obtain a unique solution.

The reduced oscillating boundary problem (9.4) has characteristics along the lines

$$\frac{x}{z} = \pm \frac{N}{\omega_0} + C \tag{9.7}$$

where C is an arbitrary constant. All solutions will be of the form

$$w = w(x \pm Nz/\omega_0) \tag{9.8}$$

Likewise, the reduced traveling boundary problem (9.6) has characteristic surfaces

$$\left.\begin{aligned} x &= C \\ z &= C \end{aligned}\right\} \qquad (9.9)$$

and solutions will have the general form

$$w = W(x)\, e^{\pm i N z / U} \qquad (9.10)$$

We shall refer to motions of the general form (9. 8) as "oscillating (hydrostatic) gravity waves", and (9. 10) as (hydrostatic) "lee waves". The problem at hand, then, is to describe how switch-on of the time dependent boundary conditions (9. 2) and (9. 6) will excite gravity wave motions that asymptote to an oscillating gravity wave or to a lee wave motion.

The vertical propagation of transient gravity wave motions excited by a sinusoidal horizontal boundary is quite similar to the horizontal propagation of 1-D transient Rossby waves. However, the forced motions excited in the present model by a switch-on localized oscillating boundary disturbance, are quite dissimilar to the Rossby waves, since there no longer exists a Brillouin front separating a region of transient decaying motions from a region where the final forced motion has ensued. Rather, the outward energy propagation depends on the scale of the disturbances. This may be attributed to the fact that the group velocity of the gravity wave motions decreases for decreasing wavelength of the motion.

Consequently, the onset of motion forced by a source exciting energy in all wavelengths occurs over a very broad transition region that propagates outward with roughly the group velocity of the dominant wavelength of the disturbance.

The motion excited by a switch-on traveling boundary perturbation is found to be confined within a Brillouin front traveling with the velocity of the boundary perturbation and another Brillouin front at a distance of 2 Ut behind the perturbation. The vertical propagation of the forced motion again depends on the spatial scale of the boundary. This problem is discussed in C.

In D. we analyze the horizontal propagation of forced gravity waves in a rotating system.

B. Gravity Waves Excited by a Slowly Oscillating Source

In this section we solve (9.1) subject to a switch-on boundary condition, (9.2). We shall first consider the special boundary condition

$$w\Big]_{z=0} = H(t)\ \pi^{-1}\frac{\ell}{x^2+\ell^2}\ e^{i\omega_0 t} \tag{9.11}$$

The x-dependence of the source chosen has a "distance scale" ℓ, and has the elementary Fourier integral representation

$$\pi^{-1}\frac{\ell}{x^2+\ell^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\kappa x-|\kappa|\ell}\,d\kappa \tag{9.12}$$

such that

$$\lim_{\ell\to 0}\ \pi^{-1}\frac{\ell}{x^2+\ell^2} = \delta(x) \tag{9.13}$$

Let W be the solution to

$$\left(\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial z^2} - N^2\kappa^2\right)W(\kappa,z,t) = 0$$

$$\text{B. C.}\quad W\Big]_{z=0} = H(t)\,e^{i\omega_0 t} \tag{9.14}$$

Then w satisfying (9.11) and (9.1) is synthesized as

$$w = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\kappa|\ell}\,W(\kappa,z,t)\,e^{i\kappa x}\,d\kappa \tag{9.15}$$

But (9.14) is solved by

$$W = (\sigma - i\omega_0)^{-1}\,e^{-N|\kappa|z/\sigma}\,\delta(t) \tag{9.16}$$

which is equivalent to solutions obtained for oscillating Rossby waves.

The Brillouin front is given by the planes

$$z/t = \pm\frac{\omega_0^2}{|\kappa|N} \tag{9.17}$$

Evaluation of (9.16) for large t gives

$$W = H\left(t - \frac{\kappa N z}{\omega_0^2}\right) W_F + W_{T_+} + W_{T_-} \tag{9.18}$$

where

$$W_F = e^{i(\omega_0 t + |\kappa| N z/\omega_0)} \tag{9.19}$$

and

$$W_{T_\pm} \simeq \frac{e^{\mp i\pi/4}}{2} \pi^{-1/2} (\kappa N z t)^{-1/4} \left(1 \mp \frac{\omega_0}{(\kappa N z/t)^{1/2}}\right)^{-1} e^{\pm 2i(N\kappa z t)^{1/2}} \tag{9.20}$$

We shall not use (9.19) and (9.20) for further reduction, since it is somewhat simpler to proceed directly from W given by (9.16). The integral (9.15) can be evaluated exactly with the result that

$$w = (\sigma - i\omega_0)^{-1} \frac{1}{2\pi} \left[\frac{1}{\ell + Nz/\sigma - ix} + \frac{1}{\ell + Nz/\sigma + ix}\right] \delta(t) \tag{9.21}$$

The pole at $\sigma = i\omega_0$ gives the final forced motion, while the other poles give transient motions with an ultimate decay like $e^{-N\ell t/(x^2+\ell^2)^{1/2}}$

That is w is evaluated as

$$w = \frac{e^{i\omega_0 t}}{2\pi} \left[\frac{1}{\ell - i\left(\frac{Nz}{\omega_0} + x\right)} + \frac{1}{\ell - i\left(\frac{Nz}{\omega_0} - x\right)}\right] + O\left(e^{-\frac{N\ell t}{(x^2+\ell^2)^{1/2}}}\right) \tag{9.22}$$

Recall that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon - ix} = \lim \frac{\epsilon + ix}{\epsilon^2 + x^2} = \pi\,\delta(x) + i/x \tag{9.23}$$

so that if we take $N z/\omega_0 = Z$, it is found that

$$\lim_{\lambda \to 0} \lim_{t \to \infty} w = \frac{1}{2} e^{i\omega_0 t} \left[\left(\delta(x+Z) + \frac{i}{\pi(x+Z)} \right) + \left(\delta(x-Z) - \frac{i}{\pi(x-Z)} \right) \right] \tag{9.24}$$

This establishes a Green's function solution for the motions excited by an oscillatory source at the boundaries, for time large so that transients have decayed to zero.

The steady solution to (9.1) for the arbitrary boundary conditions (9.2) is then given by

$$w(x,z,t) = \frac{e^{i\omega_0 t}}{2} \left[f(x + Nz/\omega_0) + f(x - Nz/\omega_0) \right] \tag{9.25}$$

where f is given by

$$f(x \pm Nz/\omega_0) = w_0(x \pm Nz/\omega_0) + \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{w_0(x')\,dx'}{Nz/\omega_0 \pm (x-x')} \tag{9.26}$$

The solution is seen to be of the functional form given by (9.8).

The solution is composed of two pieces, the first of which, $\frac{e^{i\omega_0 t}}{2} \left[w_0(x+Z) + w_0(x-Z) \right]$, by itself satisfies the inhomogeneous boundary conditions. The second piece, given as an integral, vanishes on the boundaries but is important for the determination of the pressure perturbations and hence, for the wave energy radiated from the boundary. The vertical motion defined by the first piece has a domain of dependence at a given point which is restricted to the two boundary points intersected by the characteristic lines $x/z = \pm N/\omega_0$ emanating from the point. The domain of dependence for the second piece is the entire boundary.

C. The Lee Wave Mode

We consider here the gravity wave equation (9.1) for the elementary boundary condition

$$w\Big]_{z=0} = H(t)\,\delta(x+Ut) \tag{9.27}$$

The solution may be obtained as the double Fourier integral

$$w = \frac{1}{4\pi^2 i}\iint_{-\infty}^{\infty} d\kappa\, d\gamma\; e^{i\kappa x + i\gamma z}\left[\left(\frac{e^{i\kappa Ut} - e^{iN\kappa t/\gamma}}{(\gamma - N/U)}\right) - \left(\frac{e^{i\kappa Ut} - e^{-iN\kappa t/\gamma}}{(\gamma + N/U)}\right)\right] \tag{9.28}$$

This integral may be evaluated asymptotically by deforming the γ contour into a steepest descents path with κ fixed at one of its points of stationary phase. We omit the details of this computation. The result obtained depends on the stationary phase wavenumbers

$$\left.\begin{aligned} \kappa_{sp} &= \pm N t z/x^2 \\ \gamma_{sp} &= \mp \frac{Nt}{x} \end{aligned}\right\} \tag{9.29}$$

We find that

$$w = w_F + w_T \qquad (9.30)$$

with

$$w_F = \left[\cos\left(\frac{Nz}{U}\right)\right] \delta(x+Ut) + \frac{H\left(t-\frac{|x|}{U}\right)}{(x+Ut)} \sin\frac{N}{U}z \qquad (9.31)$$

and w_T is a transient term that will decay exponentially for a boundary perturbation of nonzero horizontal distance scale. This result agrees with the steady state result of Queney(1945), except that the second wave-like term is cut off at the source and $2Ut$ downstream of the source. Roughly speaking, these cutoffs are a result of the fact that the group velocity of the motion excited is $\pm U$, so that no wave disturbance can propagate out of the region $|x| \leq Ut$. The upstream cutoff should alter the outward energy flow from the source relative to that computed from the steady state theories.

D. On the Horizontal Propagation of Atmospheric Gravity Waves

When the vertical dependence of the hydrostatic gravity wave equation is removed by separation of variables, the resulting inhomogeneous equation may be written

$$\left[\Delta - \left(\frac{\partial^2}{\partial t^2} + f_0^2\right) 1/c^2\right] w = -f(x,y,t) \qquad (9.32)$$

where C^2 is a separation of variables parameter which may be equated to gD, D being the ocean depth for the equivalent shallow water ocean wave equation. The solution for the elementary source

$$f(x,y,t) = -\delta(x)\,\delta(y)\,\delta(t) \tag{9.33}$$

is a simple exact result which is classic. That is

$$w = \frac{1}{2\pi} K_0\left[(\sigma^2+f_0^2)\left(\frac{x^2+y^2}{C^2}\right)\right]^{1/2} \delta(t) = \frac{1}{2\pi} H(t-\rho/C)\,\frac{\cos\left[f_0(t^2-\rho^2/C^2)^{1/2}\right]}{(t^2-\rho^2/C^2)^{1/2}} \tag{9.34}$$

See Appendix II, 5, and Obukov (1949).

Also classical is the result for the one-dimensional source

$$f(x,y,t) = \delta(y)\,\delta(t) \tag{9.35}$$

which is (c.f., Appendix II, 3 and Cahn (1945)).

$$w = \frac{C\, e^{-\frac{|y|}{C}(\sigma^2+f_0^2)^{1/2}}}{2\,(\sigma^2+f_0^2)^{1/2}}\,\delta(t) = \frac{C}{2} J_0\left[f_0\left(t^2-|y|^2/C^2\right)\right] \tag{9.36}$$

The solutions (9.34) and (9.36) may be used as comparison functions for the description of motions excited by other sources. Also we note that the present theory applies to the acoustic wave motions excited in a weakly stratified atmosphere for which one may use for an approximate description

$$\left[\Delta_3 - \frac{1}{c^2}\left(\omega_A^2 + \frac{\partial^2}{\partial t^2}\right)\right] w = - f(x,y,z,t) \qquad (9.37)$$

where we assume that ω_A , the acoustic oscillation frequency , is constant.

If we take $f(x,y,z,t)$ to be the elementary source $\delta(\vec{R})\,\delta(t)$ then (9.37) has the exact solution

$$w = \frac{1}{4\pi R}\, e^{-(\sigma^2+\omega_A^2)^{1/2} R/c}\, \delta(t) = \frac{-c}{4\pi R}\frac{\partial}{\partial R} H(t-R/c)\; J_0\left[\omega_A\left(t^2-\frac{R^2}{c^2}\right)^{1/2}\right] \qquad (9.38)$$

We now give here, as an example, the approximate solution of (9.32) for one dimensional switch-on periodic forcing. That is, assume $f(x,t) = H(t)\, e^{i\omega_0 t}$, so that we consider

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + f_0^2\right)\right] w = -\delta(x)\, H(t)\, e^{i\omega_0 t} \qquad (9.39)$$

with inverse

$$w = \frac{c}{2}\; \frac{e^{-(\sigma^2+f_0^2)^{1/2}|y|/c}}{(\sigma^2+f_0^2)^{1/2}}\, (\sigma - i\omega_0)^{-1}\, \delta(t) \qquad (9.40)$$

This result has the contour integral representation

$$w = \frac{c}{2}\,\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\sigma \frac{e^{\sigma t}}{(\sigma^2+f_0^2)^{1/2}}\; \frac{e^{-(\sigma^2+f_0^2)^{1/2}|y|/c}}{(\sigma - i\omega_0)} \qquad (9.41)$$

The approximate evaluation of (9.41) is by the method of steepest descents. Saddle points of the integrand are located at

$$\frac{d}{d\sigma}\left(\sigma t - (\sigma^2 + f_o^2)^{1/2} |y| /C\right) \tag{9.42}$$

with roots

$$\sigma = \sigma_{s.p.} = \frac{\pm i f_o}{\left(1 - (y/Ct)^2\right)^{1/2}} \tag{9.43}$$

We deform the path of integration in (9.41) into the steepest descents contour, as sketched in Fig. (9-2). When $y/Ct \ll 1$ each branch reduces to that considered in Appendix III, B. In the following discussion, we shall assume this is the case, so that each branch of the contour may be considered a parabola which intesects the $\mathrm{Im}\ \sigma$-axis at the four points

$$\sigma \simeq \pm i f_o \left(1 \pm \frac{1}{2}\left(\frac{y}{Ct}\right)^2\right) \tag{9.44}$$

Of these, the outer two points at

$$\left.\begin{aligned} \sigma_{sp_+} &\simeq i f_o \left(1 + \tfrac{1}{2}\left(\tfrac{y}{Ct}\right)^2\right) \\ \sigma_{sp_-} &\simeq -i f_o \left(1 - \tfrac{1}{2}(y/Ct)^2\right) \end{aligned}\right\} \tag{9.45}$$

are saddle points. It follows from the discussion of Appendices III, B. and IV, B. that we may evaluate (9.41) as

$$w = H\left(t - \left|\left(\frac{\omega_o}{f_o}\right)^{1/2} - 1\right|^{-1} \frac{|y|}{2C}\right) w_F + w_T \tag{9.46}$$

where w_F is the residue contribution

$$w_F = \begin{Bmatrix} \dfrac{C\, e^{-(f_o^2-\omega_o^2)^{1/2}|y|}}{2\,(f_o^2-\omega_o^2)^{1/2}} & \omega_o < f_o \\ \dfrac{C\, e^{i(\omega_o^2-f_o^2)^{1/2}|y| - i\pi/2}}{2\,(\omega_o^2-f_o^2)^{1/2}} & \omega_o > f_o \end{Bmatrix} \tag{9.47}$$

and w_T is the saddle point contribution. By comparison with (9.36) w_T may be written approximately as

$$w_T \simeq \frac{1/4\, H_o^{(1)}\left[f_o\,(t^2-|y|^2/c^2)^{1/2}\right]}{(\sigma_{sp+}-i\omega_o)} + \frac{1/4\, H_o^{(2)}\left[f_o\,(t^2-|y|^2/c^2)^{1/2}\right]}{(\sigma_{sp-}-i\omega_o)} \tag{9.48}$$

for $\left[f_o\,(t^2-|y|^2/c^2)^{1/2}\right] >> 1$

and $(\sigma_{sp\pm}-i\omega_o) = O(1)$

where $\sigma_{sp\pm}$ is given by (9.43).

In the neighborhood of the Brillouin front:

$$t = \left|\left(\frac{\omega_o}{f_o}\right)^{1/2} - 1\right|^{-1} \frac{|y|}{2C} \tag{9.49}$$

the result (9.48) may not be used, but an approximate expression for w in terms of an error function may be obtained as discussed in Appendix IV.

For $\omega_o < f_o$, the steady forced motion as given by (9.47) decays exponentially from the source, while for $\omega_o > f_o$, the motion is a traveling wave. In both cases this motion is confined within the Brillouin front given by (9. 49).

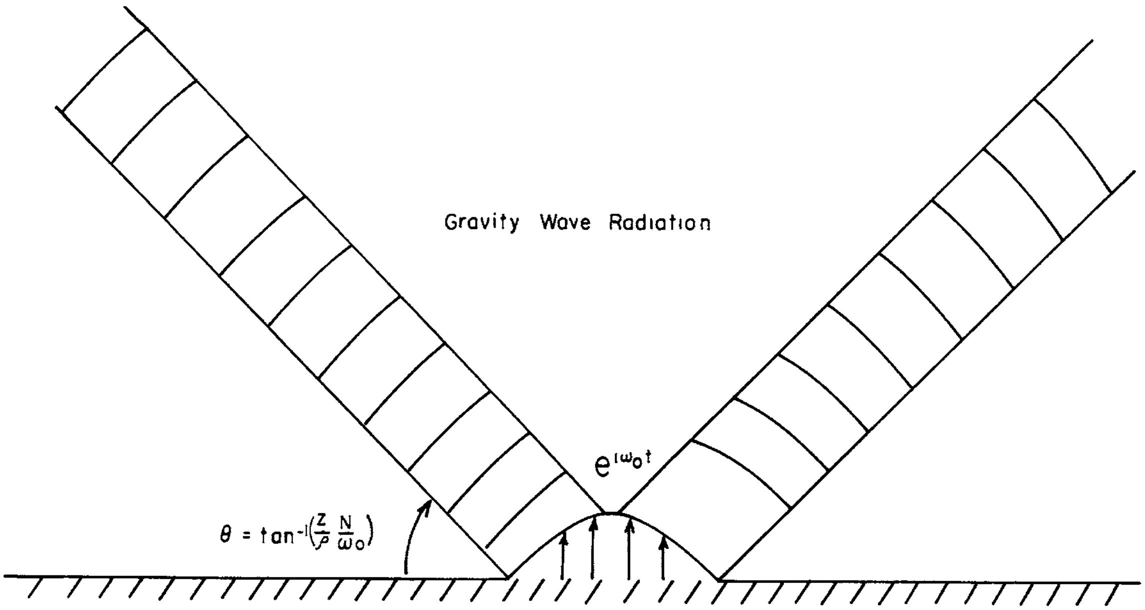


Figure 9-1. Sketch of gravity wave radiation from an oscillating boundary perturbation.

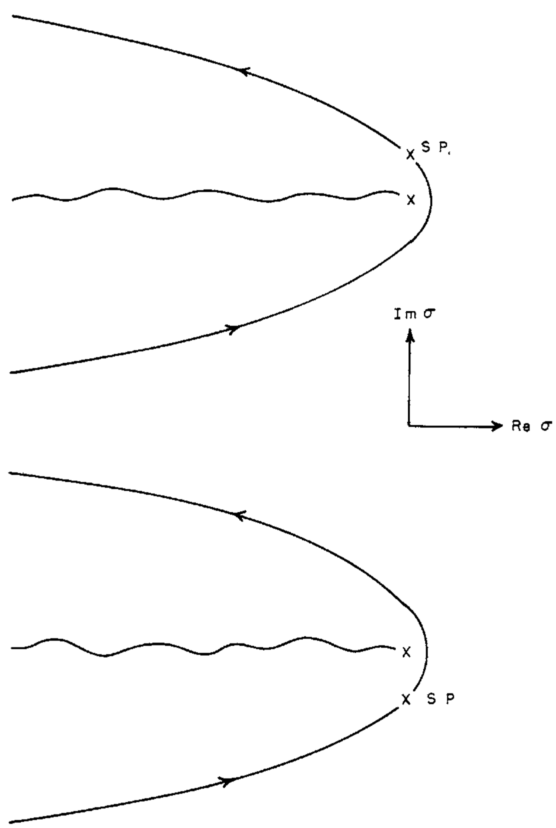


Figure 9-2. Steepest descent contour for horizontally propagating gravity-inertial wave.

X. ON THE LINEAR THEORY OF ATMOSPHERIC TIDES

A. Energetics of the Tidal Equation

The energy conservation properties of the system (3.34)-(3.37) result in several simple but useful theorems concerning the various reduced systems of equations useful in tidal theory. Assume for simplicity a domain, $\mathcal{M}$, bounded by two constant pressure surfaces, $\hat{z}_B$ and $\hat{z}_T$, and by a vertical wall, $\mathcal{B}(\lambda, \theta)$. Furthermore, assume the boundary condition at $\hat{z}_B$ is that $w = \left(\frac{\partial h}{\partial t} - \omega/\rho_0 g\right) = 0$, but that $\hat{z}_T$ and $\mathcal{B}(\lambda,\theta)$ are open. Then one finds from (3.34) -(3.37), the energy conservation equation

$$\frac{d}{dt}\,\mathcal{E} + \mathcal{F} = \Sigma \tag{10.1}$$

where $\mathcal{E}$ is the sum of the kinetic, available potential, and boundary potential energy in $\mathcal{M}$,

$$\mathcal{E} = \frac{1}{2}\int_{M}\left((\vec{c}\cdot\vec{c}) + \frac{g\,\theta^2}{\Theta\,\frac{\partial\Theta}{\partial z}}\right) dM + \frac{1}{2}\int_{\hat{z}_B} g h^2\, dS \tag{10.2}$$

The boundary flux $\mathcal{F}$ is given by

$$\mathcal{F} = -\int_{\hat{z}_T} \omega\, h\, dS + \int_{\mathcal{B}(\lambda,\theta)} g h\, \vec{c}\cdot\hat{n}\, \frac{d\Pi}{g}\, d\mathcal{B} \tag{10.3}$$

and Σ , the total rate of energy generation, is given by

$$\xi = \int_M \left(\vec{c} \cdot \vec{F} + \frac{g\,\theta\, Q}{C_p \Theta \frac{\partial \Theta}{\partial z}} \right) dM \tag{10.4}$$

which is assumed to include all nonlinear terms not explicitly given in the formulation.

In the above integrals we use dM for an element of mass in M, dS for a horizontal surface element, and dB for a horizontal line element on B.

$$\left. \begin{aligned} dM &= d\Pi/_g \; r_e^2 \cos\theta \; d\lambda\, d\theta \\ dS &= r_e^2 \cos\theta \; d\lambda\, d\theta \\ dB &= \frac{\partial B}{\partial \lambda} d\lambda + \frac{\partial B}{\partial \theta} d\theta \end{aligned} \right\} \tag{10.5}$$

If $\hat{z}_B$ and $\hat{z}_T$ are taken to be spherical shells, the only boundary flux is the wgh flux out the top.

Let us now specialize our discussion to consideration of strictly exponential type solutions to the homogeneous tidal equations. That is, assume all dependent variables have time dependence proportional to $e^{\sigma t}$. The homogeneous tidal system may then be written

$$\left. \begin{aligned} \sigma \vec{c} + f \hat{k} \times \vec{c} + \nabla g h &= 0 \\ (\rho_0 \Theta)^{-1} \sigma \theta - \left(\frac{N H}{\Pi} \right)^2 \omega &= 0 \\ \nabla \cdot \vec{c} - \frac{1}{\Pi} \frac{\partial \omega}{\partial \hat{z}} &= 0 \\ \theta/\Theta + {}^1/_H \frac{\partial h}{\partial \hat{z}} &= 0 \end{aligned} \right\} \tag{10.6}$$

Solutions to this homogeneous system, called eigen-solutions, will exist only for certain values of σ which are called the "spectrum" of the system. The spectrum of the tidal system and the resulting eigen-solutions must satisfy certain constraints which are a consequence of the energy conservation law (10.1). Let ($\vec{c}_1$, θ_1, h_1) be an eigen-solution with eigen-value σ_1 , and ($\vec{c}_2$, θ_2, h_2) an eigen-solution with eigen-value σ_2 . Let $*$ denote complex conjugate.

We define $\mathcal{E}_{12}$ and $\mathcal{F}_{12}$ by

$$\mathcal{E}_{12} = \frac{1}{2}\int_M \left(\vec{c}_1 \cdot \vec{c}_2^{\,*} + \frac{g\,\theta_1\,\theta_2^{\,*}}{\Theta\,\frac{\partial}{\partial z}\Theta} \right) dM \quad + \frac{1}{2}\int_{\hat{\Sigma}_B} g\,h_1\,h_2^{\,*}\,dS \tag{10.7}$$

$$\mathcal{F}_{12} = -\frac{1}{2}\int_{\hat{\Sigma}_T} \left(\omega_1 h_2^{\,*} + h_1 \omega_2^{\,*} \right) dS \quad + \frac{1}{2}\int_{B(\lambda,\theta)} g \left(h_1\, \vec{c}_2^{\,*}\cdot\hat{n} + h_1^{\,*}\, \vec{c}_1\cdot\hat{n} \right) \frac{d\Pi\, dB}{g} \tag{10.8}$$

Then from (10.6) one finds that the energy in the product of the two eigen-solutions is related to the flux determined by the two eigen-solutions by the Lagrange identity

$$(\sigma_1 + \sigma_2^{\,*})\,\mathcal{E}_{12} = -\mathcal{F}_{12} \tag{10.9}$$

When 1 = 2, the two eigen-functions are the same; this reduces to

$$2\,Re\ \sigma_1 = -\mathcal{F}_{11}/\mathcal{E}_{11} \tag{10.10}$$

It follows from (10.9) that when boundary conditions are assumed which result in vanishing boundary flux, $\mathcal{F}_{12} = 0$, then either the eigen-solutions are orthogonal in that $\mathcal{E}_{12} = 0$, or else $(\sigma_1 + \sigma_2^*) = 0$. Similarly, when $\mathcal{F}_{11} = 0$, it follows from the fact that $\mathcal{E}_{11}$ is positive definite that $Re\,\sigma = 0$; the spectrum lies on the imaginary axis. The same results concerning exponential solutions can be obtained directly from the vorticity-divergence system, and other useful theorems of a similar nature may be obtained without great labor. We shall, however, now proceed to the consideration of reduced tidal systems given by separation of variables.

B. Integral Theorems on the Separated Homogeneous Tidal Equations

After eliminating ω, θ from the homogeneous system (10. 6), and after separation of variables (μ = separation parameter), one finds the homogeneous system may be written as

$$\left.\begin{aligned} \sigma \vec{c} + f \hat{k} \times \vec{c} + \nabla g h &= 0 \\ \sigma \mu g h + \nabla \cdot \vec{c} &= 0 \end{aligned}\right\} \qquad (10.11)$$

and

$$\left.\begin{aligned} \left[\pi^{-1} \frac{\partial}{\partial \hat{z}} \frac{\pi}{(NH)^2} \frac{\partial}{\partial \hat{z}} + \mu \right] h(\hat{z}, \mu) &= 0 \\ \left(\frac{\partial}{\partial \hat{z}} - \frac{N^2 H}{g} \right) h = 0 \;\Big]_{\hat{z}=0} & \end{aligned}\right\} \qquad (10.12)$$

The latter Sturm-Liouville system for the vertical variation of tidal motions may be written, using $\hat{h} = \left(\frac{\pi}{NH}\right)^{1/2} h$, as

$$\left(\frac{\partial^2}{\partial \hat{z}^2} - \left(\frac{1}{4} + g(\hat{z})\right) + \mu S(\hat{z})\right)\hat{h} = 0 \tag{10.13}$$

$$\left(\frac{\partial}{\partial \hat{z}} + \alpha\right)\hat{h}\Big]_{\hat{z}=0} = 0$$

where $S = \frac{N^2 H}{g}$,

$$\alpha = \left[\frac{1}{2} - \frac{S}{H} + \frac{1}{2}\frac{\partial \ln S}{\partial \hat{z}}\right]_{\hat{z} = \hat{z}_B}$$

This system satisfies the Lagrange identity for two solutions, $\hat{h}_1(\hat{z},\mu_1)$, $\hat{h}_2(\hat{z},\mu_2)$.

$$W\left(\hat{h}_1(\hat{z}), \hat{h}_2^*(\hat{z})\right) = (\mu_1 - \mu_2)\int_0^{\hat{z}} S(z')\,\hat{h}_1(z')\,\hat{h}_2^*(z')\,dz' \tag{10.14}$$

where the Wronskian W is defined by

$$W = \left[\hat{h}_1 \frac{d\hat{h}_2^*}{d\hat{z}} - \hat{h}_2^* \frac{d\hat{h}_1}{d\hat{z}}\right]$$

In particular if we take $\mu_2 = \mu_1^*$, it follows that

$$W\left(\hat{h}(\hat{z}), \hat{h}^*(\hat{z})\right) = 2i\,\mathrm{Im}\,\mu\int_0^{\hat{z}} dz'\,S(z')\left|\hat{h}(z',\mu)\right|^2 \tag{10.15}$$

which relates the energy flow past $\hat{z}$ to the imaginary part of μ multiplied by the integral of the positive definite quantity $S(z')\left|\hat{h}(z',\mu)\right|^2$.

Another useful result is the Sturm-Liouville eigen-function expansion, constructed from the Green's function. That is, let $G(z,z',\mu)$

be the solution to

$$\left(\frac{d^2}{d\hat{z}^2} - \left(\frac{1}{4} + g(\hat{z})\right) + \mu\, S(\hat{z})\right) G(\hat{z}, z', \mu) = -\delta(\hat{z} - z') \tag{10.16}$$
$$\left(\frac{dG}{d\hat{z}} + \alpha G\right)\Big|_{\hat{z}=0} = 0$$

and which satisfies the condition of finite energy for $0 < \arg\mu < 2\pi$.

$$\int_0^\infty |G(\hat{z}, z', \mu)|^2 dz' < \infty \tag{10.17}$$

Then $G(\hat{z}, z', \mu)$ constructed by standard procedures, is an analytic function of μ for $0 < \arg(\mu - C) < 2\pi$, where C is some real constant, and hence $G(\mu)$ may be represented by Cauchy's theorem as the contour integral

$$G(\hat{z}, z', \mu) = \frac{1}{2\pi i}\int_\Gamma \frac{d\lambda}{\mu - \lambda}\, G(\hat{z}, z', \lambda) \tag{10.18}$$

The contour Γ is taken to be a large circle plus a cut along the μ-axis for $(\mu - C) > 0$. Integration around the circle gives a vanishingly small contribution as the circle is taken to infinity, so that Γ may be restricted to a loop around the cut on to both sides of (10.18). It follows from application of (10.16) to both sides of (10.18) that

$$-\frac{\delta(\hat{z} - z')}{S(\hat{z})} = \frac{1}{2\pi i}\int_{\Gamma'} d\lambda\, G(\hat{z}, z', \lambda) \tag{10.19}$$

where Γ' is a loop around the cut on the μ axis.

The singularities of G on the real μ-axis will consist of one or more poles and a branch point, so that (10. 19) may be evaluated as

$$\delta(\hat{z} - z') = \sum_{j=0}^{n} \psi_j(\hat{z}, \mu)\, \psi_j(z', \mu) + \int_K \psi_\lambda(\hat{z}, \lambda)\, \psi_\lambda(z', \lambda)\, d\lambda \qquad (10.20)$$

where the $\psi_j(\hat{z}, \mu_j)$ are normalized eigen-functions of the discrete spectrum and $\psi_\mu(\hat{z})$ are the normalized eigen-functions of the continuous spectrum. For the present expansion, the $j = 0$ mode is the Lamb wave mode, and $j = 1, 2, \cdots n$, are other possible discrete internal modes. It is sometimes useful to evaluate the branch line integral over the continuous spectrum as a sum of "leaky" normal modes. That is, one may shift the location of the branch line to uncover poles on the Riemann sheets $-2\pi < \arg(\mu - c) < 0$, or $2\pi < \arg(\mu - c) < 4\pi$, whose residues will give normal modes which "leak" energy. The normal mode expansion, (10. 20), gives separation of the $\hat{z}$ -dependence for the inhomogeneous tidal equation.

We shall not discuss further the vertical tidal equation, since there is available an extensive body of mathematical literature concerning such problems. Let us now go on to the system (10. 11) and (10. 12) giving the horizontal dependence of tidal motions. Assume a domain D bounded by $B(\lambda, \theta)$. First, assume σ to be eigen-parameter, such that there exist eigen-solutions $\vec{c}_1$, h_1, with eigen-value σ_1, and eigen-solutions $\vec{c}_2$, h_2 with eigen-value σ_2. Then one finds the

Lagrange identity

$$(\sigma_1 + \sigma_2^*) = -\mathcal{F}_{12} / \mathcal{E}_{12} \tag{10.21}$$

with

$$\mathcal{E}_{12} = \frac{1}{2} \int_D (\vec{c}_1 \cdot \vec{c}_2^* + \mu g^2 h_1 h_2^*) \, dS \tag{10.22}$$

$$\mathcal{F}_{12} = \frac{1}{2} g \int_{B(\lambda, D)} \hat{n} \cdot \left(h_2^* \vec{c}_1 + h_1 \vec{c}_2^* \right) dB \tag{10.23}$$

Then $\forall \; \vec{c}_1, h_1 \; ; \; \vec{c}_2, h_2, \ni \; \mathcal{F}_{12} = 0$, either the eigen-solutions are orthogonal in that $\mathcal{E}_{12} = 0$, or else are degenerate in that $\mathrm{Im}\,\sigma_1 = \mathrm{Im}\,\sigma_2$. Also $\forall \; \vec{c}_1, h_1, \ni \mathcal{F}_{11} = 0, \mu > 0$, the σ spectrum is pure imaginary.

Assume now that $\sigma = i\omega$, where ω is a real constant, and let μ be a complex eigen-parameter. One then obtains a Lagrange identity for two eigen-solutions as

$$(\mu_2^* - \mu_1) = \frac{\mathcal{F}_{12}}{i\omega} \div g^2 \int_M (h_2^* h_1) \, dS \tag{10.24}$$

where $\mathcal{F}_{12}$ is again given by (10.23). Thus, $\forall \; \vec{c}_1, h_1 \; ; \; \vec{c}_2, h_2, \ni \mathcal{F}_{12} = 0$ either $\int_M h_2^* h_1 \, dS = 0$, or else $\vec{c}_1, h_1 \; ; \; \vec{c}_2, h_2$ have the same eigen values, $\mu_1 = \mu_2$. Also $\forall \; \vec{c}_1, h_1, \ni \mathcal{F}_{12} = 0$, the μ

spectrum is pure real.

For free harmonic tidal oscillations, $Im\, \omega = 0$, the horizontal and vertical energy fluxes as given in (10. 3) must balance. Thus (10. 24) and the vertical equation identities (10. 14) and (10. 15) may be considered constraints imposed by this balance.

Equivalent to the reduced tidal equation (10. 11) is the homogeneous reduced equation for (Ψ, Φ, h) written

$$\left.\begin{aligned} \sigma \Delta \Psi + \frac{\partial \Psi}{\partial \lambda} + \nabla \cdot f \nabla \Phi &= 0 \\ \sigma \Delta \Phi + \frac{\partial \Phi}{\partial \lambda} - \nabla \cdot f \nabla \Psi + \Delta g h &= 0 \\ \Delta \Phi + \sigma \mu g h &= 0 \end{aligned}\right\} \qquad (10.25)$$

where we take $2\Omega_e = 1$, $r_e = 1$. If we assume for a domain of integration the entire sphere, then boundary flux terms are absent. Again let $\sigma = i\omega$, ω pure real. Then one may derive directly from (10. 25) the identity, equivalent to (10. 24),,

$$0 = (\mu_1 - \mu_2^*) \int_0^{2\pi} \int_0^{\pi} \cos\theta \, d\lambda \, d\theta \; h_1(\mu_1, \theta, \lambda) h_2^*(\mu_2, \theta, \lambda) \qquad (10.26)$$

Hence when μ is taken to be an eigen-parameter, the above orthogonality of the h's , for integrations on a sphere, makes it desirable to use h as the primary dependent variable. The h eigen-solutions for integration of the reduced tidal system (10.25), on a sphere are called Hough's functions. In the next section is given the theory of the integration of the inhomogeneous tidal system.

C. Integration of the Separated Tidal Equation on a Sphere

We discuss integration of the system

$$\left(\sigma\Delta + \frac{\partial}{\partial\lambda}\right)\Psi + \nabla\cdot f\nabla\Phi = -V_0\,\delta(\lambda-\lambda')\,\frac{\delta(\theta-\theta')}{\cos\theta}\,\delta(t) \tag{10.27}$$

$$\left(\sigma\Delta + \frac{\partial}{\partial\lambda}\right)\Phi - \nabla\cdot f\nabla\Psi + g\Delta h = -D_0\,\delta(\lambda-\lambda')\,\frac{\delta(\theta-\theta')}{\cos\theta}\,\delta(t) \tag{10.28}$$

$$\sigma\mu g h + \Delta\Phi = -\mathcal{J}_0\,\delta(\lambda-\lambda')\,\frac{\delta(\theta-\theta')}{\cos\theta}\,\delta(t) \tag{10.29}$$

which may be considered to represent the motion excited by a point impulse of vorticity of strength V_0, a point impulse of velocity divergence of strength D_0, and a point mass impulse of strength $\mathcal{J}_0$. The natural boundary conditions for a sphere are that (Ψ, Φ, h) be regular at $\theta = \pm\pi/2$, and periodic in longitude, with period 2π. From the latter condition we have the longitudinal wave number expansions

$$\delta(\lambda-\lambda') = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{im(\lambda-\lambda')} \tag{10.30}$$

$$\begin{pmatrix} \Psi \\ \phi \\ h \end{pmatrix} = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \begin{pmatrix} \Psi^m(\theta) \\ \Phi^m(\theta) \\ H^m(\theta) \end{pmatrix} e^{im(\lambda-\lambda')} \tag{10.31}$$

To simplify the notation, let

$$\sigma = \partial/\partial t = i\omega \tag{10.32}$$

$$\sin\theta = f = \eta \tag{10.33}$$

and define the differential operators D_1, D_2, D_3, by

$$\left.\begin{aligned} D_1 &= \frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta} - \frac{m^2}{1-\eta^2} \\ D_2 &= D_1 + m/\omega \\ D_3 &= \frac{\partial}{\partial\eta}\eta(1-\eta^2)\frac{\partial}{\partial\eta} - \eta\frac{m^2}{1-\eta^2} \end{aligned}\right\} \tag{10.34}$$

Using the above notation, and (10.30) and (10.31), the tidal equation may be written

$$T X^{(m)} = Y \delta(t) \tag{10.35}$$

where T is the Hermitian operator

$$T = \begin{pmatrix} \omega D_2 & -i D_3 & 0 \\ i D_3 & \omega D_2 & i D_1 \\ 0 & -i D_1 & \mu \omega \end{pmatrix} \tag{10.36}$$

$X^{(m)}$ is the m'th wave number tidal motion vector

$$X^{(m)} = \begin{pmatrix} \Psi^m(\eta) \\ \Phi^m(\eta) \\ g H^m(\eta) \end{pmatrix} \tag{10.37}$$

and Y is the vector forcing function

$$Y = \frac{i}{2\pi} \begin{pmatrix} V_0 \\ D_0 \\ \Delta_0 \end{pmatrix} \tag{10.38}$$

Because the operators D_1, D_2 have Legendre polynomials for eigen-functions, and D_3 has a simple matrix representation when Legendre polynomials are used for basis functions, the integration of (10.35) by expansion of $X^{(m)}$ in the Legendre polynomials of m'th order is quite straightforward. Before this integration is carried out, it is helpful to obtain some general theoretical results concerning the homogeneous system.

If one expands the homogeneous equation of motion system (10.11) in wave numbers, and eliminates the u velocity, one finds the homogeneous tidal system may alternately be written as

$$\frac{d}{d\eta}\begin{pmatrix}(1-\eta^2)^{1/2}V^m(\eta)\\ i g H^m(\eta)\end{pmatrix} + \frac{A(\eta)}{\omega(1-\eta^2)}\begin{pmatrix}(1-\eta^2)^{1/2}V^m(\eta)\\ i g H^m(\eta)\end{pmatrix} = 0 \qquad (10.39)$$

where $V^m(\eta)$ is the m 'th wave number, north-south velocity, and $A(\eta)$ is the variable coefficient matrix

$$A(\eta) = \begin{pmatrix} m\eta & (1-\eta^2)\omega^2\mu - m^2 \\ \eta^2-\omega^2 & -m\eta \end{pmatrix} \qquad (10.40)$$

The alternate homogeneous system given by (10. 39) is somewhat simpler to discuss theoretically than the equation $TX^{(m)} = 0$, since (10. 39) is in standard form for a first order differential equation. The following general facts are known: (See for instance Coddington and Levinson, Chapters III and IV).

1. There exist two independent solutions $V_1^m(\eta)$, $H_1^m(\eta)$; V_2^m, H_2^m, of the homogeneous system. These may be written as a solution matrix

$$M(\eta) = \begin{pmatrix} V_1^m(\eta) & V_2^m(\eta) \\ H_1^m(\eta) & H_2^m(\eta) \end{pmatrix} \qquad (10.41)$$

2. If $A(\eta)$ is analytic at a point $\eta = \eta_0$, then $M(\eta)$ will be analytic in some neighborhood of $\eta = \eta_0$.

3. If A has a singularity at η_0 which is at most a pole, then η_0 is a regular singular point and $M(\eta)$ may be expressed as

$$M(\eta) = R(\eta)\,(\eta-\eta_0)^{P-\kappa E} \tag{10.42}$$

where $R(\eta)$ is analytic at η_0, P in a constant 2 x 2 matrix, κ is an integer, and $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

4. If the singularity at η_0 is worse than a pole, the point is an irregular singular point, and $M(\eta)$ will not have the expansion (10.42).

Referring to the definition of $A(\eta)$ given by (10.40), we see that the singularities are:

$$\left.\begin{array}{ll} \text{regular singular points} & \eta = \pm 1 \\ \text{irregular singular point} & \eta = \infty \end{array}\right\} \tag{10.43}$$

Moreover, it may be shown, (see Eckart, p. 264), that the power series expansions of $M(\eta)$ about $\eta = \pm 1$ begin with

$$\lim_{\eta \to \pm 1} \begin{pmatrix} V^m(\eta) \\ H^m(\eta) \end{pmatrix} \sim (1-\eta^2)^{\pm m/2} \tag{10.44}$$

which is the same as for Legendre functions.

One may also determine directly from the other formulation given by (10.35) that $H^m \sim f(\eta)(1-\eta^2)^{\pm m/2}$, $f(\eta)$ regular at ± 1. It follows from the equivalence of the two formulations that $f(\eta)$ determined from (10.35) will be regular for the domain $[-1, 1]$

Expanding $H^m(\eta)$ in a power series about $\eta = 0$ (using either (10.35) or (10.39)) one finds a recursion formula which separates into a recursion formula for odd powers of η and another for even powers of η. It follows that the $H^m(\eta)$ will either be symmetric or antisymmetric about the equator.

Eigen-solutions to either (10.35) or (10.39) may be determined numerically by a trial and error process. That is; one solves the initial value problems

a) $H^m_\eta(0)$, $\left(\frac{dH^m}{d\eta}\right)_{\eta=0} = 1$ for odd solutions

b) $H^m_\eta(0) = 1$, $\left(\frac{dH^m}{d\eta}\right)_{\eta=0} = 0$ for even solutions

When the integration is carried out from the equator $\eta = 0$, to the pole, $\eta = 1$, the solution as $\eta \rightarrow 1$, will, in general, be of the form

$$H^m(\eta) \cong C\left((1-\eta^2)^{m/2} + A(\mu,\omega)(1-\eta^2)^{-m/2}\right) \tag{10.45}$$

and will hence go to infinity like $(1-\eta)^{-m/2}$ except at the zeros of $A(\mu,\omega)$, which are eigen-values of the tidal equation. One may vary either μ or ω until the integration is regular at $\eta = 1$, and hence determine the eigen-values. For more information concerning this procedure, see for instance the book of Hartree.

The alternate, more classical procedure, is to expand the solution in m'th degree Legendre polynomials Since these are already regular at the points $\eta = \pm 1$, the singular eigen-solutions to the tidal equation are automatically discarded. Hence we use

$$\delta(\eta-\eta') = \sum_{n=m}^{\infty} \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} P_n^m(\eta) \; P_n^m(\eta') \tag{10.46}$$

$$X^m = \sum_{n=m}^{\infty} P_n^m(\eta) \, X_n^m \tag{10.47}$$

The X_n^m are the coefficients of the tidal motion vector expanded in Legendre polynomials.

When the expansions (10.46) and (10.47) are inserted in (10.35), the tidal system reduces to the matrix equation

$$n(n+1) \, L^{(m)}_{nj} \, X_j^m = Y_n^m \; \delta(t) \tag{10.48}$$

where $L^{(m)}_{nj}$ is the matrix operator

$$L^{(m)}_{nj} = \begin{pmatrix} \left[\omega - \frac{m}{n(n+1)}\right]\delta_{nj} & -i\,F^m_{nj} & 0 \\ i\,F^m_{nj} & \left[\omega - \frac{m}{n(n+1)}\right]\delta_{nj} & -i\,\delta_{nj} \\ 0 & i\,\delta_{nj} & \frac{\mu\omega\,\delta_{nj}}{n(n+1)} \end{pmatrix} \tag{10.49}$$

Y_n^m is the forcing function matrix

$$Y_n^m = \frac{1}{2\pi i}\left(\frac{2n+1}{2}\right)\frac{(n-m)!}{(n+m)!}\begin{pmatrix} V_0 \\ D_0 \\ \zeta_0 \end{pmatrix} \tag{10.50}$$

and F^m_{nj} is the "Coriolis" matrix

$$F^m_{nj} = \frac{(n^2-1)(n-m)}{(2n-1)n(n+1)}\,\delta_{n-1,j} + \frac{(n+2)(n+1)(n+m+1)}{(2n+3)\,n(n+1)}\,\delta_{n+1,j} \tag{10.51}$$

and may be obtained from combining the well known recursion formulae

$$(1-\eta^2)\,\frac{\partial}{\partial\eta}\,P^m_n(\eta) = (n+1)\,\eta\,P^m_n - (n-m+1)\,P^m_{n+1}$$

$$\eta\,P^m_n(\eta) = \frac{n-m+1}{2n+1}\,P^m_{n+1} + \frac{n+1}{2n+1}\,P^m_{n-1}$$

to give

$$-D_3\,P^m_n(\eta) = \frac{n(n+2)(n-m+1)}{2n+1}\,P^m_{n+1} + \frac{(n+1)(n-1)(n+m)}{2n+1}\,P^m_{n-1} \tag{10.52}$$

The solution to (10.48) may be written

$$X^{(m)}_n = \frac{1}{n(n+1)}\,L^{-1}_{nj}\,Y_j \tag{10.53}$$

where the inverse matrix L^{-1} is constructed as follows. We define the 3N x 3N matrix $L^{(N)}$ by taking the first N x N terms in each of the component matrices of the matrix L defined by (10.49). Since this matrix is finite, it has the inverse

$$L^{(N)\,-1} = \frac{\left[D^N_{ij}(\omega,\mu)\right]}{D^N(\omega,\mu)} \tag{10.54}$$

where $D^N_{ij}(\omega,\mu)$ is the transpose of the matrix of cofactors of $L^{(N)}$, and $D^N(\omega,\mu) = \det.|L^{(N)}|$.

Then the inverse matrix L^{-1} is defined as

$$L^{-1} = \lim_{N \to \infty} L^{(N)^{-1}} \tag{10.55}$$

where we augment $L^{(N)^{-1}}$ with zeros so that it has the same number of elements as L^{-1}, and define convergence as convergence of each element. The elements $L^{-1}_{nj}(\omega,\mu)$ are meromorphic functions of the parameters ω and μ.

We define the "approximate" matrix $\hat{L}$ by replacing the variable Coriolis parameter f in (10.27) -(10.28) by a constant average value f_0. This matrix is given by

$$\hat{L}_{nj} = \begin{pmatrix} \left(\omega - \frac{m}{n(n+1)}\right)\delta_{nj} & -if_0 & 0 \\ if_0 & \left(\omega - \frac{m}{n(n+1)}\right)\delta_{nj} & -i\delta_{nj} \\ 0 & i\delta_{nj} & \frac{\mu\omega}{n(n+1)}\delta_{nj} \end{pmatrix} \tag{10.56}$$

and has the inverse

$$\hat{L}^{-1}_{nj} = \frac{\delta_{nj}}{\omega\left(\left[(\omega 1_B)^2 - f_0^2\right]\frac{\mu}{n(n+1)} - 1_B\right)} \begin{pmatrix} \left(\frac{\omega^2\mu 1_B}{n(n+1)} - 1\right), & \frac{if_0\mu\omega}{n(n+1)}, & -f_0 \\ -\frac{if_0\mu\omega}{n(n+1)}, & \left(\frac{\omega^2\mu 1_B}{n(n+1)} - 1\right), & i\omega 1_B \\ -f_0, & -i\omega 1_B, & \left((\omega 1_B)^2 - f_0^2\right) \end{pmatrix} \tag{10.57}$$

where

$$1_\beta = 1 - \frac{m/\omega}{n(n+1)} \tag{10.58}$$

The inverse matrix $\hat{L}^{-1}$ as a function of μ is finite except for a discrete infinity of poles on the real μ axis, where

$$\mu = \hat{\mu}_n(\omega, m) = \frac{n(n+1)\,1_\beta}{(\omega^2 1_\beta^2 - f_0^2)}; \tag{10.59}$$

The poles of L^{-1} are similarly obtained as the limit of the sequence of μ roots given by

$$D^N(\omega, \mu) = 0 \tag{10.60}$$

which defines again a discrete infinity of points in the real μ axis, $\mu_m, \mu_{m+1}, \cdots \mu_{m+j} \cdots$, which for j sufficiently large, are qualitatively similar to the $\hat{\mu}_n(\omega)$ given by (10.59).

There are two limiting cases where the roots of (10.59) and (10.60) agree exactly. These are

a) The large ω (nonrotating earth) limit. That is

$$\lim_{\omega \to \infty} \mu_n(\omega) = \lim_{\omega \to \infty} \hat{\mu}(\omega) = \frac{n(n+1)}{\omega^2} \tag{10.61}$$

b) The Haurwitz-Rossby limit for small μ . That is for $\mu = 0$ (10.59) and (10.60) have the ω - roots given by $1_\beta = 0$ or

$$\lim_{\mu \to 0} \omega(\mu) = \frac{m}{n(n+1)} \qquad (10.62)$$

The "approximate" dispersion relation (10.59) is sketched in Fig (10-1). The branches include an eastward propagating and westward propagating internal gravity wave for $\omega 1_{\beta} > f_0$, external gravity wave and Rossby wave branches for $-f_0 < \omega < 0$, and $\frac{m}{n(n+1)} < \omega < f_0$, and an internal Rossby wave branch for $0 < \omega < \frac{m}{n(n+1)}$. Little accurate information is yet available concerning the roots of (10.60) except for the westward propagating internal gravity wave mode, which is of primary interest for the discussion of the observed atmospheric thermal tides. See the monographs of Wilkes and Siebert for further information concerning this branch.

Note that for small negative μ, there will be two complex ω roots of (10.60), as seen by comparison with (10.59). These complex roots of (10.59) exist for μ in the range

$$-\frac{n(n+1)}{f_0^2} \lesssim \mu < 0 \qquad (10.63)$$

as may be seen by explicit computation (assuming $\frac{m}{n^2} \ll \omega \ll f_0$).

In order to use the inverse matrix L^{-1}, it is helpful to expand the elements L^{-1}_{nj} in partial fractions. That is

$$L^{-1}_{nj} = \sum_{k=m}^{\infty} \frac{\bar{\Theta}_{njk}}{\mu - \mu_k(\omega)} \qquad (10.64)$$

where the $\mu_{k}(\omega)$ are the roots defined by (10. 60), and

$$\overline{\Theta}_{njk} = \lim_{N \to \infty} \frac{D^N_{nj}(\omega, \mu_k(\omega))}{\frac{d}{d\mu} D^N(\omega, \mu_k)} \tag{10. 65}$$

Substituting (10. 53) in (10. 47) and summing over the Legendre polynomials gives

$$X^{(m)} = \sum_{k=m}^{\infty} \sum_{j=m}^{\infty} \frac{\Theta^m_{kj}(\eta)\, Y^m_j\, \delta(t)}{\mu - \mu_k(\omega)} \tag{10. 66}$$

where $\Theta^m_{kj}(\eta)$ is given by

$$\Theta^m_{kj}(\eta) = \sum_{n=m}^{\infty} P^m_n(\eta)\, \overline{\Theta}_{njk} \tag{10. 67}$$

The expansion (10. 66) expresses the solution as a sum over normal modes of the rotating atmosphere. It is not difficult to obtain from the $\Theta_{kj}(\eta)$ the normalized eigen functions of the tidal equation, but we omit this computation. If the $\Theta_{kj}(\eta)$ are known, there is no need for the eigen-functions. We shall call $\Theta_{kj}(\eta)$ the Hough transfer function since it transforms the forcing function Y^m_j into the solution function $X^{(m)}$.

In order to complete the discussion of the initial value problem, we expand each $\dfrac{\Theta^m_{kj}(\eta)}{\mu - \mu_k}$ in partial fractions in $\omega = -i\sigma$

$$\frac{\Theta^m_{jk}(\eta)}{\mu - \mu_k} = \sum_{\ell=1}^{3} \frac{\Theta^m_{jk}(\eta, \mu(\omega_\ell))}{\left(\frac{d\mu_k}{d\omega}\right)_{\omega = \omega_{\ell k}} (\omega - \omega_\ell)} \tag{10. 68}$$

Then (10.66) may be evaluated as

$$X^m(t) = i \sum_{\kappa=m}^{\infty} \sum_{j=m}^{\infty} \sum_{\ell=1}^{3} \frac{\Theta_{j\kappa}^m(\eta, \mu_\kappa(\omega_{\kappa\ell}^m))}{\left(\frac{d\mu_\kappa}{d\omega}\right)_{\omega_{\kappa\ell}^m} (\omega - \omega_{\kappa\ell}^m)} Y_j^m e^{i\omega_{\kappa\ell}^m t} \qquad (10.69$$

In general for each (κ, m), $\omega_{\kappa\ell}^m$ will be real and consist of two gravity wave modes and one Rossby wave mode, except for possible μ small and negative as given by (10.63), where there may be one exponentially growing and one decaying mode. The physical significance of such instabilities is not well understood.

Added Note: Lindzen (1966), Mon. Wea. Rev., 94, 295, has given a detailed analysis of the roots of (10.60) for $m > 0$, $\omega = 1$, corresponding to the forced diurnal tidal motions. This motion fits in Fig. 10-1 approximately in the neighborhood of the dashed line through the $\omega > 0$ axis. Lindzen finds that for this frequency there is both the internal gravity wave branch with largest amplitudes near the equator as well as an external branch with largest amplitudes in middle latitudes. This may be interpreted in terms of Fig. 10-1 as requiring that the effective Coriolis parameter f_0 decreases with increasing μ. Since for large $|\mu|$, both branches asymtote to f_0, it follows that at least over a certain range of ω there will be two sequences of μ roots rather than a single sequence.

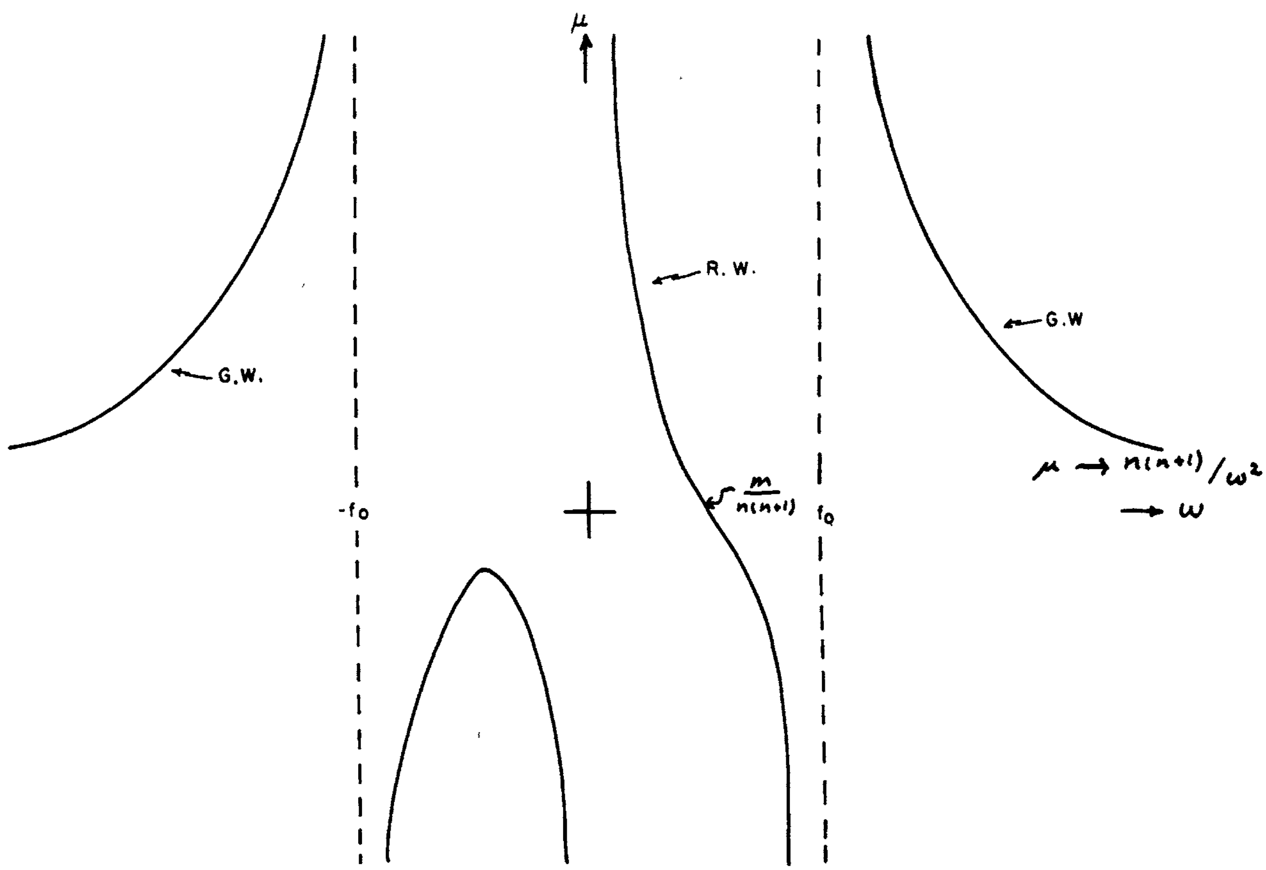


Figure 10-1. Sketch of the gravity wave and Rossby wave branches of the dispersion equation for tidal motions on a rotating sphere.

XI. CONCLUDING REMARKS

A. Survey of the Analysis of Atmospheric Wave Propagation

The present theory will not be compared in detail with observed atmospheric wave motions. It is apparent that the results presently available on atmospheric wave propagation are but a beginning in a complicated subject. The present work is primarily intended to describe, within the framework of the constant coefficient approximation, the initiation of various forced atmospheric wave motions. This has been done by the analysis of elementary examples, selected for reasons of physical interest and mathematical simplicity. Dispersive wave motions have been studied by many earlier authors by similar means, but the subject has not reached the attention of many able mathematicians capable of establishing a general mathematical theory of the subject. Such work, concentrating primarily on the mathematical details involved, would not be unwelcome. One might mention, as contributions to such a theory, the work of Lighthill (1960) and Whitham (1961), (1965), as well as the related work of Keller and his colleagues at NYU. See for instance Lewis (1965).

There remains much to be done in the development of a theoretical physical description of atmospheric wave motions by means of classical analytical methods as employed in this thesis.

Little is yet known concerning the details of energy flow through the atmosphere by atmospheric wave motions. Perhaps the best observational evidence of this phenomenon is given by studies of wave motions propagating upward out of the troposphere. On the theoretical side, there is needed an increased understanding of the importance of variable stratification in the "trapping" and "guiding" of atmospheric wave energy. The most obvious direct extension of the models considered in this thesis to variably stratified atmospheres would be to use "locally constant coefficient" solutions in the neighborhood of a source to match to asymptotic variable coefficient solutions obtained by solution of the relevant "Hamilton-Jacobi" equations.

The dynamics of the upper atmosphere, above the range of validity of the inviscid wave propagation models, requires the inclusion of molecular transport processes. It is, however, not possible to decouple the dynamics of this region from the dynamics of the lower atmosphere because of the extensive upward leakage of wave energy from below. While it is an open question as to whether this energy flux is sufficient to be of first order importance in determining the vertical temperature structure of the upper atmosphere, the available wave energy is seemingly adequate to maintain many of the observed motions of the upper atmosphere. The theoretical models of this thesis can be expected to overemphasize somewhat the propagation of wave energy away from an original point of excitation and hence into the upper atmosphere. Variable coefficients will result in wave-guide like localization of the wave energy. In many instances, these "wave guides" will "leak" energy by energy tunneling through the "wave guide" walls and by dissipation at the lower boundary, so that normal mode descriptions of the motion will in general require complex normal modes.

For the purpose of analyzing models for the quantitative description of observed atmospheric wave motions, it is necessary to have detailed observational information concerning the energy sources requisite for the excitation of the motion. It is to be recalled in this respect that these energy sources will consist of all terms entering the governing dynamics which are not explicitly incorporated into internal dynamics of the model discussed. Modern statistical techniques may be used for the organization of this data.

In conclusion, we consider a simple example to illustrate the application of the methods of this thesis to the analysis of atmospheric instabilities.

B. Remarks Concerning Atmospheric Instabilities

Several physical processes by which the atmosphere may release stored up energy are known, but the mathematical description of these phenomona is quite rudimentary. As a simple example illustrating some common mathematical types of atmospheric instabilities, consider the equation for 2-D motions of an unstably stratified atmosphere, written as,

$$\left[\left(\frac{\partial^2}{\partial t^2} + f_0^2\right)\frac{\partial^2}{\partial z^2} + \left(\frac{\partial^2}{\partial z^2} - v^2\right)\frac{\partial^2}{\partial x^2} \right] w = 0 \tag{11.1}$$

where

$$v^2 = - \frac{g}{\Theta}\frac{\partial \Theta}{\partial z}$$

is an assumed constant unstable stratification parameter. Assume the domain $[0,\infty)$ and the elementary switch-on bottom boundary condition:

$$w\Big]_{z=0} = H(t)\, e^{ikx + i\omega_0 t} \qquad , \quad k > 0$$

The operational solution to this problem is then

$$w = e^{ikx}(\sigma - i\omega_0)^{-1}\, e^{-k\left(\frac{\sigma^2 - v^2}{\sigma^2 + f_0^2}\right)^{1/2}|z|}\, \delta(t) \tag{11.2}$$

Assume that $v \gg 1$, $\omega_0 > 1$ and that: a) $f_0 = 1$; b) $f_0 = 0$.

For large time, the motion decomposes into the following modes:

$$e^{-ikx} w = w_F + w_C + w_G \tag{11.3}$$

where w_F is the motion forced by the sinusoidal forcing, w_C is the "convective mode" and w_G is an unstable gravity wave mode. The forced motion, determined by the $\sigma = i\omega_0$ pole of the solution operator, is given by

$$w_F = e^{i\omega_0 t}\, e^{-\kappa\left(\frac{\omega_0^2 + v^2}{\omega^2 - f_0^2}\right)^{1/2} z} \tag{11.4}$$

The convective mode and gravity wave mode are given by saddle points in the neighborhood of the $\sigma = v$ and $\sigma = \pm i f_0$ branch points respectively. Let $\lambda = \left(\frac{2 v \kappa^2 z^2}{f_0^2 + v^2}\right)^{1/2}$ and $\mu = \left(\frac{v^2 + f_0^2}{2 f_0}\right)^{1/2} \kappa z$. Then these motions are found to be given approximately by

$$w_c \simeq \frac{1}{v - i\omega_0}\, e^{-\lambda(\sigma - v)^{1/2}}\, \delta(t) \tag{11.5}$$

$$w_G\Big|_{f_0 = 1} \simeq \frac{1}{f_0 - \omega_0} \operatorname{Im} e^{-\mu \frac{e^{+\pi i/4}}{(\sigma - i f_0)^{1/2}}}\, \delta(t) \tag{11.6}$$

$$w_G\Big|_{f_0 = 0} \simeq -\operatorname{Im} \frac{e^{-i v z/\sigma}}{\omega_0}\, \delta(t) \tag{11.7}$$

The time dependence of w_c, $w_G\big|_{f_0=1}$, $w_G\big|_{f_0=0}$, is approximately evaluated as

$$w_c \simeq \frac{1}{v - i\omega_0}\, \frac{\lambda}{2(\pi t^3)^{1/2}}\, e^{vt - \lambda^2/4t} \tag{11.8}$$

$$w_G\Big|_{f_0 = 1} \simeq \frac{1}{(f_0 - \omega_0)} \left(\frac{\mu}{2t}\right)^{2/3} \frac{e^{3\sqrt{3}/4\,(2\mu^2 t)^{1/3}}}{(3\pi t)^{1/3}} \cos\left[\tfrac{3}{4}(2\mu^2 t)^{1/3} - \pi/3\right] \tag{11.9}$$

$$w_G\Big|_{f_0=0} \simeq \frac{1}{\omega_0}\frac{\partial}{\partial t}\left(\frac{e^{(2\nu\kappa z)^{1/2}t^{1/2}}}{\pi^{1/2}(\nu\kappa z t)^{1/2}}\cos\left[(2\nu\kappa z t)^{1/2}-\frac{3\pi}{8}\right]\right) \qquad (11.10)$$

The asymptotic time dependence of atmospheric instabilities will in general depend on the assumed form of forcing and initial conditions. One may distinguish between weak instabilities, where the growth rate of the instability is algebraic or less, and strong instabilities, where the growth rate is greater than algebraic. Weak instabilities usually occur as a consequence of some unrealistic assumption concerning the excitation of the motion, while strong instabilities appear to be related to physical processes of energy release. It seems reasonable to classify the strong instabilities according to their order of growth as entire functions of the parameter t. Thus the above convective mode instability is of exponential order (order one), while the unstable gravity wave is of order 1/3 or order 1/2, depending on whether rotation is present or absent. The decrease of the growth rate in the presence of rotation may be considered an illustration of the general principle that "rotation inhibits convection".

APPENDIX I

Partial Glossary of Special Notation Used

$Ai(x)$	Airy function
c	speed of sound, $(\gamma \Pi/\rho_0)^{1/2}$.
$\vec{c}$	horizontal velocity vector
C	speed of internal gravity waves, or some other constant defined in the text.
C_p	specific heat at constant pressure
C_v	specific heat at constant density
$\mathrm{erfc}(\;)$	complementary error function
f	Coriolis parameter $2\Omega \sin\theta$
f_0	constant Coriolis parameter $2\Omega \sin\theta_0$
$F(a,b,c;z)$	hypergeometric function
F. T.	Fourier transform
$\vec{F}$	$F_x \hat{\imath} + F_y \hat{\jmath} + F_z \hat{k}$, external force per unit mass vector
$\vec{F}_h$	$F_x \hat{\imath} + F_y \hat{\jmath}$
$F_{1,2}$	Coriolis operator
$F(\sigma)$	$F(\partial/\partial t)$
$-g\hat{k}$	force of gravity
h	geopotential height for pressure coordinates
$\hat{h}$	$(\Pi^{1/2}/NH)\,h$
$H(x)$, $H(t)$	Heaviside function. $H(x) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$
H	atmospheric scale height

$H_0^{(1)}(z)$, $H_0^{(2)}(z)$	zeroth order Hankel functions
$\hat{\imath}$	unit vector in direction of increasing longitude
$\hat{\jmath}$	unit vector in direction of increasing latitude
$J_0(z)$	zeroth order Bessel function
$\hat{k}$	unit vector upwards relative to gravity
(k_x, k_y, k_z)	wave number vector
$K_0(z)$	Bessel function of imaginary argument
N	buoyancy frequency $\left(g/\Theta \, \frac{\partial \Theta}{\partial z} \right)^{1/2}$.
p, p'	perturbation pressure
$P_n^m(\eta)$	Legendre polynomial of m'th degree, and n'th order
P. D. E.	partial differential equation
Q	rate of addition of heat per unit mass times Θ/T
r_e	mean radius of earth
Re	real part of a complex quantity
$\vec{R}$	radial vector, $\hat{\imath}x + \hat{\jmath}y + \hat{k}z$.
R	$\lvert\vec{R}\rvert$
S. D.	steepest descents
s.p., S.P.	saddle point (or stationary phase)
S	planetary stability $N^2H^2/4\Omega_e^2 r_e^2$
$\mathscr{s}$	rate of mass addition per unit volume
t	time
T	temperature $^{\circ}$K

$\vec{u} = (u, v, w)$	velocity vector for geometrical coordinate system
$W(x, y, z, t)$	vertical motion propagator
$x = r_e \cos\theta (\lambda - \lambda_0)$,	displacement from a reference longitude λ_0.
$y = r_e(\theta - \theta_0)$	, displacement from a reference latitude
z	vertical coordinate for geometric coordinate system
$\hat{z} = -\ln \Pi$	, vertical coordinate for pressure coordinate system
$\beta =$	$\frac{2\Omega_e \cos\theta}{r_e}$, gradient of planetary vorticity at a fixed point
γ	either gas constant C_p/C_v , or vertical wave number κ_z.
$\Gamma(z)$	Eckart's parameter, $\frac{1}{H}\left(\frac{1}{\gamma} - \frac{1}{2}\right) - \frac{1}{2H}\frac{\partial H}{\partial z}$
$\Gamma(n)$	Gamma (factorial) function, $\int_0^\infty e^{-\lambda} \lambda^{n-1} d\lambda$.
$\delta(x), \delta(t)$	Dirac distributions
$\delta(\vec{R})$	$\delta(x)\,\delta(y)\,\delta(z)$
∇	horizontal gradient operator ($\frac{\hat{i}}{r_e \cos\theta}\frac{\partial}{\partial\lambda} + \frac{\hat{j}}{r_e}\frac{\partial}{\partial\theta}$, for spherical coordinates)
$\nabla_{\vec{\kappa}}$	three dimensional gradient operator in wave number space
$\nabla \cdot$	horizontal divergence operator ($\frac{1}{r_e \cos\theta}\frac{\partial}{\partial\lambda}\hat{i} + \frac{1}{r_e \cos\theta}\frac{\partial}{\partial\theta}\cos\theta\,\hat{j}$, for spherical coordinates)
Δ, Δ_2	$\nabla \cdot \nabla$, horizontal Laplacian operator
Δ_3	$\Delta + \frac{\partial^2}{\partial z^2}$, three dimensional Laplacian
ϵ	asymptotic error term
$\epsilon(z), \epsilon^*(z)$	See (3. 25), (6. 1)
$\int_{-i\infty+\epsilon}^{i\infty+\epsilon} (\ )$	contour to the right of all singularities of the integrand

η	$\sin\theta$
θ	latitude or (perturbation) potential temperature
Θ	mean reference potential temperature
κ	gas constant $\frac{\gamma-1}{\gamma}$
λ	longitude
μ	separation parameter for the hydrostatic wave equation
ν	kinematic coefficient of viscosity
$\Pi(z)$	mean reference pressure
ρ	(perturbation) density, or horizontal radial distance, $(x^2+y^2)^{1/2}$.
ρ_o	mean reference density
σ	complex variable equivalent to the operator $\partial/\partial t$
ω	$\frac{d}{dt}(\Pi + p')$, vertical motion parameter
ω	frequency of motion
ω_o	frequency of forced oscillation
Ω_e	average angular velocity of the earth
$\forall$	"for all"
$\ni$	"such that"

Notation such as $(\hat{\ })$, $(\ )'$, $(\ )^*$, $(\bar{\ })$, is defined in the text to indicate modifications of the above definitions. We frequently use $(\ )_o$ to indicate a constant parameter. For the mathematical definitions of special functions employed, the reader may refer to the mathematical tables of Abramson et al.

In defining contour integrals, notation is used whose meaning should be clear from the mathematical context. For example, we use $\int_{\sigma=-\infty}^{(\sigma=i)} (\ ) \, d\sigma$ to indicate a counterclockwise loop integral from $\sigma = -\infty$ about a singularity at $\sigma = i$, and $\int_{SD} (\ ) \, d\sigma$; $\int_{\sigma=-\infty}^{(S.D.)}$ to indicate counterclockwise integration from $\sigma = -\infty$ along a steepest descents path.

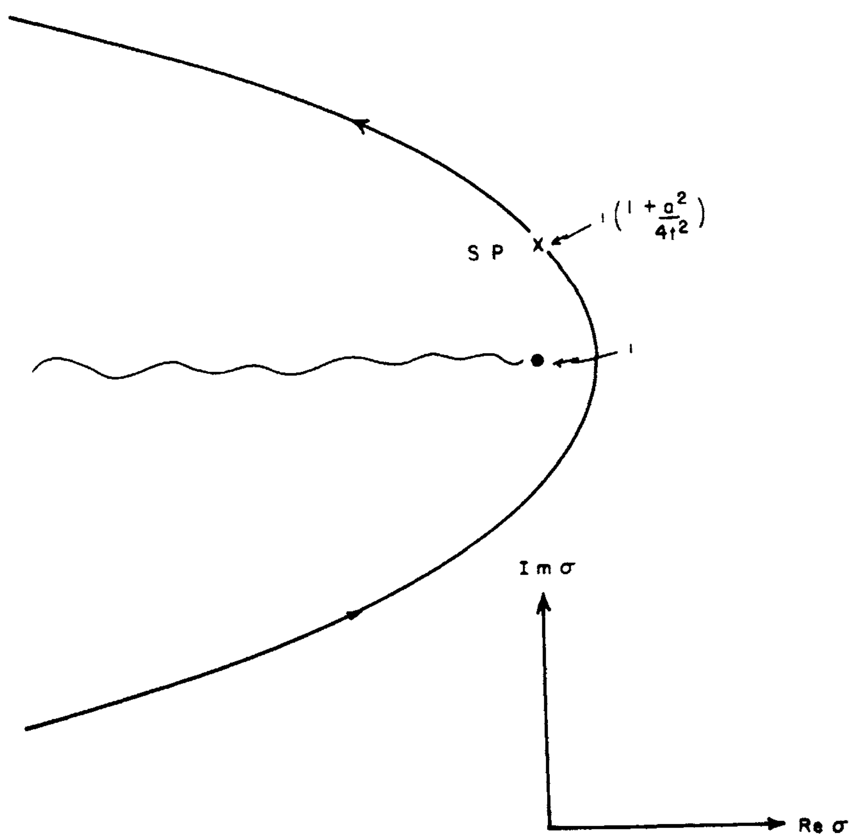


Figure A-1. Steepest descents contour for III, B.

APPENDIX II

Some Elementary Propagators, $f(t)$ and their Operational Representation $F(\sigma)\,\delta(t)$

	$f(t)$	$F(\sigma)\,\delta(t)$
1.	$H(t)(\pi t)^{-1/2}$	$\sigma^{-1/2}\delta(t)$
2.	$H(t)\ J_0(\omega t)$	$(\sigma^2+\omega^2)^{-1/2}\delta(t)$
3.	$H(t-a)\ J_0(\omega(t^2-a^2)^{1/2})$	$(\sigma^2+\omega^2)^{-1/2}e^{-a(\sigma^2+\omega^2)^{1/2}}\delta(t)$
4.	$H(t)\ J_0[2(\beta t)^{1/2}]$	$e^{-\beta/\sigma}\sigma^{-1}\delta(t)$
5.	$H(t-a)\ \dfrac{\cos[\omega(t^2-a^2)^{1/2}]}{(t^2-a^2)^{1/2}}$	$K_0[a(\sigma^2+\omega^2)^{1/2}]\ \delta(t)$
6. a)	$H(t)\ e^{-\lambda t}$	$(\sigma+\lambda)^{-1}\delta(t)$
b)	$H(t)\cos\omega t$	$(\sigma^2+\omega^2)^{-1}\sigma\ \delta(t)$
c)	$H(t)\sin\omega t$	$(\sigma^2+\omega^2)^{-1}\omega\ \delta(t)$
7.	$H(t)\ t^{-1/2}\displaystyle\sum_{j=0}^{\infty}\frac{(a^2t)^{j/2}e^{3\pi j/4}}{j!\,\Gamma(j+1/2)}$	$e^{-ae^{-i\pi/4}\sigma^{-1/2}}\sigma^{-1/2}\ \delta(t)$
8.	$H(t)(\pi t)^{-1/2}e^{i\omega(t-a^2/4t)}$	$(\sigma-i\omega)^{-1/2}e^{-ae^{i\pi/4}(\sigma-i\omega)^{1/2}}\delta(t)$
9.	$H(t)\ e^{i\alpha^2 t}\,\mathrm{erfc}\,[e^{i\pi/4}\alpha\sqrt{t}]$	$\sigma^{-1/2}(\sigma^{1/2}+\alpha e^{i\pi/4})^{-1}\delta(t)$
10.	$H(t)\dfrac{a\,e^{-a^2/4t}}{2(\pi t^3)^{1/2}}$	$e^{-a\sigma^{1/2}}\delta(t)$

APPENDIX III

On the Asymptotic Evaluation of $F(\sigma)\,\delta(t)$

To evaluate $F(\sigma)\,\delta(t)$, we use the contour integral representation

$$F(\sigma)\,\delta(t) = \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} e^{\sigma t} F(\sigma)\, d\sigma \qquad \text{(III. 1)}$$

This integral may be evaluated by the steepest descents technique. A path of steepest descent is defined as a contour along which

$$\operatorname{Im}\left(\sigma t + \log F(\sigma)\right)$$

remains constant. Assume t to be sufficiently large that $t > \operatorname{Re}\frac{1}{\sigma}\log F(\sigma)$ so that the s. d. path terminates in the $\operatorname{Im}\sigma < 0$ plane. Assume that $F(\sigma)$ has n poles at σ_j, $j = 1 \cdots n$, that lie between the steepest descent contour and the path in (III. 1). Denote the residue at the j'th pole, σ_j, by R_j;

$$R_j = \lim_{\sigma \to \sigma_j} (\sigma - \sigma_j)\, F(\sigma) \qquad \text{(III. 2)}$$

Then by Cauchy's theorem, $F(\sigma)\,\delta(t)$ may be written as

$$F(\sigma)\,\delta(t) = \sum_{j=0}^{n} \frac{R_j}{\sigma - \sigma_j}\,\delta(t) + \frac{1}{2\pi i}\int_{S.D.} F(\sigma)\, e^{\sigma t}\, d\sigma \qquad \text{(III. 3)}$$

where $\int_{S.D.} (\ )\, d\sigma$ denotes integration along the steepest descent contour.

Assume further that in the vicinity of the κ'th saddle point, $\sigma = \sigma_{sp\,\kappa}$, that $F(\sigma)$ has an expression

$$F(\sigma) = \alpha(\sigma)\ G_\kappa(\sigma)$$

where $G_\kappa(\sigma)$ is one of the elementary propagators given in Appendix II that has the same saddle point as $F(\sigma)$ at $\sigma = \sigma_{sp,\kappa}$, and $\alpha(\sigma)$ is regular at $\sigma = \sigma_{sp,\kappa}$; $\alpha = 1 + \alpha_1(\sigma - \sigma_{sp,\kappa}) + \cdots$.

It follows that $F(\sigma)\ \delta(t)$ is given by

$$F(\sigma)\,\delta(t) = \sum_{j=0}^{n} \frac{R_j}{\sigma-\sigma_j}\,\delta(t) + \sum_{\kappa}\Big(1 + O(\sigma-\sigma_{sp,\kappa})\Big)\,G_\kappa(\sigma)\,\delta(t) \qquad \text{(III. 4)}$$

More detailed examples of the evaluation of $\alpha(\sigma)\ G_\kappa(\sigma)\,\delta(t)$ are given below.

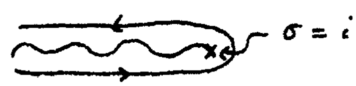
<u>Example A.</u>

$$\alpha(\sigma) = 1 + \sum_{j=1}^{\infty} \alpha_j\,(\sigma - i)^j \quad , \quad G_\kappa(\sigma) = (\sigma - i)^{-1/2} \qquad \text{(III. 5)}$$

The S. D. contour for $(\sigma - i)^{-1/2}\delta(t)$ is a loop integral about the b.p. $\sigma = i$.

$$(\sigma - i)^{-1/2}\,\delta(t) = \frac{1}{2\pi i}\int_{\sigma=-\infty}^{(i)} \frac{e^{\sigma t}}{(\sigma - i)^{1/2}} = \frac{e^{it}}{(\pi t)^{1/2}}$$

where the notation indicates the branch line contour,



The higher order terms in $(\sigma - i)$ may be similarly evaluated. One finds

$$\alpha(\sigma)\ G_\kappa(\sigma)\ \delta(t) = \frac{(1+\epsilon)}{(\pi t)^{1/2}}\,e^{it}\sum_{j=0}^{N}(-)^j\,\alpha_j\left(\tfrac{1}{2}\cdot\tfrac{3}{2}\cdots\tfrac{2j-1}{2}\right)t^{-j} \qquad \text{(III. 6)}$$

where $\epsilon = O\left(t^{-(N+1)}\right)$

Example B

$$\alpha(\sigma)\, G_K(\sigma) = \frac{\alpha(\sigma)\, e^{-a e^{i\pi/4}(\sigma-i)^{1/2}}}{(\sigma-i)^{1/2}} \qquad \text{(III. 7)}$$

The steepest descent contour for $\exp\left[\sigma t - a e^{i\pi/4}(\sigma-i)^{1/2}\right]$ is approximately the parabola $\left(\sigma_i - \left(1+\frac{a^2}{4t^2}\right)\right) = \frac{\sigma_r a^2}{2t}$, with a saddle point occuring at $\sigma = \sigma_{sp} = i\left(1+\frac{a^2}{4t^2}\right)$

See Fig. A-1

The value of $G_K(\sigma)\,\delta(t)$ is given by (II. 8), so it follows from steepest descent integration that

$$\alpha(\sigma)\; G_K(\sigma)\; \delta(t) = \frac{1+O(t^{-1/2})}{(\pi t)^{1/2}}\, \alpha(\sigma_{sp})\; e^{i(t - a^2/4t)} \qquad \text{(III. 8)}$$

Example C

$$\alpha(\sigma)\, G_K(\sigma) = \alpha(\sigma)\, e^{-a e^{i\pi/4}/\sigma^{1/2}} \Big/ \sigma^{1/2} \qquad \text{(III. 9)}$$

The saddle points, $\frac{d}{d\sigma}\left[\sigma t - a e^{-i\pi/4}/\sigma^{1/2}\right] = 0$ are located at

$$\sigma = \begin{cases} \sigma_+ = \left(\frac{a}{2t}\right)^{2/3} e^{i\pi/2} \\ \sigma_- = \left(\frac{a}{2t}\right)^{2/3} e^{-5i\pi/6} \end{cases}$$

We use

$$\frac{1}{2}\frac{d^2}{d\sigma^2}\left[\sigma t - a e^{-i\pi/4}/\sigma^{1/2}\right]\Bigg|_{\sigma_\pm} = \frac{3t}{4\sigma_\pm}$$

We integrate $\alpha(\sigma)\, G_K(\sigma)\, \frac{1}{2\pi i}\, e^{\sigma t}$ along the S. D. contour sketched in Fig. A-2 to obtain

$$\alpha(\sigma)\, G_K(\sigma)\, \delta(t) = \left(1 + O\left(\frac{\partial}{\partial t}\right)^{1/3}\right)\left(I_+(t) + I_-(t)\right) \tag{III. 10}$$

where

$$I_\pm(t) = \frac{\alpha(\sigma_\pm)}{(3\pi t)^{1/2}}\, e^{\left[\frac{3}{2}(2a^2t)^{1/3}\left(e^{-\pi i/6 \pm \frac{2\pi i}{3}}\right)\right]} \tag{III. 11}$$

For $-7\pi/4 < \arg a < \pi/4$, $I_-/I_+ = O\, t^{-N}$

for N arbitrarily large. Hence for real a , $I_- \sim 0$.

The exact power series evaluation of $G_K(\sigma)\,\delta(t)$, is given by (II. 7)

<u>Example D</u>

$$\alpha(\sigma)\, G_K(\sigma) = \alpha(\sigma)\, e^{-a/\sigma} \tag{III. 12}$$

The steepest descent contour for $e^{\sigma t - a/\sigma}$ is sketched in Fig. A-3. Saddle points occur for

$$\sigma = \pm i\,(a/t)^{1/2}$$

It follows by comparison with (II. 4) that

$$\alpha(\sigma)\, G_K(\sigma)\, \tilde{J}(t) = (1+\epsilon)\ \mathrm{Re}\ \alpha\left(i\left(\frac{a}{t}\right)^{1/2}\right) H_0^{(1)}\left[2(at)^{1/2}\right] \qquad \text{(III. 13)}$$

where $H_0^{(1)}\left[2(at)^{1/2}\right]$ is a Hankel function

$$H_0^{(1)}\left[2(at)^{1/2}\right] = \frac{(1+\epsilon)}{2\pi^{1/2}(at)^{1/4}}\, e^{i\left[2(at)^{1/2} - \pi/4\right]}$$

and the asymptotic error ϵ is

$$\epsilon = O\,(at)^{-1/2}$$

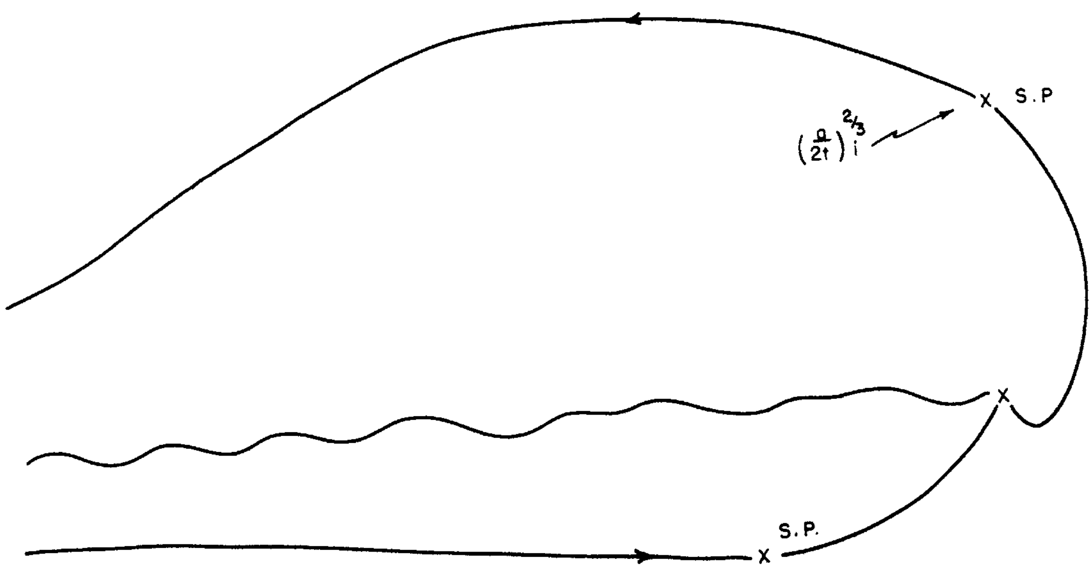


Figure A-2. Sketch of the steepest descent contour for III, C.

APPENDIX IV

On the Asymptotic Evaluation of $F(\sigma)\,\delta(t)$ for a Saddle Point of $F(\sigma)$ Near a Pole of $F(\sigma)$

This appendix describes the application of the "Pauli-Greenspan" modification of the steepest descents integration, which is applied when the steepest descents contour of (III. 3) "coalesces" with one of the poles, $\sigma = \sigma_j$. Assume for simplicity that

$$F(\sigma)\,\delta(t) = \frac{G_K(\sigma)}{\sigma - i\omega_0}\,\delta(t) = \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\sigma\, e^{\sigma t}\frac{G_K(\sigma)}{(\sigma - i\omega_0)} \qquad \text{(IV. 1)}$$

where $G_K(\sigma)\,\delta(t)$ is one of the elementary propagators considered in Examples B, C, or D or III.

Three possibilities are then distinguished.

a) The pole at $\sigma = i\omega_0$ lies within the steepest descents contour such that $|\sigma_{sp} - i\omega_0| \gtrsim t^{-1}$

b) The s. p. has coalesced with the pole. That is $|\sigma_{sp} - i\omega_0| \ll t^{-1}$

c) The s. p. contour has crossed the pole and so the pole is outside the s. p. contour, and furthermore $|\sigma_{sp} - i\omega_0| \gtrsim t^{-1}$

For a) the pole gives no residue contribution to the integral, which is hence given only by the s. p. integration. For c) the integral is evaluated as the residue $G_K(i\omega)\,e^{i\omega t}$ plus the s. p. integral. For b) the asymptotic error resulting from the usual method of s. p. integration is unacceptable, and so the steepest descents integration is carried out with $(\sigma - i\omega_0)^{-1}$ taken to vary along the **S. p.** path.

When $G_k(\sigma)e^{\sigma t}$ is expanded about the saddle point, one then obtains integrals of the form

$$I(b,\alpha) = \frac{1}{2\pi i}\int_{SD} \frac{2\, e^{-bi\lambda^2}}{\lambda + i\alpha}\, d\lambda \tag{IV. 2}$$

where λ measures the distance from the saddle point and

$$\alpha = \left(|\sigma_{sp}| - \omega_0 \right) \tag{IV. 3}$$

is the distance of the saddle point from the pole.

Setting $\lambda = \sqrt{p}\, e^{i\pi/4}$, the integral reduces to

$$I(b,\alpha) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{dp}{\sqrt{p}}\, \frac{e^{bp}}{\sqrt{p} + \alpha e^{i\pi/4}} = \sigma^{-1/2}\left(\sqrt{\sigma} + \alpha e^{i\pi/4}\right)^{-1} \delta(b) \tag{IV. 4}$$

which may be evaluated by (II. 9).

The term b will be $b = \frac{1}{2}\frac{d^2}{d\sigma^2}\Phi(\sigma_{sp})$ where $\Phi(\sigma) = \sigma t + \ln G_k(\sigma$

The lettering of the examples given below corresponds to that used for Appendix III.

Example B

$$F(\sigma) = \frac{\sigma^{-1/2}}{\sigma - i\omega_0} \, e^{-a e^{i\pi/4}(\sigma - i)^{1/2}} \tag{IV. 6}$$

Using $\quad \sigma_{sp} = i\,(1 + a^2/4t^2)\,, \quad b = \left|\tfrac{1}{2}\,\phi''(\sigma_{sp})\right| = t^3/a^2$

Hence for $\left[(1 + a^2/4t^2) - \omega_0\right] \ll t^{-1}$, one finds

$$F(\sigma)\,\delta(t) \simeq \frac{e^{i[\omega_0 t - a(\omega_0 - 1)^{1/2} - \pi/4]}}{\omega_0^{1/2}} \, \tfrac{1}{2}\,\mathrm{erfc}\left[e^{i\pi/4}\frac{t^{3/2}}{a}\left(1 + \frac{a^2}{4t^2} - \omega_0\right)\right] \tag{IV. 7}$$

Example C

$$F(\sigma) = \frac{\sigma^{-1/2}\, e^{-a e^{-i\pi/4}/\sigma^{1/2}}}{(\sigma - i\omega_0)} \tag{IV. 8}$$

Using $\quad \sigma_{sp} = \sigma_+ = i\left(\frac{a}{2t}\right)^{2/3}, \quad b = \frac{3t}{4\sigma_+} = \left(\frac{2t}{a}\right)^{2/3}\frac{3t}{4}$

Hence for $\left(\left(\frac{a}{2t}\right)^{2/3} - \omega_0\right) \ll t^{-1}$, we have

$$F(\sigma)\,\delta(t) \simeq \frac{e^{i\left(\omega_0 t + \frac{a}{\omega_0^{1/2}} - \pi/4\right)}}{\omega_0^{1/2}} \, \tfrac{1}{2}\,\mathrm{erfc}\left[e^{i\pi/4}\left(\frac{2t}{a}\right)^{1/3}\left(\frac{3t}{4}\right)^{1/2}\left(\left(\frac{a}{2t}\right)^{2/3} - \omega_0\right)\right] \tag{IV. 9}$$

Example D.

$$F(\sigma) = \frac{1}{\sigma - i\omega_0}\,\frac{e^{-a/\sigma}}{\sigma} \tag{IV. 10}$$

we use $\quad \sigma_{sp} = \pm i\left(\frac{a}{t}\right)^{1/2}, \quad b = \frac{t}{\sigma_{sp}} = t^{3/2}/a^{1/2}$

Hence for $\left(\left(\frac{a}{t}\right)^{1/2} - \omega_0\right) \ll 1$, we have

$$F(\sigma)\,\delta(t) \simeq \frac{i\, e^{i(\omega_0 t + a/\omega_0)}}{\omega_0} \, \tfrac{1}{2}\,\mathrm{erfc}\left[e^{i\pi/4}\left(\frac{t^3}{a}\right)^{1/4}\left(\left(\frac{a}{t}\right)^{1/2} - \omega_0\right)\right] \tag{IV. 11}$$

To interpret the above evaluations, one may use

$$\operatorname{erfc}(x e^{i\pi/4}) = \frac{2}{\pi^{1/2}} \int_{x e^{i\pi/4}}^{\infty} e^{-t^2} dt \sim \begin{cases} \dfrac{1}{\pi^{1/2}|x|} e^{-ix^2 - i\pi/4}, & x \to \infty \\ 1, & x \to 0 \\ 2 + \dfrac{e^{-ix^2 - i\pi/4}}{\pi^{1/2}|x|}, & x \to -\infty \end{cases} \qquad \text{(IV. 12)}$$

Applying the $x \to \pm\infty$ limit of $\operatorname{erfc}(x e^{i\pi/4})$ to the approximate solutions (IV. 7), (IV. 9) and (IV. 11), and using (IV. 5), it may be shown that above expressions provide a smooth transition from the result obtained by s. d. integration with the pole inside the contour, to the s. d. plus residue evaluation which is obtained when the pole lies outside the s. d. contour.

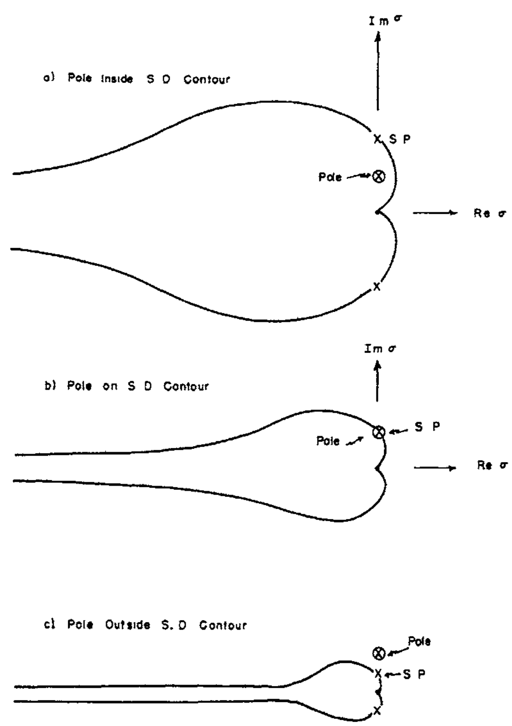


Figure A-3. Sketches of steepest descent contours for III and IV, D. ("forced Rossby waves"), illustrating the shrinkage of the S. D. countour with inceasing time.

APPENDIX V

On the Evaluation of Fourier Integrals Occuring in Forced Atmospheric Wave Propagation Problems.

It is sometimes convenient to employ Fourier integrals to solve atmospheric wave propagation problems. Restricting ourselves to 1-D problems, we consider integrals of the form

$$I(x,t) = \int_{-\infty}^{\infty} f(k)\, e^{ikx + i\omega(k)t} \qquad \text{(V. 1)}$$

Such integrals may be decomposed into elementary integrals in a fashion similar to that described in Appendix III. Since the procedure to be followed in the evaluation of all such elementary integrals is similar, we may for definiteness restrict our discussion to the evaluation of the forced Rossby wave problem

$$v(x,t) = \int_{-\infty}^{\infty} \frac{f(k)\, e^{ikx}}{Uk^2-\beta}\left[e^{ikUt} - e^{i\beta t/k}\right] dk \qquad \text{(V. 2)}$$

(For another example related to water waves, see Greenspan (1956)).

The integral is evaluated by deforming the path of integration into a steepest descent path of the term $e^{i(kx + \beta t/k)}$. This path has saddle points for $x > 0$

$$k = \pm\left(\frac{\beta t}{x}\right)^{1/2} \qquad \text{(V. 3)}$$

while for $x < 0$, at

$$\kappa = \pm i\left(\frac{\beta t}{x}\right)^{1/2} \tag{V. 4}$$

Thus for $x>0$, we deform the contour into a double lobed steepest descents contour from $I_m\,\kappa = \infty$ through the $\kappa = -\left(\frac{\beta t}{x}\right)^{1/2}$ s.p. into the origin and then both around through the $\kappa = \left(\frac{\beta t}{x}\right)^{1/2}$ s.p. to $I_m\,\kappa = \infty$ This contour is then connected to the $Re\,\kappa$ axis by arcs at infinity in the $I_m\,\kappa > 0$ plane. For $x < 0$, we deform the contour into the $I_m\,\kappa < 0$ axis path of integration.

Having deformed the original contour into a steepest descents contour, the second term in (V. 2) may be integrated by a saddle point integration, and the first term evaluated by Cauchy's theorem.

First assume $x>0$; then for $(x+Ut) > 0$, poles lying inside the steepest descents countour give residue contributions, and when $(x+Ut) < 0$ poles lying outside the contour give residue contributions. Likewise when $x < 0$, poles off the $I_m\,\kappa$ axis give residue contributions for $(x+Ut) > 0$, and poles on the $I_m < 0$ axis, when $(x+Ut) < 0$.

The following table summarizes the information needed on the residues of the integrand.

	Pole	Domain	Location rel. to contour	Residue
	$\kappa = i\left(\frac{\beta}{U}\right)^{1/2}$	$(x+Ut) > 0,\ x > 0$	inside	yes
		$(x+Ut) < 0,\ x > 0$	inside	no
		$(x+Ut) < 0,\ x < 0$	outside	no
$\underline{U < 0}$	$\kappa = -i\left(\frac{\beta}{U}\right)^{1/2}$	$(x+Ut) > 0,\ x > 0$	outside	no
		$(x+Ut) < 0,\ x > 0$	outside	yes
		$x < 0$	inside	no

For $U > 0$ the poles are located at $\kappa = \pm(\beta/U)^{1/2}$

Domain	Location	Residue
$x > Ut$	outside	no
$0 < x < Ut$	inside	yes
$-Ut < x < 0$	outside	yes
$x < -Ut$	outside	no

Carrying out the computation described above we find:

for $U < 0$

$$v = \frac{H(x)\, e^{-\left(\frac{\beta}{U}\right)^{1/2}|x+Ut|}}{2(\beta U)^{1/2}} - \frac{1}{2\pi}\int_{SD} \frac{d\kappa\, e^{i\kappa x + i\beta t/\kappa}}{U\kappa^2 - \beta} \tag{V. 6}$$

for $U > 0$

$$v = \frac{1}{(\beta U)^{1/2}} H(x+Ut)\, H\left(t - \frac{x}{U}\right) \sin\left(\frac{\beta}{U}\right)^{1/2}(x+Ut) - \frac{1}{2\pi}\int_{S.D.} \frac{d\kappa\, e^{i\kappa x + i\beta t/\kappa}}{U\kappa^2 - \beta} \tag{V. 7}$$

The evaluation of the steepest descents integral gives

$$\frac{1}{2\pi}\int_{SD} \frac{d\kappa\, e^{i\kappa x + i\beta t/\kappa}}{U\kappa^2 - \beta} \simeq \frac{1}{\pi^{1/2}}(1+6)\left(\frac{\beta x}{t^3}\right)^{1/4} \frac{\cos\left[2(\beta x t)^{1/2} - \frac{3\pi}{4}\right]}{\beta\left(\frac{x}{t} - U\right)} \tag{V. 8}$$

for $x > 0$

$$\simeq 0 \qquad , \quad x < 0$$

The error term ϵ is

$$\epsilon = O\,(\beta x t)^{-1/2}\left(\frac{x}{Ut} - 1\right)^{-1}$$

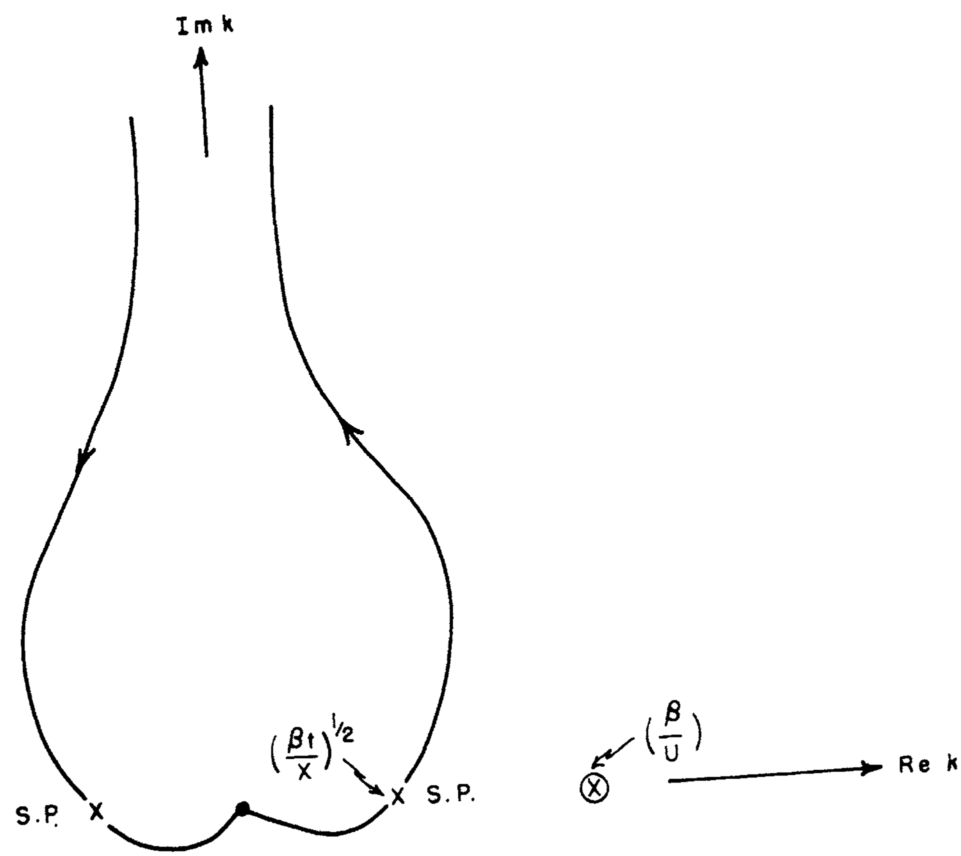


Figure A-4. Sketch of steepest descent contour for Rossby waves forced by a traveling switch-on disturbance.

APPENDIX VI

On Multiple Saddle Point Integration

This appendix discusses the application of steepest descents integration to integrals, when the integrand has multiple saddle points which may coalesce for some values of x and t. The example of propagation of 1-D modified Rossby waves illustrates the procedure. (See for instance the section of Jefferies and Jefferies on dispersive wave motions for further details).

In the theory of the propagation of 1-D modified Rossby waves, there occur integrals of the form

$$I(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{i\left[kx + \frac{\beta k t}{k^2+a^2}\right]} \tag{VI. 1}$$

Let $\phi(k) = kx + \beta k t/(k^2+a^2)$. Saddle points of $e^{i\phi(k)}$ are located where $\frac{d\phi}{dk} = 0$. That is

$$\frac{x}{\beta t} = \frac{k^2-a^2}{(k^2+a^2)^2} \tag{VI. 2}$$

First assume $(\beta t/x) \gg a^2$. Then (VI. 2) has the roots

$$k = k_{sp} = \left\{\begin{array}{l} k_1 = \pm\left(\frac{\beta t}{x}\right)^{1/2}\left(1 - \frac{3}{2}\frac{a^2 x}{\beta t} + \cdots\right) \\ k_{2>} = \pm\, a\left(1 + \frac{4a^2 x}{\beta t} + \cdots\right)\end{array}\right\} \quad x>0$$

$$k = k_{sp} = k_{2<} = \pm\, a\left(1 + \frac{4a^2 x}{\beta t} + \cdots\right), \quad x<0$$

It follows that for $(\beta t/x) >> a^2$, $t >> 1$, the integral

$I(x,t)$ may be asymptotically evaluated as

$$I(x,t) = H(x)\, I_1(x,t) + I_2(x,t) \tag{VI. 3}$$

where

$$I_1(x,t) = \frac{1}{2\pi}\int_{k_1\ s.p.} dk\, e^{ikx - \frac{\beta k t}{k^2+a^2}} \tag{VI. 4}$$

$$I_2(x,t) = \frac{1}{2\pi}\int_{k_2\ s.p.} dk\, e^{ikx - \frac{\beta k t}{k^2+a^2}} \tag{VI. 5}$$

We use

$$\frac{1}{2}\left(\frac{d^2\phi}{dk^2}\right)_{k=k_1} = \frac{x^{3/2}}{(\beta t)^{1/2}}\left(1 - \frac{3}{2}\frac{a^2 x}{\beta t} + \cdots\right)$$

$$\frac{1}{2}\left(\frac{d^2\phi}{dk^2}\right)_{k=k_2} = \frac{-t}{8a^3}\,\beta\left(1 - 13\, a^2\, x/\beta t + \cdots\right)$$

Asymptotic evaluation of I_1 and I_2 then gives

$$I_1(x,t) = \frac{(\beta t)^{1/4}}{\pi^{1/2}\, x^{3/4}}\left(1 + \frac{3a^2 x}{4\beta t} + \cdots\right)\cos\left[2(\beta x t)^{1/2} + \pi/4\right]$$

$$I_2(x,t) = 4a\left(\frac{a}{2\pi\beta t}\right)^{1/2}\left(1 + \frac{6a^2 x}{\beta t} + \cdots\right)\cos\left[\left(1 + \frac{4a^2 x}{\beta t} - \pi/4\right]\right. \tag{VI. 6}$$

When $\beta t/x << a^2$, the integrand (VI. 1) has no s. p. on the real axis so that for large t, the solution is in this case asymptotically zero. When $\beta t/_x \sim a^2$ the two pairs of s. p. coalesce and the decomposition (VI. 3) is not possible. The procedure then, is to expand the exponent of the integrand about the point on the κ- plane where the two pairs of saddle points coincide. That is, at the "turning point" κ_{TP}, where

$$\frac{d^2\phi}{d\kappa^2} = \frac{2\beta\kappa t\,(\kappa^2 - 3a^2)}{(\kappa^2+a^2)^3} = 0 \qquad \text{(VI. 7)}$$

$$\Rightarrow \quad \kappa = \kappa_{TP} = \sqrt{3}\,a$$

The third derivative of the phase at this point is

$$\left(\frac{d^3\phi}{d\kappa^3}\right)_{\kappa = \kappa_{T.P.}} = \frac{3}{16}\,\frac{\beta t}{a^4}$$

so that

$$I(x,t) \simeq \frac{1}{2\pi}\,\mathrm{Re}\, e^{i\sqrt{3}a\left(x+\frac{\beta t}{4a^2}\right)} \int_{-\infty}^{\infty} e^{i\lambda\left(x-\frac{\beta t}{8a^2}\right) + i\beta t\lambda^3/32a^4}\, d\lambda \qquad \text{(VI. 8)}$$

Recalling the definition of the Airy function, $A_i(x)$, is

$$A_i(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\lambda x + \lambda^3/3)}\, d\lambda$$

we find

$$I(x,t) \simeq \left(\frac{32a^4}{3\beta t}\right)^{1/3} A_i\left[\left(\frac{32a^4}{3\beta t}\right)^{1/3}\left(x-\frac{\beta t}{8a^2}\right)\right]\cos\left[\sqrt{3}\,a\left(x+\frac{\beta t}{4a^2}\right)\right] \qquad \text{(VI. 9)}$$

But

$$\left.\begin{aligned} Ai(\bar{x}) &= \frac{(1+\epsilon)}{2\pi^{1/2}\bar{x}^{1/4}}\, e^{-\frac{2}{3}\bar{x}^{3/2}}, \quad x>0 \\ Ai(\bar{x}) &= \frac{(1+\epsilon)}{\pi^{1/2}\bar{x}^{1/4}} \cos\left(\tfrac{2}{3}|\bar{x}|^{3/2} - \pi/4\right), \quad x<0 \end{aligned}\right\} \tag{VI. 10}$$

with $\quad \epsilon = O(\bar{x}^{-3/2})$

Hence $I(x,t)$ decays as $e^{-\frac{2}{3}\left(\frac{32a^4}{3\beta t}\right)^{1/2}\left(x - \frac{\beta t}{8a^2}\right)^{3/2}}$ for $x > \beta t/8a^2$,

while for $\beta t/8a^2 > x$, $I(x,t)$ is proportional to the product of two wave trains

$$I(x,t) \sim \left(\frac{32a^4}{3\beta t}\right)^{1/4}\left(x - \frac{\beta t}{8a^2}\right)^{-1/4}\frac{1}{\pi^{1/2}}\cos\left[(\beta a)^{1/2}\left(x + \frac{\beta t}{4a^2}\right)\right]\cos\left[\frac{2}{3}\left(\frac{32a^4}{3\beta t}\right)^{1/2}\left(x - \frac{\beta t}{8a^2}\right)^{3/2} - \frac{\pi}{4}\right]$$

The single wave (VI. 9) gradually merges into the I_1, I_2 waves, (VI. 6), as $\beta t/a^2 x \rightarrow \infty$

The transition point

$$x/t = \beta/8a^2 \tag{VI. 11}$$

between exponentially decaying motions, and oscillating motions, is called the "Airy front".

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