PULSE BROADENING DUE TO

MULTIPATH SCATTERING OF RADIO WAVES BY

THE INTERSTELLAR MEDIUM

BY

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PULSE BROADENING DUE TO MULTIPATH SCATTERING

OF RADIO WAVES BY THE INTERSTELLAR MEDIUM

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Lawrence Bell

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of Science.

ABSTRACT

The formal solution to the problem of finding the pulse shape in the time domain received when an impulse is transmitted by a pulsating radio source and deflected at an arbitrary number of screens between the source and the observer is found. A Gaussian distribution of scattering angles is assumed at each scattering screen. The general solution is found for N-1 screens in the form of an integral over 2N-3 variables. This expression is integrated analytically for N=2 (single screen case). A Monte-Carlo technique is used to find an approximate solution for the pulse shape in the case of N=2,3,4,5, 6, and 10.

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CHAPTER 1

INTRODUCTION

The random effects of the interstellar medium on the propagation of radio waves have been discussed at some length by Hollweg, Lovelace, and others. (Hollweg, 1967; Lovelace, 1970), who have developed models of scintillation phenomena and angular broadening. The discovery of pulsars in recent years has made it possible to investigate the properties of the interstellar medium in a new way, by studying its effect in the time domain on a brief pulse of radiation.

The best known effect is the delay in reception time of a radio pulse as a function of radio frequency due to dispersion. This delay is proportional to the inverse second power of the frequency and to the dispersion constant, which is related to the free electron concentration integrated along the path from the pulsar to the earth. The subject of this paper is another frequency dependent effect, the broadening of pulses at lower frequencies by an amount approximately proportional to the inverse fourth power of the frequency and believed to be the result of multipath scattering in the interstellar medium.

Because of scattering in the interstellar medium, radio waves from a pulsar, a star, or any radiating object can reach the earth by many different paths, and, in fact, the total radiation from such an object observed at the earth is the sum over all possible paths of that part of the total radiation associated with each path.

For the purpose of constructing a manageable model for this behavior, scattering is assumed to occur at distinct thin screens with rays travelling in straight line paths between them. If some random distribution of position, size, or other scattering properties of the scattering elements in the screen is assumed, then scattering at a screen is a random process and some probability can be associated with each value of the angle of deflection. It is assumed in the following treatment of the problem that the angle of deflection at a screen has a Gaussian probability distribution with a mean of zero and a standard deviation σ, which is a function of wavelength and properties of the scattering elements.

A useful method of reducing the complexity of the problem can be demonstrated in the simpler problem of finding the overall intensity at the earth of an isotropically radiating object after scattering by the interstellar medium. The radiation leaving the pulsar in a single direction $\overrightarrow{AB}$ (fig. 1a) produces on a plane through the earth C, perpendicular to the line of centers, some intensity distribution. So at C the intensity has some value represented by CD. The radiation leaving at another direction $\overrightarrow{A'B'}$ (fig. 1b) produces a similar intensity distribution with its center shifted. The intensity at C' has the value represented by C'D'. Likewise the radiation leaving in a third direction $\overrightarrow{A''B''}$ produces a pattern shifted further and so forth. The total intensity at the earth then is the sum of all the intensities produced by radiation leaving the isotropic source in all possible directions, which is the same as the intensity pattern of a single ray direction integrated over the

plane on which it is projected (fig. 1d). A similar method is used in the following treatment, but taking into consideration not only the intensity, but also the different times of travel along different paths.

CHAPTER 2

THE FORMAL SOLUTION

The problem is to determine the intensity as a function of time at point D' (fig. 2) due to an isotropic impulse source at A'. The radiation from the source is randomly scattered at N-1 screens (in fig. 2 N-1=2) spaced evenly between the source and the observer. The scattering angles (θ_2, θ_3, ... θ_n) are independent and each has a Gaussian probability distribution given by

$$\text{pdf}(\theta_i) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2\sigma_i^2}\theta_i^2} \tag{1}$$

So with each ray path is associated a joint probability density function in (θ_2, θ_3, ... θ_n) of the form

$$\text{pdf}(\theta_2,\theta_3,...\theta_n) = \frac{1}{(2\pi)^{\frac{n-1}{2}}\sigma_2\sigma_3...\sigma_n} e^{-\frac{1}{2}(\frac{\theta_2^2}{\sigma_2^2}+\frac{\theta_3^2}{\sigma_3^2}+...+\frac{\theta_n^2}{\sigma_n^2})} \tag{2}$$

and if we let $\sigma_2=\sigma_3=...=\sigma_n=\sigma$ then

$$\text{pdf}(\theta_2,\theta_3,...\theta_n) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{n-1} e^{-\frac{1}{2\sigma^2}(\theta_2^2+\theta_3^2+...+\theta_n^2)} \tag{3}$$

Associated with each ray is also a time of travel which for small angles is approximately equal to

$$t = (t_0 + \frac{t_0}{2}(-\theta_1)^2) + (t_0 + \frac{t_0}{2}(-\theta_1+\theta_2)^2) + (t_0 + \frac{t_0}{2}(-\theta_1+\theta_2+\theta_3)^2) +$$
$$... + (t_0 + \frac{t_0}{2}(-\theta_1+\theta_2+\theta_3+...+\theta_n)^2)$$

$$= t_0\left\{n + \frac{1}{2}[(-\theta_1)^2 + (-\theta_1+\theta_2)^2 + ... + (-\theta_1+\theta_2+...+\theta_n)^2]\right\} \tag{4}$$

in which t_0 is the shortest transit time between screens and with

the constraint $n\theta_1 = (n-1)\theta_2 + (n-2)\theta_3 + \ldots + \theta_n$.

Now expanding t

$$t = \frac{t_0}{2}\Big[2n + n\theta_1^2 - 2\theta_1\theta_2 - 2\theta_1(\theta_2 + \theta_3) - 2\theta_1(\theta_2 + \theta_3 + \theta_4) -$$
$$\ldots - 2\theta_1(\theta_2 + \theta_3 + \ldots + \theta_n)$$
$$+ (\theta_2)^2 + (\theta_2 + \theta_3)^2 + \ldots + (\theta_2 + \theta_3 + \ldots + \theta_n)^2 \Big]$$

$$= \frac{t_0}{2}\Big[2n + \theta_2^2 + (\theta_2 + \theta_3)^2 + \ldots + (\theta_2 + \theta_3 + \ldots + \theta_n)^2$$
$$+ n\theta_1^2 - 2\theta_1\Big(\theta_2 + (\theta_2 + \theta_3) + \ldots + (\theta_2 + \theta_3 + \ldots + \theta_n)\Big) \Big] \quad (5)$$

But $\Big(\theta_2 + (\theta_2 + \theta_3) + \ldots + (\theta_2 + \theta_3 + \ldots + \theta_n)\Big) = n\theta_1$

So

$$\frac{t}{t_0} = \Big\{ n + \frac{1}{2}\big[\theta_2^2 + (\theta_2 + \theta_3)^2 + \ldots + (\theta_2 + \theta_3 + \ldots \theta_n)^2 \big] - \frac{n}{2}[\theta_1]^2 \Big\} \quad (6)$$

Compare this result with that of the geometry in Figure 3. The scattering angles in Figure 3 are the same as those in Figure 2. For small angles, except for a rotation about A or A' by an angle θ_1, the only difference between the two geometries is that the straight line distance A'D' is equal to $AD\cos\theta_1$ or approximately $AD(1-\frac{1}{2}\theta_1^2)$. The time of travel for the geometry in Figure 2 is given by

$$\frac{t}{t_0} = \Big\{ n + \frac{1}{2}\big[\theta_2^2 + (\theta_2 + \theta_3)^2 + \ldots + (\theta_2 + \theta_3 + \ldots + \theta_n)^2 \big] \Big\} \quad (7)$$

where t_0 is again the shortest transit time between screens. To the second power in θ's this is too big by just the factor $(1-\frac{1}{2}\theta_1^2)^{-1}$, i.e., the geometry in Figure 3 gives the same answer as that in Figure 2 if the time for Figure 2 is multiplied by $(1-\frac{1}{2}\theta_1^2)$.

The effect of this is to scale the straight line path $\overline{AD}$ down to the length of $\overline{AE}$. So the original problem with the constraint that rays leave A'and arrive at D' is equivalent to the problem of integrating over the plane ED the distribution of a single initial ray. So in terms of the scattering angles

$$\frac{t}{t_0} = n + \frac{1}{2}\left[\theta_2^2 + (\theta_2+\theta_3)^2 + \ldots + (\theta_2+\theta_3+\ldots+\theta_n)^2\right]$$
$$-\frac{n}{2}\left[\frac{\theta_2+(\theta_2+\theta_3)+\ldots+(\theta_2+\theta_3+\ldots+\theta_n)}{n}\right]^2 \qquad (8)$$

The cross product terms in the expansion of this expression result in complications in taking the gradient in the following calculations, so a change in variables was performed. Instead of using the deflection angle, the angle with respect to the local normal was used (see fig. 4). These angles are related to the scattering angles by the following formulae.

$$\alpha_2 = \theta_2 \qquad\qquad \theta_2 = \alpha_2$$
$$\alpha_3 = \frac{\theta_2}{2} + \theta_3 \qquad\qquad \theta_3 = \alpha_3 - \frac{1}{2}\alpha_2$$
$$\alpha_4 = \frac{\theta_2}{3} + \frac{2\theta_3}{3} + \theta_4 \qquad\qquad \theta_4 = \alpha_4 - \frac{2}{3}\alpha_3 \qquad (9)$$
$$\alpha_5 = \frac{\theta_2}{4} + \frac{\theta_3}{2} + \frac{3\theta_4}{4} + \theta_5 \qquad\qquad \theta_5 = \alpha_5 - \frac{3}{4}\alpha_4$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$\alpha_n = \frac{\theta_2}{n-1} + \frac{2\theta_3}{n-1} + \frac{3\theta_4}{n-1} + \ldots + \theta_n \qquad \theta_n = \alpha_n - \frac{n-2}{n-1}\alpha_{n-1}$$

So, in terms of these angles

$$\theta_2 = \alpha_2$$
$$\theta_2 + \theta_3 = \frac{\alpha_2}{2} + \alpha_3 \qquad\qquad (10)$$
$$\theta_2 + \theta_3 + \theta_4 = \frac{\alpha_2}{2} + \frac{\alpha_3}{3} + \alpha_4$$
$$\vdots$$
$$\theta_2 + \theta_3 + \ldots \theta_n = \frac{\alpha_2}{2} + \frac{\alpha_3}{3} + \ldots + \frac{\alpha_{n-1}}{n-1} + \alpha_n$$

and

$$\frac{t}{t_0} = n + \frac{1}{2}\left[\alpha_2^2 + \left(\frac{\alpha_2}{2}+\alpha_3\right)^2 + \left(\frac{\alpha_2}{2}+\frac{\alpha_3}{3}+\alpha_4\right)^2 \right.$$

$$\left. + \ldots + \left(\frac{\alpha_2}{2}+\frac{\alpha_3}{3}+\frac{\alpha_4}{4}+\ldots+\alpha_n\right)^2 \right] - \frac{n}{2}\left[\frac{\alpha_2}{2}+\frac{\alpha_3}{3}+\ldots\frac{\alpha_n}{n}\right]^2$$

$$\frac{t}{t_0} = n + \frac{1}{2}\left[\frac{\alpha_2^2}{2} + \frac{2\alpha_3^2}{3} + \frac{3\alpha_4^2}{4} + \ldots + \frac{(n-1)\alpha_n^2}{n} \right] \tag{11}$$

Now to find the probability density function in terms of these angles

such that

$$pdf(\alpha_2, \alpha_3, \ldots \alpha_n)\, d\alpha_2 d\alpha_3 \ldots d\alpha_n =$$

$$\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{n-1} e^{-\frac{1}{2\sigma^2}(\theta_2^2 + \theta_3^2 + \ldots \theta_n^2)}\, d\theta_2 d\theta_3 \ldots d\theta_n \tag{12}$$

with

$$\theta_2^2 + \theta_3^2 + \ldots + \theta_n^2 =$$

$$\alpha_2^2 + \left(\alpha_3 - \frac{1}{2}\alpha_2\right)^2 + \left(\alpha_4 - \frac{2}{3}\alpha_3\right)^2 + \ldots + \left(\alpha_n - \frac{n-2}{n-1}\alpha_{n-1}\right)^2$$

$$\tag{13}$$

where $d\alpha_2 d\alpha_3 d\alpha_4 \ldots d\alpha_n$ is a volume element in N-1 space

in terms of α's and $d\theta_1 d\theta_2 d\theta_3 \ldots d\theta_n$ is a volume element in

N-1 space in terms of θ's, and the relationship between them is

given by the Jacobian of the transformation, the following substitution

must be made:

$$d\alpha_2 d\alpha_3 \ldots d\alpha_n = \det(\underline{J})\, d\theta_2 d\theta_3 \ldots d\theta_n \tag{14}$$

where the element in the ith row and j^{th} column of the matrix $\underline{J}$ is
$\frac{\partial \alpha_i}{\partial \theta_j}$.

From the relationship between α's and θ's (eqn. 9) $\underline{J}$ is found to be

given by:

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \cdots & 0 \\ \vdots & & & & & \\ \frac{1}{n-1} & \frac{2}{n-1} & \frac{3}{n-1} & \frac{4}{n-1} & \cdots & 1 \end{bmatrix} \quad (15)$$

and $\det(J) = 1$

so

$$pdf(\alpha_2, \alpha_3 \ldots \alpha_n) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{n-1} e^{-\frac{1}{2\sigma^2}\left[\alpha_2^2 + (\alpha_3 - \frac{1}{2}\alpha_2)^2 + \ldots + (\alpha_n - \frac{n-2}{n-1}\alpha_{n-1})^2\right]}$$

$$(16)$$

So far this ray path has been restricted to lie in a plane.
If the third dimension is included, the probability density
function is given by

$$pdf(\alpha_2, \beta_2, \alpha_3, \beta_3, \ldots \alpha_n, \beta_n) =$$

$$\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{2(n-1)} \exp\left\{-\frac{1}{2\sigma^2}\left[\alpha_2^2 + \beta_2^2 + (\alpha_3 - \frac{1}{2}\alpha_2)^2 + (\beta_3 - \frac{1}{2}\beta_2)^2 + \right.\right.$$

$$\left.\left. \ldots + (\alpha_n - \frac{n-2}{n-1}\alpha_{n-1})^2 + (\beta_n - \frac{n-2}{n-1}\beta_{n-1})^2\right]\right\} \quad (17)$$

and

$$\frac{t}{t_\bullet} = n + \frac{1}{2}\left[\frac{\alpha_z^2}{2} + \frac{\beta_z^2}{2} + \frac{2\alpha_3^2}{3} + \frac{2\beta_3^2}{3} + \ldots + \frac{n-1}{n}\alpha_n^2 + \frac{n-1}{n}\beta_n^2 \right] \qquad (18)$$

Now the problem is to find the probability density function with

the time t/t_0 as the variable.

If we let t/t_0 be a parameter which is held constant, then

equation 18 describes a family of surfaces, for different values of

the parameter t/t_0, in 2(N-1) dimensions. The parameter t/t_0 has

the same value everywhere on the surface. If t/t_0 is given a

slight increment $d(t/t_0)$, the surface is expanded by an increment

everywhere perpendicular to the old surface, given by the inverse

of the magnitude of the gradient of the function $t/t_0(\alpha,\beta)$ as given

in eqn. 18. (This is true since the gradient is perpendicular to

the surface and equal to the increment in t/t_0 associated with a unit

increment in the direction perpendicular to the surface t/t_0=constant.)

So to get the probability associated with an increment $d(t/t_0)$,

we must integrate the probability density function pdf(α,β) times

the inverse of the gradient over the surface t/t_0=constant.

The integral to be performed is

$$\int_{\text{surface } S} \frac{\text{PDF}(\alpha_z,\beta_z,\ldots\beta_{n-1},\alpha_n,\frac{t}{t_\bullet})\ dS}{(\text{gradient})} \qquad (19)$$

where pdf$(\alpha_z,\beta_z,\ldots\beta_{n-1},\alpha_n,\frac{t}{t_\bullet})$ is given by eqns. 17 and 18 and

the magnitude of the gradient is

$$\left[\left(\frac{\partial \frac{t}{t_\bullet}}{\partial \alpha_z}\right)^2 + \left(\frac{\partial \frac{t}{t_\bullet}}{\partial \beta_z}\right)^2 + \left(\frac{\partial \frac{t}{t_\bullet}}{\partial \alpha_3}\right)^2 + \ldots + \left(\frac{\partial \frac{t}{t_\bullet}}{\partial \alpha_n}\right)^2 + \left(\frac{\partial \frac{t}{t_\bullet}}{\partial \beta_n}\right)^2 \right]^{1/2} \qquad (20)$$

and the surface element is given by

$$\left[\left(\frac{\partial \beta_n}{\partial \alpha_2}\right)^2 + \left(\frac{\partial \beta_n}{\partial \beta_2}\right)^2 + \left(\frac{\partial \beta_n}{\partial \alpha_3}\right)^2 + \ldots + \left(\frac{\partial \beta_n}{\partial \alpha_n}\right)^2 + 1\right]^{1/2} d\alpha_2 d\beta_2 \ldots d\alpha_n \quad (21)$$

So the integration must be performed over 2N-3 variables, $\alpha_2, \beta_2, \alpha_3, \beta_3, \ldots \beta_{n-1}, \alpha_n$. The dependent variable β_n must be taken from eqn. 18 with both the positive square root and the negative square root, and the limits of integration must be taken as follows:

α_2 from
$$-(2)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{\beta_2^2}{2} - \frac{2\alpha_3^2}{3} - \frac{2\beta_3^2}{3} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{1/2}$$

to
$$+(2)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{\beta_2^2}{2} - \frac{2\alpha_3^2}{3} - \frac{2\beta_3^2}{3} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{1/2}$$

β_2 from
$$-(2)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{2\alpha_3^2}{3} - \frac{2\beta_3^2}{3} - \frac{3\alpha_4^2}{4} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{1/2}$$

to
$$+(2)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{2\alpha_3^2}{3} - \frac{2\beta_3^2}{3} - \frac{3\alpha_4^2}{4} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{1/2}$$

(22)

α_3 from
$$-\left(\frac{3}{2}\right)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{2\beta_3^2}{3} - \frac{3\alpha_4^2}{4} - \frac{3\beta_4^2}{4} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{1/2}$$

to
$$+\left(\frac{3}{2}\right)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{2\beta_3^2}{3} - \frac{3\alpha_4^2}{4} - \frac{3\beta_4^2}{4} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{1/2}$$

β_3 from
$$-\left(\frac{3}{2}\right)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{3\alpha_4^2}{4} - \frac{3\beta_4^2}{4} - \frac{4\alpha_5^2}{5} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{1/2}$$

to
$$+\left(\frac{3}{2}\right)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{3\alpha_4^2}{4} - \frac{3\beta_4^2}{4} - \frac{4\alpha_5^2}{5} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{1/2}$$

$\cdot$
$\cdot$
$\cdot$

α_n from
$$-\left(\frac{n}{n-1}\right)^{1/2}\left[2\frac{t}{t_o} - 2n\right]^{1/2}$$

to
$$+\left(\frac{n}{n-1}\right)\left[2\frac{t}{t_o} - 2n\right]^{1/2}$$

Now from equation 18

$$\beta_n = \pm \left(\frac{n}{n-1}\right)^{1/2} \left[2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \frac{2}{3}\alpha_3^2 - \frac{2}{3}\beta_3^2 - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{1/2} \quad (23)$$

so

$$pdf(\alpha_2, \beta_2, \alpha_3, \beta_3, \ldots \alpha_n, t/t_o) = exp\left[-\frac{1}{2\sigma^2}\Delta\right] \quad (24)$$

where

$$\Delta = \left[\alpha_2^2 + \beta_2^2 + (\alpha_3 - \frac{1}{2}\alpha_2)^2 + (\beta_3 - \frac{1}{2}\beta_2)^2 + (\alpha_4 - \frac{2}{3}\alpha_3)^2 +\right.$$

$$\left. \ldots + (\beta_n - \frac{n-2}{n-1}\beta_{n-1})^2\right]$$

$$= \left\{\frac{5}{4}\alpha_2^2 + \frac{5}{4}\beta_2^2 + \frac{13}{9}\alpha_3^2 + \frac{13}{9}\beta_3^2 + \ldots + [1 + (\frac{n-2}{n-1})^2]\beta_{n-1}^2 + \alpha_n^2 \right.$$

$$- \alpha_2\alpha_3 - \beta_2\beta_3 - \frac{4}{3}\alpha_3\alpha_4 - \frac{4}{3}\beta_3\beta_4 - \frac{6}{4}\alpha_4\alpha_5 - \ldots - \frac{2(n-2)}{n-1}\alpha_{n-1}\alpha_n$$

$$+ \left[\left(\frac{n}{n-1}\right)\left(2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - 2\alpha_3^2 - 2\frac{\beta_3^2}{3} - \ldots - \frac{n-1}{n}\alpha_n^2\right)\right]$$

$$\left. \mp \frac{2(n-2)}{n-1}\beta_{n-1}\left(\frac{n}{n-1}\right)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{1/2}\right\} \quad (25)$$

THE GRADIENT

$$\frac{\partial}{\partial\alpha_2}(t/t_o) = \frac{\alpha_2}{2} \quad , \quad \frac{\partial}{\partial\beta_2}(t/t_o) = \frac{\beta_2}{2} \quad , \quad \frac{\partial}{\partial\alpha_3}(t/t_o) = \frac{2}{3}\alpha_3 \quad , \quad \frac{\partial}{\partial\beta_3}(t/t_o) = \frac{2}{3}\beta_3 \quad ,$$

$$\frac{\partial}{\partial\alpha_4}(t/t_o) = \frac{3}{4}\alpha_4 \quad , \ldots \quad , \quad \frac{\partial}{\partial\alpha_n}(t/t_o) = \frac{n-1}{n}\alpha_n \quad , \quad \frac{\partial}{\partial\beta_n}(t/t_o) = \frac{n-1}{n}\beta_n \quad (26)$$

So the magnitude of the gradient as given by eqn. 20 is

$$\left[\frac{\alpha_2^2}{4} + \frac{\beta_2^2}{4} + \frac{4\alpha_3^2}{9} + \frac{4\beta_3^2}{9} + \frac{9\alpha_4^2}{16} + \ldots + (\frac{n-1}{n})^2\alpha_n^2 + (\frac{n-1}{n})^2\beta_n^2\right]^{1/2} \quad (27)$$

Combining eqn. 23 and eqn. 27, the gradient is given by

$$\left\{\frac{\alpha_2^2}{4} + \frac{\beta_2^2}{4} + \frac{4\alpha_3^2}{9} + \ldots + (\frac{n-1}{n})^2\alpha_n + \frac{n-1}{n}\left[2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \ldots - \frac{n-1}{n}\alpha_n^2\right]\right\}^{1/2} \quad (28)$$

THE SURFACE ELEMENT

$$\frac{\partial \beta_n}{\partial \alpha_2} = -\left(\frac{n}{n-1}\right)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{-1/2}\frac{\alpha_2}{2}$$

$$\frac{\partial \beta_n}{\partial \beta_2} = -\left(\frac{n}{n-1}\right)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{-1/2}\frac{\beta_2}{2}$$

$$\frac{\partial \beta_n}{\partial \alpha_3} = -\left(\frac{n}{n-1}\right)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{-1/2}\frac{2}{3}\alpha_3 \qquad (29)$$

$$\vdots$$

$$\frac{\partial \beta_n}{\partial \alpha_n} = -\left(\frac{n}{n-1}\right)^{1/2}\left[2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \ldots - \frac{n-1}{n}\alpha_n^2\right]^{-1/2}\frac{n-1}{n}\alpha_n$$

So the surface element is

$$\left\{\frac{\left(\frac{n}{n-1}\right)\left[\frac{\alpha_2^2}{4} + \frac{\beta_2^2}{4} + \frac{4}{9}\alpha_3^2 + \ldots + \left(\frac{n-1}{n}\right)^2\alpha_n^2 + \frac{n-1}{n}\left[2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \ldots - \frac{n-1}{n}\alpha_n^2\right]\right]}{2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \ldots - \left(\frac{n-1}{n}\right)\alpha_n^2}\right\}^{1/2}$$

$$(30)$$

But the numerator in this expression is just n/(n-1) times the

magnitude of the gradient, so the integral becomes

$$\left(\frac{1}{2\pi\sigma^2}\right)^{n-1}\int_S \exp\left[-\frac{\Delta}{2\sigma^2}\right]\left[\frac{\frac{n}{n-1}}{2\frac{t}{t_o} - 2n - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \ldots - \frac{n-1}{n}\alpha_n^2}\right]^{1/2}d\alpha_2 d\beta_2 \ldots d\alpha_n$$

$$(31)$$

If we use the following notation for simplification

$$P = \alpha_2^2 + \beta_2^2 + \alpha_3^2 + \beta_3^2 + \alpha_4^2 + \beta_4^2 + \cdots + \alpha_n^2$$

$$Q = \frac{\alpha_2^2}{2} + \frac{\beta_2^2}{2} + \frac{\alpha_3^2}{3} + \frac{\beta_3^2}{3} + \frac{\alpha_4^2}{4} + \frac{\beta_4^2}{4} + \cdots + \frac{\alpha_n^2}{n}$$

$$R = \frac{\alpha_2^2}{4} + \frac{\beta_2^2}{4} + \frac{\alpha_3^2}{9} + \frac{\beta_3^2}{9} + \frac{\alpha_4^2}{16} + \frac{\beta_4^2}{16} + \cdots + \frac{\alpha_n^2}{n^2} \qquad (32)$$

$$S = \frac{1}{2}\alpha_2\alpha_3 + \frac{1}{2}\beta_2\beta_3 + \frac{2}{3}\alpha_3\alpha_4 + \frac{2}{3}\beta_3\beta_4 + \frac{3}{4}\alpha_4\alpha_5$$

$$+ \cdots + \frac{n-2}{n-1}\alpha_{n-1}\alpha_n$$

Then the integral becomes

$$\left(\frac{1}{2\pi\sigma^2}\right)^{n-1} \int_S \exp\left[-\frac{\Delta_\pm}{2\sigma^2}\right] \left[\frac{\frac{n}{n-1}}{2\frac{t}{t_o}-2n-(P-Q)}\right]^{1/2} d\alpha_2 d\beta_2 \cdots d\alpha_n$$

$$(33)$$

where

$$\Delta_\pm = 2P - 2Q + R - 2S - \left(\frac{n-1}{n}\right)^2 \alpha_n^2 + \frac{n}{n-1}\left[2\frac{t}{t_o}-2n-(P-Q)\right]$$

$$\mp 2\beta_{n-1}\frac{(n-2)n^{1/2}}{(n-1)^{3/2}}\left[2\frac{t}{t_o}-2n-(P-Q)\right]^{1/2}$$

$$(34)$$

The problem now is to integrate this function with the limits
of integration given by eqns. 22. Taking the negative sign before
the last term in eqn. 34 is the result of taking β_n as the pos-
itive square root in equation 23 and is equivalent to integrating
over only one half of the surface to be integrated over. The
integral must be repeated with the positive sign or simply let

$$\exp\left[-\frac{\Delta_\pm}{2\sigma^2}\right] = \exp\left[-\frac{\Delta_+}{2\sigma^2}\right] + \exp\left[-\frac{\Delta_-}{2\sigma^2}\right] \qquad (35)$$

The result would be the probability density function in t

integrated over the observation plane, which is what is desired.

The integral is in general too complicated for a simple analytical

solution. The special case of one screen (N=2) is easily calculated.

SINGLE SCREEN CASE

$$P = \alpha_2^2$$
$$Q = \frac{\alpha_2^2}{2}$$
$$R = \frac{\alpha_2^2}{4}$$
$$S = 0$$

$$\Delta = \alpha_2^2 + \frac{\alpha_2^2}{4} + 2\left(2\frac{t}{t_0} - 4 - \frac{\alpha_2^2}{2}\right) - \frac{1}{4}\alpha_2^2$$
$$= 4\left(\frac{t}{t_0} - 2\right)$$

$$pdf(t) = \frac{4}{2\pi\sigma^2} \exp\left[-\frac{2(\frac{t}{t_0}-2)}{\sigma^2}\right] \int_0^{2(\frac{t}{t_0}-2)^{1/2}} \frac{2\,d\alpha_2}{\left[4(\frac{t}{t_0}-2)-\alpha_2^2\right]^{1/2}}$$

$$\alpha_2 = 2\left(\frac{t}{t_0}-2\right)^{1/2}\sin x$$
$$d\alpha_2 = 2\left(\frac{t}{t_0}-2\right)^{1/2}\cos x$$

$$pdf(t) = \frac{4}{2\pi\sigma^2}\exp\left[-\frac{2(\frac{t}{t_0}-2)}{\sigma^2}\right]\int_0^{\pi/2} 2\,dx$$

$$= \frac{2}{\sigma^2}\exp\left[-\frac{2(\frac{t}{t_0}-2)}{\sigma^2}\right] \qquad t \geq 2t_0 \qquad (36)$$

Since t_0 is the shortest time between screens, $nt_0 = d/c$ where d is

the distance from the pulsar to the earth and c is the speed of light.

Equation 36 can therefore be re-written as

$$pdf(t) = \frac{2}{\sigma^2}\exp\left[-\frac{4}{\sigma^2}\left(\frac{t-\frac{d}{c}}{\frac{d}{c}}\right)\right] \qquad (37)$$

But for two screens (N=3)

$$P = \alpha_2^2 + \beta_2^2 + \alpha_3^2$$

$$Q = \frac{\alpha_2^2}{2} + \frac{\beta_2^2}{2} + \frac{\alpha_3^2}{3}$$

$$R = \frac{\alpha_2^2}{4} + \frac{\beta_2^2}{4} + \frac{\alpha_3^2}{9}$$

$$S = \frac{1}{2} \alpha_2 \alpha_3$$

$$\Delta_\pm = 3\frac{t}{t_0} - 9 - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} \mp \beta_2 \left(\frac{3}{2}\right)^{1/2} \left(2\frac{t}{t_0} - 6 - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - 2\frac{\alpha_3^2}{3}\right)^{1/2}$$

$$\left(\frac{1}{2\pi\sigma^2}\right)^2 \iiint \exp\left[-\frac{\Delta_\pm}{2\sigma^2}\right] \left[\frac{3/2}{2\frac{t}{t_0} - 6 - \frac{\alpha_2^2}{2} - \frac{\beta_2^2}{2} - \frac{2}{3}\alpha_3^2}\right]^{1/2} d\alpha_2 \, d\beta_2 \, d\alpha_3$$

which is already too complex for a simple analytical solution.

In the next section an approximate solution is found with the use of a Monte-Carlo technique.

CHAPTER 3

COMPUTER APPROXIMATION

An approximation to the solution of the problem can be found by using a computer and a source of random numbers to produce values for the scattering angles $\theta_2, \theta_3, \ldots \theta_n$, that have a Gaussian probability distribution, to calculate the time associated with a set of values and after repeating the process many times to sort the times calculated into several bins and thus produce a histogram, approximating the probability versus time curve.

The specific program and subroutines used are included in the appendix. Histograms were produced for 1,2,3,4,5, and 10 screens. The abscissa is the dimensionless quantity $(t-d/c)/(d/c)$ and the bin size is $.55 \times 10^{-4}$. The ordinate is the fraction of the total sample that falls into each bin. The total sample consists of one thousand different paths, with a standard deviation for each deflection angle of 0.02. The results of these calculations are shown in Figures 5-11.

The results for N=2 (one screen) fit a truncated exponential reasonably well and so agree with the analytically integrated result. The result for N=3 (two screens) appears to fit reasonably well with the curve $Y=Axe^{-Bx}$. The results for higher N are a little more difficult to fit. As the number of screens increases, the pulse broadens, the maximum comes later, and a larger tail develops.

CHAPTER 4

CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

The general solution for N-1 screens above may not be analytically

integrable for other than the single screen case, but numerical

integration or a Monte-Carlo technique such as that used in this

treatment of the problem can produce the desired result.

Several pulsars have been observed to exhibit exponential pulse

broadening (Lang,1971) consistent with the single screen effect.

The pulsar in the Crab Nebula, NP0532, has been observed to exhibit

broadening which is better described by an xe^{-x} broadening function

(Rankin et al.,1970;Counselman et al.,1970) consistent with the

two screen effect.

Further work is suggested in both the experimental and theoretical

aspects of this problem. A larger sample of pulsars and observations

may indicate multiple screen scattering effects and changes over

periods of time produced by the distribution and movement of clouds

in the interstellar medium. Detailed analysis of the results from

a Monte-Carlo technique may result in analytic expressions for the

broadening effect of many screens.

The treatment discussed here is for an isotropic pulse source.

If the pulsing phenomena are due to rotation, then the additional

term $\frac{2\pi X\theta_1}{P t_0}$ must be added to the equation for the time of travel,

where P is the period of the pulsar and X is a function of the

orientation of the axis of rotation of the pulsar. This calculation

and the others discussed above could provide useful results for

interpreting the behavior of pulsars and the effects of the inter-
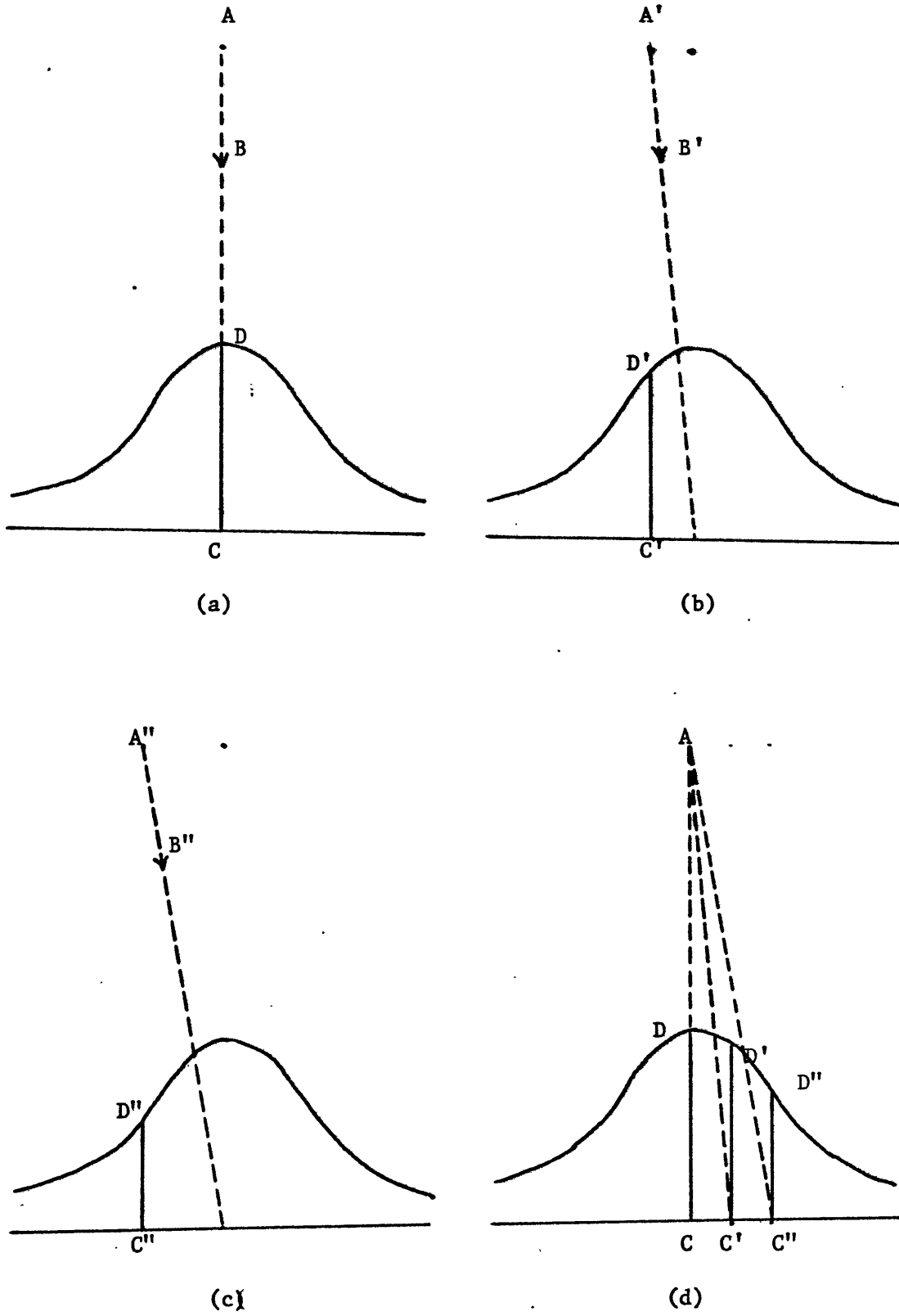
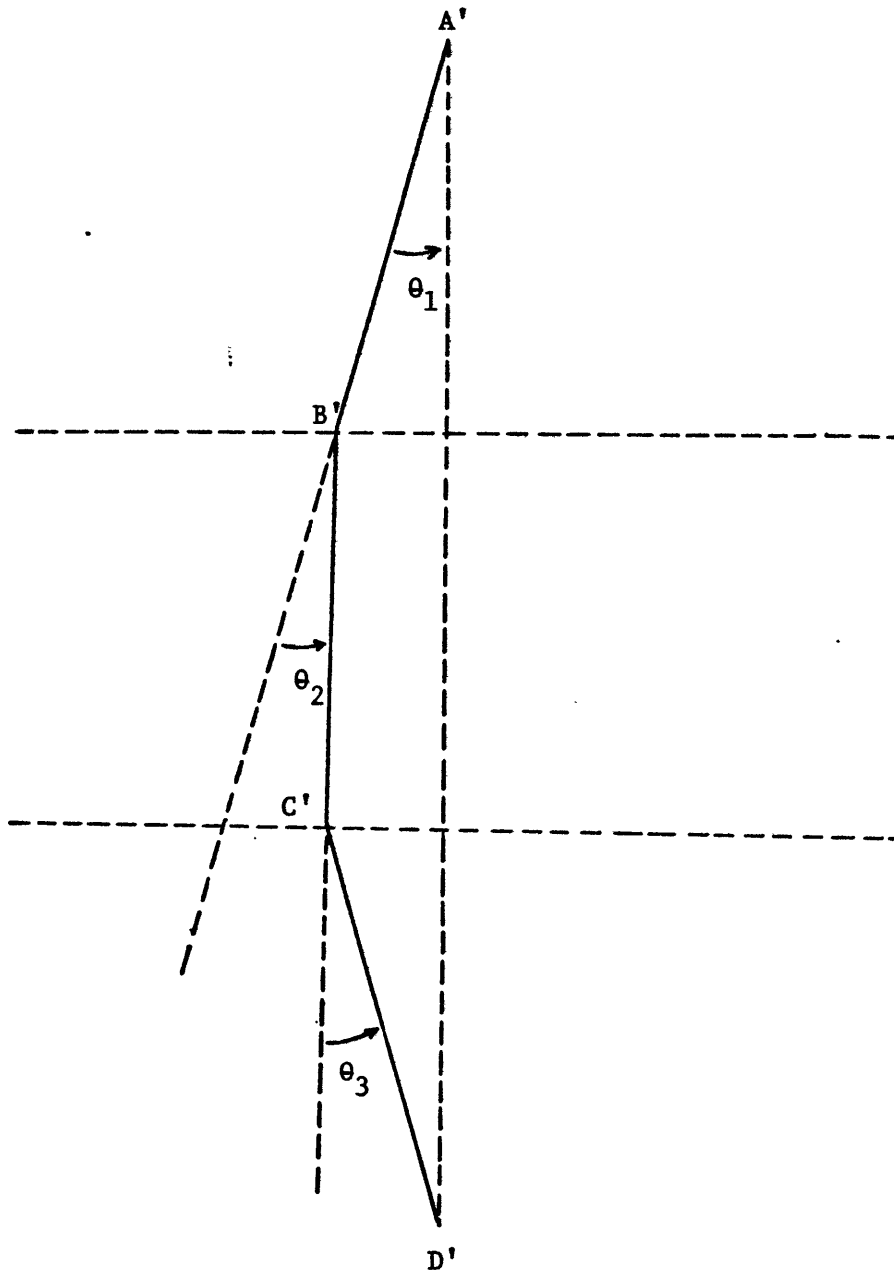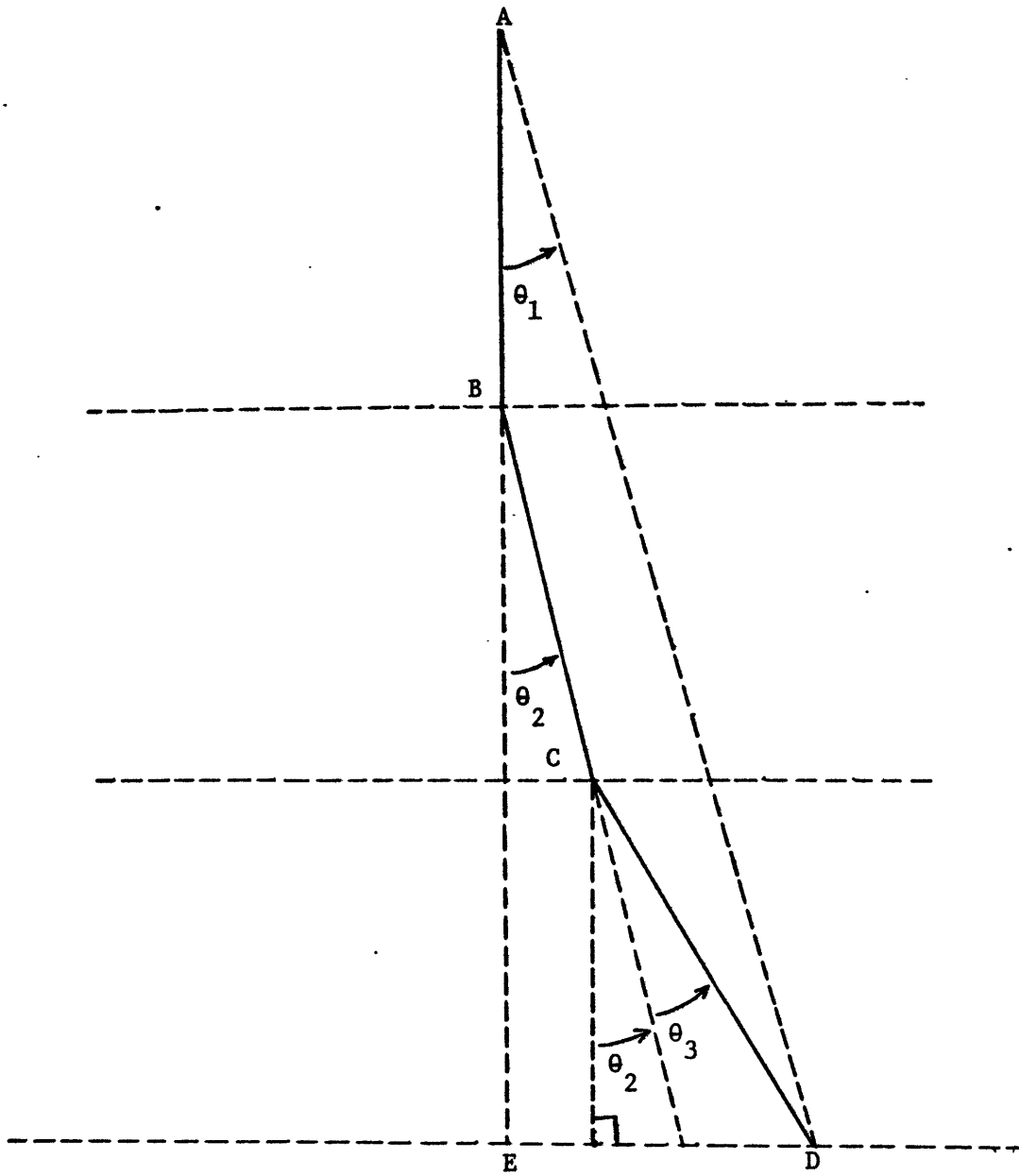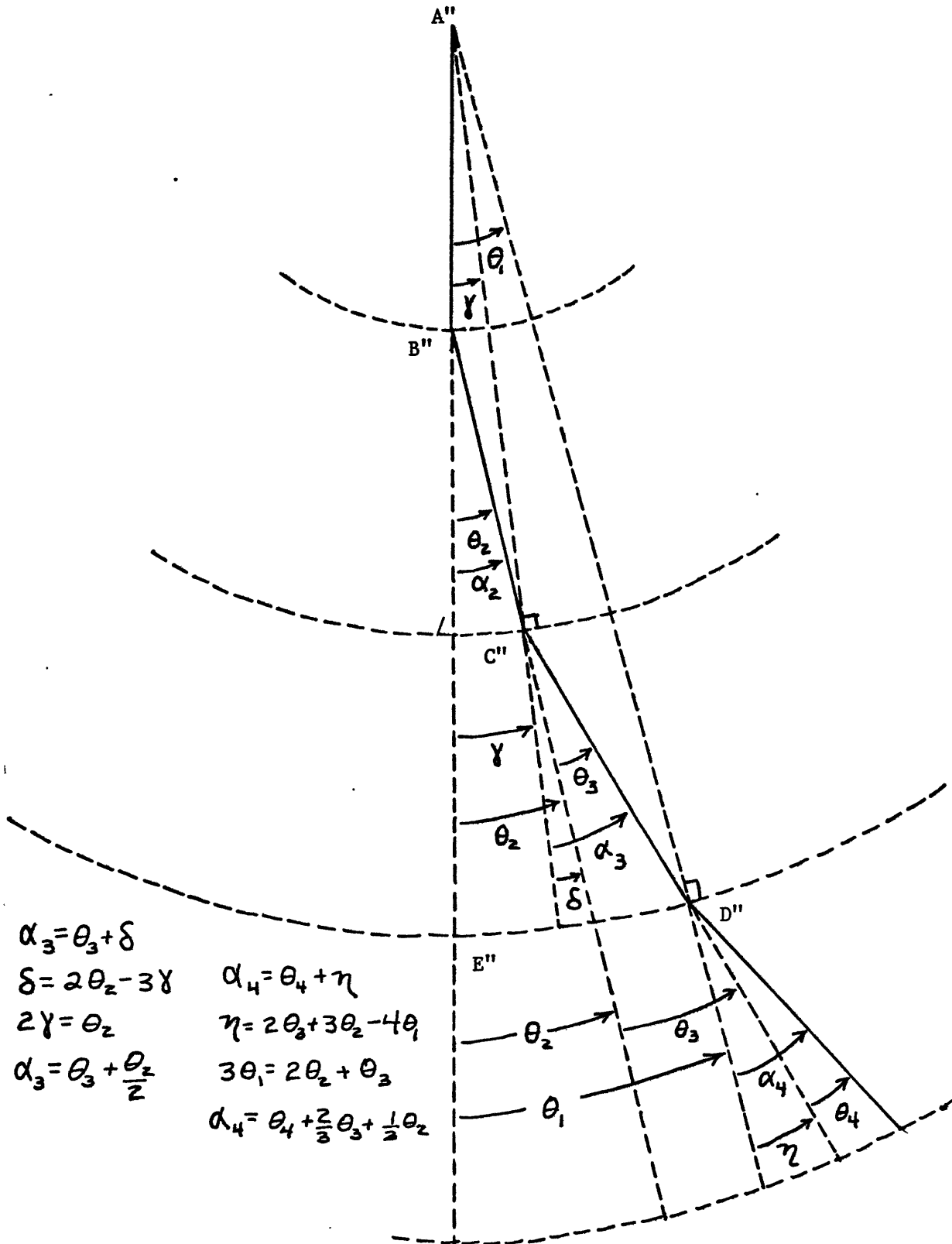stellar medium on radio waves.

FIGURE 1

FIGURE 2

FIGURE 3

$$\alpha_3 = \theta_3 + \delta$$
$$\delta = 2\theta_2 - 3\gamma \qquad \alpha_4 = \theta_4 + \eta$$
$$2\gamma = \theta_2 \qquad \eta = 2\theta_3 + 3\theta_2 - 4\theta_1$$
$$\alpha_3 = \theta_3 + \frac{\theta_2}{2} \qquad 3\theta_1 = 2\theta_2 + \theta_3$$
$$\alpha_4 = \theta_4 + \frac{2}{3}\theta_3 + \frac{1}{2}\theta_2$$

FIGURE 4

PULSE SHAPE HISTOGRAM

ONE SCREEN

solid curve is $Y=.54e^{-1.05x}$

$x\ 10^{-4}\ =\ (t-d/c)/(d/c)$

FIGURE 5

PULSE SHAPE HISTOGRAM

TWO SCREENS

solid curve is $Y = .75xe^{-1.23x}$

$x \ 10^{-4} = (t-d/c)/(d/c)$

FIGURE 6

PULSE SHAPE HISTOGRAM

THREE SCREENS

.500

.400

.300

.200

.100

.000

0   .55   1.1   1.65   2.2   2.75   3.3   3.85   4.4   4.95   5.5   6.05   6.6   7.15   7.7

$x \ 10^{-4} = (t-d/c)/(d/c)$

FIGURE 7

PULSE SHAPE HISTOGRAM

FOUR SCREENS

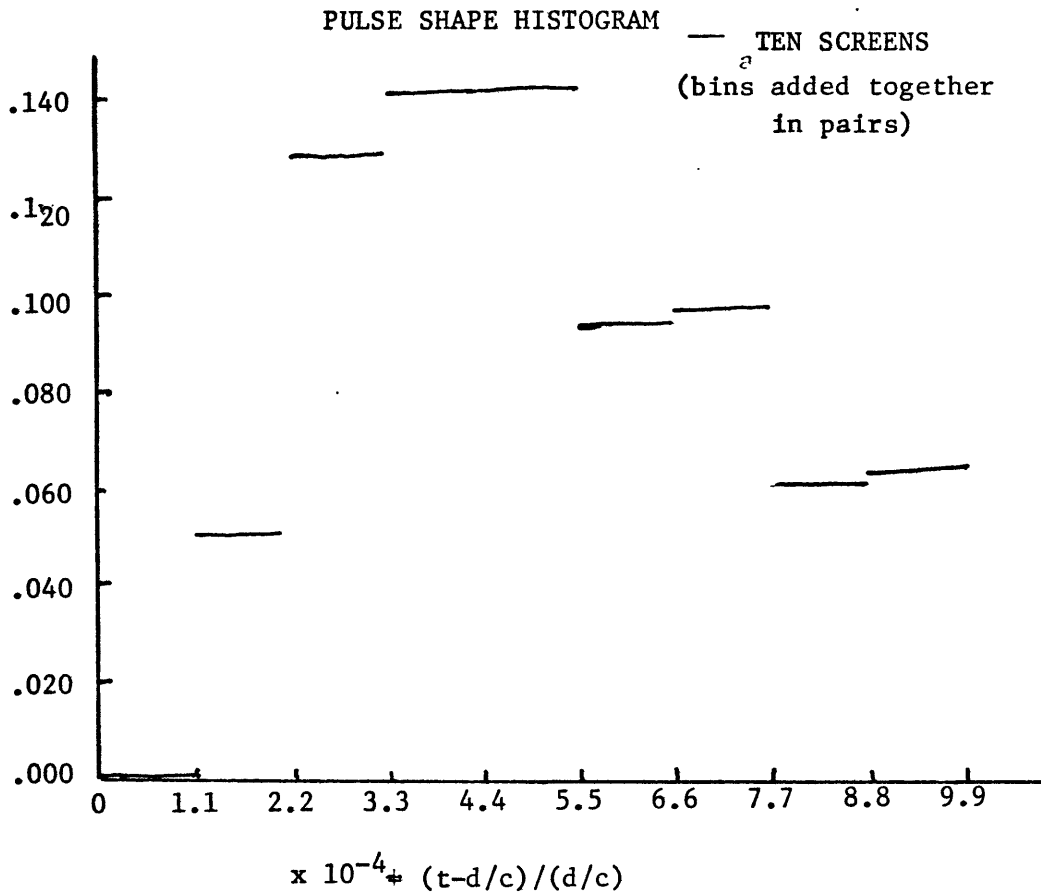$x\ 10^{-4}\ =\ (t-d/c)/(d/c)$
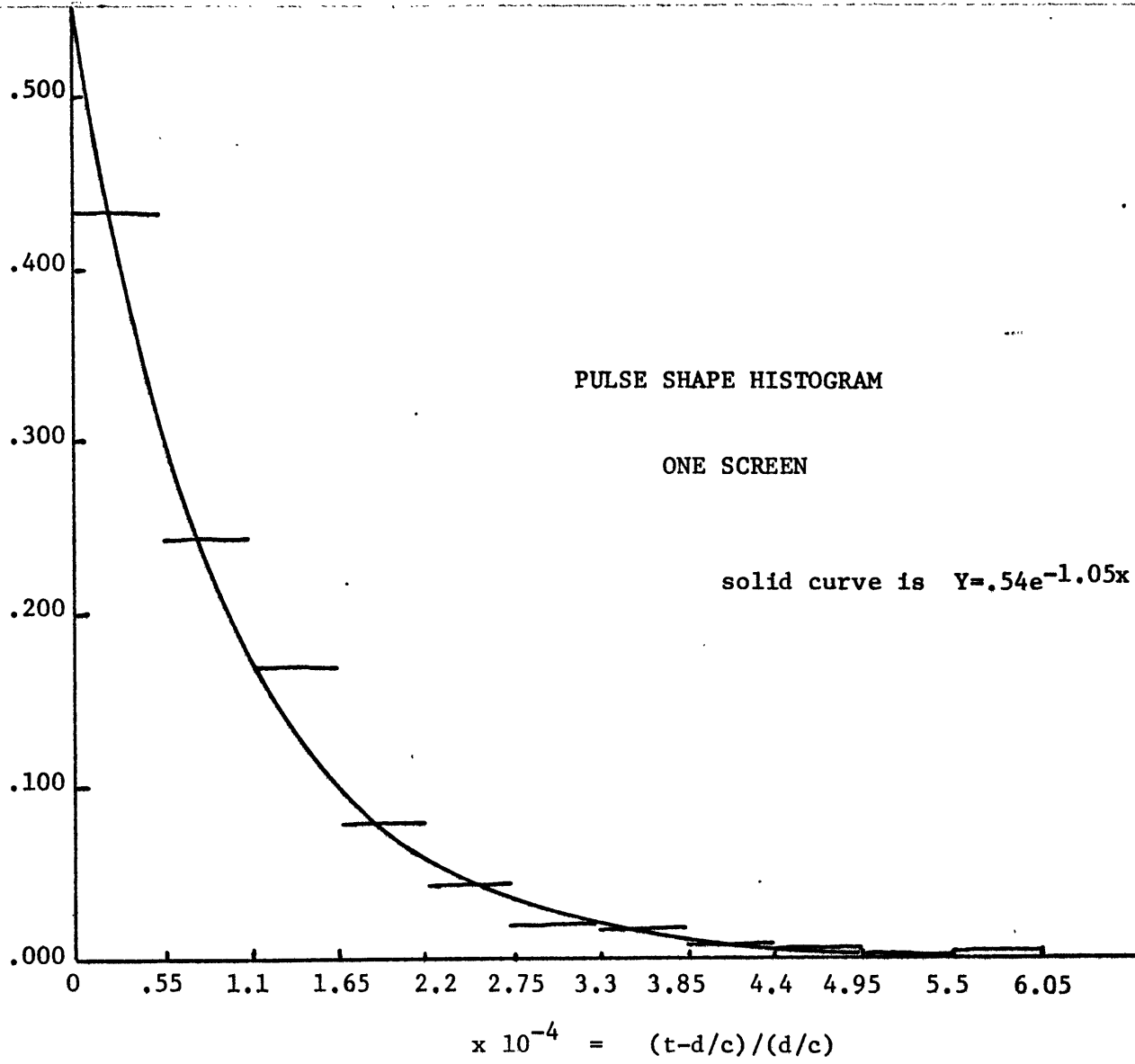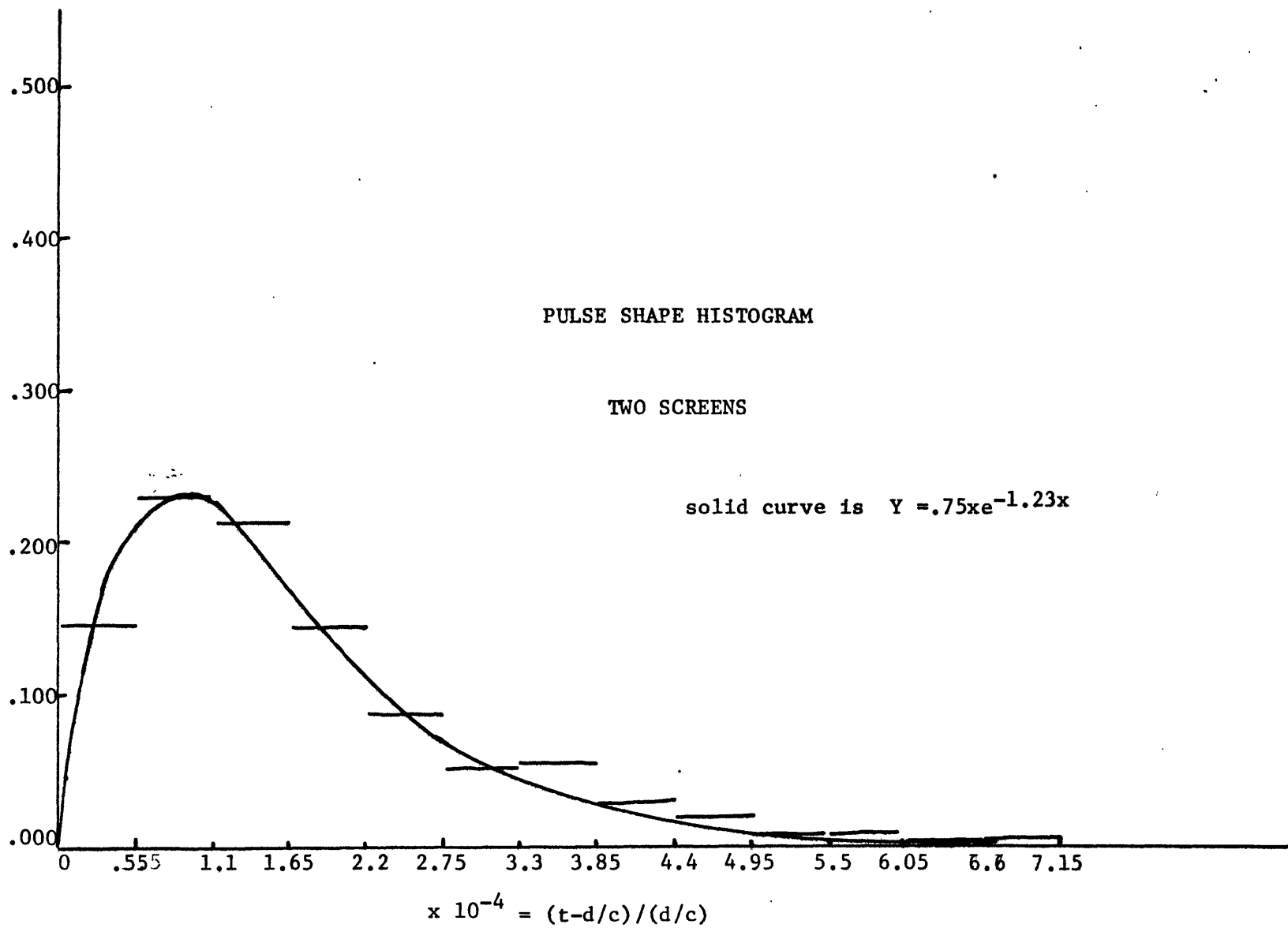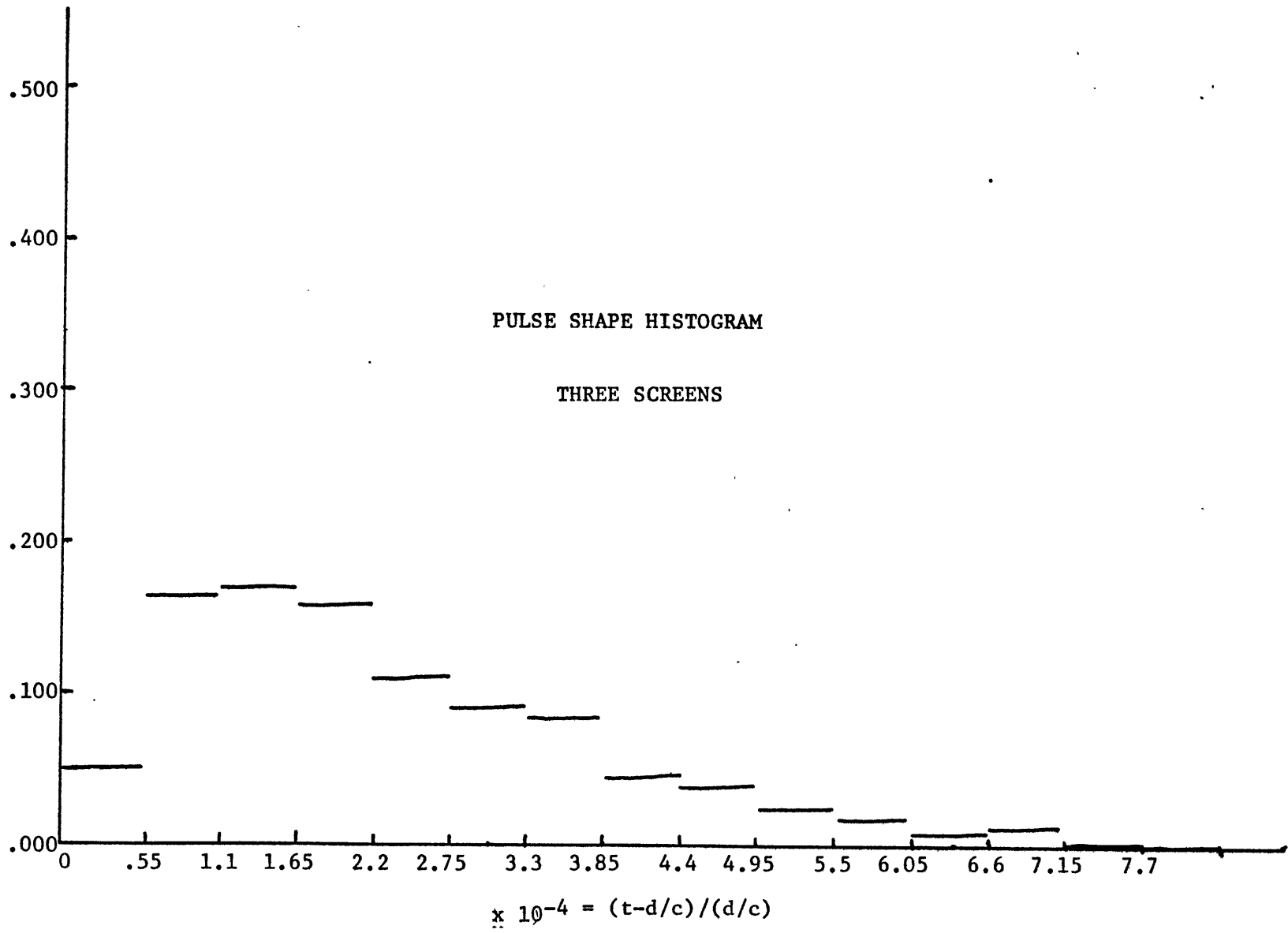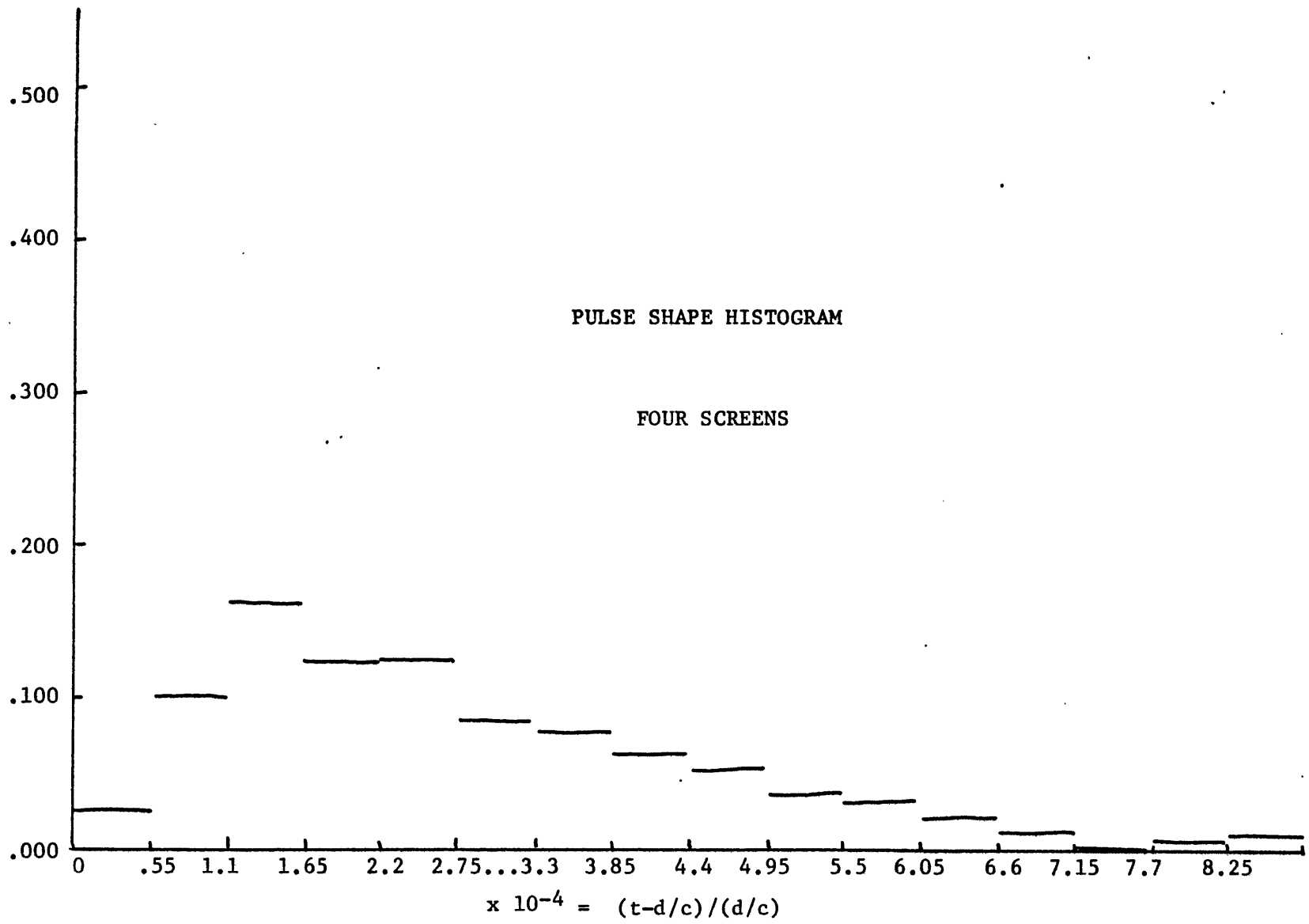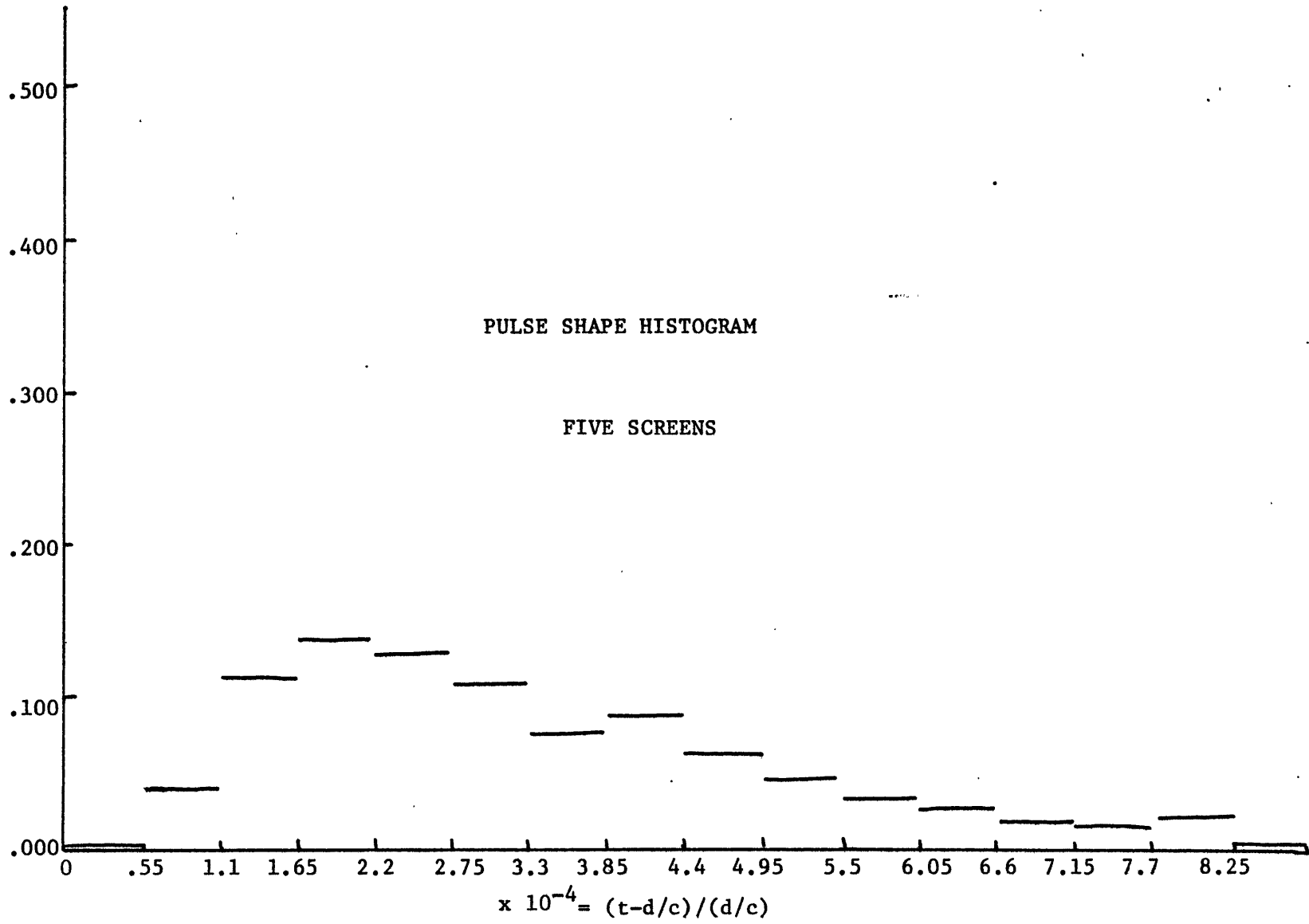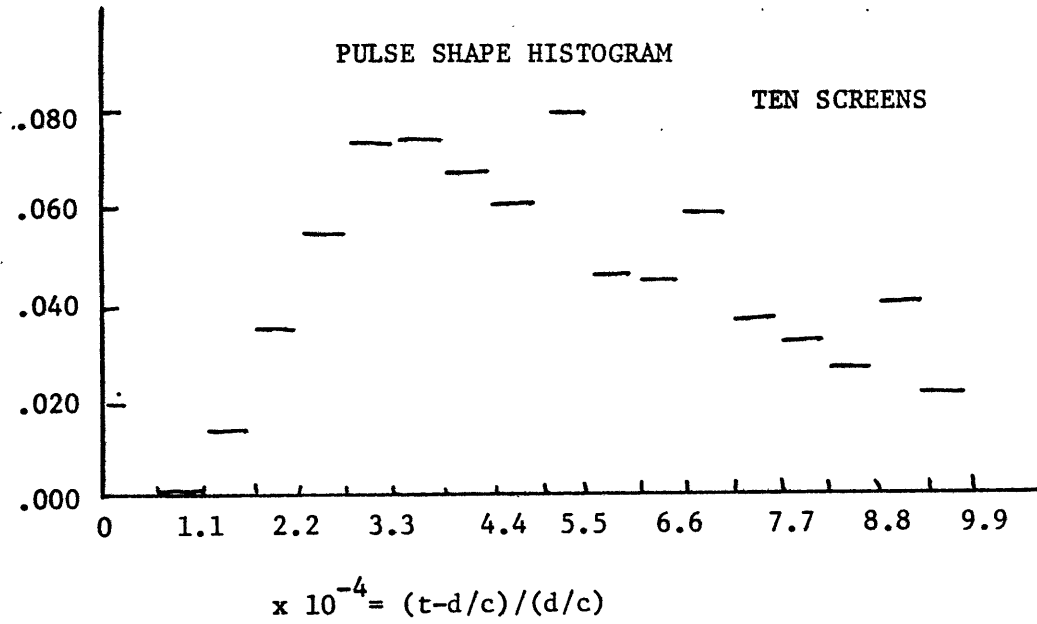
FIGURE 8

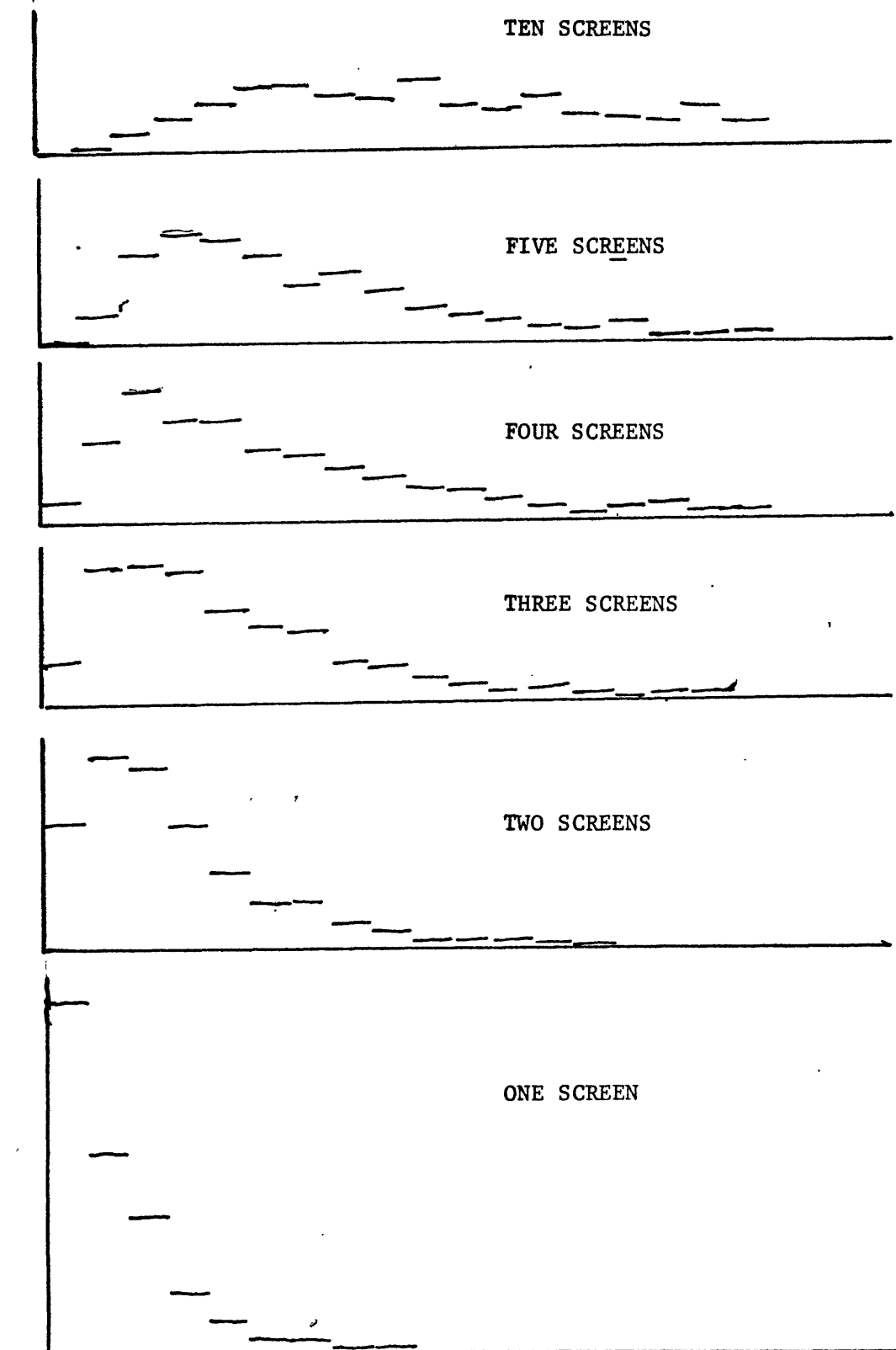FIGURE 9

FIGURE 10 a

FIGURE 10 b

RELATIVE PULSE SHAPES

FIGURE 11

APPENDIX

The program used for the Monte-Carlo calculation is included here for completeness and is not intended to represent the best or most efficient method of performing the calculation. The program was written to perform the special calculations desired without input at execution time. Minor modifications would make it suitable for a wider range of calculations by allowing parameters to be read at execution time.

The program is written in Fortran IV and the subroutines used are included in I.B.M.'s Scientific Subroutine Packet.

```
1      REAL UBON(3),COORD(20),TYME(1000),FREK(20),PCNT(20),Stat(5),MYNUS

2      UBON(1)=0.0

3      UBON(2)=20.0

4      UBON(3)=0.001

5      IXY=12345

6      EN=1.0

7      DO 75 N=1,5

8      EN=EN+1.0

9      DO 35 J=1,1000

10     TYME(J)=0.0

11     DO 35 L=1,2

12     MYNUS=0.0

13     THETA=0.0

14     DO 25 I=1,N

15     CALL GAUSS(IXY,0.02,0.0,VM)

16     THETA=THETA+VM

17     TYME(J)=TYME(J)+((THETA**2)/(2.0*EN))

18  25 MYNUS=MYNUS+THETA

19  35 TYME(J)=TYME(J)-(((MYNUS/EN)**2)/2.0)

20     CALL TAB1(TYME,TYME,1,UBON, FREK,PCNT,STAT,1000,1)

21  75 WRITE(6,22) FREK

22  22 FORMAT (10F10.6/10F10.6///)

23     RETURN

24     END
```

To get the result for Ten screens cards 6,7,8 were replaced by

```
    N=10

    EN=11.0
```

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