DYNAMICS OF DISTURBANCES ON THE
INTERTROPICAL CONVERGENCE ZONE

by

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DYNAMICS OF DISTURBANCES ON THE INTERTROPICAL
CONVERGENCE ZONE

John Raphael Bates

Submitted to the Department of Meteorology on September 4, 1969 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The dynamics of disturbances on a theoretically derived Intertropical Convergence Zone are studied.

Radiative cooling over a hemisphere is parameterised by a Newtonian cooling law, relative to a radiative equilibrium temperature which decreases from equator to pole. Release of latent heat of condensation in the tropics is treated as a function of convergence in the planetary boundary layer.

The zonally symmetric field of motion which evolves in response to these sources of energy shows a concentrated region of rising motion near, but not at, the equator. The associated low-level wind field possesses a strong cyclonic shear.

Asymmetric perturbations periodic in longitude are introduced by means of truncated Fourier series. The wavelengths are chosen to correspond to maximum instability in the ITCZ, and as such are too small to permit baroclinic instability in middle latitudes. In this way, the stability of the low latitude flow is examined while excluding middle latitude perturbations.

The low-level wind field in the vicinity of the ITCZ is found to be barotropically unstable, the wavelength of maximum growth rate being about 2000 km. The corresponding e-folding time is found to be of the order of two days, depending on the frictional and heating coefficients.

The perturbations are allowed to grow to finite amplitude. In the initial stages of growth, the Reynolds stresses supply most of the perturbation energy. At the mature stage, the energy is provided mainly by direct conversion of condensationally produced eddy available potential energy. Further growth is then limited by frictional dissipation of kinetic energy.
The mature disturbance shows a highly asymmetric cell of concentrated rising motion, bounded by regions of weak sinking motion, propagating towards the west at about 13 kts. The disturbance is 'warm-core', having a much larger amplitude at the lower than at the upper level.

The mean flow is in turn influenced by the disturbance through the mechanisms of Reynolds stresses, eddy conduction and the modification of the mean flow condensational heating through boundary-layer pumping. The influence is seen on the mean temperature and zonal wind fields, and may extend to latitudes poleward of where the perturbation amplitude in situ has decreased to zero.

A framework is thus provided for viewing tropical disturbances as an integral component of the general circulation of the atmosphere.

Thesis Supervisor: Jule G. Charney
Title: Sloan Professor of Meteorology
DEDICATION

To my Father and Mother
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DEFINITIONS OF SYMBOLS

Variables and Constants - Roman Alphabet

\( \alpha \) = mean radius of the earth
\( a_j \) = wave number for jth spectral component
\( A \) = Austausch coefficient
\( C \) = phase speed
\( C' \) = dimensionless ground friction coefficient for mean flow
\( C_p \) = drag coefficient
\( C_p' \) = coefficient of specific heat at constant pressure
\( C \) = scaling magnitude for the wind
\( D\hat{\omega} \) = characteristic magnitude of \( \hat{\omega} \) used in \( \hat{\omega} \)
\( \bar{E}_m \) = dimensionless mean kinetic energy
\( E_m' \) = dimensionless perturbation kinetic energy
\( E_p \) = dimensionless eddy conduction of heat
\( E_p' \) = dimensionless mean potential energy
\( E_p' \) = dimensionless perturbation potential energy
\( \hat{\omega} \) = Coriolis parameter \( 2\Omega \sin\varphi \)
\( \hat{\omega}_c \) = Coriolis parameter at central latitude for perturbations
\( \hat{\omega}_s \) = smoothing function for condensational heating
\( F_j(\lambda) \) = spectral component
\( F_e \) = frictional force per unit mass in eastward direction
\( F_g \) = frictional force per unit mass in northward direction
\( g \) = acceleration of gravity
\( \hat{e} \) = unit vector in eastward direction
\( \hat{t} \) = unit vector in northward direction
\( J \) = number of spectral (sine and cosine) components used in expansion of perturbation quantities

\( \vec{k} \) = unit vector in vertical direction

\( k_i \) = internal friction coefficient

\( \bar{K} \) = dimensionless lateral viscosity coefficient

\( K' \) = dimensionless internal friction coefficient

\( K'' \) = dimensionless radiational cooling coefficient

\( \hat{K} \) = dimensionless ground friction coefficient for the perturbations

\( L \) = basic linear periodicity at equator

\( L_L \) = scale length

\( M \) = angular momentum per unit mass

\( P \) = pressure

\( P_{oo} \) = 1000 mb

\( q \) = specific humidity

\( Q \) = rate of heat addition per unit mass

\( \hat{Q} \) = dimensionless condensational heating function

\( r \) = distance from earth's centre

\( R \) = gas constant for dry air

\( R_i \) = Richardson number

\( \alpha \) = Rossby number

\( R_m \) = dimensionless difference of Reynolds stresses at upper and lower levels

\( R_P \) = dimensionless sum of Reynolds stresses at upper and lower levels

\( t \) = time

\( T \) = absolute temperature ... also dimensionless temperature relative to radiative equilibrium value at the equator

\( T_e \) = equator to pole difference of radiative equilibrium temperature
\( u \) = component of velocity in eastward direction

\( \mathcal{U} \) = dimensionless sum of eastward velocities at upper and lower levels

\( \mathcal{U} \) = representative value of low level wind speed

\( v \) = component of velocity in the northward direction

\( \mathcal{V} \) = dimensionless northward velocity at upper level

\( \sigma \) = component of velocity in the vertical direction

\( \mathcal{W} \) = dimensionless difference of eastward velocities at upper and lower levels

\( x \) = coordinate towards the east

\( y \) = coordinate towards the north ... also sine of latitude (transformed coordinate)

\( \gamma_c \) = value of \( \sin \varphi \) at central latitude of the perturbation

\( \gamma_d \) = value of \( \sin \varphi \) north of which \( \gamma \) decreases rapidly

\( \gamma_s \) = value of \( \sin \varphi \) used as reference for the \( \sqrt{y} \) variation of the boundary layer height

\( z \) = height of an isobaric surface above sea level
Variables and Constants - Greek Alphabet

\( \alpha \) = specific volume ... also, wave number in \( \lambda \) direction

\( \beta \) = Rossby parameter, \( \frac{1}{\alpha} \frac{\partial \beta}{\partial \eta} \)

\( \beta^* \) = dimensionless Rossby parameter

\( \delta_{\eta} \) = characteristic magnitude of \( y \) in the decrease of \( \eta \) north of \( \eta_\alpha \)

\( \Delta \lambda \) = basic periodicity in \( \lambda \) of the perturbations

\( \Delta t \) = time step

\( \varepsilon_j \) = amplitude of \( j \)th spectral component in the expansion of the perturbation condensational heating

\( \mathcal{S} \) = vertical component of the relative vorticity

\( \eta \) = condensational heating function

\( \eta_\circ \) = condensational heating parameter

\( \Theta \) = potential temperature

\( \kappa \) = static stability parameter

\( \lambda \) = longitude, measured eastwards

\( \mu \) = internal viscosity coefficient ... also, \( \mu^* \) = dimensionless static stability parameter

\( \rho \) = density of the air

\( \sigma \) = growth rate for exponentially growing perturbations

\( \tau_a \) = ground stress

\( \varphi \) = latitude

\( \varphi^* \) = geopotential

\( \psi \) = geostrophic stream function

\( \psi_j \) = amplitude of \( j \)th spectral component in the expansion of \( \psi \)

\( \psi_j^* \) = non-dimensionalised value of \( \psi_j \)
\( \omega = \text{pressure velocity, } \frac{dp}{dt} \)

\( \tilde{\omega}_j = \text{amplitude of } j\text{th spectral component in the expansion of } \omega \)

\( \tilde{\omega}_j = \text{non-dimensionalised value of } \tilde{\omega}_j \)

\( \omega_L = \text{pressure velocity at the top of the boundary layer} \)

\( \hat{\omega}_L = \text{non-dimensionalised value of } \omega_L \)
**SUBSCRIPTS AND SUPERSCRIPTS**

- \(c\) : the value at the central latitude of the perturbations
- \(s\) : the value at the latitude north of which \(n\) decreases rapidly
- \(\alpha\) : Dimensional value
- \(i\) : the value at the \(i\)th grid point in \(y\)
- \(j\) : the value of the \(j\)th spectral component
- \(L\) : the value at the top of the boundary layer
- \(n\) : the value at the \(n\)th time step
- \(\phi\) : the value at the top level, or at the equator, or at the central latitude of the perturbation
- \(\zeta\) : the value at the reference latitude for the \(\sqrt{y}\) variation of the boundary layer height
- \(s\) : the standard atmospheric value
- \(\ast\) : the radiative equilibrium value
- \(\tilde{\phi}\) : the value averaged through the depth of the boundary layer
DIFFERENTIAL OPERATORS

\[ \mathcal{D}(\ ) = \frac{1}{\alpha \cos \varphi} \frac{\partial^2}{\partial \varphi^2} [ \ ( \ ) \cos \varphi ] \]

\[ \nabla^2(\ ) = \frac{\partial^2}{\partial x^2}(\ ) + \frac{\partial^2}{\partial y^2}(\ ) + \]

\[ \text{Also} \quad \frac{1}{\alpha^2 \cos^2 \varphi} \frac{\partial^2}{\partial x^2}(\ ) + \mathcal{D} \left( \frac{1}{\alpha} \frac{\partial}{\partial \varphi} (\ ) \right) \]

\[ \nabla^2(\ )_j = \mathcal{D} \left( \frac{1}{\alpha} \frac{\partial}{\partial \varphi} (\ )_j \right) - \frac{\alpha^2}{\alpha^2 \cos^2 \varphi} (\ )_j \]

\[ \hat{\nabla}^2(\ )_j = \frac{2}{\partial y} \left[ (1-y^2) \frac{\partial}{\partial y} (\ )_j \right] - \frac{\alpha^2}{1-y^2} (\ )_j \]
Chapter 1. Introduction

While the dynamics of middle and high latitude atmospheric motions have undergone intensive investigation, both observational and theoretical, during the past quarter century and are now fairly well understood, the Tropics remain an area of scant observation, conflicting hypotheses and plain ignorance.

The increasing realisation of the all-important role the Tropics play in the general circulation of the atmosphere and the desire to understand the genesis of hurricanes have recently led increasing numbers of research workers into this area.

With the advent of the meteorological satellite, giving frequent pictorial coverage of vast stretches of tropical ocean which were previously outside the network of meteorological observations, clearer ideas have begun to form regarding the nature of tropical motions. The outstanding feature shown by the satellite photographs is the presence of one or more bands of cloudiness, roughly parallel to the equator, stretching round the whole earth. These are especially evident in pictures averaged over periods of a week or more. They provide visual evidence of what has become known as the Intertropical Convergence Zone (ITCZ), a region (or regions) of concentrated rising motion with bands of cumulonimbus clouds extending to the tropopause. The ITCZ is the location where most of the enormous quantity of latent heat evaporated from the tropical oceans into the Trade Winds is converted
into sensible heat.

The ITCZ is almost always situated away from the equator, often as far away as 10° or more. Its mean position varies with the season, advancing furthest from the equator in the summer hemisphere. It is most stable in the eastern parts of the oceans, showing a marked tendency to migrate, break down into isolated disturbances and reform in the western parts. There are in many cases two distinct ITCZ's, one in either hemisphere.

In addition to the ITCZ cloud bands, other patterns have been recognised on the satellite photographs of the Tropics. These consist of westward moving 'Inverted V's', 'Blobs' and vortices. In late summer, some of these disturbances amplify into hurricanes or Typhoons in the western parts of the oceans. In the Atlantic, the 'Inverted V' patterns can almost always be traced back to the coast of Africa and are undoubtedly linked with continental influences. They generally lie to the north of the ITCZ and become less intense in moving over the ocean (Frank, 1969).

Many tropical disturbances form directly on the ITCZ. It is a matter of conjecture whether, if the influence of continents and extratropical disturbances were excluded, all tropical disturbances would be associated with the ITCZ.

On looking into the literature on the synoptic structure of tropical disturbances, one finds that sufficient data for detailed analysis are available only in the western parts of the Atlantic and Pacific. A relatively small number of detailed studies have been made, based on data from the Caribbean islands and the Marshall
Islands. Various empirical models have been proposed which attempt to synthesize the synoptic experience gained from these studies. The best known are the 'Equatorial Wave' model of Palmer (1952), based on Marshall Island data and the 'Easterly Wave' model of Riehl (1954), based on data from the Caribbean chain.

That the easterly wave concept has fallen short of universal acceptance can be gleaned from the title of a talk given at the National Center for Atmospheric Research in 1966 - "The easterly wave - the greatest hoax in tropical meteorology" (Sadler, 1966). Numerous other disagreements of a less irate nature pervade the literature.

Both Riehl and Palmer agree that the wavelength of tropical disturbances is about 2000 km. on the average and that they propagate towards the west at about 13 kts. Riehl's model shows the upward motion and precipitation occurring to the east of the wave axis, with descending motion ahead of the axis. Yanai and Nitta (1967) agree with this and state that the maximum upward velocity is about 4 cm/sec. Palmer, on the other hand, claims that in the Pacific disturbances heavy cloud and precipitation occur to the west of the axis.

The maximum amplitude of the Easterly Wave in Riehl's model lies between the 700 and 500 mb levels. At high levels, disturbances with an entirely different wind field may prevail. Riehl also regards easterly waves as in general having a cold core.

Elsberry (1966) has studied a Caribbean disturbance which had maximum amplitude at 925 mb and was warm cored. He suggests
that many more disturbances are warm cored than was previously believed. The assumption that warm core disturbances are inevitably in the process of amplifying is now known to be false - many warm core disturbances do not amplify beyond the wave stage.

The occurrence of tropical disturbances has a strong seasonal dependence. Palmer (1952) states that the Marshall Islands area is affected by stable waves during the greater part of the year, but from July to September the waves in this area tend to amplify, while the region in which only stable waves are found moves upstream. According to Riehl (1954), well developed easterly waves seldom occur in winter or early spring in the Atlantic.

Concerning the question of where tropical disturbances get their energy, there is no general agreement among synoptic meteorologists.

In theoretical investigations of tropical motions, most success has been obtained in the study of hurricanes. The biggest problem facing theoreticians has been the mechanism of moist convection and its relation to the large scale flow. Unlike the situation at higher latitudes, the lower half of the tropical atmosphere in its mean state is potentially unstable, so that upward motion which persists for any length of time inevitably leads to the release of instability in the form of cumulus or cumulonimbus clouds. Charney and Eliassen (1964) postulated that the heating due to moist convection is proportional to the large scale convergence of moisture in the atmospheric boundary layer. From this they derived the theory of 'conditional instability of the second
kind', showing how a circularly symmetric system would amplify on a scale and at a rate consistent with the observed characteristics of hurricanes.

Nitta (1964) applied the boundary layer convergence mechanism to the study of a disturbance which is sinusoidal in the east-west direction. In his theory, the on-off nature of the heating is neglected by assuming negative condensational heating in regions of negative boundary layer pumping. The mean flow is assumed to be zero and the $\beta$-effect is neglected. With lateral viscosity included, he shows how disturbances of wavelength up to 1000 km may amplify.

Linear studies of tropical motions have been carried out by Rosenthal (1965), Matsuno (1966) and Koss (1967). In these studies, the heating is zero and the mean flow is either zero or constant, so that it does not provide a source of perturbation energy.

The possibility that tropical motions are forced, the perturbation energy being derived from lateral coupling with extratropical motions, has been investigated by Mak (1969). This gives some interesting results which are in accord with observation, even though heating by condensation is excluded from the model.

In the case of low level tropospheric motions in the tropics, however, condensational heating undoubtedly plays an important role, and it seems unlikely that lateral forcing could be a predominant mechanism. For one thing, low level perturbations in the Tropics are most pronounced in summer while forcing from extratropical perturbations is greatest in winter.
Charney (1963) has presented scale arguments showing that, in the absence of condensation, tropical motions are uncoupled in the vertical. Holton (1969) has pointed out, however, that Charney's analysis applies only to motions whose vertical extent is of the order of the atmospheric scale height. For motions having a smaller vertical scale, it is possible to have vertical propagation and strong coupling.

Barotropic instability as the source of perturbation energy in the Tropics has been considered by Nitta and Yanai (1969). Using the barotropic vorticity equation, and with observed zonal wind profiles from low-level Marshall Island data taken as basic state, they investigated the growth of barotropic disturbances. The wavelength of maximum growth rate was found to be $2000\text{km}$, with a corresponding e-folding time of 5.2 days. In their study, no heating or friction was included.

The present thesis is an attempt to investigate theoretically the dynamics of tropical disturbances from a more fundamental point of view than has previously been taken. Tropical disturbances are considered as an element of the general circulation with energy sources related to the large scale dynamics.

The basis of this study is the zonally symmetric ITCZ theory of Charney (1968). This was the first theoretical investigation in which due emphasis was given to the role of the ITCZ in the general circulation. Charney showed how the baroclinicity on which middle latitude disturbances depend for their energy is intimately related to the ITCZ. The main feature distinguishing Charney's
model from other general circulation models is his parameterizing of the condensational heating in terms of the pumping of moisture out of the boundary layer. Mintz (1964), in his numerical model, uses the temperature lapse rate as the sole criterion determining whether moist convection will occur. Manabe, Smagorinsky and Strickler (1965 and 1967) compute the moisture field and allow convective adjustment to take place when the relative humidity reaches 100% and the lapse rate exceeds the moist adiabatic lapse rate. There is a widespread feeling among tropical synoptic meteorologists, based on day to day observation, that lapse rates cannot be regarded as the main criteria for moist convection in the Tropics. An extreme example in support of this point of view is the hurricane, where convection continues unabated in spite of the fact that the lapse rate is practically moist adiabatic.

Kasahara and Washington (1967), in the NCAR general circulation model, allow the release of latent heat whenever there is upward motion, the amount of heat released being proportional to the vertical velocity in the interior of the atmosphere. When applied to a potentially unstable atmosphere, this is a questionable procedure.

As in Charney's study, the model used in this thesis covers an entire hemisphere. All meridional motions are required to vanish at the equator. The model uses two levels in the vertical and a series of grid points to represent variations in the north-south direction. All perturbations are taken to be periodic in longitude, the east-west variation being represented by a truncated
Fourier series, The Fourier components interact with each other and with the mean flow. Longitudinal periodicities which correspond to the observed scales of tropical motions are chosen. The corresponding wavelengths in middle latitudes are baroclinically stable. Thus tropical perturbations, which depend on barotropic and condensational energy sources, can be studied in isolation.

Balance equations are used to calculate the mean flow while the perturbations are assumed to be quasi-geostrophic. The heating function consists of condensational and radiational components. Three kinds of friction are included, surface, internal and lateral.

The details of the mathematical model are described in Chapter 2. Some results derived by linearising the equations are given in Chapter 3. Chapter 4 describes some preliminary numerical experiments while the main results of the thesis are contained in Chapter 5. A summary and discussion of the results are given in the final chapter.
Chapter 2. The Mathematical Model

2.1 The Two-Level Model

The primitive equations of atmospheric motion, referred to spherical coordinates rotating with the earth, and with pressure as vertical coordinate, can be written in the following form (see for example, Lorenz, 1967, Chapter 2):

Equations of motion

\[ \frac{du}{dt} = - \frac{1}{a \cos \phi} \frac{d \Phi}{d \lambda} + f v + \frac{uv J \cos \Phi}{a} + F_\lambda \]  \hspace{1cm} (2.1.1)

\[ \frac{dv}{dt} = - \frac{1}{a \cos \phi} \frac{d \Phi}{d \Phi} - f u - \frac{u^2 J \cos \Phi}{a} + F_\phi \]  \hspace{1cm} (2.1.2)

Hydrostatic equation

\[ \frac{d \Phi}{d \phi} = - \alpha \]  \hspace{1cm} (2.1.3)

Continuity equation

\[ \frac{1}{a \cos \phi} \frac{d u}{d \lambda} + \nabla (v) + \frac{\partial \omega}{\partial \phi} = 0 \]  \hspace{1cm} (2.1.4)

Thermodynamic energy equation

\[ \frac{d \Theta}{dt} = \frac{\Theta}{\gamma T} Q \]  \hspace{1cm} (2.1.5)

Equation of state

\[ p \alpha = RT \]  \hspace{1cm} (2.1.6)
where
\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{\alpha \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{\alpha} \frac{\partial}{\partial \phi} + \frac{\omega}{\alpha} \frac{\partial}{\partial \rho}
\]
and
\[
\mathcal{D}(\lambda) = \frac{1}{\alpha \cos \phi} \frac{\partial}{\partial \phi} \left( \cos \phi \right)
\]

For the purposes of the present study, a two-level model in the vertical is used. (See Fig. 2.1) The equations of motion are resolved into mean zonal and perturbation components which are approximated differently, the mean components being reduced to a set of balanced equations and the perturbation components to a set of quasi-geostrophic equations. This is done in such a way as to preserve correct energetic interaction between the mean and perturbation components of the flow.

Equations (2.1.1), (2.1.2) and (2.1.4), expressed at levels 1 and 3, give

\[
\begin{align*}
\frac{u_1}{\alpha \cos \phi} \frac{\partial u_1}{\partial \lambda} + \frac{v_1}{\alpha} \frac{\partial v_1}{\partial \phi} + \frac{\omega_1}{\alpha} (u_1 - u_2) &= \frac{1}{\alpha \cos \phi} \frac{\partial \theta}{\partial \lambda} + \frac{1}{\alpha} + \frac{1}{\alpha \cos \phi} \frac{\partial \phi}{\partial \lambda} + (F_1) \\
\frac{u_2}{\alpha \cos \phi} \frac{\partial u_2}{\partial \lambda} + \frac{v_2}{\alpha} \frac{\partial v_2}{\partial \phi} + \frac{\omega_2}{\alpha} (u_2 - u_3) &= \frac{1}{\alpha \cos \phi} \frac{\partial \theta}{\partial \lambda} + \frac{1}{\alpha} + \frac{1}{\alpha \cos \phi} \frac{\partial \phi}{\partial \lambda} + (F_2) \\
\frac{u_3}{\alpha \cos \phi} \frac{\partial u_3}{\partial \lambda} + \frac{v_3}{\alpha} \frac{\partial v_3}{\partial \phi} + \frac{\omega_3}{\alpha} (u_3 - u_1) &= \frac{1}{\alpha \cos \phi} \frac{\partial \theta}{\partial \lambda} + \frac{1}{\alpha} + \frac{1}{\alpha \cos \phi} \frac{\partial \phi}{\partial \lambda} + (F_3)
\end{align*}
\]

The hydrostatic and thermodynamic equations, expressed at level 2, give

\[
\begin{align*}
\frac{1}{\alpha \cos \phi} \frac{\partial v_1}{\partial \lambda} + \mathcal{D}(v_1) + \frac{\omega_1}{\alpha} &= 0 \\
\frac{1}{\alpha \cos \phi} \frac{\partial v_3}{\partial \lambda} + \mathcal{D}(v_3) - \frac{\omega_3}{\alpha} &= 0
\end{align*}
\]
\[ p_0 = 0 \]
\[ p_1 = 250 \]
\[ p_2 = 500 \]
\[ p_3 = 750 \]
\[ p_4 = 1000 \]

\[ \omega_0 = 0 \]
\[ \Delta p \]
\[ \bar{u}_1, \bar{v}_1, \bar{\Phi}_1, \psi_1' \]
\[ \bar{w}_2, \omega_2' \]
\[ \Delta p \]
\[ \bar{u}_3, \bar{v}_3, \bar{\Phi}_3, \psi_3' \]

**FIGURE 2.1** THE TWO LEVEL MODEL, SHOWING QUANTITIES EXPRESSED AT EACH LEVEL.
\[
\frac{\Phi_1 - \Phi_3}{\Delta \rho} = \alpha_2
\]  \hspace{1cm} (2.1.13)

\[
\frac{\partial \Theta}{\partial t} + \left( \frac{u_1 + u_5}{2} \right) \frac{1}{\alpha \Theta \delta t} + \left( \frac{v_1 + u_5}{2} \right) \frac{1}{\alpha \Theta \delta \lambda} + \omega_2 \left( \frac{\partial \Theta}{\partial \rho} \right)_2 = \left( \frac{\Theta}{\Theta} \right)_2
\]  \hspace{1cm} (2.1.14)

By using the equation of state, \( \alpha_2 \) is eliminated from (2.1.13), giving

\[
(\Phi_1 - \Phi_3) = \frac{\Delta \rho}{R} \frac{\rho_p}{R} \frac{1}{\rho_p} \Theta_2
\]

Hence

\[
\Theta_2 = \frac{R}{\Delta \rho} \frac{1}{\rho_p} \left( \frac{\rho_o}{\rho_p} \right)^{\frac{\rho_p}{k_p}} (\Phi_1 - \Phi_3)
\]  \hspace{1cm} (2.1.15)

The static stability \( (\Theta \Theta \rho)_2 \) is taken as a constant in this model and assigned its standard atmospheric value \( (\Theta \Theta \rho)_5 \). Two-level models can be devised in which this parameter is allowed to vary (Lorenz, 1960), but this would require knowledge of the heating rates at levels 1 and 3. Since it is not known how condensational heating due to large convective cells (the predominant mode of heating in the tropics) is distributed in the vertical, it seems best to assume uniform heating and to express the thermodynamic equation simply at level 2. Changes in the static stability will, however, be indirectly allowed for in the heating term, as will be seen later (§2.4).

The presence of a boundary layer has not been explicitly referred to in the equations so far; the only place it will appear will be in the parameterization of \( \Theta_2 \). Anticipating this (see
§2.4), the heating term will be written

\[
\left( \frac{\Theta}{c_p T} \rho \right)_2 = \frac{p}{\Delta p} \left( \frac{\rho u}{\rho z} \right) \left( -\kappa \eta \omega + k_{r} (\bar{\phi} - \phi_3)^* - (\bar{\phi} - \phi_3) \right)
\] 

(2.1.16)

Using (2.1.15), the thermodynamic equation then becomes

\[
\frac{\partial}{\partial t} (\bar{\phi} - \phi_3) + \left( \frac{\mu_1 + \mu_3}{2} \right) \frac{1}{\alpha \omega_{3 \phi}} \frac{\partial}{\partial \phi_{3 \phi}} (\bar{\phi} - \phi_3) + \left( \frac{\mu_1 + \mu_3}{2} \right) \frac{\partial}{\partial \phi_{3 \phi}} (\bar{\phi} - \phi_3) \\
+ \kappa \left[ \eta \omega - \omega_\sigma \right] = k_{r} \left[ (\bar{\phi} - \phi_3)^* - (\bar{\phi} - \phi_3) \right]
\]

(2.1.17)

where

\[
\kappa = \left( \frac{L}{\frac{1}{2} \frac{\Delta \phi_{3 \phi}}{\phi_{3 \phi}}} \right)
\]

(2.1.18)

These equations serve as the basis for the description of the mean and perturbation flow, which will now be discussed separately.
2.2 The Mean Equations

In this study, the variation of all quantities in the longitudinal direction is assumed to be periodic, with basic angular period $\Delta \lambda$. Thus, there is a basic linear period which varies as $\cos \varphi$.

The mean is defined as

$$\bar{() = \frac{1}{\Delta \lambda} \int_{0}^{\Delta \lambda} (\ ) d\lambda}$$

$$= \frac{1}{\int_{0}^{\Delta \lambda} (\ ) \ dx} \tag{2.2.1}$$

where

$$\Delta \lambda = \frac{L}{\alpha} \tag{2.2.2}$$

Deviations from this mean are designated by a prime, i.e.

$$() = (\bar{() + (\ )} \tag{2.2.3}$$

Taking the mean of equations (2.1.7) - (2.1.14) and (2.1.17) gives

$$\bar{u}_{1t} + \frac{\bar{u}_{1}}{\alpha} \frac{d\bar{u}_{1}}{d\varphi} + \frac{\bar{u}_{1}'}{\alpha} \frac{d\bar{u}_{1}'}{d\varphi} + \frac{\omega_{3}}{2} (\bar{u}_{3} - \bar{u}_{1}) + \frac{\omega_{3}'}{2} (\bar{u}_{3}' - \bar{u}_{1}')$$

$$= \int \bar{u}_{1} + (\bar{u}_{1} + \bar{u}_{1}') \Delta \varphi + \bar{F}_{1} \tag{2.2.4}$$

$$\bar{u}_{3t} + \frac{\bar{u}_{2}}{\alpha} \frac{d\bar{u}_{2}}{d\varphi} + \frac{\bar{u}_{2}'}{\alpha} \frac{d\bar{u}_{2}'}{d\varphi} + \frac{\omega_{3}}{2} (\bar{u}_{3} - \bar{u}_{2}) + \frac{\omega_{3}'}{2} (\bar{u}_{3}' - \bar{u}_{2}')$$

$$= \int \bar{u}_{3} + (\bar{u}_{2} + \bar{u}_{2}') \Delta \varphi + \bar{F}_{1} \tag{2.2.5}$$

$$\bar{u}_{1t} + \frac{\bar{u}_{1}'}{\alpha} \frac{d\bar{u}_{1}'}{d\varphi} + \frac{\bar{u}_{1}'}{\alpha} \frac{d\bar{u}_{1}'}{d\varphi} + \frac{\omega_{3}}{2} (\bar{u}_{3} - \bar{u}_{1}) + \frac{\omega_{3}'}{2} (\bar{u}_{3}' - \bar{u}_{1}')$$

$$= -\frac{\omega_{3}}{\alpha} \Delta \varphi - \int \bar{u}_{1} - (\bar{u}_{1} + \bar{u}_{1}^{\prime}) \Delta \varphi + \bar{F}_{1} \tag{2.2.6}$$
\[
\begin{align*}
\bar{u}_{3t} + \frac{u_1'}{\alpha \sigma_{1g}} \bar{u}_1 + \frac{u_2'}{\alpha \sigma_{2g}} \bar{u}_2 + \frac{u_3'}{\alpha \sigma_{3g}} \bar{u}_3 + \frac{\omega_k}{2} (\bar{u}_2 - \bar{u}_1) + \frac{\omega_k'}{\alpha} (\bar{u}_3 - \bar{u}_1') & \\
= - \frac{1}{\alpha} \frac{\partial \Phi_3}{\partial \phi} - \int \bar{u}_1 - (\bar{u}_2 + \bar{u}_1') \frac{\partial \omega_k}{\partial \phi} + (F_{\phi})_3 & \quad (2.2.7)
\end{align*}
\]

\[
\begin{align*}
\mathcal{D} (\bar{u}_1) + \frac{\omega_k}{\alpha} & = 0 & \quad (2.2.8)
\end{align*}
\]

\[
\begin{align*}
\mathcal{D} (\bar{u}_3) - \frac{\omega_k'}{\alpha} & = 0 & \quad (2.2.9)
\end{align*}
\]

\[
\begin{align*}
(\Phi_i - \Phi_3) + \left( \frac{u_1' + u_3'}{2} \right) \frac{1}{\alpha \sigma_{1g}} \frac{\partial}{\partial \lambda} (\Phi_i' - \Phi_3') + \left( \frac{u_3 + u_2}{2} \right) \frac{1}{\alpha \sigma_{3g}} (\Phi_3' - \Phi_3)
\end{align*}
\]

\[
\begin{align*}
= & \quad \mathcal{H} \left[ (\Phi_i - \Phi_3)' - (\Phi_i - \Phi_3) \right] & \quad (2.2.10)
\end{align*}
\]

Adding (2.2.8) and (2.2.9), integrating \((\int_0^\phi C_1 \cos \phi \frac{d \phi}{\partial \phi})\) and using the fact that \(\bar{u}_1 (\text{equator}) = \bar{u}_3 (\text{equator}) = 0\)
gives

\[
\bar{u}_1 + \bar{u}_3 = 0 & \quad (2.2.11)
\]

Using this and the perturbation form of the continuity equations (2.1.11) and (2.1.12), the equations (2.2.4) - (2.2.7) and (2.2.10) become

\[
\begin{align*}
\bar{u}_{1t} + \bar{u}_1 \mathcal{D} (\bar{u}_1) + \frac{\omega_k}{2} \left( \frac{u_3 - \bar{u}_1}{\alpha} \right) - \int \mathcal{D} \left( \bar{u}_1 \bar{u}_1' \right) - \frac{\bar{u}_1' \bar{u}_1'}{\alpha} & \\
+ \frac{\omega_k'}{\alpha} \left( \frac{u_1' + u_3'}{2} \right) & = (F_{1})_1 & \quad (2.2.12)
\end{align*}
\]

\[
\begin{align*}
\bar{u}_{3t} + \bar{u}_3 \mathcal{D} (\bar{u}_3) + \frac{\omega_k}{2} \left( \frac{u_3 - \bar{u}_1}{\alpha} \right) - \int \mathcal{D} \left( \bar{u}_3 \bar{u}_3' \right) - \frac{\bar{u}_3' \bar{u}_3'}{\alpha} & \\
+ \frac{\omega_k'}{\alpha} \left( \frac{u_1' + u_3'}{2} \right) & = (F_{1})_3 & \quad (2.2.13)
\end{align*}
\]
The vertical Reynolds stresses due to synoptic scale motions in the atmosphere are negligible by comparison with the horizontal Reynolds stresses, so the terms \( \frac{\omega_x}{\alpha} \left( \frac{u''_1 + u''_3}{2} \right) \) in (2.2.12) and (2.2.13) are dropped, leaving

\[
\bar{u}_{1t} + \bar{u}_1 \mathcal{D}(\bar{u}_1) + \frac{\omega_x}{2} \left( \frac{\bar{u}_3 - \bar{u}_1}{\alpha} \right) - \frac{1}{\alpha} \frac{\partial \bar{u}_1}{\partial \phi} + \mathcal{D}(\bar{u}_1, \bar{u}_1) - \frac{u''_1 \bar{u}_1' \bar{u}_1 \tan \phi}{\alpha} = \bar{F}_1, \tag{2.2.17}
\]

\[
\bar{u}_{3t} + \bar{u}_3 \mathcal{D}(\bar{u}_3) + \frac{\omega_x}{2} \left( \frac{\bar{u}_3 - \bar{u}_1}{\alpha} \right) - \frac{1}{\alpha} \frac{\partial \bar{u}_3}{\partial \phi} + \mathcal{D}(\bar{u}_3, \bar{u}_3) - \frac{u''_3 \bar{u}_3' \bar{u}_3 \tan \phi}{\alpha} = \bar{F}_3. \tag{2.2.18}
\]

The mean zonal flow is assumed to be in a state of balance, so the zonal equations (2.2.14) and (2.2.15) reduce to

\[
\frac{\bar{u}_1' \bar{u}_1' \tan \phi}{\alpha} + \frac{1}{\alpha} \frac{\partial \bar{u}_1}{\partial \phi} = -\frac{1}{\alpha} \frac{\partial \bar{u}_1}{\partial \phi}, \tag{2.2.19}
\]

\[
\frac{\bar{u}_3' \bar{u}_3' \tan \phi}{\alpha} + \frac{1}{\alpha} \frac{\partial \bar{u}_3}{\partial \phi} = -\frac{1}{\alpha} \frac{\partial \bar{u}_3}{\partial \phi}. \tag{2.2.20}
\]
This approximation is justified by consideration of the observed scales of atmospheric motion and, a posteriori, from the results of numerical integrations with the model.

The terms \( \frac{\partial u}{\partial z} \) are necessary to obtain a correct energy equation. In numerical integrations, however, they are dropped, being very small. (see §2.7(c))

Equations (2.2.8), (2.2.11), (2.2.17), (2.2.18), (2.2.19) and (2.2.20) are now a complete set of equations governing the mean flow.

An angular momentum principle can be derived from (2.2.17), (2.2.18) as follows:

The angular momentum about the earth's axis per unit mass is given by

\[
M = (u + \Omega a \cos \varphi)a \cos \varphi
\]

with mean and perturbation components

\[
\bar{M} = (\bar{u} + \Omega a \cos \varphi)a \cos \varphi
\]

\[
M' = u' a \cos \varphi
\]

Therefore

\[
\frac{\partial}{\partial t} (\bar{M} + \bar{m}) = a \cos \varphi \left( \bar{u}_t + \bar{u}_{st} \right)
\]

\[
= a \cos \varphi \left[ -\bar{u}_i D(\bar{u}) - \bar{u}_3 D(\bar{u}_3) - \frac{\partial}{\partial \rho} (\bar{u}_3 - \bar{u}) 
- D(\bar{u}_i \bar{u}_i + \bar{u}_3 \bar{u}_3) + \bar{u}_i \bar{u}_i + \bar{u}_3 \bar{u}_3 \right] 
\]

\[
+ \left( F_\lambda_1 + F_\lambda_3 \right)
\]

\[
= -D \left[ \bar{u}_i \bar{M}_i + \bar{u}_3 \bar{M}_3 + \bar{u}_i \bar{u}_i' + \bar{u}_3 \bar{u}_3' \right] 
\]

\[
+ a \cos \varphi \left[ (F_\lambda_1 + F_\lambda_3) \right]
\]
Integrating gives

\[
\frac{d}{dt} \int_{\theta_1}^{\theta_2} (\bar{m}_1 + \bar{m}_3) \cos \theta \, d\phi = - \frac{d}{d\theta} \left[ \bar{u}_1 \bar{m}_1 + \bar{u}_3 \bar{m}_3 + \bar{u}_1^' \bar{m}_1^' + \bar{u}_3^' \bar{m}_3^' \right]_{\theta_1}^{\theta_2} \\
+ \int_{\theta_1}^{\theta_2} a \cos \theta \left[ (\bar{f}_x)_{\theta_1}^{\theta_2} + (\bar{f}_y)_{\theta_1}^{\theta_2} \right] \cos \theta \, d\phi
\] (2.2.24)

i.e. the rate of change of angular momentum within a region bounded by latitude walls equals the transport into the region by the mean and eddy flow, plus the integrated torque of the frictional forces about the earth's axis.

Due to the different physical mechanisms of vertical and lateral diffusion of momentum in the atmosphere, it is desirable to separate the frictional force \((\bar{f}_x, \bar{f}_y)\) into vertical and horizontal components with separate coefficients.

This is done by writing

\[
\overrightarrow{F} = -g \overrightarrow{\nabla}_p + A \overrightarrow{\nabla}_v \overrightarrow{V}
\] (2.2.25)

where

\[
\overrightarrow{\nabla}_p = \left\{ \begin{array}{ll}
\mu \frac{\partial \overrightarrow{V}}{\partial \theta} & \text{in the interior} \\
c_D \rho_v |v_3| \overrightarrow{v}_3 & \text{at the ground}
\end{array} \right.
\] (2.2.26)

The horizontal operator \(\overrightarrow{\nabla}_v\) is obtained by setting

\[
\frac{\partial}{\partial \theta} = 0, \quad \overrightarrow{V} = \overrightarrow{\alpha}
\]
in the general expression for $\nabla^2$ in spherical coordinates, giving

$$\nabla^2 \mathbf{V} = \mathbf{V}_x \left[ \frac{1}{a^2 \cos \phi} \frac{\partial^2 u}{\partial \lambda^2} + \frac{1}{a^2 \cos \phi} \left( D(u) \right) - \frac{2 \sin \phi}{a \cos \phi} \frac{\partial u}{\partial \lambda} \right] + \mathbf{V}_y \left[ \frac{1}{a^2 \cos \phi} \frac{\partial^2 u}{\partial \lambda^2} + \frac{1}{a^2 \cos \phi} \left( D(v) \right) + \frac{2 \sin \phi}{a \cos \phi} \frac{\partial u}{\partial \lambda} \right]$$

(2.2.27)

The following expressions, derived from the above, are then used for the mean frictional force:

$$\overline{(F_1)}_1 = -k_1 (\overline{u}_1 - \overline{u}_3) + \frac{A}{\alpha} \frac{\partial}{\partial \phi} \left( D(\overline{u}_1) \right)$$

(2.2.28)

$$\overline{(F_1)}_3 = k_1 (\overline{u}_1 - \overline{u}_3) - \frac{c}{\alpha} \left[ \rho_1 T_1 \overline{u}_3 / \overline{u}_1 \right] + \frac{A}{\alpha} \frac{\partial}{\partial \phi} \left( D(\overline{u}_3) \right)$$

(2.2.29)

Over the ocean, the drag coefficient $C_D$ varies on the average between .0015 and .0025 (Palmen and Holopainen, 1962). A value of .002 is used here.

An estimate of $k_1$ can be obtained from the value of Bjerknes and Venkateswaran (1957) for the internal coefficient of viscosity of the atmosphere, viz., $\mu = 200 \frac{m}{cm \cdot sec}$. Hence $k_1 \equiv \frac{9 \mu}{\alpha \rho_0} \approx 4.8 \times 10^{-6} \frac{cm}{sec}$ (as well as all other static parameters of the atmosphere, is taken from Jordan's data (1958).)

No dependable valuables of the Austausch coefficient $A$ are known; in fact, to represent atmospheric eddy diffusion by a Fickian mechanism at all is to stretch the imagination. Richardson (1926) suggests a "non-Fickian" diffusion of the form

$$\frac{\partial u}{\partial t} = \frac{2}{\ell} \left\{ F(\ell) \frac{\partial u}{\partial \ell} \right\}$$

with $F(\ell) = 0.6 \ell^{5/2} cm^2/sec$ for eddies of 1 metre to 10 km in the atmosphere. Assuming a characteristic eddy scale, $\ell$, of 50 km
and neglecting \( \frac{\partial F}{\partial t} \), this would give an Austausch coefficient of 5 \( \times 10^8 \) cm\(^2\)/sec. Islitzer and Slade (1968) suggest a value of 4 \( \times 10^8 \) cm\(^2\)/sec, an average of results derived from experiments with smoke plumes, multiple balloon releases and clouds from nuclear detonations. In the present study, values ranging from 4 \( \times 10^8 \) to 10\(^9\) cm\(^2\)/sec (the value used by Phillips (1956)) are used.

Energy equations for the mean flow are obtained by multiplying (2.2.17) by \( \tilde{u}_1 \), (2.2.18) by \( \tilde{u}_3 \), (2.2.16) by \( (\bar{E}_i - \bar{E}_3) \), and integrating. This gives, making use of (2.2.19), (2.2.20),

\[
\frac{\partial}{\partial t} \int_{\Omega} \left( \frac{\tilde{u}_1^2 + \tilde{u}_3^2}{2} \right) a^2 \cos \varphi \, d\varphi = -\int_{\Omega} \frac{\partial}{\partial p} (\bar{E}_i - \bar{E}_3) a^2 \cos \varphi \, d\varphi \\
+ \int_{\Omega} \left[ \tilde{u}_1 \tilde{u}_1' (\tilde{E}(\tilde{u}_1) + \frac{2\tilde{u}_1 \bar{E}_3}{\bar{a}}) + \tilde{u}_3 \tilde{u}_3' (\tilde{E}(\tilde{u}_3) + \frac{2\tilde{u}_3 \bar{E}_3}{\bar{a}}) \right] a^2 \cos \varphi \, d\varphi \\
- \int_{\Omega} \frac{\partial}{\partial p} \left[ \tilde{u}_1 \tilde{u}_1' + \tilde{u}_3 \tilde{u}_3' \right] a^2 \cos \varphi \, d\varphi \\
- \int_{\Omega} A \left[ \left( \frac{\partial \tilde{u}_1}{\partial \tilde{u}_3} \right)^2 + \left( \frac{\partial \tilde{u}_3}{\partial \tilde{u}_3} \right)^2 \right] a^2 \cos \varphi \, d\varphi
\]

\[
\frac{\partial}{\partial t} \int_{\Omega} \left( \frac{1}{\bar{a}} (\bar{E}_i - \bar{E}_3) \right) a^2 \cos \varphi \, d\varphi = \int_{\Omega} \frac{\partial}{\partial p} (\bar{E}_i - \bar{E}_3) a^2 \cos \varphi \, d\varphi \\
- \int_{\Omega} \frac{\partial}{\partial p} (\bar{u}_1') a^2 \cos \varphi \, d\varphi \\
+ \int_{\Omega} A \left[ \left( \frac{\partial \tilde{u}_1}{\partial \tilde{u}_3} \right)^2 + \left( \frac{\partial \tilde{u}_3}{\partial \tilde{u}_3} \right)^2 \right] \frac{\partial}{\partial \varphi} \left( \bar{E}_i - \bar{E}_3 \right) a^2 \cos \varphi \, d\varphi
\]

In deriving the above equations, the conditions of zero meridional motion at the equator, i.e.

\[
\tilde{u}_1 = \tilde{u}_3 = u_1' = u_3' = 0, \quad \varphi = 0
\]
and of symmetry, i.e.

\[ \frac{1}{\alpha} \frac{\partial \bar{u}}{\partial x} = \frac{1}{\alpha} \frac{\partial \bar{u}^3}{\partial x} = 0 , \quad \varphi = 0 \]

have been used.

From these equations, the mechanisms by which the perturbations interact with the mean flow are clearly seen:

(i) The Reynolds stresses act on the horizontal shear of the mean flow to convert kinetic energy in either direction, depending on the barotropic stability properties of the mean flow.

(ii) The eddy conduction acts on the horizontal gradient of mean temperature (vertical shear of mean wind) to convert potential energy in either direction, depending on the baroclinic stability of the mean flow.

(iii) The perturbations affect the mean flow through the condensational heating, since \( (\eta \omega_L) = \bar{\eta} \bar{\omega}_L + \bar{\eta} \bar{\omega}_L'. \) This mechanism is related to "conditional instability of the second kind" (Charney and Eliassen, 1964).

Some consideration is later given to each of these effects in isolation (see Chapter 3). In final numerical integrations with the model, all three are present simultaneously.
2.3 The Perturbation Equations

The quasi-geostrophic system of equations can be written as follows:

Vorticity equation
\[
\frac{\partial \Omega}{\partial t} = - \nabla \cdot \nabla \Omega - \beta v + \frac{\partial \phi}{\partial 
abla \cdot \mathbf{F}} \]
(2.3.1)

Thermodynamic equation
\[
\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{v} \cdot \nabla \phi + \omega \frac{\partial \phi}{\partial x} = \frac{\partial \Omega}{\partial t} \quad \text{Q} \quad (2.3.2)
\]

Hydrostatic equation
\[
\frac{\partial \psi}{\partial t} = - \frac{R}{\rho} \left( \frac{P}{p_{oo}} \right) \nabla \phi \quad \text{Q} \quad (2.3.3)
\]

where
\[
\psi = \frac{\partial \phi}{\partial \theta} = \frac{\phi}{\theta} \quad (2.3.4)
\]

\[
\nabla \phi = \nabla \times \nabla \phi = \left( - \frac{1}{\alpha \theta \frac{\partial \phi}{\partial \theta}} \right) \frac{\partial \phi}{\partial \lambda} \left( \frac{1}{\alpha \theta \frac{\partial \phi}{\partial \theta}} \right) \frac{\partial \phi}{\partial \theta} \quad (2.3.5)
\]

\[
\mathbf{f} = \nabla \psi \quad \text{Q} \quad (2.3.6)
\]

These can be derived from the primitive equations (2.1.1) - (2.1.6) subject to the following conditions (see, for example, Phillips, 1963):

1. \[ \mathcal{R}_0 \equiv \frac{c}{L} \ll 1 \]
\( R_o^2 R_i = R_i \left( \frac{g \frac{\partial^2 \theta}{\partial z^2}}{f_0^2} \right) \sim 1 \)

\( L_i / \alpha \ll 1 \)

\[ R_o = \text{Rossby number}, \quad R_i = \text{Richardson number}, \quad L_y = \text{north-south scale of the motions under consideration}, \quad C = \text{velocity scale}, \quad L = \text{length scale}. \]

The quasi-geostrophic equations have been extensively used in the study of extratropical motions. At lower latitudes, if the velocity scale were to remain the same as in middle latitudes, the Rossby number would become intolerably large due to the decrease of \( f_0 \) and \( L \). The mean zonal winds in the region of tropical disturbances tend to be small, however, and if the amplitude of the disturbances is not too large, the Rossby number is still reasonably small.

For example, at \( \phi = 20^\circ \), for \( C = 10 \text{ m/sec} \) and \( L = 500 \text{ km} \),

\[ R_o = 0.4 \]

(By comparison, in the westerlies at \( \phi = 45^\circ \) and for typical scales of \( C = 30 \text{ m/sec} \) and \( L = 1000 \text{ km} \),

\[ R_o = 0.29 \]

The quasi-geostrophic equations can therefore only be expected to indicate the salient features of the dynamics of tropical disturbances.

Also, at \( \phi = 20^\circ \), with \( L = 500 \text{ km} \), and using the static parameters given by Jordan's (1958) data, it is found that

\[ R_o^2 R_i \sim \frac{g \frac{\partial^2 \theta}{\partial z^2}}{f_0^2} \frac{\Delta \theta}{\theta} = 1.3 \]

The third condition, \( L_i / \alpha \ll 1 \), and the concomitant feature of the equations that \( \frac{1}{f_0} \) and \( \beta \) must be chosen as constants referring
to a particular latitude (unlike the situation with the mean equations), mean that attention must be confined to perturbation motions within a fairly narrow latitude region. Although the model extends over a whole hemisphere, tropical perturbations may be studied in isolation by choosing values of $\lambda$ corresponding to wavelengths which will grow to finite amplitude in the tropics but which are baroclinically stable in middle latitudes; hence, the allowed middle latitude perturbations will decay.

Taking the perturbation form of (2.3.1), expressed at levels 1 and 3, gives

\[
\frac{\partial \delta_j'}{\partial t} = -\left(\frac{u_1 + u_3'}{a} \right) \frac{\partial \delta_j'}{\partial \lambda} + \frac{u'_1}{a} \frac{\partial \delta_j'}{\partial \lambda} - \left(\frac{u_1 + u_3'}{a} \right) \frac{\partial \delta_j'}{\partial \phi} + \frac{u'_1}{a} \frac{\partial \delta_j'}{\partial \phi} \\
- \sigma_i' \left( \frac{1}{a} \frac{\partial \delta_j'}{\partial \phi} + \beta \right) + \frac{\partial \phi}{\partial \phi} \\
+ \frac{i}{a \sigma_0} \frac{\partial}{\partial \lambda} ((F_3)'') - \partial ((F_3)'') \\
\frac{\partial \delta_j'}{\partial t} = -\left(\frac{u_3 + u_3'}{a} \right) \frac{\partial \delta_j'}{\partial \lambda} + \frac{u'_3}{a} \frac{\partial \delta_j'}{\partial \lambda} - \left(\frac{u_3 + u_3'}{a} \right) \frac{\partial \delta_j'}{\partial \phi} + \frac{u'_3}{a} \frac{\partial \delta_j'}{\partial \phi} \\
- \sigma_3' \left( \frac{1}{a} \frac{\partial \delta_j'}{\partial \phi} + \beta \right) - \phi \frac{\partial \phi}{\partial \phi} \\
+ \frac{i}{a \sigma_0} \frac{\partial}{\partial \lambda} ((F_3)''') - \partial ((F_3)''') \tag{2.3.7}\tag{2.3.8}
\]

The perturbation heating function (again anticipating § 2.4) is parameterized as

\[
\left( \frac{\partial}{\partial \phi} \frac{\partial}{\partial \lambda} \phi \right)' = \frac{p_1}{\Delta \rho} \left( \frac{p_{01}}{p_1} \right)^{N_1} \left[ -X (\eta \omega)' - k \left( \delta_1' - \delta_2' \right) \right] \tag{2.3.9}
\]

Then, using the perturbation forms of (2.3.2) and (2.3.3) expressed
at level 2, the perturbation thermodynamic equation is obtained:

\[
\frac{2}{\rho_1} \left( \psi'_1 - \psi'_3 \right) + \left( \frac{\bar{u} + \bar{u}_3}{2} \right) \frac{i}{\alpha \omega_l \rho} \frac{\partial}{\partial \lambda} \left( \psi'_1 - \psi'_3 \right) + \left( \frac{u'_1 + u'_3}{2} \right) \frac{i}{\alpha \omega_l \rho} \frac{\partial}{\partial \lambda} \left( \psi'_1 - \psi'_3 \right)
\]

\[
- \frac{(u'_1 + u'_3)}{2} \frac{i}{\alpha \omega_l \rho} \frac{\partial}{\partial \lambda} \left( \psi'_1 - \psi'_3 \right) + \left( \frac{u'_1 + u'_3}{2} \right) \frac{i}{\epsilon} \frac{\partial}{\partial \theta} \left( \bar{\theta}_1 - \bar{\theta}_3 \right) + \left( \frac{u'_1 + u'_3}{2} \right) \frac{i}{\epsilon} \frac{\partial}{\partial \phi} \left( \psi'_1 - \psi'_3 \right)
\]

\[
- \frac{(u'_1 + u'_3)}{2} \frac{i}{\epsilon} \frac{\partial}{\partial \phi} \left( \psi'_1 - \psi'_3 \right) + \frac{\chi}{\epsilon} \left[(\psi'_0)' - \omega'_1 \right] = - k_r \left[ \psi'_1 - \psi'_3 \right]
\]

(2.3.10)

where

\[
\mathcal{J}' = \nabla^2 \psi
\]

(2.3.11)

\[
u_1' = - \frac{i}{\alpha} \frac{\partial \psi'}{\partial \theta}
\]

(2.3.12)

\[
u_2' = \frac{i}{\alpha \omega_l \rho} \frac{\partial \psi}{\partial \lambda}
\]

(2.3.13)

and similarly for \( \mathcal{J}_2', u_2', v_2' \).

In order that the perturbation equations may have energetics consistent with the mean equations, the terms \( \frac{\bar{u} \partial \mathcal{J}_1'}{\alpha \omega_l \rho} \) and \( \frac{\bar{u}_2 \partial \mathcal{J}_2'}{\alpha \omega_l \rho} \) must be omitted from (2.3.7) and (2.3.8) respectively. This is in accord with the reduction of the mean meridional equations to the forms (2.2.19), (2.2.20), and is not a serious approximation, due to the smallness of \( \bar{u}_1 \) and \( \bar{u}_2 \).

In conformity with the expressions (2.2.25), (2.2.26) and (2.2.27) for the friction, the perturbation friction term at level 1 becomes

\[
\frac{1}{\alpha \omega_l \rho} \frac{\partial}{\partial \lambda} \left( \left(F_\phi,\right)' \right) - \mathcal{D} \left( \left(F_\lambda,\right)' \right) = \frac{1}{\alpha \omega_l \rho} \frac{\partial}{\partial \lambda} \left[ -k_i \left( u'_1 - u'_3 \right) + A \left\{ \frac{1}{\alpha \omega_l \rho} \frac{\partial u'_3}{\partial \lambda} + \frac{1}{\epsilon} \frac{\partial}{\partial \phi} \psi' \right\} \right.
\]

\[
+ \frac{2 \sin \phi}{\alpha \omega_l \rho} \frac{\partial \psi'}{\partial \lambda} \left] \right\} - \mathcal{D} \left[ -k_i \left( u'_1 - u'_3 \right) + A \left\{ \frac{1}{\alpha \omega_l \rho} \frac{\partial u'_3}{\partial \lambda} + \frac{1}{\epsilon} \frac{\partial}{\partial \phi} \psi' \right\} \right]
\]

\[
= - k_i \nabla^2 \left( \psi'_1 - \psi'_3 \right) + A \nabla^2 \left( \psi'_1 \right)
\]

(2.3.14)
For simplicity, the perturbation ground friction is linearized, so that

\[
\begin{align*}
(\tau_y)_y' &= c_0 \rho' |\bar{u}| u_y', \\
(\tau_y)_y &= c_0 \rho' |\bar{u}| v_y'
\end{align*}
\]  \hspace{1cm} (2.3.15)

where $|\bar{u}|$ is a constant, representative of the average value of $|V_3|$. The perturbation friction term at level 3 then becomes

\[
\begin{align*}
\frac{1}{\alpha \cos \Phi} \frac{\partial}{\partial \lambda} ((F_y)_y) - \mathcal{D} ((F_y)_3) \\
= \lambda \nabla^2 (\psi_1 - \psi_1') + A \nabla^2 (\nabla^2 \psi_1') - (\Omega K) \nabla^2 \psi_1'
\end{align*}
\]  \hspace{1cm} (2.3.16)

where $\hat{K}$ is a dimensionless parameter defined by

\[
\hat{K} = \frac{g}{\alpha P} \frac{c_0 \rho' |\bar{u}|}{\omega_2}
\]  \hspace{1cm} (2.3.17)

Equations (2.3.7) and (2.3.8) then become

\[
\begin{align*}
\frac{\partial S_1'}{\partial t} &= - \left( \frac{\bar{u}_i + u_i'}{\alpha \cos \phi} \right) \frac{\partial S_1'}{\partial \lambda} + \frac{\bar{u}_i'}{\alpha \cos \phi} \frac{\partial S_1'}{\partial \phi} - \frac{u_i' \partial S_1'}{\alpha \partial \phi} + \frac{u_i' \partial S_1'}{\alpha \partial \phi} \\
&\quad - u_i' \left( \frac{1}{\alpha} \frac{\partial S_1'}{\partial \phi} + \beta \right) + \frac{E'}{\alpha P} \frac{\partial S_1'}{\partial \phi} \\
&\quad - k_1 (s_1' - s_1') + A \nabla^2 s_1'
\end{align*}
\]  \hspace{1cm} (2.3.18)

\[
\begin{align*}
\frac{\partial S_2'}{\partial t} &= - \left( \frac{\bar{u}_3 + u_3'}{\alpha \cos \phi} \right) \frac{\partial S_2'}{\partial \lambda} + \frac{\bar{u}_3'}{\alpha \cos \phi} \frac{\partial S_2'}{\partial \phi} - \frac{u_3' \partial S_2'}{\alpha \partial \phi} + \frac{u_3' \partial S_2'}{\alpha \partial \phi} \\
&\quad - u_3' \left( \frac{1}{\alpha} \frac{\partial S_2'}{\partial \phi} + \beta \right) - \frac{E'}{\alpha P} \frac{\partial S_2'}{\partial \phi} \\
&\quad + k_1 (s_2' - s_2') + A \nabla^2 (\psi_2') - (\Omega \hat{K}) \psi_2'
\end{align*}
\]  \hspace{1cm} (2.3.19)
In order to be consistent with the quasi-geostrophic formulation, the perturbation values of frictional force and heating must satisfy the following inequalities (Phillips, 1963):

\[
F' \leq R_0 \frac{1}{k} C \quad (2.3.20)
\]

\[
(\omega l)' \leq \frac{C^2 \ell^2}{n/\ell} \quad (2.3.21)
\]

Applied to each component of the friction separately, (2.2.20) demands that

\[
\hat{\eta} \leq (2.5 \omega_0 g_0) R_0 \quad (2.3.22)
\]

\[
\kappa_i \leq (2.5 \omega_0 g_0) R_0 \quad (2.3.23)
\]

\[
A \leq (2.5 \omega_0 g_0) \ell^2 R_0 \quad (2.3.24)
\]

The values of the frictional coefficients used are consistent with these demands.

Applied to each component of the perturbation heating separately, (2.2.21) demands that

\[
\left| (\eta \omega_l)' \right| \leq \frac{C^2 \ell^2}{n/\ell} \quad (2.3.25)
\]

i.e.,

\[
(\eta \omega_l)' \leq \frac{C^2 \ell^2}{n/\ell} \frac{1}{\rho_0 g_0} \approx 0.5 \frac{\omega_0}{\sec}
\]

and

\[
\left| k_r (\psi^* \psi^*) \right| \leq C^2 f_0 \quad (2.3.26)
\]
The results of some numerical experiments show that $(\eta \omega_2)'$ is greater than .5 cm/sec, but is always easily within that order of magnitude.

In §2.5, the perturbation equations will be resolved into a series of spectral components; first, the parameterisation of the heating is discussed.
2.4 Parameterisation of the Heating

The diabatic heating and cooling of the atmosphere is accomplished by absorption and emission of long wave heat radiation, direct absorption of short wave solar radiation, release of latent heat of condensation and pickup of sensible heat from the surface of the oceans and continents.

In this model, only the predominant mechanisms of long wave radiative cooling (which is active at all latitudes) and heating due to the release of latent heat of condensation (which occurs mainly in the Intertropical Convergence Zone) are taken into account.

In parameterising the radiative effect, the earth is regarded as being completely covered by a ocean of fixed temperature which varies with latitude according to the formula

$$T_0(\phi) = T_c(\phi) - T_\infty \sin^{n} \phi, \quad n=2,0,3$$  (2.4.1)

This gives a fair approximation to the observed variation of ocean temperature.

Calculations of the radiative equilibrium temperature profile of the atmosphere by Manabe and Müller (1961), assuming a fixed temperature at the surface and taking into account the long wave radiation by water vapour, carbon dioxide and ozone and the absorption of solar radiation by these three gases, have shown that the equilibrium profile closely parallels, throughout the troposphere, the dry adiabat from the surface.
This would give, in the present case, a radiative equilibrium temperature at level 2 with latitudinal variation described by

\[ T_2^*(\phi) = T_2^*(0) - T_m \sin \phi \]  
(2.4.2)

The greater influence of solar radiation at lower than at higher latitudes, on the average, would lead to a value of \( T_m \) in (2.4.2) greater than that in (2.4.1).

A Newtonian law of cooling relative to the radiative equilibrium temperature is then assumed for the atmosphere, expressed by

\[ (\mathcal{Q}_2)_r = \kappa_r \frac{C_p}{\gamma} [T_2^* - T_2] \]  
(2.4.3)

A rationale for such a mechanism, based on a linearization of the Boltzmann fourth-power law of heat radiation, has been given by Charney (1968), who arrives at a value of the radiative relaxation constant, \( \kappa_r \), given by

\[ \kappa_r = \frac{4 \pi \sigma T_0^4}{C_p \rho g \Delta z_{13}} \]

\[ = \frac{1}{14 \text{ days}} \]

(\( \sigma \) = Boltzmann constant).

Using the hydrostatic equation, (2.4.3) can be written

\[ (\mathcal{Q}_2)_r = \kappa_r \frac{C_p}{\gamma} \frac{R}{4 \rho} \frac{1}{R} \left[ \left( \bar{\theta}_2 - \bar{\theta}_3 \right) - \left( \bar{\theta}_1 - \bar{\theta}_3 \right) \right] \]

\[ = \kappa_r \frac{C_p}{\gamma} \frac{R}{4 \rho} \frac{1}{R} \left[ \left( \bar{\theta}_1 - \bar{\theta}_3 \right) - \left( \bar{\theta}_1 - \bar{\theta}_3 \right) - \int_0^1 (\psi \cdot \psi_1) \right] \]  
(2.4.4)
The heating of the atmosphere due to the release of latent heat of condensation, which, from the global point of view, occurs mainly in the 'hot towers' of the Intertropical Convergence Zone, will be parameterised in terms of the pumping of moisture out of the boundary layer. An elaboration of the Charney and Eliassen (1964) formulation will be used here.

When there is positive pumping of moisture out of the boundary layer, all the latent heat will be assumed to be released and distributed uniformly in the vertical, giving rise to a condensational heating per unit mass

\[ (Q_c)_C = \frac{L \rho_g g \omega_L}{2 \Delta p g} \]  

(2.4.6)

For negative pumping, the condensational heating will be zero.

It is convenient to define a heating coefficient \( \eta \) as follows:

\[ \frac{\Delta p}{\mu} \frac{R}{c_p} (Q_c)_C = -\kappa \eta \omega_L \]  

(2.4.7)

i.e.

\[ \eta = \left( \frac{L q_L}{c_p T_L} \right) \frac{2 (\frac{\Delta q_L}{\Delta T})}{(\frac{\Delta q_L}{\Delta T})_0} \]  

(2.4.8)

Evaporation from the ocean surface in situ, which supplements the moisture advected by the large-scale flow, will be allowed for by increasing the value of \( q_4 \). To remove the discontinuity in \( \eta \) in going from a region of positive to one of negative pumping, a multiplicative factor \( \frac{1}{\mu} \) is employed, defined by
If $\frac{w_0}{\partial w}$ is large and positive

$$f_\nu \to 1$$

while if $\frac{w_0}{\partial w}$ is large and negative

$$f_\nu \to 0$$

The boundary layer specific humidity, $\phi_\nu$, is itself a function of latitude. According to Ekman theory, the depth of the boundary layer varies with latitude as $\sqrt{\sin \phi}$. The air which is injected into the base of the 'hot towers' is then taken from higher levels as the equator is approached. Assuming a linear decrease of $\phi_\nu$ with height (the actual rate of decrease is more nearly exponential), it is appropriate to multiply $\phi_\nu$ by a factor $\sqrt{\sin \phi}$

where $\phi$ corresponds to some fixed latitude away from the equator.

In addition to the latitudinal variation of $\phi$ due to the change of boundary layer height, there are two other factors whose influence causes $\phi$ to decrease with increasing latitude. In the first place, the static stability $\left(\frac{\partial \phi_\nu}{\partial \phi}\right)$ increases with latitude and in the second place, $\phi_\nu$ is diminished due to decreasing sea surface temperature going to higher latitudes. Both of these effects will be allowed for by multiplying $\phi$ by a factor $\frac{1}{\left[1 + e^{-\frac{w_0}{\partial w}}\right]}$ which is approximately 1 for $\phi < \phi_\nu$, and tends to zero for $\phi > \phi_\nu$.

As seen previously, the static stability in the mean and perturbation thermodynamic equations (2.2.16) and (2.3.10) is not allowed to vary with time. The relative effect of the condensational heating term in these equations decreases as the static stability...
increases, however, and it is essential to take this into account in some way. This is done in an indirect manner by multiplying $\eta$ by a factor $(\Delta \theta_{LW})/\Delta \theta_{AV}$.

The final form of $\eta$ is then

$$\eta = \left[ \frac{(L \theta_{LW})/\Delta \theta_{LW}}{(2 \theta_{AV})/\Delta \theta_{AV}} \sin \theta \right] \left[ \frac{(\Delta \theta_{LW})/\Delta \theta_{AV}}{\sin \theta} \right] \left[ \frac{(\Delta \theta_{LW})/\Delta \theta_{AV}}{\sin \theta} \right]$$

(2.4.9)

It remains to arrive at an expression for the boundary layer pumping, $\omega_L$.

Charney (1968)\(^1\), in his zonally symmetric ITCZ study, used

---

1. In the reference quoted, Charney did not give the details of the derivation of the expression (2.4.10). His derivation is as follows:

Assuming that the time scale of adjustment in the boundary layer is small in comparison to that of the flow under consideration, the zonal momentum equation for the boundary layer on a sphere may be approximated by

$$\left( \frac{\rho_s v \cos^2 \theta}{a \cos \phi} \right)_\phi + \left( \frac{\rho_s w \cos^2 \theta}{a \cos \phi} \right)_\theta - \frac{1}{\rho_s} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} \right) = \zeta_L^{(1)}$$  \(1\)

Integration of the above equation and the continuity equation

$$\left( \frac{\rho_s v \cos^2 \theta}{a \cos \phi} \right)_\phi + \left( \frac{\rho_s w \cos^2 \theta}{a \cos \phi} \right)_\theta = 0$$  \(2\)

through the boundary layer then gives

$$\left( \frac{\rho_s v \cos^2 \theta}{a \cos \phi} \right)_\phi + \rho_s w_L u_L - \frac{1}{\rho_s} \frac{B}{\sin \phi} = \zeta_L^{(1)} - \zeta_L^{(2)} = -\zeta_L^{(1)}$$  \(3\)

$$\left( \frac{\rho_s \cos \phi \cos \theta}{a \cos \phi} \right)_\phi + \rho_s w_L = 0$$  \(4\)

where the subscripts "L" denote quantities at the top of the boundary layer, $\hat{u}$ is the weighted average of $u$ in the boundary layer, and
the expression

$$\omega_L = -\frac{1}{\rho_L} \mathcal{D} \left[ \frac{\tilde{\gamma}^{(\lambda)}}{T + T_a} \right]$$

(2.4.10)

where

$$\tilde{\gamma}^{(\lambda)}$$

denotes the weighted average through the depth of the boundary layer.

In this study, where \(\lambda\)-variations are included, a corresponding expression, which reduces to Charney's when all perturbation quantities are zero, is used, viz.,

$$\omega_L = -\frac{1}{\rho_L} \mathcal{D} \left[ \frac{\tilde{\gamma}^{(\lambda)}}{T + T_a} \right] + \frac{1}{\rho_L a \omega} \frac{\partial}{\partial \lambda} \left[ \frac{\tilde{\gamma}^{(\lambda)}}{T + T_a} \right]$$

(2.4.11)

with

$$\tilde{\gamma}^{(\lambda)} = -\mathcal{D} (\tilde{u})$$

$$\tilde{\gamma}^{(\lambda)} = \frac{1}{a \omega} \frac{\partial \tilde{u}}{\partial \lambda}$$

In conjunction with (4), this leads to the expression (2.4.10).

1. Continued

\(B = \int_{\tilde{\gamma}}^{\tilde{\gamma}} d\tilde{\gamma}\) is the meridional mass transport in the boundary layer.

Eliminating \(\rho_L \omega_L\), the following differential equation in \(B\) is obtained:

$$\left(\tilde{u} - \omega_L\right) \frac{\partial \tilde{\gamma}^{(\lambda)}}{\partial \tilde{\gamma}} - \left(\tilde{f} + T_a\right) B = -\tilde{\gamma}^{(\lambda)}$$

(5)

where \(\tilde{\gamma}^{(\lambda)} = -\mathcal{D} (\tilde{u})\).

Assuming the first term is small, this equation reduces to

$$\left(\tilde{f} + T_a\right) B = \tilde{\gamma}^{(\lambda)}$$

(6)

In conjunction with (4), this leads to the expression (2.4.10).
Here, it is assumed that
\[ \tilde{u} = \overline{u}_3 + u_3' \]
\[ \tilde{v} = v_3' \]

[Results of numerical integrations show that \( S_a \) and \( S_c \) are small terms in the denominator of (2.4.11).]

The expressions for \( \gamma_j^{(x)} \) and \( \gamma_j^{(y)} \) are taken as follows:

\[ \gamma_j^{(x)} = c_0 \rho \pi \sqrt{\overline{u}_3 + u_3'} \overline{u}_3 + u_3' \]
\[ \gamma_j^{(y)} = c_0 \rho + \sqrt{\overline{u}_3 + u_3'} \overline{u}_3 + u_3' \]

\( \overline{u}_3 \) is neglected because of its smallness (verified a posteriori) and because retaining it would give rise to problems in solving the equations, as will be seen later (§2.7(e)).
2.5 Spectral Resolution of the Perturbation Equations

The $\Delta \lambda$-variation of each perturbation quantity, $\mathcal{G}$, within the basic period $\Delta \lambda$, will be represented by a truncated Fourier series:

$$\mathcal{G}(\lambda, g, t) = \sum_{j=1}^{J} \mathcal{G}(g, t) F_j(\lambda)$$  \hspace{1cm} (2.5.1)

where

$$F_j(\lambda) = \left\{ \sqrt{\frac{2}{\Delta \lambda}} \cos(j \alpha_j \lambda) \right\}$$  \hspace{1cm} (2.5.2)

$$\alpha_j = \left\{ \frac{\pi j}{\Delta \lambda} \right\}$$  \hspace{1cm} (2.5.3)

Note: The convention will be used that whenever two subscripted quantities appear within curly brackets, the upper will refer to even values of the subscript, the lower to odd values.

It is easily seen that the functions $F_j$ so defined have the following properties:

(a) Zero mean:

$$\bar{F}_j \equiv \frac{1}{\Delta \lambda} \int_{\alpha_j}^{\alpha_j+\Delta \lambda} F_j(\lambda) d\lambda = 0$$

(b) Orthonormality:

i.e. $$\bar{F}_j^* F_i = \delta_{ij}$$

where $\delta_{ij}$ is the Kronecker delta

(c) $$\frac{\partial F_j}{\partial \lambda} = \left\{ -\alpha_j F_{j-1} \right\} = \left\{ -\alpha_j F_{j+i} \right\}$$
\( \frac{\partial^2 F_i}{\partial \lambda^2} = -a_i^2 F_i \)

\( F_{i j k} = \frac{1}{\sqrt{2}} \) if \((i, j, k)\) are all even and any one of them equals the sum or difference of the other two.

\( F_{i j k} = \frac{1}{\sqrt{2}} \) if one of \((i, j, k)\) is even, while the other two are odd, and the even one equals the difference of the other two.

\( F_{i j k} = -\frac{1}{2} \) if one of \((i, j, k)\) is even, while the other two are odd, and the even one equals the sum of the others plus 2.

\( F_{i j k} = 0 \) in all other cases.

The quantities \( \psi' \), \( \psi' \), \( \omega' \) are expanded to give:

\[ \psi' = \sum_{j \neq i} \psi_{ij} (\varphi_j t) F_j(\lambda) \quad (2.5.4) \]

\[ \psi' = \sum_{j \neq i} \psi_{ij} (\varphi_j t) F_j(\lambda) \quad (2.5.5) \]

\[ \omega' = \sum_{j \neq i} \omega_{ij} (\varphi_j t) F_j(\lambda) \quad (2.5.6) \]

From (2.3.12), (2.3.13), it then follows that:

\[ u' = -\sum_{j \neq i} \frac{1}{\alpha} \frac{\partial \varphi_j}{\partial \psi_j} F_i(\lambda) \quad (2.5.7) \]

\[ v_i' = \sum_{j \neq i} \frac{1}{\alpha_{ij}} \left\{ a_{ij} \psi_{ji} - \alpha_{ij} \psi_{ji} \right\} F_j(\lambda) \quad (2.5.8) \]

\[ \nabla^2 \psi' = \sum_{j \neq i} \nabla^2 \psi_{ij} F_j(\lambda) \quad (2.5.9) \]
where
\[ \nabla^2 (\phi)_{ij} = \Delta \left( \frac{1}{\alpha} \frac{\partial^2 \phi}{\partial x^2} \right) - \frac{q^2_{ij}}{\alpha^2 \cos \phi} (\phi)_{ij} \]

Similarly for \( u'_3, u'_4, \nabla^2 v'_3, \).

The non-linear terms in the perturbation equations are also represented by truncated Fourier series:
\[
\frac{u'_i}{\alpha \cos \phi} \frac{\partial^2 (\nabla^2 \psi')}{\partial \phi^2} = \sum_{j=1}^{J} c_{ij} (x,t) F_j(\lambda)
\]
\[
\frac{u'_3}{\alpha \cos \phi} \frac{\partial^2 (\nabla^2 \psi')}{\partial \phi^2} = \sum_{j=1}^{J} b_{ij} (x,t) F_j(\lambda)
\]
\[
\frac{v'_i}{\alpha} \frac{\partial^2 (\nabla^2 \psi')}{\partial \phi^2} = \sum_{j=1}^{J} b_{ij} (x,t) F_j(\lambda)
\]
\[
\frac{(u'_i u'_j)}{2 \alpha \cos \phi} \frac{\partial^2 (\nabla^2 \psi')}{\partial \phi^2} = \sum_{j=1}^{J} d_{ij} (x,t) F_j(\lambda)
\]
\[
\frac{v'_i v'_j}{\alpha} \frac{\partial^2 (\nabla^2 \psi')}{\partial \phi^2} = \sum_{j=1}^{J} e_{ij} (x,t) F_j(\lambda)
\]

The interaction coefficients are found by multiplying across by \( F_j(\lambda) \), taking the mean and using the orthonormality of the functions \( F_j(\lambda) \). Hence
\[
c_{ij} = \frac{1}{2} \sum_{k=1}^{N} \sum_{k'=1}^{N} \left\{ \frac{1}{\alpha} \frac{\partial^2 \phi}{\partial x^2} \right\} (\nabla^2 \psi'_{k}) \frac{F_{k} \frac{\partial F_{k'}}{\partial x}}{\alpha \cos \phi} \frac{F_{j} \frac{\partial F_{j'}}{\partial x}}{\alpha \cos \phi}
\]
\[
b_{ij} = \frac{1}{2} \sum_{k=1}^{N} \sum_{k'=1}^{N} \left\{ \frac{1}{\alpha} \frac{\partial^2 \phi}{\partial x^2} \right\} (\nabla^2 \psi'_{k}) \frac{F_{k} \frac{\partial F_{k'}}{\partial x}}{\alpha \cos \phi} \frac{F_{j} \frac{\partial F_{j'}}{\partial x}}{\alpha \cos \phi}
\]
\[
d_{ij} = \frac{1}{2} \sum_{k=1}^{N} \sum_{k'=1}^{N} \left\{ \frac{1}{\alpha} \frac{\partial^2 \phi}{\partial x^2} \right\} (\nabla^2 \psi'_{k}) \frac{F_{k} \frac{\partial F_{k'}}{\partial x}}{\alpha \cos \phi} \frac{F_{j} \frac{\partial F_{j'}}{\partial x}}{\alpha \cos \phi}
\]
\[
e_{ij} = \frac{1}{2} \sum_{k=1}^{N} \sum_{k'=1}^{N} \left\{ \frac{1}{\alpha} \frac{\partial^2 \phi}{\partial x^2} \right\} (\nabla^2 \psi'_{k}) \frac{F_{k} \frac{\partial F_{k'}}{\partial x}}{\alpha \cos \phi} \frac{F_{j} \frac{\partial F_{j'}}{\partial x}}{\alpha \cos \phi}
\]
With this representation, the averaged non-linear terms of
\[ \frac{u^i}{\alpha_{\text{avg}}} \frac{2}{\delta \lambda} \left( \nabla^2 \psi \right) \]
eq etc., all disappear from the perturbation equations, as can be seen by taking the means of (2.5.10) and using property (a) of the functions \( F_j \).

The perturbation frictional terms are likewise resolved:

**Lateral friction**
\[ A \nabla^2 \left( \nabla^2 \psi \right) = A \sum_{j=1}^{3} \nabla^2 \left( \nabla^2 \psi \right) F_j(\lambda) \]
\[ A \nabla^2 \left( \nabla^2 \psi \right) = A \sum_{j=1}^{3} \nabla^2 \left( \nabla^2 \psi \right) F_j(\lambda) \]

**Internal vertical friction**
\[ \lambda \nabla^2 \left( \psi_i - \psi_j \right) = \lambda \sum_{j=1}^{3} \nabla^2 \left( \psi_i - \psi_j \right) F_j(\lambda) \]

**Ground friction**
\[ \omega \nabla^2 \psi \]

The perturbation condensational heating term is represented as
\[ \left( \eta \omega \right)' = \sum_{j=1}^{3} \xi_j(\varphi, t) F_j(\lambda) \]

where
\[ \xi_j(\varphi, t) = \frac{(\eta \omega)'}{F_j(\lambda)} \]

Since \( \eta \omega \) is a highly non-linear function of the perturbation variables, the coefficients \( \xi_j \) are evaluated numerically.

The perturbation vorticity and thermodynamic equations, (2.3.18), (2.3.19) and (2.3.10),
can now be separated into a set of equations for the spectral coefficients, $\psi_{ij}, \psi_{ij}$, by equating the coefficients of $\psi_j(\lambda)$.

Thus

\begin{equation}
\frac{2}{a} \frac{\partial}{\partial t} \nabla^2 \psi_{ij} = - \frac{\bar{u}_i}{\alpha \cos \phi} \left\{ a_{ij} \psi_{ij} \right\} - c_{ij} - b_{ij}
- \frac{(\frac{1}{a} \frac{\partial^2}{\partial x^2} + \beta)}{\alpha \cos \phi} \left\{ a_{ij} \psi_{ij} \right\} + \frac{L}{\Delta \phi} \Delta \psi_{ij}
- \frac{k}{a} \nabla^2 (\psi_{ij} - \psi_{ij}) + A \nabla^2 (\nabla^2 \psi_{ij})
\tag{2.5.15}
\end{equation}

\begin{equation}
\frac{2}{a} \frac{\partial}{\partial t} \nabla^2 \psi_{ij} = - \frac{\bar{u}_j}{\alpha \cos \phi} \left\{ a_{ij} \psi_{ij} \right\} - c_{ij} - b_{ij}
- \frac{(\frac{1}{a} \frac{\partial^2}{\partial y^2} + \beta)}{\alpha \cos \phi} \left\{ a_{ij} \psi_{ij} \right\} - \frac{L}{\Delta \phi} \Delta \psi_{ij}
+ \frac{k}{a} \nabla^2 (\psi_{ij} - \psi_{ij}) + A \nabla^2 (\nabla^2 \psi_{ij}) - \nabla^2 \nabla^2 \psi_{ij}
\tag{2.5.16}
\end{equation}

\begin{equation}
\frac{2}{a} \frac{\partial}{\partial t} (\psi_{ij} - \psi_{ij}) = - \left( \frac{\bar{u}_i + \bar{u}_j}{2} \right) \frac{1}{\alpha \cos \phi} \left\{ a_{ij} \psi_{ij} - a_{ij} \psi_{ij} \right\}
- \frac{1}{\alpha \cos \phi} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \frac{1}{\alpha \cos \phi} \left\{ a_{ij} \psi_{ij} - a_{ij} \psi_{ij} \right\}
- \frac{\partial^2}{\partial x^2} \left[ e_{ij} - d_{ij} \right] - \frac{\partial^2}{\partial y^2} \left[ e_{ij} - d_{ij} \right]
\tag{2.5.17}
\end{equation}

Taking the sum and difference of (2.5.15), (2.5.16) and using (2.5.17) to eliminate $\Delta \psi_{ij}$, gives the two prognostic equations for $(\psi_{ij} + \psi_{ij}), (\psi_{ij} - \psi_{ij})$:

\begin{equation}
\frac{2}{a} \frac{\partial}{\partial t} \nabla^2 (\psi_{ij} + \psi_{ij}) = - \frac{\bar{u}_i}{\alpha \cos \phi} \left\{ a_{ij} \psi_{ij} \right\} - \left( \frac{1}{a} \frac{\partial^2}{\partial x^2} + \beta \right) \frac{1}{\alpha \cos \phi} \left\{ a_{ij} \psi_{ij} \right\}
- c_{ij} - b_{ij} + A \nabla^2 (\nabla^2 \psi_{ij})
- \frac{\bar{u}_j}{\alpha \cos \phi} \left\{ a_{ij} \psi_{ij} \right\} - \left( \frac{1}{a} \frac{\partial^2}{\partial y^2} + \beta \right) \frac{1}{\alpha \cos \phi} \left\{ a_{ij} \psi_{ij} \right\}
- c_{ij} - b_{ij} + A \nabla^2 (\nabla^2 \psi_{ij}) - \nabla^2 \nabla^2 \psi_{ij}
\tag{2.5.18}
\end{equation}
Examining equations (2.5.15) - (2.5.17), it can be seen that

1. The lateral friction, ground friction and radiative terms involve no interaction between different spectral components. The vertical internal friction involves interaction between corresponding components at different levels.

2. The terms involving the mean motion and $\beta$ give interaction between sine and cosine modes of the same wavelength, i.e., they cause progression of a wave component.

3. The non-linear terms $c_{ij}, c_{ij}, b_{ij}, d_{ij}, e_{ij}$ give selective
interaction between modes of different wavelength, depending on whether \( \frac{F_i F_j}{\alpha} \frac{\partial F_i}{\partial \alpha} \) is non-vanishing. In order that there be any non-linear interaction, \( J \) must be greater than 2.

(4) The highly non-linear term \( \xi_j \) gives interaction between all components, so that no matter what initial conditions are chosen, every component is excited.

Energy Equations

In order to obtain perturbation energy equations, \( \nabla \psi^2 \left( \frac{x}{\alpha}, \nabla \psi \right) F_i \) is multiplied by \( \psi' \left( \frac{x}{\alpha}, \psi F_i \right) \), \( \nabla \psi^2 \left( \frac{x}{\alpha}, \nabla \psi \right) F_i \) is multiplied by \( \psi_j' \left( \frac{x}{\alpha}, \psi_j F_i \right) \), and the products are added and integrated \( \left( \int_{\alpha=0}^{\alpha} \right) a^2 \cos \phi \, d\phi \). Making use of equations (2.5.15) and (2.5.16), together with the orthogonality property of the \( F_j \)'s and the boundary conditions

\[
\psi_{ij} = \psi_{ij} = 0, \quad \phi = 0, \quad 0 \leq j \leq J
\]  

(2.5.20)

it is found that

\[
\frac{d}{dt} \int_{\alpha=0}^{\alpha} \int_{\phi=0}^{\phi} \frac{1}{2} \left\{ \left( \frac{\partial \psi_{ij}^2}{\partial \phi} \right)^2 + \left( \frac{\partial \psi_{ij}^2}{\partial \alpha} \right)^2 + \frac{\partial^2}{\partial \alpha^2} \left( \psi_{ij}^2 + \psi_{ij}^2 \right) \right\} a^2 \cos \phi \, d\phi 
= - \int_{\alpha=0}^{\alpha} \int_{\phi=0}^{\phi} \left[ \left( \frac{\partial \psi_{ij}^2}{\partial \phi} \right)^2 + \left( \frac{\partial \psi_{ij}^2}{\partial \alpha} \right)^2 + \frac{\partial^2}{\partial \alpha^2} \left( \psi_{ij}^2 + \psi_{ij}^2 \right) \right] a^2 \cos \phi \, d\phi 
- \int_{\alpha=0}^{\alpha} \int_{\phi=0}^{\phi} \left[ \left( \frac{\partial \psi_{ij}^2}{\partial \phi} \right)^2 + \left( \frac{\partial \psi_{ij}^2}{\partial \alpha} \right)^2 + \frac{\partial^2}{\partial \alpha^2} \left( \psi_{ij}^2 + \psi_{ij}^2 \right) \right] a^2 \cos \phi \, d\phi 
+ A \left\{ \left( \frac{\partial \psi_{ij}^2}{\partial \phi} \right)^2 + \left( \frac{\partial \psi_{ij}^2}{\partial \alpha} \right)^2 + \frac{\partial^2}{\partial \alpha^2} \left( \psi_{ij}^2 + \psi_{ij}^2 \right) \right\} a^2 \cos \phi \, d\phi
\]  

(2.5.21)
It is to be noted that the contributions from the non-linear terms $\xi_{ij}, \gamma_{ij}, \delta_{ij}, \beta_{ij}$, and from the $\left( \frac{1}{\alpha} \frac{\partial}{\partial t} + \beta \right)$ terms have integrated to zero.

Similarly, on multiplying (2.5.17) by $(\psi_j - \psi_{ij})$ and integrating, it is found that

\[
\frac{2}{\beta} \int_{q=0}^{\pi} \sum_{j=1}^{P} \left( \frac{\partial^{2}}{\partial P^{2}} \right) \left( \frac{\psi_i - \psi_{ij}}{2} \right) \cos \theta dq
\]

\[
= -\int_{q=0}^{\pi} \frac{1}{\alpha} \theta \left[ (\phi_i - \phi_j) \left( \frac{\psi_i - \psi_{ij}}{2} \right) \right] \cos \theta dq
\]

\[
+ \int_{q=0}^{\pi} \sum_{j=1}^{P} \left[ \frac{\partial^{2}}{\partial P^{2}} \left( \psi_i - \psi_{ij} \right) - \frac{1}{\alpha} \left( \psi_i - \psi_{ij} \right) \right] \cos \theta dq
\]

\[
(2.5.22)
\]

Again, the contributions from the non-linear terms have integrated to zero.

These perturbation energy equations are consistent with the mean energy equations (2.2.30), (2.2.31).
2.6 The Governing Equations in Dimensionless Form

For convenience in solving the equations and in order to see what dimensionless parameters are important for the motion, the mean and perturbation equations are non-dimensionalized. In addition, a transformation of the latitudinal co-ordinate is effected by defining

\[ y = \sin \theta \]  \hspace{1cm} (2.6.1)

A dimensionless time variable, \( t' \), and dimensionless dependent variables \( U, W, V, T, \hat{\psi}_j, \hat{\psi}_j, \hat{\xi}_j \) are defined as follows:

\[
\begin{align*}
\frac{t'}{t} &= \frac{1}{\Delta} t' \\
\left( \frac{\bar{u}_1 + \bar{u}_0}{\Delta} \right) &= aL \sqrt{1 - \beta^2} U \\
\left( \frac{\bar{u}_1 - \bar{u}_0}{\Delta} \right) &= aL \sqrt{1 - \beta^2} W \\
\frac{\nu_j}{\Delta} &= aL \sqrt{1 - \beta^2} V \\
(\tilde{\phi}_j - \tilde{\phi}_0) &= (\tilde{\phi}_j - \tilde{\phi}_0)^* + 4 + \beta^2 \Delta^2 T \\
\psi_j &= \Delta \hat{\psi}_j \\
\psi_j &= \Delta \hat{\psi}_j \\
\xi_j &= \Delta \hat{\xi}_j
\end{align*}
\]  \hspace{1cm} (2.6.2)

Additional dimensionless quantities, which are convenient to employ, are:

\[
T^* = \frac{1}{4L^2 \Delta^2} \left[ (\tilde{\phi}_j)^* - (\tilde{\phi}_0)^* \right]
\]

\[
T_0 = \frac{1}{4L^2 \Delta^2} \left[ (\tilde{\phi}_1 - \tilde{\phi}_0)_0 - (\tilde{\phi}_1 - \tilde{\phi}_0)^* \right]
\]
\[ R_p = \frac{1}{a^2 \rho^2 \sqrt{1 - \eta^2}^2} \left( \frac{u'_i u'_i + u'_j v'_j}{2} \right) \]
\[ R_m = \frac{1}{a^2 \rho^2 \sqrt{1 - \eta^2}^2} \left( \frac{u'_i u'_i - u'_j v'_j}{2} \right) \]
\[ E_p = \frac{1}{4 a^2 \rho^2 \sqrt{1 - \eta^2}^2} \left( \Phi'_i - \Phi'_j \right) \left( \frac{u'_i + v'_j}{2} \right) \]
\[ \dot{\Omega} = -\frac{\chi}{4 a^3 \rho^2} (\eta \Omega_L) \]
\[ \Omega_L = \frac{\chi}{4 a^3 \rho^2} \Omega_L \]
\[ \dot{C}_{ij} = \frac{i}{\rho} C_{ij} \]
\[ \dot{C}_{ij} = \frac{i}{\rho} C_{ij} \]
\[ \dot{b}_{ij} = \frac{i}{\rho} b_{ij} \]
\[ \dot{b}_{ij} = \frac{i}{\rho} b_{ij} \]
\[ \dot{\alpha}_j = \frac{i}{\alpha \rho^2} \alpha_j \]
\[ \dot{\xi}_j = \frac{i}{\alpha \rho^2} \xi_j \]
\[ \dot{\xi}_j = \frac{2 A e_2}{\alpha \rho^2} \xi_j \]
\[ K' = 2 k_i / \mu \]
\[ K'' = k_r / \mu \]
\[ \tilde{K} = \frac{\tilde{A}}{\tilde{a}^2} \]
\[ \tilde{K} = \frac{\tilde{A}}{\tilde{a}^2} \]
\[ c' = \frac{3}{2 \rho} \frac{e_{\rho \rho} / \alpha}{a} \]
\[ \mu^2 = 4 \pi a^2 / 24 \rho \]
\[ (\tilde{W}_3 = \text{typical value of mean wind at level 3}) \]
In all that follows, the prime in $t'$ is dropped.

**Mean Flow Equations**

The sum and difference of (2.2.17), (2.2.18) give, on non-dimensionalization

$$u_t = -V \left[ (\nu+\nu')w \right]_y - W \left[ (\nu+\nu')V \right]_y + \kappa \left[ (\nu+\nu')w \right]_{yy}$$

$$- c' \sqrt{1+\nu} |u-w| (u-w)$$

$$- \left[ (\nu+\nu') \rho_p \right]_y \sqrt{1+\nu} \quad (2.6.4)$$

$$w_t = -V \left[ (\nu+\nu')w \right]_y + 2 \nu V - \kappa w + \kappa \left[ (\nu+\nu')w \right]_{yy}$$

$$+ c' \sqrt{1+\nu} |u-w| (u-w) - \left[ (\nu+\nu') \rho_m \right]_y \sqrt{1+\nu} \quad (2.6.5)$$

The dimensionless thermal wind equation, derived from (2.2.19) minus (2.2.20), becomes

$$\bar{T}_y = -y w (1+u) \quad (2.6.6)$$

The thermodynamic equation (2.2.16) becomes

$$\bar{T}_e = - \left[ (\nu+\nu') \rho \right]_y \sqrt{\kappa} + \kappa' \left[ \bar{r} - \frac{ar{T}}{\kappa} \right]$$

$$- \left[ (\nu+\nu') \rho \right]_y \hat{Q} \quad (2.6.7)$$
Perturbation Equations

The non-dimensional forms of (2.5.18), (2.5.19) and (2.5.17) can be written

\[
\hat{v}_j \frac{\partial}{\partial \hat{t}} (\hat{v}_j + \hat{w}_j) = \hat{Z}_1 + \hat{Z}_2 \quad \quad \quad \quad (2.6.9)
\]

\[
(\hat{v}_j - 2 \mu z \hat{z}_j) \frac{\partial}{\partial \hat{t}} (\hat{v}_j + \hat{w}_j) = \hat{Z}_1 - \hat{Z}_2 + \hat{Z}_3 \quad \quad \quad \quad (2.6.10)
\]

\[
\hat{S}_{ij} = \frac{1}{4 \mu_0} \left[ 2 \mu_0 z_0 \frac{\partial}{\partial \hat{t}} (\hat{v}_j - \hat{w}_j) + \hat{Z}_3 + \kappa' \hat{v}_j (\hat{v}_j - \hat{w}_j) \right] \quad \quad \quad \quad (2.6.11)
\]

where

\[
\hat{Z}_1 = -(u+w) \left\{ \begin{array}{l} a_{j-1} \hat{v}_j + \hat{w}_j \\ -a_{j-1} \hat{v}_j + \hat{w}_j \end{array} \right\} + \left[ \frac{1}{2} \left\{ (u+w) - \frac{2 \sqrt{1-y_c^2}}{\sqrt{1-y_c^2}} \right\} \right] \left\{ \begin{array}{l} a_{j-1} \hat{w}_j \\ -a_{j-1} \hat{w}_j \end{array} \right\}
\]

\[
\hat{Z}_2 = -(u-w) \left\{ \begin{array}{l} a_{j-1} \hat{v}_j + \hat{w}_j \\ -a_{j-1} \hat{v}_j + \hat{w}_j \end{array} \right\} + \left[ \frac{1}{2} \left\{ (u-w) - \frac{2 \sqrt{1-y_c^2}}{\sqrt{1-y_c^2}} \right\} \right] \left\{ \begin{array}{l} a_{j-1} \hat{w}_j \\ -a_{j-1} \hat{w}_j \end{array} \right\}
\]

\[
\hat{Z}_3 = \hat{S}_{ij} - \kappa' \hat{v}_j (\hat{v}_j - \hat{w}_j)
\]

\[
+ 2 \mu z_0 \hat{z}_j \left\{ \hat{S}_{ij} + \hat{S}_{ij} + \mu \left\{ \begin{array}{l} a_{j-1} (\hat{w}_{j-1} - \hat{w}_{j+1}) \\ -a_{j+1} (\hat{w}_{j+1} - \hat{w}_{j-1}) \end{array} \right\} - \frac{y_c}{y_c} \left\{ \begin{array}{l} a_{j-1} (\hat{w}_{j-1} + \hat{w}_{j+1}) \\ -a_{j+1} (\hat{w}_{j+1} + \hat{w}_{j-1}) \end{array} \right\} + \kappa'' (\hat{v}_j - \hat{w}_j) \right\}
\]
and
\[ \hat{\nabla}^2 (\nu) = \frac{1}{\nu^2} \left[ (1-y^2) \frac{\partial^2 \psi}{\partial y^2} \right] - \frac{a_j^2}{(1-y^2)} (\psi) \]

\( y_L \) = value of \( y \) at which \( \frac{\partial}{\partial y} \) and \( \beta \) are expressed.

The dimensionless forms of the interaction coefficients, derived from (2.5.11) are

\[
\begin{align*}
\hat{c}_{ij} &= - \sum_{k=1}^{n} \sum_{K=1}^{n} \left( \frac{\partial \psi}{\partial y} \right)(\hat{\nabla}^2 \psi_{jk}) \left\{ - \frac{a_k}{\nu} \frac{F_i F_j F_{k-1}}{a_n} \right\} \\
\hat{c}_{ij} &= - \sum_{k=1}^{n} \sum_{K=1}^{n} \left( \frac{\partial \psi}{\partial y} \right)(\hat{\nabla}^2 \psi_{jk}) \left\{ - \frac{a_k}{\nu} \frac{F_i F_j F_{k-1}}{a_n} \right\} \\
\hat{b}_{ij} &= \sum_{k=1}^{n} \sum_{K=1}^{n} \left( \frac{\partial \psi}{\partial y} \right)(\hat{\nabla}^2 \psi_{jk}) \left\{ - \frac{a_k}{\nu} \frac{F_i F_j F_{k-1}}{a_n} \right\} \\
\hat{b}_{ij} &= - \sum_{k=1}^{n} \sum_{K=1}^{n} \left( \frac{\partial \psi}{\partial y} \right)(\hat{\nabla}^2 \psi_{jk}) \left\{ - \frac{a_k}{\nu} \frac{F_i F_j F_{k-1}}{a_n} \right\} \\
\hat{d}_{ij} &= \sum_{k=1}^{n} \sum_{K=1}^{n} \left( \frac{\partial \psi}{\partial y} \right)(\hat{\nabla}^2 \psi_{jk}) \left\{ - \frac{a_k}{\nu} \frac{F_i F_j F_{k-1}}{a_n} \right\} \\
\hat{e}_{ij} &= \sum_{k=1}^{n} \sum_{K=1}^{n} \left( \frac{\partial \psi}{\partial y} \right)(\hat{\nabla}^2 \psi_{jk}) \left\{ - \frac{a_k}{\nu} \frac{F_i F_j F_{k-1}}{a_n} \right\}
\end{align*}
\]

Condensational Heating Term

The non-dimensionalized boundary layer pumping is given by

\[
\hat{\omega}_L \equiv \frac{X}{\nu^2 \alpha^2}, \omega_L \equiv \frac{X}{\nu^2 \alpha^2} \left[ \mathcal{D} \left( \frac{g(r_j) L}{X + \frac{3}{2} \alpha} \right) - \frac{1}{\alpha \cos \theta} \frac{1}{\nu} \left( \frac{g(r_j) L}{X + \frac{3}{2} \alpha} \right) \right] \]

\[
= \frac{X}{\nu^2 \alpha^2} \left[ \frac{1}{\nu^2} \left\{ \frac{X}{\nu^2} \left[ \frac{(1-X)^3}{X + \frac{3}{2} \alpha} \right] \right\} ^2 - \alpha \left\{ \frac{X}{\nu^2} \left[ \frac{(1-X)^3}{X + \frac{3}{2} \alpha} \right] \right\} ^2 \right] \]
where
\[
\hat{c} = \sqrt{(u - w' + \frac{u_y'}{\alpha n \sqrt{1 - \gamma}})^2 + \left(\frac{u_y'}{\alpha n \sqrt{1 - \gamma}}\right)^2}
\]
\[
\frac{\delta_y}{\alpha} = -\frac{\partial}{\partial y} \left[ (u - w' + \frac{u_y'}{\alpha n \sqrt{1 - \gamma}})(1 - y^2) \right]
\]
\[
\frac{\delta_i}{\alpha} = \frac{1}{\alpha n \sqrt{1 - \gamma}} \frac{\partial u_i'}{\partial \lambda}
\]
(2.6.14)

\[
\frac{u_y'}{\alpha} = -\frac{1}{\sum_{j=1}^{J} \sqrt{1 - \eta^2}} \sum_{j=1}^{J} \frac{\partial \eta_j}{\partial y} \left\{ \frac{\partial \eta_j}{\partial \lambda} \right\} F_i(\lambda)
\]
\[
\frac{u_y'}{\alpha} = \frac{1}{\sum_{j=1}^{J} \sqrt{1 - \eta^2}} \left\{ -\frac{\partial \eta_j}{\partial \lambda} \right\} \left\{ \frac{\partial \eta_j}{\partial \lambda} \right\} F_i(\lambda)
\]

In order to non-dimensionalize the condensational heating factor, \( \eta \), the static stability factor \( \frac{(\Delta \Theta_{ij})_s}{\Delta \Theta_{ij}} \) is expressed as follows:

\[
\frac{(\Delta \Theta_{ij})_s}{\Delta \Theta_{ij}} = \frac{\Theta_s - \Theta_{ij}}{\Theta_s - \Theta_4}
\]
\[
= \frac{1}{1 + \frac{(\Theta_s - \Theta_5) + (\Theta_{ij} - \Theta_4)}{\Theta_{ij} - \Theta_4}}
\]
\[
= \frac{1}{1 + \left( \frac{\Theta_s - \Theta_{ij}}{\Delta \Theta_{ij}} \right) \frac{1}{\Delta \Theta_{ij}} \left[ \frac{1}{\Delta \Theta_{ij}} \left[ \frac{1}{\Delta \Theta_{ij}} \left[ T - T_0 - T' \right] \right] \right]}
\]
\[
= \frac{1}{1 + \frac{\partial^2 (\Delta \Theta_{ij})}{\partial \Theta_{ij}} \left[ T - T_0 - T' \right]}
\]
(2.6.15)

Here
\[
T = T + T'
\]
\[
= \bar{T} + \frac{\frac{1}{\alpha} (\bar{\Theta} - \Theta_4)}{\Delta \Theta_{ij}}
\]
\[
= \bar{T} + \frac{u_y'}{\sum_{j=1}^{J} \left( \eta_j - \eta_4 \right) F_i(\lambda)}
\]
Hence

\[ \eta = \frac{\eta_0 \sqrt{y}}{\left[ 1 + \epsilon \left( \frac{y}{y_0} \right) \right] \left[ 1 + \left( \frac{\Delta \theta_3}{\Delta \theta_4} \right)_5 \mu (T - T_0 - T^*) \right]} \]  

(2.6.16)

where

\[ \mu = \frac{1}{1 + \epsilon \frac{\partial \lambda}{\partial \alpha}} \]

and

\[ \eta_0 = \left[ \left( \frac{L_{c}}{\frac{L_{c}}{2}} \right) / \left( 2 \frac{\Delta \theta_1}{\Delta \theta_4} \right)_5 \right] \frac{1}{\sqrt{y}} \]  

(2.6.17)

The mean and perturbation condensational heating terms are then

\[ \hat{\Omega} = -\left( \eta \hat{\omega}_c \right) \]  

(2.6.18)

\[ \hat{\xi}_j = \frac{2 \eta_0}{\Omega^2 \Delta \rho} \xi_j \]  

(2.6.19)

\[ = 4 \mu^2 y_c (\eta \hat{\omega}_c) F_i(\lambda) \]  

(2.6.10)

Energy Equations

In dimensionless form, the mean and perturbation energy equations (2.2.30), (2.2.31), (2.5.21) and (2.5.22) can be written

\[ \frac{\partial \tilde{E}_k}{\partial t} = \left\{ \tilde{E}_p \cdot \tilde{E}_k \right\} - \left\{ \tilde{E}_k \cdot \tilde{E}_k \right\} - \left\{ \tilde{E}_k \cdot k_i \right\} - \left\{ \tilde{E}_k \cdot \tilde{q}_j \right\} - \left\{ \tilde{E}_k \cdot A \right\} \]

\[ \frac{\partial \tilde{E}_p}{\partial t} = -\left\{ \tilde{E}_p \cdot \tilde{E}_p \right\} - \left\{ \tilde{E}_p \cdot \tilde{E}_p' \right\} + \left\{ \tilde{Q}_c \cdot \tilde{E}_p \right\} + \left\{ \tilde{Q}_r \cdot \tilde{E}_p \right\} \]  

(2.6.21)
\[
\frac{\partial \bar{E}_k'}{\partial t} = \left\{ \bar{E}_k' \cdot \bar{E}_k' \right\}_t \left\{ \bar{E}_p' \cdot \bar{E}_u' \right\} - \left\{ \bar{E}_u' \cdot \bar{L}_i \right\} - \left\{ \bar{E}_k' \cdot \bar{Q}_i' \right\} - \left\{ \bar{E}_u' \cdot \bar{A} \right\} \\
\frac{\partial \bar{E}_p'}{\partial t} = \left\{ \bar{E}_p' \cdot \bar{E}_p' \right\} - \left\{ \bar{E}_p' \cdot \bar{E}_u' \right\} + \left\{ \bar{Q}_i' \cdot \bar{E}_p' \right\} + \left\{ \bar{Q}_i' \cdot \bar{E}_p' \right\}
\]

(2.6.21, cont.)

where \( \bar{E}_k, \bar{E}_k', \bar{E}_p, \bar{E}_p' \) are, respectively, the mean and perturbation kinetic and potential energies, given by

\[
\bar{E}_k = \int_0^1 (1-y') (u'^2 + w'^2) \, dy' \]
\[
\bar{E}_p = \int_0^1 \frac{\bar{T}}{2} y'^4 \left[ \bar{T} + \frac{(\bar{\bar{\theta}}_c - \bar{\bar{\theta}}_c)^2}{\lambda} \right] \, dy'
\]
\[
\bar{E}_k' = \int_0^1 \sum_{j=1}^4 \left\{ \left( \frac{(y'^2)_{i,j}}{2} \right)^2 \right\} \, dy'
\]
\[
\bar{E}_p' = \int_0^1 \sum_{j=1}^4 \left[ \left( \frac{\hat{\psi}_j - \hat{\psi}_j}{\lambda} \right)^2 \right] \, dy'
\]

The transformation terms, in dimensionless form, become

\[
\left\{ \bar{E}_p' \cdot \bar{E}_k' \right\} = \int_0^1 4 \bar{T} \frac{\partial}{\partial y} \left[ (1-y') v' \right] \, dy'
\]
\[
\left\{ \bar{E}_p' \cdot \bar{E}_p' \right\} = \int_0^1 4 \bar{\bar{T}}^2 \frac{\partial}{\partial y} \left[ (1-y') \bar{E}_p \right] \, dy'
\]
\[
\left\{ \bar{E}_p' \cdot \bar{E}_u' \right\} = - \int_0^1 \sum_{j=1}^4 2 \bar{y}_c \bar{\theta}_j \left( \hat{\psi}_j - \hat{\psi}_j \right) \, dy'
\]
\[
\left\{ \bar{E}_p' \cdot \bar{A}_i \right\} = - \int_0^1 2 \left( 1-y' \right)^3 \left[ R_p \frac{\partial U}{\partial y} + R_w \frac{\partial W}{\partial y} \right] \, dy'
\]

(2.6.23)
\[
\begin{align*}
\{ E_k \cdot h_i \} &= - \int_0^1 2 \kappa' (1-y^2) W^2 \, dy \\
\{ E_k \cdot \overrightarrow{g} \} &= - \int_0^1 2 c' (1-y^2) |U-W| (U-W)^2 \, dy \\
\{ E_k \cdot A \} &= - \int_0^1 \mathbf{R} \left\{ (1-y^2) \left[ \frac{2}{s_j} (\sqrt{1+y} (U+W)) \right]^2 \\
&\quad + (1-y) \left[ \frac{2}{s_j} (\sqrt{1-y} (U-W)) \right]^2 \\
&\quad + 2 (U^2 + W^2) \right\} \, dy \\
\{ E_k \cdot h_i \} &= - \int_0^1 \sum_{j=1}^J K' \left\{ (1-y^2) \left[ \frac{2}{s_j} (\psi_i - \psi_j) \right]^2 + \frac{a_j^2}{(1-y)} (\psi_i - \psi_j)^2 \right\} \, dy \\
\{ E_k \cdot \gamma_i' \} &= - \int_0^1 \sum_{j=1}^J \left\{ (1-y) \left( \frac{1}{s_j} \right)^2 + \frac{a_j^2}{(1-y)} (\gamma_j)^2 \right\} \, dy \\
\{ E_k \cdot A \} &= - \int_0^1 \sum_{j=1}^J \mathbf{R} \left\{ \left[ \frac{2}{s_j} (1-y^2) \psi_i \right]^2 + \left[ \frac{2}{s_j} (1-y) \psi_i \right]^2 \right\} \\
&\quad + \frac{a_j^2}{(1-y)} (a_j - 2(1-y)) (\psi_i \hat{a} + \psi_i \hat{a}^2) \\
&\quad + 2 a_j^2 \left( \left( \frac{1}{s_j} \right)^2 + \left( \frac{1}{s_j} \right)^2 \right) \right\} \, dy \\
\{ \overline{Q}_c \cdot \overline{E}_p \} &= \int_0^1 4 \mu^a Q \left[ \mathbf{T} + \frac{(\hat{\phi}_f - \hat{\phi}_1)}{4 \mu a^2} \right] \, dy \\
\{ \overline{Q}_r \cdot \overline{E}_p \} &= \int_0^1 4 \mu^a \kappa' \left( \mathbf{T} - \mathbf{T}(\frac{(\hat{\phi}_f - \hat{\phi}_1)}{4 \mu a^2}) \right) \, dy \\
\{ \overline{Q}_c \cdot \overline{E}_p \} &= - \int_0^1 \sum_{j=1}^J \hat{\varepsilon}_j (\psi_i - \psi_j) \, dy \\
\{ \overline{Q}_r \cdot \overline{E}_p \} &= - \int_0^1 \sum_{j=1}^J \kappa' \mu^a y_i^2 (\psi_i - \psi_j)^2 \, dy
\end{align*}
\]
The notation used here corresponds to that of Phillips (1956), bracketed terms signifying a transformation of energy from the first quantity to the second.

In the combined energy equation, obtained by adding all the components of (2.6.21), the conversion terms (2.6.23) cancel.

The terms (2.6.24) represent the frictional dissipation of kinetic energy, while the terms (2.6.25) represent the generation of potential energy due to the diabatic heating and cooling.
2.7 Algorithm for Solving the Equations

In order to study the evolution of the mean and perturbation flow, the differential equations (2.6.4) - (2.6.7) and (2.6.8) - (2.6.10) are replaced by finite-difference equivalents which can be integrated using an electronic computer.

The integration scheme will now be described.

Note: In all that follows, the superscript n will refer to the time step, while the subscripts i and j will refer respectively to the grid-point in y and the order of the spectral component.

(a) Calculation of $\bar{T}^{n+1}$

Using $v^n$, $\bar{T}^n$, $R_p^n$ and $Q^n$, $(\frac{\partial \bar{T}}{\partial t})^n$ is found from (2.6.7).

Centered space differences are used at all interior points, forward space differences at the equator and backward space differences at the pole. $\bar{T}^{n+1}$ is then obtained by forward time stepping:

\[ \bar{T}^{n+1} = \bar{T}^n + (\frac{\partial \bar{T}}{\partial t})^n \Delta t \]

(b) Calculation of $U^{n+1}$

Using $V^n$, $W^n$, $U^n$ and $R_p^n$, $(\frac{\partial U}{\partial t})^n$ is found from (2.6.4). Again, centered space differences are used at all interior points and forward space differences at the equator. $U$ is set permanently equal to zero at the pole. $U^{n+1}$ is then obtained by forward time-stepping

\[ U^{n+1} = U^n + (\frac{\partial U}{\partial t})^n \Delta t \]
(c) Calculation of $W^{n+1}$

For consistence with Charney's (1968) zonally symmetric ITCZ, equation (2.6.6) is simplified to

$$\bar{T}_j = -yW$$

(2.7.3)

The neglected term is very small since $U \simeq 5 \times 10^{-2}$. This equation is then used to give $W^{n+1}$ directly:

$$W^{n+1} = -\frac{1}{x} \left( \frac{\partial \bar{T}}{\partial x} \right)^{n+1}$$

(2.7.4)

Centered space differences are used. At the equator, the above formula becomes indeterminate due to the symmetry of $\bar{T}$, necessitating the use of L'Hospital's Rule. This gives

$$W^{n+1}(\theta) = -\left( \frac{\partial \bar{T}}{\partial x} \right)^{n+1}_\theta$$

$$= -\frac{2}{\Delta y} \left( \bar{T}^{n+1}(\theta) - \bar{T}^{n+1}(i) \right)$$

At the pole, $W$ is set permanently to zero.
(d) Calculation of $\hat{\psi}_{ij}^{n+1}, \hat{\psi}_{3j}^{n}$

Using $U^n, W^n, T^n, \hat{\psi}_{ij}^n$ and $\hat{\psi}_{3j}^n$, the perturbation equations (2.6.9), (2.6.10) are solved for $\left[ \frac{\partial}{\partial t} (\hat{\psi}_{ij} + \hat{\psi}_{3j}) \right]^{n+1}$, $\left[ \frac{\partial}{\partial t} (\hat{\psi}_{ij} - \hat{\psi}_{3j}) \right]^{n}$.

In order to ensure numerical stability, it was found necessary to employ a combination of forward, centered and implicit time differences (the numerical stability analysis is given in the appendix).

Expressing the t-derivative as

$$\left[ \frac{\partial}{\partial t} \right]^{n} = \frac{\left( \frac{\partial}{\partial t} \right)^{n+1} - \left( \frac{\partial}{\partial t} \right)^{n-1}}{2At}$$

the lateral viscosity term is separated into forward-differenced and implicitly-differenced parts:

$$\bar{\kappa} \hat{\nabla}_y \left( \hat{\nabla}_y \right) = \bar{\kappa} \frac{\partial}{\partial y} \left[ (\nabla^2) \frac{\partial}{\partial y} \hat{\nabla}_y \right]^{n-1} - \bar{\kappa} \left( \frac{\partial^2}{\partial y^2} \right) \hat{\nabla}_y \left( \hat{\nabla}_y \right)^{n+1}$$

$$= \bar{\kappa} \frac{\partial}{\partial y} \left[ (\nabla^2) \frac{\partial}{\partial y} \hat{\nabla}_y \right]^{n-1} - \bar{\kappa} \left( \frac{\partial^2}{\partial y^2} \hat{\nabla}_y \right)^{n-1}$$

$$+ 2At \left( \frac{\partial}{\partial t} \right)^n$$
Hence, (2.6.9) and (2.6.10) can be written

\[
\left( 1 + 2\Delta t \frac{\partial}{\partial t} \right) \hat{V}_i \left( \frac{\partial}{\partial t} \left[ \hat{\psi}_j + \hat{\phi}_j \right]^n \right) = Z'_i + Z'_j
\]

\[
\left( 1 + 2\Delta t \frac{\partial}{\partial t} \right) \hat{V}_j \left( \frac{\partial}{\partial t} \left[ \hat{\psi}_j - \hat{\phi}_j \right]^n \right) = Z'_i - Z'_j + Z'_j
\]

i.e.

\[
(1 - y^2) \frac{\partial^2}{\partial y^2} \left( \frac{\partial}{\partial t} \left[ \hat{\psi}_j + \hat{\phi}_j \right]^n \right) - 2y \frac{\partial}{\partial y} \left( \frac{\partial}{\partial t} \left[ \hat{\psi}_j + \hat{\phi}_j \right]^n \right) - \frac{\alpha_j^2}{(1 - y^2)} \frac{\partial}{\partial t} \left[ \hat{\psi}_j + \hat{\phi}_j \right]^n
\]

\[
= \frac{1}{f_j} \left[ Z'_i + Z'_j \right]
\]

(2.7.5)

\[
(1 - y^2) \frac{\partial^2}{\partial y^2} \left( \frac{\partial}{\partial t} \left[ \hat{\psi}_j - \hat{\phi}_j \right]^n \right) - 2y \frac{\partial}{\partial y} \left( \frac{\partial}{\partial t} \left[ \hat{\psi}_j - \hat{\phi}_j \right]^n \right) - \left( \frac{\alpha_j^2}{(1 - y^2)} + \frac{2 \delta n_{k1}}{f_j} \right) \frac{\partial}{\partial t} \left[ \hat{\psi}_j - \hat{\phi}_j \right]^n
\]

\[
= \frac{1}{f_j} \left[ Z'_i - Z'_j + Z'_j \right]
\]

(2.7.6)

where

\[
f_j = \left( 1 + 2\Delta t \frac{\partial}{\partial t} \right) \frac{\alpha_j^2}{1 - y^2}
\]

(2.7.7)

\[
Z'_i = - (u^n + W^n) \left\{ \frac{a_{ij}}{\alpha_{ij}} \hat{V}^n \hat{\psi}_{ji} \right\} + \left[ \frac{\partial^2}{\partial y^2} \left( (1 - y^2) (u^n + W^n) \right) \right]
\]

\[
- \frac{2 \sqrt{1 - y^2}}{1 - y^2} \left\{ \frac{a_{ij}}{\alpha_{ij}} \hat{\psi}_{ji} \right\}^n \right] - (\hat{C}_j)^n - (\hat{G}_j)^n
\]

\[
+ \kappa \frac{\partial}{\partial y} \left[ (1 - y^2) \frac{\partial}{\partial y} \hat{V}^n \hat{\psi}_j \right]^{n-1} - \frac{\kappa a_{ij}^2}{(1 - y^2)} \hat{V}^n \hat{\psi}_j^{n-1}
\]

(2.7.8)

\[
Z'_j = - (u^n - W^n) \left\{ \frac{a_{ij}}{\alpha_{ij}} \hat{V}^n \hat{\psi}_{ji} \right\} + \left[ \frac{\partial^2}{\partial y^2} \left( (1 - y^2) (u^n - W^n) \right) \right]
\]

\[
- \frac{2 \sqrt{1 - y^2}}{1 - y^2} \left\{ \frac{a_{ij}}{\alpha_{ij}} \hat{\psi}_{ji} \right\}^n \right] - (\hat{C}_j)^n - (\hat{G}_j)^n
\]

\[
+ \kappa \frac{\partial}{\partial y} \left[ (1 - y^2) \frac{\partial}{\partial y} \hat{V}^n \hat{\psi}_j \right]^{n-1} - \frac{\kappa a_{ij}^2}{(1 - y^2)} \hat{V}^n \hat{\psi}_j^{n-1} - \kappa \hat{V}^2 \hat{\psi}_j^{n-1}
\]
The above quantities are calculated at all interior grid points using centered space differences. It is assumed that

\[ \hat{\nabla}^2 \hat{\psi}_j = \hat{\nabla}^2 \hat{\psi}_j = 0 \quad \text{at equator and pole.} \]

Equations (2.7.5) and (2.7.6) are now solved using a method given by Richtmyer (1957). In finite difference form, these equations become

\[
- \lambda_p (i,j) \mathcal{P}(i,j) - \lambda_p (i,j) \mathcal{P}(i,j) - \lambda_p (i,j) \mathcal{P}(i,j) = \mathcal{D}_p (i,j)
\]

\[
- \lambda_m (i,j) \mathcal{M}(i,j) + \lambda_m (i,j) \mathcal{M}(i,j) - \lambda_m (i,j) \mathcal{M}(i,j) = \mathcal{D}_m (i,j)
\]

where

\[
\mathcal{P}(i,j) = \frac{\partial}{\partial t} \left( \hat{\psi}_j + \hat{\psi}_j \right)_i
\]

\[
\mathcal{M}(i,j) = \frac{\partial}{\partial t} \left( \hat{\psi}_j - \hat{\psi}_j \right)_i
\]

\[
\lambda_p (i,j) = \left[ \frac{(\psi_j - \psi_j)}{\Delta y} - \frac{y_i}{\Delta y} \right]
\]

\[
\lambda_m (i,j) = \left[ \frac{(\psi_j + \psi_j)}{2 \Delta y} + \frac{y_i}{\Delta y} \right]
\]

\[
\mathcal{D}_p (i,j) = - \frac{1}{\Delta y} \left[ \mathcal{Z}_j + \mathcal{Z}_j \right]_{i,j}
\]
\[ A_m(i,j) = \left[ \frac{(1-\gamma^2)}{\rho^2} - \frac{y_j}{\partial y} \right] \]
\[ B_m(i,j) = \left[ \frac{2x(1-\gamma^2)}{\rho^2} + \frac{y_j^2}{(1-\gamma^2)} + \frac{2\rho^2 y_j}{\partial_j} \right] \]  
\[ C_m(i,j) = \left[ \frac{(1-\gamma^2)}{\rho^2} + \frac{y_j}{\partial y} \right] \]
\[ D_m(i,j) = -\frac{1}{\partial_j} \left[ Z_1 - Z_2 + Z_3 \right] \]

The boundary conditions at equator and pole are

\[ P(e_p,i,j) = P(pole,i,j) = 0 \]
\[ M(e_p,i,j) = M(pole,i,j) = 0 \]  

The solution is then given by

\[ P(i,j) = E_p(i,j) P(i+1,j) + B_p(i,j) \]
\[ M(i,j) = E_m(i,j) M(i+1,j) + D_m(i,j) \]  

where

\[ E_p(o,j) = F_p(o,j) = E_m(o,j) = F_m(o,j) = 0 \]

and, at interior points

\[ E_p(i,j) = \frac{A_p(i,j)}{B_p(i,j) - C_p(i,j) E_p(i+1,j)} \]
\[ F_p(i,j) = \frac{D_p(i,j) + C_p(i,j) F_p(i+1,j)}{B_p(i,j) - C_p(i,j) E_p(i+1,j)} \]
The inequalities among the coefficients necessary for a valid solution, viz.,
\[ A_p > 0, \ B_p > 0, \ C_p > 0, \ B_p > A_p + C_p \]
\[ A_m > 0, \ B_m > 0, \ C_m > 0, \ B_m > A_m + C_m \]
are seen to be satisfied.

The components of the perturbation stream function, are now stepped forward in time:

\[
\begin{align*}
(\hat{\psi}_j + \hat{\psi}_j)^{n+1} &= (\hat{\psi}_j + \hat{\psi}_j)^{n-1} + \hat{\rho}(i,j) \times 2\Delta t \\
(\hat{\psi}_j - \hat{\psi}_j)^{n+1} &= (\hat{\psi}_j - \hat{\psi}_j)^{n-1} + \hat{M}(i,j) \times 2\Delta t
\end{align*}
\]

From (2.7.14), the Reynolds stress and eddy conduction terms immediately follow:

\[
\begin{align*}
(\bar{R}_p)^{n+1} &= \frac{1}{\alpha^2\sqrt{1-\eta}} \left( \frac{u_i'u_i' + u_j'u_j'}{2} \right)^{n+1} \\
&= - \frac{1}{2\sqrt{1-\eta}} \sum_{j=1}^{3} \left[ \frac{\partial \hat{\psi}_j^{n+1}}{\partial y} \right] \left[ \alpha_{ij} \hat{\psi}_j^{-1} \right] + \left[ \frac{\partial \hat{\psi}_j^{n+1}}{\partial y} \right] \left[ -\alpha_{ij} \hat{\psi}_j^{n+1} \right] \\
(\bar{R}_m)^{n+1} &= \frac{1}{\alpha^2\sqrt{1-\eta}} \left( \frac{u_i'u_i' - u_j'u_j'}{2} \right)^{n+1} \\
&= - \frac{1}{2\sqrt{1-\eta}} \sum_{j=1}^{3} \left[ \frac{\partial \hat{\psi}_j^{n+1}}{\partial y} \right] \left[ \alpha_{ij} \hat{\psi}_j^{-1} \right] - \left[ \frac{\partial \hat{\psi}_j^{n+1}}{\partial y} \right] \left[ -\alpha_{ij} \hat{\psi}_j^{n+1} \right]
\end{align*}
\]
(e) Calculation of $\hat{\mathbb{Q}}^{n+1}$, $\hat{\mathbb{E}}_j^{n+1}$

Using $\mathbb{U}^{n+1}$, $\mathbb{W}^{n+1}$, $\mathbb{T}^{n+1}$, $(\hat{\psi}_j^{n+1}, \hat{\eta}_j^{n+1})^{n+1}$, the condensational heating terms $\hat{\mathbb{Q}}^{n+1}$, $\hat{\mathbb{E}}_j^{n+1}$ are now calculated from (2.6.18), (2.6.20).

Since $\mathbb{V}^{n+1}$ is not known at this stage, the difficulty, referred to in §2.4, which would be caused by including $\mathbb{V}_3$ in the calculations of $\mathbb{W}_L$ (and hence $\hat{\mathbb{Q}}$, $\hat{\mathbb{E}}_j$) becomes evident.

(f) Calculation of $\mathbb{V}^{n+1}$

Taking the t-derivative of (2.7.3) and eliminating $\mathbb{T}_t$ and $\mathbb{W}_t$ by means of (2.6.7) and (2.6.5) gives a diagnostic equation for $\mathbb{V}$:

\[
\frac{1}{\mu^2} \left[ (1-\gamma) \mathbb{V} \right]_yy - y \mathbb{V} \left[ 2y - \{ (1-\gamma) \mathbb{U} \}_y \right] = \frac{\partial \hat{\mathbb{Q}}}{\partial y} - y \mathbb{K} \left( \mathbb{W} - \mathbb{W}^* \right) - y \mathbb{U} \mathbb{W} + c' \gamma \sqrt{1-\gamma} |u-w| (u-w) + y \mathbb{K} \left[ (1-\gamma) \mathbb{W} \right]_yy \\
- \left[ (1-\gamma) \mathbb{E}_f \right]_yy - \left[ (1-\gamma) \mathbb{R}_w \right]_y - y \mathbb{K}_w \left. \left( \frac{y}{\sqrt{1-\gamma}} \right) \right]
\]

(2.7.16)
Using (2.4.2),

\[ W^* = - \frac{1}{y} \frac{\partial T^*}{\partial y} \]

where

\[ W^* = \begin{cases} \lambda \quad \text{if } n = n+1 \\ \frac{3}{2} k y \quad \text{if } n = n+1 - 1 \end{cases} \]

(2.7.17)

Using \( U^{n+1}, W^{n+1}, Q^{n+1}, E^{n+1}, F^{n+1} \), equation (2.7.16) is solved for \( V^{n+1} \). Again, the method given by Richtmyer (1957) is suitable.

Expressed in finite difference form, (2.7.16) becomes

\[ - A_v(i) V^{n+1}(i) + B_v(i) V^{n+1}(i) - C_v(i) V^{n+1}(i-1) = D_v(i) \]

where

\[ A_v(i) = \frac{(1-y^3)_i}{(\Delta y)^2} \]

\[ B_v(i) = \frac{2(1-y^3)_i}{(\Delta y)^2} + \frac{1}{y_i} \left\{ 2y_i \frac{[e - [y^3] U^{n+1}_y]}{y} \right\} \]

(2.7.18)

\[ C_v(i) = \frac{(1-y^3)_{i-1}}{(\Delta y)^2} \]
\begin{align*}
D_v(i) &= \mu \left[ \frac{d^2}{dy^2} - \gamma \kappa''(w'' - w'') - \gamma \kappa' w''' \right. \\
& \quad - \gamma \sqrt{\eta - 1} \left[ u - w \right]''' \left( u - w \right)'''' \\
& \quad + \gamma \tilde{\kappa} \left[ (1 - y^{
u}) w''' \right] - \left[ (1 - y^{
u}) \kappa''' \right] \bigg|_y \\
& \quad - \frac{y}{\sqrt{1 - 1}} \left\{ [ (1 - y^{
u}) R_m''' ] \bigg|_y - \gamma R_m'''' \right\}
\end{align*}

(2.7.18, cont.)

At equator and pole, the boundary condition is

\[
V''' = 0
\]

The solution is then given by

\[
V''(i) = E_v(i)V''(i+1) + F_v(i)V''(i)
\]

(2.7.19)

where

\[
E_v = F_v = 0 \quad \text{at equator and pole}
\]

and

\[
E_v(i) = \frac{A_v(i)}{B_v(i) - C_v(i) E_v(i-1)}
\]

\[
F_v(i) = \frac{D_v(i) + C_v(i) F_v(i-1)}{B_v(i) - C_v(i) E_v(i-1)}
\]

at all interior points.

Again, the coefficients \( A_v, B_v, C_v, D_v \) must satisfy

the inequalities
In order that these may hold, it is necessary that

\[ A_v > 0, \quad B_v > 0, \quad C_v > 0, \quad \beta_v > \beta_v + C_v \]

or, written in dimensional form,

\[ 2y - \left[ (l-y^2)u \right]_y > 0 \] \hspace{1cm} (2.7.20)

or, written in dimensional form,

\[ \mathcal{D}(\ddot{u} + \bar{u}) > 0 \] \hspace{1cm} (2.7.21)

Physically, the interpretation of this requirement is that the mean of the zonal velocities at the upper and lower levels must be inertially stable (Eliassen and Kleinschmidt, 1957). This model is incapable of following the evolution of inertially unstable motions. In the computer program, a test for this phenomenon was included, giving a printout whenever it occurred. This was almost invariably followed by computational blowup within a small number of iterations.
3.1 The Linearized Equations

In order to obtain as much insight as possible into the dynamics of the model, the perturbation equations will be linearized and expressed in Cartesian co-ordinates. Analytical solutions can then be obtained for some special cases.

Thus simplified, the perturbation equations (2.3.18), (2.3.19) and (2.3.10) become

\[
\frac{\partial}{\partial t} \nabla^2 \psi' = -\bar{u}_i \frac{\partial}{\partial x} \left( \nabla^2 \psi' \right) - \psi' \left( \beta - \bar{u}_{3y} \right) + \frac{1}{\Omega_p} \frac{\omega_p'}{\Delta_p} \\
- k_i \nabla^2 (\psi' - \psi_3') + \Lambda \nabla^2 (\nabla^2 \psi') \tag{3.1.1}
\]

\[
\frac{\partial}{\partial t} \nabla^2 \psi_3' = -\bar{u}_3 \frac{\partial}{\partial x} \left( \nabla^2 \psi_3' \right) - \psi_3' \left( \beta - \bar{u}_{3y} \right) - \frac{1}{\Omega_p} \frac{\omega_p'}{\Delta_p} \\
+ k_i \nabla^2 (\psi' - \psi_3') + \Lambda \nabla^2 (\nabla^2 \psi_3') - \Omega_p \nabla^2 \psi_3' \tag{3.1.2}
\]

\[
\frac{\partial}{\partial t} (\psi' - \psi_3') + \left( \frac{\bar{u}_i + u_3}{2} \right) \frac{\partial}{\partial x} (\psi' - \psi_3') - \left( \frac{\psi' + \psi_3'}{2} \right) \frac{\partial}{\partial y} (\bar{u}_i - \bar{u}_3) \\
+ \frac{\chi_c}{q_0} \left[ (\eta \omega_x' - \omega_3') \right] = -k_c (\psi' - \psi_3') \tag{3.1.3}
\]

where

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\
u_i' = \frac{\partial \psi'}{\partial x}, \quad u_3' = \frac{\partial \psi_3'}{\partial y} \\
u_i' = \frac{\partial \psi'}{\partial x}, \quad u_3' = \frac{\partial \psi_3'}{\partial x} \\
\bar{u}_i = -\frac{1}{q_0} \frac{\partial \phi_i}{\partial y}, \quad \bar{u}_3 = -\frac{1}{q_0} \frac{\partial \phi_3}{\partial y} \tag{3.1.4}
\]
The parameter $f$ is not necessarily the same as $f_0$, reflecting the fact that, in the mean equations, $f$ is a variable.

The condensational heating term $(\eta \omega)'$ is simplified by assigning $\eta$ a constant value $\eta_0$ and linearizing equation (2.4.11) to give

$$\omega' = \frac{\partial}{\partial y} \left( \frac{g (\tau_y)'}{\eta_0} \right) - \frac{\partial}{\partial x} \left( \frac{g (\tau_x)'}{\eta_0} \right)$$

with

$$\left( \tau_y \right)' = c_p \rho_v \left| \bar{u}_l \right| u_3'$$

(3.1.5)

$$\left( \tau_x \right)' = c_p \rho_v \left| \bar{u}_l \right| v_3'$$

(3.1.6)

Hence

$$\omega' = - (c_0 \rho_v \left| \bar{u}_l \right| \frac{g}{\eta_0}) \nabla^2 \psi'$$

(3.1.7)

Taking the sum and difference of (3.1.1), (3.1.2) and using (3.1.3) to eliminate $\omega_1'$ gives

$$\frac{2}{\partial t} \nabla^2 (\omega' + \psi') = - \bar{u}_l \frac{\partial^2}{\partial z} (\nabla^2 \omega') - \bar{u}_3 \frac{\partial^2}{\partial x} (\nabla^2 \psi')$$

$$- \bar{u}_l' (\beta - \bar{u}_l \eta) - \bar{u}_3' (\beta - \bar{u}_3 \eta)$$

(3.1.8)

$$+ A \nabla^2 (\nabla^2 (\omega' + \psi')) - \nabla^2 \nabla^2 \psi'$$
These equations are now non-dimensionalized.

Dimensionless independent variables \( x', y' \) and dimensionless stream functions \( \hat{\psi}', \hat{\psi}_3 \), are defined by

\[
(x, y) = \alpha(x', y')
\]
\[
t = \frac{1}{\eta} t'
\]
\[
(\psi', \psi_3') = \alpha^2 \eta (\hat{\psi}, \hat{\psi}_3)
\]

Hence, (3.1.8) and (3.1.9) become

\[
\frac{\partial^2}{\partial t'} \hat{\nabla}^2 (\hat{\psi}_1 + \hat{\psi}_3) = -\hat{u}_1 \frac{\partial^2}{\partial x'} (\hat{\nabla}^2 \hat{\psi}_1) - \hat{u}_3 \frac{\partial^2}{\partial x'} (\hat{\nabla}^2 \hat{\psi}_3)
\]

\[
+ \frac{\partial \hat{\psi}_1}{\partial x'} (\hat{u}_1 y' - \hat{\beta}) + \frac{\partial \hat{\psi}_3}{\partial x'} (\hat{u}_3 y' - \hat{\beta})
\]

\[
+ \kappa \hat{\nabla}^2 (\hat{\nabla} (\hat{\psi}_1 + \hat{\psi}_3)) - \kappa \hat{\nabla}^2 \hat{\psi}_3
\]
\[ \frac{\partial}{\partial t} \nabla^2 (\psi - \hat{\psi}) = \lambda_3 \frac{\partial^2}{\partial x^2} \left( \frac{\partial \psi}{\partial x} \right) + \lambda_1 \frac{\partial}{\partial x} \left( \nabla^2 \psi - \nabla^2 \hat{\psi} \right) + \lambda_2 \frac{\partial^2}{\partial x^2} \left( \nabla^2 \psi - \nabla^2 \hat{\psi} \right) \]

(3.1.12)

where

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

\[ \hat{\omega}_i = \frac{\partial \bar{\omega}_i}{\partial x}, \quad \hat{\omega}_j = \frac{\partial \bar{\omega}_j}{\partial y} \]

\[ \hat{\beta} = \frac{\partial \bar{\beta}}{\partial x} \]

\[ \varphi_L = \text{latitude at which } \nabla^2 \text{ is evaluated}. \] (3.1.13)

The parameters \( \kappa', \kappa, \kappa', \kappa \) are as defined in (2.6.3).
3.2 The Baroclinic Case

The purely baroclinic case will first be considered. In this case, the condensational heating is zero ($\eta_0 = 0$) and there is no lateral shear in the mean wind ($\hat{u}_y' = \hat{u}_y'' = 0$). The only source of energy then resides in the vertical shear of the mean wind ($\hat{u}_i - \hat{u}_i$), or equivalently, in the North-South gradient of mean temperature.

The stability of such a flow with respect to wavelike perturbations was first studied by Charney (1947) and by Eady (1949).

The stability properties in the present case are modified by the presence of the various frictional terms and the factor $\sqrt{\nu}$.

Seeking solutions of (3.1.11), (3.1.12) of the form

\[
\begin{pmatrix}
\hat{\psi}_1 \\
\hat{\psi}_2
\end{pmatrix}
= 
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
\exp(\lambda(x' - ct'))
\]

(3.2.1)

where $\alpha$ and $c$ are the dimensionless wave numbers and (possibly complex) phase speed

\[
\lambda = \frac{\pi}{L^2}
\]

\[
\zeta = \zeta_0 + ic = \frac{c}{\sqrt{\alpha}}
\]

gives

\[
A \left[ i \kappa^2 (c-\hat{u}_0) + i \hat{\sigma} - \hat{u} \kappa^3 \right] + B \left[ i \kappa^2 (c-\hat{u}_0) + i \hat{\sigma} - \hat{u} \kappa^3 - \hat{u} \kappa \right] = 0
\]

\[
A \left[ i \kappa^2 (c-\hat{u}_0) + i \hat{\sigma} - \hat{u} \kappa^3 - \hat{u} \kappa + 2 \kappa^2 \hat{u} \kappa^2 \zeta \left\{ i \kappa - i \left( \frac{\hat{u}_y' + \hat{u}_y''}{2} \right) + i \frac{\kappa}{4} \left( \frac{\hat{u}_y' + \hat{u}_y''}{2} \right)^2 \right\} \right] \\
+ B \left[ -i \kappa^2 (c-\hat{u}_0) - i \hat{\sigma} + \hat{u} \kappa^3 + \kappa \kappa + \hat{u} \kappa - 2 \kappa^2 \hat{u} \kappa^2 \zeta \left\{ i \kappa - i \left( \frac{\hat{u}_y' + \hat{u}_y''}{2} \right) - i \frac{\kappa}{4} \left( \frac{\hat{u}_y' + \hat{u}_y''}{2} \right)^2 \right\} \right] \\
= 0
\]
The discriminant gives a quadratic in $c$:

$$\lambda c^2 + mc + n = 0$$

i.e. $$c = \frac{-m \pm \sqrt{m^2 - 4\lambda n}}{2\lambda}$$ (3.2.3)

where

$$\lambda = 2\alpha^2 (\alpha^2 + 2\mu^2 \xi^2 \xi_c)$$

$$m = -2\alpha^2 (\mu + \lambda_3)(\alpha^2 + 2\mu^2 \xi^2 \xi_c) + (2\alpha^2 + 2\mu^2 \xi^2 \xi_c)(2\beta + 2i \kappa \alpha^2 + i\kappa)$$

$$n = (\beta + i\kappa \alpha^3)[-(\mu + \lambda_3)(\alpha^2 + 2\mu^2 \xi^2 \xi_c) + 2\beta + 2i \kappa \alpha^3$$

$$+ i\kappa \mu + 2i \kappa' \alpha + 2i \mu^2 \xi^2 \xi_c \frac{\lambda \mu''}{\alpha}]$$ (3.2.4)

For any combination of parameters, this gives the linear stability properties of the flow.

(a) In the special case where

$$\mu = \lambda_3 = \kappa = \kappa' = \kappa'' = 0$$
the solution reduces to

\[
\mathcal{C} = \begin{cases} 
- & \frac{\hat{\beta}}{\alpha^2} \\
- & \frac{\hat{\beta}}{\alpha^2 + 2\mu \kappa \sin^2 \gamma c}
\end{cases} 
\] \quad (3.2.5)

These non-amplifying values of the phase speed correspond respectively to the non-divergent and divergent Rossby wave.

At a latitude of 15°, with a typical wavelength of 2000 km,

\[
\frac{\hat{\beta}}{\alpha^2} = 4 \times 10^{-3}
\]

\[
\mathcal{C}_{\text{min}} = (\alpha \omega \lambda) c = -2.25 \text{ m/sec}
\]

(b) In the absence of friction and radiative cooling \((\kappa = \hat{\kappa} = \kappa'' = 0)\), the solution (3.2.3) reduces to

\[
\mathcal{C} = \left(\frac{\hat{w}_1 + \hat{w}_2}{3}\right) - \frac{\hat{\beta} (2 \mu \kappa \sin^2 \gamma c)}{2 \kappa^2 (\alpha^2 + 2\mu \kappa \sin^2 \gamma c)} \pm \sqrt{\delta} \quad (3.2.6)
\]

where

\[
\delta = \frac{\hat{\beta}^2 (2 \mu \kappa \sin^2 \gamma c)^2}{4 \alpha^4 (\alpha^2 + 2\mu \kappa \sin^2 \gamma c)^2} - \left(\frac{\hat{w}_1 - \hat{w}_2}{\alpha^2}\right)^2 \left(\frac{2 \mu \kappa \sin^2 \gamma c}{\alpha^2 + 2\mu \kappa \sin^2 \gamma c}\right) \quad (3.2.7)
\]

If \(f = f_0\), this reduces to the case discussed by Thompson (1961), and the flow is baroclinically stable \((\delta > 0)\) for all wavelengths of perturbation if...
With \( f > f_0 \), instability can occur for smaller values of the shear. This is, of course, a peculiarity of the present model which has no basis in physical reality. It turns out that, in spite of this, wavelengths of maximum instability in the vicinity of the ITCZ are still baroclinically stable at higher latitudes. In addition, since perturbations are confined to latitudes in the neighbourhood of \( \varphi_c \), this effect does not introduce any serious distortion in numerical integrations.

Again, with \( f = f_0 \), the dominant wavenumber, i.e. that for which the flow first becomes unstable as the vertical shear is increased from zero, is given by

\[
\alpha = \frac{1}{2} \sqrt{\frac{4}{\mu^2 \sin^2 \varphi_c}}
\]

(3.2.9)

These formulas will be used to indicate the baroclinic stability properties of the flow preliminary to performing numerical experiments.
3.3 The Case with Condensational Heating

In this case, the mean wind is taken to be zero, so that the only source of perturbation energy is condensational heating.

Instability associated with this mechanism was first studied by Charney and Eliassen (1964) for a circularly symmetric system with friction confined to the boundary layer. It was proposed that the phenomenon should be known as 'conditional instability of the second kind'.

In this section, sinusoidal solutions will be examined which differ from the case of Charney and Eliassen in the additional sense that negative condensational heating occurs where the boundary layer pumping is negative.

Seeking solutions of (3.1.11), (3.1.12) of the form (3.2.1) it is found that

\[
A \left[ i\alpha^2 c + i\beta - \overline{\nu} \alpha^3 \right] + B \left[ i\alpha^2 c + i\beta - \overline{\nu} \alpha^2 - \overline{\nu} \alpha \right] = 0
\]

\[
A \left[ i\alpha^2 c + i\beta - \overline{\nu} \alpha^3 + 2\mu^2 \sin^2 \phi c \left\{ i\alpha - \frac{\nu}{\alpha} \right\} \right]
+ B \left[ -i\alpha^2 c - i\beta + \overline{\nu} \alpha + \overline{\nu} \alpha^3 - \overline{\nu} \alpha (2\eta_0 - 1) - 2\mu^2 \sin^2 \phi \left\{ i\alpha - \frac{\nu}{\alpha} \right\} \right] = 0
\]

The discriminant then gives

\[
\ell c^2 + mc + n = 0
\]

\[
\ell c^2 + mc + n = 0
\]

\[
\ell = 2\alpha^2 (\alpha^2 + 2\mu^2 \sin^2 \phi c)
\]

\[
m = 2 \left( \beta + i\alpha^2 \right) \left( 2\alpha^2 + 2\mu^2 \sin^2 \phi c \right)
+ 2i\alpha^2 \left( \overline{\nu} \alpha + 2\mu^2 \sin^2 \phi c \frac{\nu}{\alpha} \right)
+ i\overline{\nu} \alpha \left( 2\mu^2 \sin^2 \phi c - 2\alpha^2 (\eta_0 - 1) \right)
\]

\[
\ell = \frac{m \pm \sqrt{\ell^2 - 4\ell n}}{2}
\]

(3.3.1)
\[ n = (\hat{\beta} + i\bar{\mu} \alpha^3) \left[ 2(\hat{\beta} + i\bar{\mu} \alpha^3) + 2i\kappa'\alpha - i\bar{\mu} \alpha \left( 2\eta_0 - 1 \right) + 2\mu^2 \bar{\mu}^2 \gamma_c \left( \frac{2i\kappa''}{\alpha} \right) \right] \]
\[ + i\bar{\mu} \alpha \left[ \hat{\beta} + i\bar{\mu} \alpha^3 + i\kappa'\alpha + 2\mu^2 \bar{\mu}^2 \gamma_c \left( \frac{i\kappa''}{\alpha} \right) \right] \]

Special Cases

(a) Let
\[ \hat{\beta} = \bar{\kappa} = \kappa' = \kappa'' = 0 \]
It is then found that
\[ c = \frac{i\bar{\mu} \alpha}{\bar{\kappa}^3} \left[ \frac{\kappa^2(\eta_0 - 1) - \mu^2 \bar{\mu}^2 \gamma_c}{\alpha^2 + 2\mu^2 \bar{\mu}^2 \gamma_c} \right] \]
\[ = ic_i \]

Hence, a necessary condition for instability is
\[ \eta_0 > 1 + \frac{\mu^2 \bar{\mu}^2 \gamma_c}{\alpha^2} \]

If this is satisfied, it is easily seen that
\[ \frac{2}{\mu \kappa} (\alpha c_i) > 0 \]
i.e. the growth rate increases with increasing wavenumber.

For \( \alpha = \infty \),
\[ (\alpha c_i) = \hat{\kappa}(\eta_0 - 1) \]
giving a dimensional e-folding time for the instability of
\[ \frac{1}{\eta \hat{\kappa}(\eta_0 - 1)} \].
(b) Let

\[ \hat{\beta} = \kappa' = \kappa'' = 0, \quad \bar{\kappa} \neq 0 \]

This case has been studied by Nitta (1964), who shows that the inclusion of lateral viscosity shifts the wavelength of maximum growth rate to a finite value. For a lateral viscosity of \(10^9 \text{ cm}^2/\text{sec}\), he finds the wavelength of maximum growth rate to be of the order of 1000 km.
3.4 A Special Case of Barotropic Instability

In this case, all baroclinic and condensational heating effects are neglected. The motion is assumed to be horizontal, non-divergent and barotropic. The only source of perturbation energy then resides in the horizontal shear of the mean zonal flow.

The stability of such a flow to wavelike disturbances has been studied by Kuo (1949) and, subsequently, by many other investigators.

A necessary condition for instability is the existence of critical points where the absolute vorticity has an extreme value, i.e.

\[ \beta - u_{yy} = 0 \]  

must be satisfied somewhere.

Lipps (1962) has studied the barotropic stability of an idealized wind profile given by

\[ u(y) = A \, \text{sech}^2 by + B \]  

where A is positive (westerly jet).

The barotropic stability of this profile is quite different if A is negative (easterly jet). This can be seen immediately by substituting (3.4.2) in (3.4.1), giving, as the necessary condition for instability:

\[ 2A \, \text{sech}^2 by \left\{ 2 - 3 \, \text{sech}^2 by \right\} > \beta \]  

(3.4.3)
The function \( \frac{\text{sech}^2 y \{ 2 - 3 \text{sech}^2 y \} }{y} \) attains a maximum value of 1/3 where \( y = \pm \frac{1}{6} \text{sech}^{-1}(A) \) and a minimum value of (-1) where \( y = 0 \). Hence, with \( A \) positive, the necessary condition for instability is

\[ \chi > \frac{1}{\lambda} \]  

(3.4.4)

while, with \( A \) negative, the necessary condition is

\[ |\chi| > \frac{1}{\lambda} \]  

(3.4.5)

where

\[ \chi = \frac{1}{3} \frac{A b^2}{\beta} \]  

(3.4.6)

Thus, the condition for instability is much easier to satisfy for an easterly jet.

In this study, it was found (see §5.1) that the computed mean zonal wind profile at the lower level in the vicinity of the ITCZ has a resemblance to a jet of this type. In order to obtain some additional insight into the barotropic stability properties of the flow, Lipps' investigation is here extended to cover the case of \( A \) negative.

Under the conditions assumed here, and with all frictional terms neglected, the vorticity equations, (3.1.1) or (3.1.2), become

\[ \frac{\partial}{\partial t} \nabla \psi = -u \frac{\partial}{\partial x} (\nabla \psi) - \frac{\partial^2}{\partial x} \left( \psi \left( \beta - u_y \right) \right) \]  

(3.4.7)
Setting $\psi(x,y,t) = \phi(y) e^{i \alpha (x-ct)}$, this becomes

$$\phi'' - \alpha^2 \phi + \left\{ \frac{\beta - u''}{u - c} \right\} \phi = 0 \quad (3.4.8)$$

For ease of comparison of results, this equation is non-dimensionalized in a manner consistent with Lipps. For this purpose, the following dimensionless variables are defined

$$x^* = 6x, \quad y^* = 6y, \quad t^* = 6At, \quad c^* = \frac{c - B}{A}$$

$$l^* = \frac{x^*}{\beta}, \quad \chi = \frac{1}{\beta} \frac{\Delta^2}{A}, \quad u^* = \text{Sech}^2 y^* , \quad \phi^* = \frac{\phi l^*}{A} \quad (3.4.9)$$

Without asterisks, the non-dimensionalized form of (3.4.8) becomes

$$\phi'' - l^2 \phi + \left\{ \frac{\chi'' - u''}{u - c} \right\} \phi = 0 \quad (3.4.10)$$

Neutral solutions occur where $u - c = 0$ (result of Kuo (1949)). The phase velocities then follow:

$$\frac{1}{3} \chi'' - u'' = 0$$

$$= 6 \text{ Sech}^2 y - 4 \text{ Sech}^2 y + \frac{1}{3} \chi''$$

$$= 6 \left( u - c_1 \right) \left( u - c_2 \right)$$

where

$$c_1 = \frac{1}{3} \left\{ 1 + \left( 1 - \frac{1}{3} \chi'' \right)^{\frac{1}{4}} \right\}$$

$$c_2 = \frac{1}{3} \left\{ 1 - \left( 1 - \frac{1}{3} \chi'' \right)^{\frac{1}{4}} \right\} \quad (3.4.11)$$
In the case of $c$ negative, $c^2$ is not an allowable value of the phase speed for a neutral wave (because negative $c^2$ would give a dimensional phase speed outside the velocity limits of the mean flow). Taking $c = c_1$, Equation (3.4.10) becomes

$$
\phi'' - \ell^2 \phi + 6 \left( u - c_2 \right) \phi = 0
$$

i.e.,

$$
\phi'' + \left\{ 6 \sec^2 \gamma - 6 c_2 - \ell^2 \right\} \phi = 0
$$

(3.4.12)

Changing the independent variable to $\xi$, where

$$
\xi = \tanh \gamma
$$

gives

$$
(1 - \xi^2) \frac{d^2 \phi}{d\xi^2} - 2\xi \frac{d\phi}{d\xi} + \left[ 6 - \frac{6c_2 + \ell^2}{1 - \xi^2} \right] \phi = 0
$$

(3.4.13)

The only solutions to this equation satisfying boundary conditions

$$
\phi = 0 \quad \text{at} \quad \xi = \pm 1
$$

(3.4.14)

are

$$
\phi(\xi) = k P^m_2(\xi), \quad m = 1, 2
$$

(3.4.15)

where $k$ is a constant, $m = \sqrt{6c_2 + \ell^2}$ and $P^m_2(\xi)$ are the associated Legendre Polynomials of degree 2 and order $m$. 
m = 1: This gives an antisymmetric solution

\[ \phi = \sqrt{1 - z^2} \]

\[ \mathcal{L}^2 = 2 \sqrt{1 - \frac{c^2}{\alpha^2}} - 1 \]  

(3.4.16)

m = 2: This gives a symmetric solution

\[ \phi = (1 - z^2) \]

\[ \mathcal{L}^2 = 2 \left\{ 1 + \sqrt{1 - \frac{c^2}{\alpha^2}} \right\} \]  

(3.4.17)

The curves of neutral stability are given in Fig. 3.1, with Lipps' results for the westerly jet included for comparison.

It is to be noted that

1) As \(|\alpha|\) increases, the easterly jet first becomes unstable, the threshold value of \(|\alpha|\) being the same for both symmetric and anti-symmetric disturbances. In the case of the westerly jet, the threshold value of \(\alpha\) for symmetric disturbances is less than that for antisymmetric disturbances.

2) As \(|\alpha| \to \infty (\beta \to 0)\), the stability curves for easterly and westerly jets tend to the same asymptotes.

3) For the easterly jet, wavelengths less than the neutral wavelength are stable, wavelengths greater than the neutral wavelength are unstable.
FIGURE 3.1  STABILITY OF EASTERNLY AND WESTERNLY JETS.
Chapter 4. Some Preliminary Experiments

In this chapter, some preliminary experiments with the model will be described. These involve integrations for the case of zero condensational heating with a wave perturbation interacting with the zonal flow, and for the case with condensational heating but with no wave perturbations.

4.1. The Case of Zero Condensational Heating

If the condensational heating is omitted, a radiatively driven circulation is set up in response to the latitudinal variation of radiative equilibrium temperature. Such a case, insofar as the earth's atmosphere is concerned, is purely hypothetical. It fails to correspond to the actual atmosphere in the basic sense that it gives rise to a broad region in equatorial latitudes where the temperature $T_2$ is below the radiative equilibrium value, whereas, in fact, $T_2$ exceeds the radiative equilibrium value. (See, for example, Manabe and Müller, 1961.)

Nevertheless, this case is of interest in that it provides a framework within which later results can be viewed.

Starting with the model atmosphere at rest, integrations were performed for the zonally symmetric case. Taking a radiative equilibrium wind shear

$$ W^* = \frac{3}{2} \kappa y $$

(4.1.1)
as given by (2.7.17), with $k = .04$, the circulation was allowed
to evolve to a steady state, the values of the other parameters
being chosen as follows:

$$
\Delta y = \frac{1}{40}
$$

$$
\Delta t = 2.
$$

$$
\mu^2 = 220.
$$

$$
\kappa' = .016
$$

$$
K'' = 1 \times 10^{-2}
$$

$$
\overline{K} = .339 \times 10^{-4}
$$

$$
\hat{K} = .06
$$

$$
\zeta' = 1.4
$$

$$
\frac{\left(\Phi_1 - \Phi_3\right)}{4 \sigma^2 a^2} = .082
$$

The zonal winds at levels 1 and 3 when the circulation has become
steady are shown in Figure 4.17. It can be seen that the winds
remain strong up to the vicinity of the pole, decreasing abruptly
to zero at the northernmost gridpoint. As well as being unrealistic,
this is undesirable from the point of view of maintaining computational stability in the perturbation equations (see equation A.11 in the appendix).

In order to improve the situation, $\hat{W}^*$ was modified so as to tend to zero near the pole by setting

$$W^* = \frac{\frac{3}{2} ky}{1 + e^{-0.85 \cdot 0.02}} \tag{4.1.2}$$

The corresponding radiative equilibrium temperature is shown in Figure 4.1, with that corresponding to the unmodified $W^*$ included for comparison.

The zonally symmetric circulation was again allowed to evolve. The zonal winds after a simulated time of 63.65 days, when a steady state was being approached, are shown in Figure 4.3 (curves (c) and (d)). It can be seen that the winds now tend to zero much more gradually towards the pole. This was chosen as a more suitable basic flow on which to study perturbations.

A single-wave perturbation ($J = 2$) was introduced and allowed to interact with the mean flow, the dominant wavelength for baroclinic instability being chosen, as given by (2.3.9) with $\gamma_c = 45^\circ$. This turns out to be 3250 km, whence $\Delta_\lambda = \frac{L(\gamma_c)/\cos \gamma}{\alpha} = 0.724$. 
In order to satisfy the criterion for numerical stability, the time increment $\Delta t$ was reduced to 0.16.

The perturbation energy grew exponentially with an e-folding time of 1.7 days. After reaching its first maximum, it oscillated with an irregular period of 5 to 11 days as shown in Figure 4.2. This 'index cycle' type of oscillation was accompanied by oscillations of the mean zonal kinetic energy, which remained greater than the perturbation kinetic energy by a factor of at least 2.

These irregular oscillations are due to the non-linear nature of the Reynolds-stress and eddy conduction interactions between the perturbations and the mean zonal flow.

Figure 4.1 shows, in addition to the radiative equilibrium temperatures, the temperature $T_2$ for the zonally symmetric case 63.65 days after it has started from rest, and $T_2$ for the perturbed case 96.5 days after the perturbation is introduced.

For the same times, the vertical velocities are shown in Figure 4.4. The Reynolds stresses and eddy conduction for the latter time are given by Figure 4.5.

It can be seen that the temperatures in the zonally symmetric and the perturbed cases are below radiative equilibrium in equatorial latitudes and above radiative equilibrium in polar latitudes,
Relative Equilibrium Temperature Corresponding to Equation (4.1.1)

Radiative Equilibrium Temperature Corresponding to Equation (4.1.2)

Temperature in Cases of Zonally Symmetric Circulation

--- Temperature at 96.5 Days After Introduction of Perturbations

**Figure 4.1** Temperature at level 2 as function of latitude.
(a) ZONAL KINETIC ENERGY
(b) PERTURBATION KINETIC ENERGY

FIGURE 4.2 KINETIC ENERGY.
FIGURE 4.3 ZONAL WINDS

(a) $\bar{U}_1$ AT 96.5 DAYS AFTER INITIALIZATION OF PERTURBATIONS
(b) $\bar{U}_3$ AT 96.5 DAYS AFTER INITIALIZATION OF PERTURBATIONS
(c) $\bar{U}_1$ AT INITIALIZATION OF PERTURBATIONS
(d) $\bar{U}_3$ AT INITIALIZATION OF PERTURBATIONS
FIGURE 4.4 VERTICAL VELOCITY AT LEVEL 2.

(a) $\bar{w}_2$ AT INITIALIZATION OF PERTURBATIONS

(b) $\bar{w}_2$ AT 96.5 DAYS AFTER INITIALIZATIONS OF PERTURBATIONS
**Figure 4.5** Dimensionless Reynolds stresses and eddy conduction.
the cross-over point being in the neighborhood of 30°. The equator to pole temperature difference is less than the radiative equilibrium temperature difference in both cases, being less for the perturbed case than for the zonally symmetric case. This is a consequence of the poleward transport of heat by the baroclinic eddies.

The effect of the eddies on the zonal wind field is to increase the maximum westerlies at level 1, giving two separate maxima. At level 3, the zonally symmetric regime with easterlies at low latitudes and westerlies in polar latitudes is modified to give a strong band of westerlies centred around 48°, bordered to north and south by easterlies. The low latitude easterlies are considerably strengthened over the zonally symmetric case.

The vertical velocity field, which in the zonally symmetric case consists of a region of rising motion at low latitudes and a region of sinking motion at high latitudes, gives way in the perturbed case to three separate regions of rising motion and three regions of sinking motion.

The contours of the perturbation stream functions $\psi'_1$ and $\psi'_3$ (divided by the non-dimensionalising factor $\sqrt{\alpha \beta}$) are shown in Figures 4.15 and 4.16. It can be seen that the baroclinic regime has two separate latitudes of maximum development, one around 30° and the other around 50°. The intensity is greatest at the upper level and the highs and lows tilt towards the west with height.

The Reynolds stresses are such as to transfer momentum
northward except for a small region around 39° and the region north of the westerly maxima at 48°, where the transport is southward.

Further experiments were performed varying the frictional parameters $\bar{K}$, $K'$, $C'$, and $\hat{K}$. With weak friction, the motion tended to become inertially unstable after the introduction of the perturbations. The vertical friction parameter $K'$ was found to be the most critical one in determining whether or not inertial instability would occur.

This experiment raises the question of whether inertial instability occurs in the real atmosphere and, if so, of what kind of motion it gives rise to.
4.2 The Zonally Symmetric ITCZ

Charney (1968) has investigated the zonally symmetric case with condensational heating. He finds that the effect of condensational heating manifests itself as an instability. For \( \eta_0 \) less than a certain critical value, a 'weak ITCZ' forms, where the circulation does not differ markedly from the 'dry' circulation. As the critical value of \( \eta_0 \) is reached, the whole circulation changes character; a 'strong ITCZ' forms, consisting of a region of concentrated rising motion at a latitude between the equator and 15°, but never at the equator. The width of the region of rising motion is approximately 300 km, and its location can be varied by changing the initial conditions.

In this section, a particular case of a zonally symmetric ITCZ is chosen and presented in detail, its response to changes in a number of the parameters being noted.

This case will be used as the basic state for the study of asymmetric perturbations in Chapter 5.

The radiative equilibrium wind shear is taken as a slightly modified version of (2.7.17) with \( n = 1 \), viz.,

\[
W^* = \frac{k}{1 + \varepsilon^{\frac{q}{0.02}}}
\]

(4.2.1)

The factor in the denominator reduces \( W^* \) to a small value near the pole. A value of .04 is chosen for \( k \), corresponding to \( T_m = 60° \). The corresponding radiative equilibrium temperature in dimensional form is shown by the solid curve of Figure 4.7.
FIGURE 4.6 GROWTH CURVE FOR ZONAL KINETIC ENERGY OF DRY CIRCULATION.
Values chosen for the other parameters are

\[ \Delta y = \frac{1}{80} \]
\[ L^2 = 100 \]
\[ \kappa' = 0.15 \]
\[ \kappa'' = 0.128 \]
\[ \bar{\kappa} = 0.15 \times 10^{-4} \]
\[ c' = 1.0 \]
\[ \Delta t = 0.5 \]
\[ \frac{(\bar{E}_1 - \bar{E}_2)}{4LN^2a^2} = 0.082 \]

At first, \( \eta_0 \) is set to zero and the 'dry' circulation, without perturbations, is allowed to evolve and approach a steady state. The growth curve for the (dimensionless) zonal kinetic energy, \( \bar{E}_K \), is shown in Figure 4.6. The temperature, vertical motion and zonal wind fields after 95.5 days are shown in Figures 4.7 to 4.10, while the energetics are given in Figure 4.11.

Condensational heating is then introduced, setting

\[ T_0 = 0.3 \times 10^{-2} \]
\[ \left( \frac{\Delta \theta_{\theta}}{\Delta \theta_{w/5}} \right) = 1.25 \]
\[ \gamma_d = 0.28 \]
\[ \delta_{\theta} = 0.1 \]
\[ \omega = 1 \times 10^{-6} \]

and increasing \( \eta_0 \) in steps to a value of 3.5.
A 'weak ITCZ' develops. The temperature, vertical motion at level 2 and zonal wind fields after the circulation has reached a steady state are again shown in Figures 4.7 to 4.10, while the energetics are given in Figure 4.11. Figure 4.12 shows the vertical motion out of the boundary layer and Figure 4.13 the resultant values of the heating parameter $\eta$.

The value of $\eta_0$ is then increased to 5. A 'strong ITCZ' develops and the change in the character of the circulation can be seen by comparing the curves for this case, after it has reached a steady state, with the other curves in Figures 4.7 to 4.13.

It can be seen that, in the 'strong ITCZ' case, the field of vertical motion at the middle level becomes concentrated in a narrow region centred around 14°. The distribution of the boundary layer pumping, as can be seen from Figure 4.12, is very similar.

Only in the case of the 'strong ITCZ' does the temperature field exceed the radiative equilibrium values at low latitudes. It can be seen that, whereas the equator-to-pole temperature difference in the case of the dry circulation and of the 'weak ITCZ' is less than the radiative equilibrium equator-to-pole temperature difference, the opposite is the case for the 'strong ITCZ'. In all cases, the temperature gradient at lower latitudes is less than the gradient of radiative equilibrium temperature.

The wind fields at the upper level show a strengthening of the westerlies as $\eta_0$ is increased. At the lower level, the easterlies and westerlies are considerably stronger for the case of the 'strong ITCZ' than for the dry circulation or the weak ITCZ. In addition,
FIGURE 4.7 TEMPERATURE AT LEVEL 2 AS FUNCTION OF LATITUDE.
FIGURE 4.8 VERTICAL VELOCITY AT LEVEL 2.
FIGURE 4.9 ZONAL WINDS AT LEVEL 1.
FIGURE 4.10 ZONAL WINDS AT LEVEL 3.
\[
\begin{align*}
\{ \bar{Q}_c \cdot \bar{E}_p \} &= 0 \\
\{ \bar{Q}_r \cdot \bar{E}_p \} &= 1.28 \times 10^{-4} \\
\{ \bar{E}_p \cdot \bar{E}_p \} &= 1.143 \\
\{ \bar{E}_p \cdot \bar{E}_k \} &= 0.137 \times 10^{-4} \\
\{ \bar{E}_k \cdot \bar{E}_k \} &= 0.974 \times 10^{-3} \\
\{ \bar{E}_k \cdot \bar{E}_q \} &= 0.002 \times 10^{-4} \\
\{ \bar{E}_k \cdot \bar{A} \} &= 0.002 \times 10^{-4} \\
\end{align*}
\]
TOTAL = \(0.139 \times 10^{-4}\)

\[
\begin{align*}
\{ \bar{Q}_c \cdot \bar{E}_p \} &= 0.128 \times 10^{-3} \\
\{ \bar{Q}_r \cdot \bar{E}_p \} &= -1.12 \times 10^{-3} \\
\{ \bar{E}_p \cdot \bar{E}_p \} &= 1.153 \\
\{ \bar{E}_p \cdot \bar{E}_k \} &= 0.158 \times 10^{-4} \\
\{ \bar{E}_k \cdot \bar{E}_k \} &= 1.19 \times 10^{-3} \\
\{ \bar{E}_k \cdot \bar{E}_q \} &= 0.003 \times 10^{-4} \\
\{ \bar{E}_k \cdot \bar{A} \} &= 0.003 \times 10^{-4} \\
\end{align*}
\]
TOTAL = \(0.164 \times 10^{-4}\)

\[
\begin{align*}
\{ \bar{Q}_c \cdot \bar{E}_p \} &= 6.16 \times 10^{-4} \\
\{ \bar{Q}_r \cdot \bar{E}_p \} &= -5.91 \times 10^{-4} \\
\{ \bar{E}_p \cdot \bar{E}_p \} &= 1.190 \\
\{ \bar{E}_p \cdot \bar{E}_k \} &= 0.2427 \times 10^{-4} \\
\{ \bar{E}_k \cdot \bar{E}_k \} &= 1.838 \times 10^{-3} \\
\{ \bar{E}_k \cdot \bar{E}_q \} &= 0.0136 \times 10^{-4} \\
\{ \bar{E}_k \cdot \bar{A} \} &= 0.0096 \times 10^{-4} \\
\end{align*}
\]
TOTAL = \(0.2549 \times 10^{-4}\)

(a) DRY CASE  
(b) WEAK ITCZ  
(c) STRONG ITCZ

FIGURE 4.11 ENERGY CONVERSIONS
FIGURE 4.12 BOUNDARY LAYER PUMPING $w_L$. 

(a) WEAK ITCZ
(b) STRONG ITCZ
FIGURE 4.13 CONDENSATIONAL HEATING PARAMETER $\eta$. 

(a) WEAK ITCZ
(b) STRONG ITCZ
a strong shear in the easterlies is associated with the 'strong ITCZ'.

The quasi-discontinuous nature of the condensational heating parameter $\eta$ in going from a region of positive to one of negative boundary pumping is clearly shown in Figure 4.13.

Examination of the energetics in Figure 4.11 shows an increasing generation, conversion and dissipation of energy as $\eta$ is increased.

To test the effect of changing the lateral viscosity, integration for the 'strong ITCZ' case is resumed with $K$ increased from $1.15 \times 10^{-4}$ to $2 \times 10^{-4}$. The resultant field of vertical motion after the circulation has reached a steady state is shown by curve (b) in Figure 4.14, with the original 'strong ITCZ' field of $w_2$ (curve (a)) included for comparison.

It is seen that the effect of increasing $K$ is to shift the ITCZ towards the equator while decreasing slightly the value of the maximum vertical velocity. To the north of the ITCZ, the field of vertical velocity remains almost identical.

The parameter $\gamma_d$ is now increased from .28 to .35 and the integration is continued until the circulation again reaches a steady state. The resulting $w_2$ is shown by curve (c) in Figure 4.1. It can be seen that the ITCZ moves away from the equator, being centred at $17^\circ 27'$ ($y = .3$).

Varying all other parameters in turn, it was found that changes in the position and strength of the ITCZ were slight, leading to the conclusion that the zonally symmetric ITCZ, once formed, possesses a high degree of stability.
Figure 4.14  Vertical velocity at level 2
Fig. 4.15: CONTOURS OF $(\psi'_1 / \sqrt{\pi} \cdot a^2 \cdot \mathcal{N})$. 
Fig. 4.16 CONTOURS OF \( \frac{\Psi_3'}{\sqrt{2}} \alpha^2 \Omega \)
Figure 4.17  Zonal winds at upper and lower levels.
Chapter 5. Disturbances on the ITCZ

In this chapter, wave perturbations on the ITCZ will be studied, taking as basic state the zonally symmetric ITCZ which has been described in Chapter 4, and whose properties have been portrayed in Figures 4.7 to 4.13.

First, the idealised case of a single wave disturbance on the 'strong ITCZ' will be examined, omitting friction and heating terms from the perturbation equations and allowing the wavelength of the disturbance to assume a series of values. Friction and heating will then be introduced in turn and their effects noted. Finally, non-sinusoidal disturbances will be studied by allowing several wave components to grow and interact.

A single case of a disturbance on the 'weak ITCZ' will also be examined.

5.1 Single-Wave Perturbations Without Friction and Condensational Heating

As a preliminary to performing any numerical integrations, some idea of the dynamic stability properties of the zonal winds in the neighbourhood of the ITCZ can be obtained by employing the results of Chapter 3.

The zonal wind at the lower level in the neighbourhood of the 'strong ITCZ' is compared in Figure 5.1 with the idealised wind profile

\[ u(y) = A \text{Sech}^2 by + B \]
discussed in §3.4. Choosing \( A = -7 \text{ m/sec}, B = -1 \text{ m/sec}, \beta = \left( \frac{1}{181.2} \right) \text{ km}^{-1} \) it is seen that the two profiles bear a resemblance, especially south of 14°.

From (3.4.5) and (3.4.6), the necessary condition for barotropic instability of the idealised profile is

\[
|\chi| = \left| \frac{\frac{\beta}{A^2}}{\beta} \right| > \frac{1}{6}
\]

Choosing \( \beta = 2.2 \times 10^{-13} \text{ cm}^{-1} \text{ sec}^{-1} \) (corresponding to a latitude of 15°) and using the values of \( A \) and \( B \) given above, it is found that

\[
|\chi| = 3.24
\]

so that the necessary condition for instability is easily satisfied.

The neutral wavelength for symmetric perturbations is given by (3.4.17), viz.,

\[
l^2 = 2 \left\{ 1 + \sqrt{1 - \frac{\beta}{k^2}} \right\}
\]

\[
= 4.148
\]

Hence

\[
l = \left( \frac{2\pi}{k} \right) \frac{1}{6} = 2.04
\]

i.e.

\[
l = 560 \text{ km}.
\]

Thus, perturbations of wavelength less than 560 km will be barotropically stable, while perturbations of wavelength greater
than this will be barotropically unstable.

Considering the baroclinic stability properties of the flow, equations (3.2.6) and (3.2.7) show that a necessary condition for baroclinic instability is

$$2\mu^2 \sin^2 \theta \frac{\partial}{\partial \phi} - \alpha^2 > 0$$

easily.

$$L/a > \frac{2\pi}{\sqrt{2\mu^2 \sin^2 \theta}}$$

For a given \( f_0 \), the larger \( \mu \), the smaller is the cut-off wavelength.

Choosing \( \mu = 100, \theta = 15^\circ, f = 2.0 \), it is found that

$$L = 5500 \text{ km}.$$  

Thus, the minimum wavelength for baroclinic instability is much greater than the neutral wavelength for barotropic instability.

Numerical experiments are now performed to test the stability of the 'strong ITCZ' zonal winds to perturbations of various wavelengths. All frictional and condensational heating terms are, for the present, omitted from the perturbation equations.

The initial perturbations are:

$$\left\{ \begin{array}{c}
\frac{\partial w_1}{\partial y} \\
\frac{\partial w_3}{\partial y}
\end{array} \right\} = 10^{-5} \cos \left[ \left( \frac{y - 212}{212} \right) \frac{\pi}{2} \right] , \quad 0 \leq y \leq 424$$

$$\left\{ \begin{array}{c}
\frac{\partial w_1}{\partial y} \\
\frac{\partial w_3}{\partial y}
\end{array} \right\} = 0 \quad , \quad y > 424$$
Friction and heating terms in the perturbation equations are set to zero. The following values of the parameters are chosen:

\[
\begin{align*}
J &= 2 \\
y_c &= 2.6 \\
\Delta t &= 0.15
\end{align*}
\]

all other parameters being the same as in §4.2.

Values of \(\Delta \lambda\) corresponding to wavelengths varying from 1000 km \((\Delta \lambda = 0.157)\) to 3000 km \((\Delta \lambda = 0.472)\) are then chosen and a numerical integration is performed for each.

The growth curves of the perturbation kinetic energy are shown in Figures 5.2 and 5.3. It can be seen from Figure 5.2 that as the wavelength increases from 1000 km to 1750 km, the growth rate of \(E'\) also increases. For a wavelength of 2000 km, the growth rate is almost indistinguishable from that for 1750 km over most of the range, but eventually the 2000 km case predominates and reaches a greater amplitude. The e-folding time for \(E'\) in this case is 1.2 days.

As the wavelength is further increased, Figure 5.3 shows that the growth rate decreases, so that 2000 km is the dominant wavelength.

In all cases, the growth of \(E'\) is due to barotropic conversion by Reynolds stresses from mean to eddy energy. In Table 5.1, the Reynolds stress conversion \(\{\overline{E_k \cdot E'_k}\}\) and the baroclinic eddy conversion \(\{\overline{E_p \cdot E'_p}\}\) at 7.16 days after initialisation are presented for comparison. It can be seen that \(\overline{E_k \cdot E'_k}\) outweighs
<table>
<thead>
<tr>
<th>Wavelength (km)</th>
<th>${ \overline{E_K} \cdot E'_K }$</th>
<th>${ \overline{E_P} \cdot E'_P }$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>5.85x10^{-8}</td>
<td>-2.69x10^{-10}</td>
</tr>
<tr>
<td>1250</td>
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<td>-4.11x10^{-10}</td>
</tr>
<tr>
<td>1500</td>
<td>5.38x10^{-7}</td>
<td>-4.15x10^{-1}</td>
</tr>
<tr>
<td>1750</td>
<td>7.59x10^{-7}</td>
<td>+9.33x10^{-10}</td>
</tr>
<tr>
<td>2000</td>
<td>8.13x10^{-7}</td>
<td>-4.59x10^{-10}</td>
</tr>
<tr>
<td>2250</td>
<td>7.09x10^{-7}</td>
<td>-1.38x10^{-9}</td>
</tr>
<tr>
<td>2500</td>
<td>5.37x10^{-7}</td>
<td>-1.37x10^{-9}</td>
</tr>
<tr>
<td>2750</td>
<td>3.75x10^{-7}</td>
<td>-9.07x10^{-10}</td>
</tr>
<tr>
<td>3000</td>
<td>2.46x10^{-7}</td>
<td>-4.17x10^{-10}</td>
</tr>
</tbody>
</table>

**Table 5.1** Comparison of barotropic and baroclinic energy conversions at 7.16 days after initialisation.
\{ \overline{E_p} \cdot E' \} by more than two orders of magnitude. In all cases except for the wavelength of 1750 km, the baroclinic conversion is stabilising.

Examining the 2000 km wavelength case in more detail, Figures 5.4 and 5.5 show the stream function components at upper and lower levels at 23.9 days after initialisation, when \( E'_\kappa \) has reached its second maximum. It is seen that the perturbations are confined to tropical latitudes where the mean motion at level 3 is barotropically unstable. As is to be expected, the perturbation amplitudes are much greater at level 3 than at level 1. The perturbations die away north of 30°, due to the baroclinic stability of the flow for the wavelength involved. The maximum amplitude of the perturbation wind at 23.9 days is approximately 7 m/sec.

The effect of the perturbations on the mean flow at 19.1 days (when \( E'_\kappa \) is at a minimum) and at 23.9 days (when \( E'_\kappa \) is at its second maximum) after initialisation is shown in Figures 5.6 to 5.9.

It can be seen from Figure 5.6 that the perturbations have a strong effect on the field of mean temperature, raising it several degrees above the zonally symmetric values in tropical latitudes. This effect is exerted mainly through the influence of the perturbation wind velocity at level 3 on the mean condensational heating term \( (\eta \omega) \).

The perturbations also tend to broaden the mean ITCZ, as can be seen from Figure 5.7, though leaving it centred around
approximately the same latitude. This is especially true at 23.9 days, though in this case the magnitude of the vertical velocity has decreased. At 19.1 days, where less broadening has taken place, the maximum vertical velocity has increased greatly. It is to be noted that the perturbations lead to the disappearance of the weak region of rising motion ('secondary ITCZ') situated adjacent to the equator.

The perturbations also increase the shear of the mean zonal wind at level 3 in the neighbourhood of the ITCZ. This is somewhat surprising; it again arises from the effect of the perturbation velocity on the mean condensational heating.

At level 1, the mean zonal winds are changed little in tropical latitudes but the maximum north of 30° is strengthened. It is to be noticed that the effect of the perturbations is felt at latitudes where the actual perturbation amplitude in situ has decreased to zero.
5.2 Single-Wave Perturbations With Friction But Without Condensational Heating

To test the effect of friction on the perturbations, the vertical and lateral frictional coefficients $K'$ and $\bar{K}$ are now given the same values in the perturbation equations as in the mean equations, viz.

$$K' = 0.15$$
$$\bar{K} = 1.5 \times 10^{-4}$$

The perturbation ground friction is linearised as described in §2.3, the relevant parameter being $\hat{K}$, given by

$$\hat{K} = \frac{9}{2p} \frac{C_{D}}{\sqrt{\rho_{w} |\bar{u}|}}$$

In deriving this, it was assumed that the wind at the surface is the same as at level 3, which may lead to an overestimate of the surface frictional dissipation.

Choosing

$$C_{D} = 2.2 \times 10^{-3}$$
$$|\bar{u}| = 5 \text{ m/sec}$$

it is found that

$$\hat{K} = 0.03$$

Taking all other parameters as in §5.1, and omitting condensational heating in the perturbation equations, a perturbation of the dominant wavelength for barotropic instability,
2000 km, was initialised as before and allowed to evolve.

It was found that $E_k'$ decayed slowly. At 2.39 days after initialisation, the eddy dissipation terms had the following values:

$$
\begin{align*}
\{ E_k' \cdot k_i \} &= 5.38 \times 10^{-10} \\
\{ E_u' \cdot \gamma_4 \} &= 1.64 \times 10^{-9} \\
\{ E_k' \cdot A \} &= 1.72 \times 10^{-9}
\end{align*}
$$

while the Reynolds stress conversion was

$$
\{ \overline{E_k} \cdot E_k' \} = 2.56 \times 10^{-9}
$$

It is evident that the production of eddy kinetic energy by barotropic conversion is outweighed by the sum of the frictional dissipations.

Keeping the vertical and lateral friction coefficients as before (i.e. the same as in the mean equations), the ground friction coefficient is halved, i.e.

$$
\hat{\kappa} = 0.015
$$

and the run is repeated.

The perturbation kinetic energy grows slowly, as shown in Figure 4.10, curve (b). The e-folding time for $E_k'$ is 4.8 days. This is to be compared with an e-folding time of 1.2 days for the frictionless case. In addition, the final amplitude of in this case is about an order of magnitude less than in the frictionless case, as can be seen by comparing curves (a) and (b) in Figure 5.10.
From these results, it is seen that, when there is no condensational heating of the perturbations, the friction may prevent the barotropic instability of the mean flow from being released.
5.3 Single-Wave Perturbations with Friction and Condensational Heating

Self-heating of the perturbations is now allowed. Referring to equation (2.5.13), it is seen that for the case of a single-wave disturbance \( J = 2 \), the perturbation condensational heating function \( (\eta \omega_L)^' \) is of sinusoidal form. To approximate the actual 'on-off' nature of the function \( (\eta \omega_L) \) in going from a region of positive to one of negative boundary-layer pumping, a higher number of wave components in the perturbation part would be required. For the present, attention is confined to the single-wave case in order to get a first look at self-heating disturbances. Later, in §5.4, multiple-wave disturbances, which require a great increase in computing time, will be considered and more detailed results presented.

The friction coefficients chosen are those used in the earlier part of §5.2, viz.,

\[
\kappa' = 0.015 \\
\overline{K} = 0.15 \times 10^{-4} \\
\hat{K} = 0.03
\]

In the numerical evaluation of the heating components \( \hat{\xi}_j \), 12 grid points are used in the east-west direction. With all other parameters as before, and with the same method of initialisation, a perturbation of 2000 km wavelength \( (\Delta \lambda = 0.314) \) is allowed to evolve.

The perturbation kinetic energy grows with an e-folding time
of 3 days, as shown in Figure 5.10, curve (c). This is intermediate between the growth rate of the free barotropic wave, where $E_k'$ had an e-folding time of 1.2 days, and of the wave which had frictional retardation (with $K = 0.015$) but no self-heating, in which case the e-folding time for $E_k'$ was 4.8 days.

The energetics of the disturbance in the growing stage (at 11.95 days) and in the mature stage (at 23.9 days) are shown in Figure 5.11. It can be seen that, in the case of the growing disturbance, the kinetic energy comes mainly from the zonal flow by Reynolds stress conversion, though a considerable amount also comes by direct conversion from eddy potential energy which is generated by the perturbation condensational heating. The dissipation is mainly accomplished by surface friction, though at this stage this is still small by comparison with the Reynolds stress conversion. When the disturbance has evolved to the mature stage, the Reynolds stress conversion is small by comparison with the dissipation terms, which are now balanced mainly by the conversion from eddy potential energy.

The stream function amplitudes for the growing and mature disturbance are shown in Figures 5.12 and 5.13. Again, the perturbations are confined to latitudes south of 30°, having maximum amplitude at around 13°. The amplitudes at level 3 are about an order of magnitude greater than at level 1. At 23.9 days, the amplitude of the perturbation wind $u_3'$ is 2.7 m/sec.

The phase of the level 3 stream function $\psi_3'$, where

$$\hat{\psi}_3 = (\hat{\psi}_3)_{j=1} \sin (\alpha \lambda) + (\hat{\psi}_3)_{j=2} \cos (\alpha \lambda)$$
at 13° latitude increases by 115°54' from 22.71 to 23.9 days, showing a westward propagation of the disturbance at 6.2 m/sec.

The vertical velocities $\omega_T$ and $\omega_L$, the perturbation temperature $T'_2$ and the geopotentials $\phi'_1$ and $\phi'_3$, all at 23.9 days and at latitude 13°, are shown in Figures 5.14 and 5.15. It can be seen that the maxima of $\omega_T$ and $\omega_L$ coincide with the low-level trough and that the disturbance is 'warm-core'. A weak ridge at the upper level lies 420 km to the east of the low-level trough.

The effect of the perturbations on the mean flow at 47.8 days after initialisation (at which stage $E'_K = 0.186 \times 10^{-5}$) can be seen by referring to curve (e) in Figure 5.6 and curves (d) in Figures 5.7, 5.8 and 5.9. The effect on the mean temperature field or the zonal winds at level 1 is not as great as in the case of the free perturbations. The ITCZ is considerably broadened and its maximum strength and the shear across it are reduced.

A further experiment was performed with a different formulation of the perturbation ground friction, all else being exactly as above. Instead of the linearised form described in §2.3 with

$$ \kappa = \frac{g}{\Delta p} \frac{c_o \rho_o |\bar{u}|}{\Omega} $$

and $|\bar{u}|$ taken as a constant, $|\bar{u}|$ is now set equal to $|\bar{u}_3(i)|$, the actual value of the low-level zonal wind at the point in question.

The growth rate of perturbation kinetic energy is slightly
greater than previously, having now an e-folding time of 2.43 days. The final amplitude attained is also greater, as can be seen from Figure 5.16.

The energy conversions for the growing and mature disturbance, with the zonal conversions also included, are shown in Figure 5.17. The qualitative features of the energetics remain unchanged.
5.4 Multiple-Wave Disturbances with Heating and Friction

Up to this point, all disturbances have been constrained to be sinusoidal in character. The most serious deficiency of this procedure lies in the approximation of the perturbation part of the condensational heating function \( (\eta L) \) by a single sinusoidal wave. A higher number of wave components would be required to represent this function accurately.

All perturbation quantities will now be allowed to assume a non-sinusoidal form by increasing the number of Fourier components in their expansion. In this section, \( J = 6 \) is the value chosen.\(^1\)

A basic periodicity of 2000 km is again used (\( \Delta = 3/4 \)), so that now a disturbance will consist of interacting components of wavelength 2000, 1000 and 666.6 km. All components are similarly initialised, after the manner described in \( \S \) 5.1.

In the numerical evaluation of the heating components \( \hat{\varepsilon}_j \), 36 grid points in the east-west direction are now used.

---

\(^1\) The immediate problem to be faced in increasing the value of \( J \) was the rapid increase in computing time required. Using the fastest computer available, the IBM 360/95 at the NASA Institute for Space Studies in New York, the time required for 100 iterations, with \( J = 6 \), was 4.28 minutes. In a normal run, the program was allowed to proceed for up to 2000 iterations.

For \( J = 12 \), the computing time increased to 23.2 minutes per 100 iterations.
The linearised form of the perturbation ground friction, with $\hat{K} = .03$, is reverted to. The time-step, $\Delta t$, is chosen to be .063, a value which satisfies all the numerical stability criteria (see Appendix). All other parameters are the same as in §5.3.

The interacting set of wave-components is then allowed to evolve and reach the mature stage. Figure 5.18 shows the growth of the perturbation kinetic energy, with the corresponding curve for the case of $J = 2$ included for comparison.

It can be seen that the e-folding time for $E_K$ is approximately the same in both cases, i.e. 3 days. The final amplitude attained in the case of $J = 6$, however, is somewhat less than in the case of $J = 2$.

A breakdown of the distribution of $E_K$ between the different components in the growing stage (10 days) and at the mature stage (24 days) is shown in Table 5.2. It can be seen that nearly all of the energy is contained in the longest wavelength ($J = 1$ and 2).

Figure 5.19 shows the energy conversions at 10 and at 20 days. In the growing stage, the predominant feature is the conversion of kinetic energy from the zonal flow to the perturbations by the Reynolds stresses. At the mature stage, this is superceded by the direct conversion of condensationally produced eddy potential energy to eddy kinetic energy. The dissipation of the eddy kinetic energy is accomplished mainly by surface friction.
### TIME = 10 DAYS

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<tr>
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<th>$E_k'(j)$</th>
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<tbody>
<tr>
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<td>.559x10^{-7}</td>
</tr>
<tr>
<td>2</td>
<td>.136x10^{-6}</td>
</tr>
<tr>
<td>3</td>
<td>.645x10^{-9}</td>
</tr>
<tr>
<td>4</td>
<td>.778x10^{-9}</td>
</tr>
<tr>
<td>5</td>
<td>.146x10^{-10}</td>
</tr>
<tr>
<td>6</td>
<td>.776x10^{-11}</td>
</tr>
</tbody>
</table>

### TIME = 24 DAYS

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<th>$E_k'(j)$</th>
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<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
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</tr>
<tr>
<td>5</td>
<td>.770x10^{-8}</td>
</tr>
<tr>
<td>6</td>
<td>.644x10^{-9}</td>
</tr>
</tbody>
</table>

**Table 5.2** Distribution of $E_k'$ among spectral components.
Figures 5.20 to 5.23 show the effects of the perturbations on the mean flow. Unlike the case of the free perturbations, the mean zonal temperatures are now slightly lower than the unperturbed values in tropical latitudes. The ITCZ has broadened and moved closer to the equator, with a diminishment of the vertical velocity in the ascending region. The shear in the zonal winds at the lower level has decreased. There is no significant change in the upper zonal wind profile.

The field of geopotential (mean plus perturbation) at level 3 is shown in Figures 5.24 and 5.25, corresponding to times of 23.5 and 24 days respectively. The variation of the geopotential at levels 1 and 3 at the latitude of maximum perturbation amplitude, 13°, is shown in Figures 5.26 and 5.27. It can be seen that the amplitude of the perturbation at level 1 is 1/6 that at level 3. The upper level ridge lies to the east of the surface trough. The wave propagates towards the west at 6.4 m/sec. The maximum amplitude of the perturbation wind in the meridional direction at level 3 is 2.65 m/sec. (The mean zonal wind, $\bar{u}_3$, at 13° is 5.8 m/sec.)

Figure 5.28 shows the field of $w_2$, with the detailed variation at 13° shown in Figure 5.29. It can be seen that the variation is no longer even approximately sinusoidal. The ITCZ has broken down into a highly asymmetric pattern of rising motion with a concentrated maximum, surrounded by a broad region of weak sinking motion. For comparison, $w_2$ in the growing stage (at 12 days) is shown in Figure 5.31. At this stage, the ITCZ has much more of a zonally symmetric character.
The level 2 temperature field at 24 days is shown in Figure 5.32. The system is seen to be 'warm-core'.

The eddy transports of momentum and energy are shown in Figure 5.33. The Reynolds stress at the lower level transports momentum northward to the south of the central latitude of the perturbation, and southward to the north of this latitude. The Reynolds stress transport at the upper level could not be represented on the same scale, being about two orders of magnitude smaller. The eddy conduction of energy is northward, having its maximum value near the central latitude of the perturbation. All eddy processes decay to zero north of 24°.

The condensational heating function is now examined to see if the spectral representation with \( J = 6 \) gives a good approximation to the 'on-off' nature of the function. The condensational heating is proportional to \( H \), as defined by

\[
H = - (\eta \omega_L)
\]

\[
= - \left( \overline{\eta \omega_L} + (\eta \omega_L)' \right)
\]

Using (2.5.13) and (2.6.3), this becomes

\[
H = - \left( \overline{\eta \omega_L} + \sum_{j=1}^{J} \eta_j (\xi_j T) F_j(\lambda) \right)
\]

\[
= \left( \rho \omega L \right) \overline{\eta \omega_L} - \frac{\Delta \rho \sigma^2}{2 \rho_0} \sum_{j=1}^{J} \hat{\xi}_j F_j
\]

\[
= \rho \omega L \left( \overline{\eta \omega_L} - \frac{\sigma^2 \Delta \rho}{2 \rho_0^2 \rho \omega L} \sum_{j=1}^{J} \hat{\xi}_j F_j \right)
\]
<table>
<thead>
<tr>
<th>$i$</th>
<th>$\hat{\mathcal{E}}_i$</th>
<th>$F_i(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.744 \times 10^{-2}$</td>
<td>$\sqrt{2} \sin \left( \frac{2\pi \lambda}{\Delta \lambda} \right)$</td>
</tr>
<tr>
<td>2</td>
<td>$0.119 \times 10^{-1}$</td>
<td>$\sqrt{2} \cos \left( \frac{2\pi \lambda}{\Delta \lambda} \right)$</td>
</tr>
<tr>
<td>3</td>
<td>$0.335 \times 10^{-2}$</td>
<td>$\sqrt{2} \sin \left( \frac{4\pi \lambda}{\Delta \lambda} \right)$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.731 \times 10^{-3}$</td>
<td>$\sqrt{2} \cos \left( \frac{4\pi \lambda}{\Delta \lambda} \right)$</td>
</tr>
<tr>
<td>5</td>
<td>$0.127 \times 10^{-2}$</td>
<td>$\sqrt{2} \sin \left( \frac{6\pi \lambda}{\Delta \lambda} \right)$</td>
</tr>
<tr>
<td>6</td>
<td>$-0.784 \times 10^{-3}$</td>
<td>$\sqrt{2} \cos \left( \frac{6\pi \lambda}{\Delta \lambda} \right)$</td>
</tr>
</tbody>
</table>

**Table 5.3** Spectral amplitudes of the perturbation condensational heating function at $\phi = 14^\circ 30'$ and Time = 24 days.
Table 5.3 gives the values of $\hat{\mathcal{E}}_j$ at a central latitude for the perturbations ($\phi = 14^\circ 30'$) at 24 days, with the accompanying functions $F_j(\lambda)$. The corresponding value of ($\eta \overline{\omega_L}$) is $0.936$. Both of the quantities ($H/\rho_0$) and $\omega_L$ are shown in Figure 5.34.

It can be seen that the distribution of the heating is non-sinusoidal, but is still positive, though small, in regions of negative boundary layer pumping at this particular latitude. It is obvious that, in order to get a more accurate representation of the heating, a higher number of spectral components would be required.

In a posteriori justification of the neglect of the mean meridional velocity in arriving at the balance equations (2.2.19), (2.2.20), and the expression for the boundary layer pumping, $\overline{\mathcal{U}}_i$ for time 24 days is shown in Figure 5.36 (note: $\overline{\mathcal{U}}_i = -\overline{\mathcal{U}}_i$). It is seen that $\overline{\mathcal{U}}_i$, with a maximum value of 0.6 m/sec, is indeed small by comparison with the mean zonal velocities and the components of the perturbation velocity. Its magnitude in other integrations was similar, justifying the approximations made.
5.5 Perturbations on the 'Weak ITCZ'

Up to this point, the perturbations studied have had as their basic state the 'strong ITCZ' of Chapter 4. The perturbations have derived their kinetic energy from the barotropically unstable flow at the lower level, and by direct conversion of condensationally produced eddy potential energy.

A perturbation similar to that described in 5.3, with variable $\hat{K}$ (proportional to $|\bar{u}_3|$), will now be imposed on the 'weak ITCZ' circulation of Chapter 4, the only difference being that now $\eta_0$ has the value 3.5.

It is found that the perturbation energy decays. Neither Reynolds stress conversion from the mean flow kinetic energy nor perturbation condensational heating is sufficient to overcome the dissipative effects of friction. The energy conversions at 4.8 days after initialization of the perturbations are shown in Figure 5.35. It can be seen that the Reynolds stresses are stabilizing and that $\mathbb{E}_p'\mathbb{E}_u'$ is much smaller than the dissipation of $\mathbb{E}_u'$.

Thus, the high value of $\eta_0$ and the associated strong shear of $\bar{u}_3$ are necessary to give growing ITCZ disturbances.

A possibility for further useful studies with the model would be to increase the condensational heating effect in the perturbation equations, leaving its value in the mean equations unchanged, so as to retain the weak ITCZ as basic zonal state. This would give some insight into disturbances which grow due to condensational heating in a situation where the Reynolds stresses are stabilizing.
Figure 5.1 Comparison of zonal wind profiles.

where

\[ q = \frac{1}{b} \ \text{km}^{-1} \]

\[ b = 1 \ \text{m/sec} \]

\[ q = 7 \ \text{m/sec} \]

(b) Idealized profile \( q^2 = A \ \text{sech}^2 B y + B \)

(6) \( q^2 \) for Strong ITCZ
Figure 5.2 Growth of Perturbation Kinetic Energy.
FIGURE 5.3 GROWTH OF PERTURBATION KINETIC ENERGY.
FIGURE 5.4 STREAM FUNCTION COMPONENTS (DIMENSIONLESS) AT LEVEL 1 FOR THE CASE $\Delta \lambda = 0.314$ AFTER 23.9 DAYS.
FIGURE 5.5 STREAM FUNCTION COMPONENTS (DIMENSIONLESS) AT LEVEL 3 FOR THE CASE $\Delta \lambda = .314$ AFTER 23.9 DAYS.
(a) RADIATIVE EQUILIBRIUM TEMP.
(b) STRONG ITCZ-BASIC STATE
(c) PERTURBED STATE, $\Delta \lambda = .314$, ZERO HEATING AND FRICTION IN PERTURBATION EQUATIONS, 19.1 DAYS AFTER INITIALIZATION
(d) AS (c), 23.9 DAYS AFTER INITIALIZATION
(e) PERTURBED STATE, $\Delta \lambda = .314$, WITH HEATING AND FRICTION, 47.8 DAYS AFTER INITIALIZATION

FIGURE 5.6 MEAN TEMPERATURE AT LEVEL 2, (RELATIVE TO RADIATIVE EQUILIBRIUM VALUE EQUATOR) SHOWING EFFECTS OF WAVE PERTURBATIONS
FIGURE 5.7 MEAN VERTICAL VELOCITY AT LEVEL 2, SHOWING EFFECTS OF WAVE PERTURBATIONS.
FIGURE 5.8 MEAN ZONAL WINDS AT LEVEL I, SHOWING EFFECTS OF WAVE PERTURBATIONS.
STRONG ITCZ-BASIC STATE.

(b) PERTURBED STATE, Δλ = .314, ZERO HEATING AND FRICTION IN PERTURBATION EQUATIONS, 19.1 DAYS AFTER INITIALIZATION.

(c) AS (b), 23.9 DAYS AFTER INITIALIZATION.

(d) PERTURBED STATE, Δλ = .314, WITH HEATING AND FRICTION, 47.8 DAYS AFTER INITIALIZATION.

FIGURE 5-4 MEAN ZONAL WINDS AT LEVEL 3 SHOWING EFFECTS OF WAVE PERTURBATIONS.
(a) $\Delta \lambda = .314$, no friction or condensational heating in perturbation equations.

(b) $\Delta \lambda = .314$, friction ($K = .015$) but no condensational heating in perturbation equations.

(c) $\Delta \lambda = .314$ all friction and heating terms included.

Figure 5.10 Growth of perturbation kinetic energy.
FIGURE 5.11 ENERGETICS OF THE DISTURBANCE.
FIGURE 5.12 STREAM FUNCTION AMPLITUDES AT LEVEL 1.
FIGURE 5.13 STREAM FUNCTION AMPLITUDES AT LEVEL 3.
Figure 5.14  Vertical velocities and perturbation temperature.
Figure 5.15  Perturbation geopotential.
CASE WHERE $\hat{K}$ IS CONSTANT

(b) CASE WHERE $\hat{K}$ VARIES WITH $|\vec{u}_3|$

FIGURE 5.16 GROWTH OF PERTURBATION KINETIC ENERGY.
(a) THE GROWING STAGE (11.95 DAYS)
(b) THE MATURE STAGE (23.9 DAYS)

FIGURE 5.17 ENERGETICS OF THE GROWING AND MATURE DISTURBANCE FOR THE CASE OF VARIABLE K.
FIGURE 5.18 GROWTH OF PERTURBATION KINETIC ENERGY.
FIGURE 5.19 ENERGY CONVERSIONS OF THE J=6 DISTURBANCE.
FIGURE 5.20 MEAN TEMPERATURE AT LEVEL 2 (RELATIVE TO RADIATIVE EQUILIBRIUM VALUE AT THE EQUATOR).
FIGURE 5.21 VERTICAL VELOCITY AT LEVEL 2.
FIGURE 5.21 ZONAL WIND AT LEVEL 1.
FIGURE 5.23 ZONAL WINDS AT LEVEL 3.
Figure 5.24  Geopotential at level 3 (meters),
relative to value at equator.
Time = 23.5 days.
Figure 5.25  Geopotential at level 3 (meters), relative to value at equator.  
Time = 24 days.
Figure 5.26  Geopotential at level 3 at 13° latitude (meters).
Figure 5.27 Geopotential at level 1 at 13° latitude (meters).
Figure 5.28  Vertical velocity (cm/sec) at level 2.  
Time = 24 days.
Figure 5.29 Vertical velocity at level 2 at latitude 13°, time = 24 days.
Figure 5.30 Vertical Pumping out of boundary layer (cm/sec).
Time = 24 days.
Figure 5.31 Vertical velocity (cm/sec) at level 2.
Time = 12 days.
FIGURE 5.32 TEMPERATURE AT LEVEL 2 (RELATIVE TO RADIATIVE EQUILIBRIUM VALUE AT $\phi = 0$). TIME = 24 days
FIGURE 5.33  EDDY TRANSPORT OF MOMENTUM AND ENERGY. TIME = 24 DAYS
FIGURE 5.34  DISTRIBUTION OF CONDENSATIONAL HEATING AND BOUNDARY LAYER PUMPING AT 14° 30', TIME = 24 DAYS.
\[ \begin{align*}
\{q'_c, e'_p\} &= 1.292 \times 10^{-3} \\
\{q'_r, e'_p\} &= 1.130 \times 10^{-3} \\
\text{TOTAL} &= 1.153 \\
\{e'_p, e'_p\} &= -0.812 \times 10^{-12} \\
\{q'_c, e'_p\} &= 0.350 \times 10^{-10} \\
\{q'_r, e'_p\} &= -0.003 \times 10^{-10} \\
\text{TOTAL} &= 0.1496 \times 10^{-10} \\
\{e'_p, e'_p\} &= 0.343 \times 10^{-10} \\
\{e'_p, e'_p\} &= 0.1824 \times 10^{-8} \\
\{e'_p, e'_p\} &= 0.158 \times 10^{-4} \\
\text{TOTAL} &= 1.292 \times 10^{-10}
\end{align*} \]

Fig. 5.35. ENERGY CONVERSIONS AT 4.8 DAYS.
Figure 5.36  Mean Meridional Velocity.
Chapter 6. Summary and Conclusion

In this thesis, the dynamics of perturbations on a theoretically derived ITCZ have been studied. The model has been of an ocean-covered hemisphere with no asymmetries due to continental influences included. The driving mechanisms for the circulation have been radiational cooling, which depends on the temperature at the middle level of the atmosphere, and conden-sational heating in the Tropics, which depends on the low level wind field.

In the case where the motion was constrained to be zonally symmetric, it was seen how an ITCZ developed whose influence extended well into middle latitudes. As compared with the purely radiatively driven circulation, the effect of the strong conden-sational heating was to concentrate the rising motion into a narrow band near, but not at, the equator. The position of this band could be varied by changing the frictional and heating parameters of the model. The basic state which was chosen for the study of perturbations had the ITCZ at a greater distance from the equator than is normally observed, corresponding perhaps to an extreme excursion of the ITCZ into the summer hemisphere. This was de-sirable since the dynamics of the perturbations were being studied in the quasi-geostrophic framework.

The effect of the condensational heating on the low level wind field was to increase both the easterlies at low latitudes
and the westerlies at higher latitudes. To the south of the latitude of maximum easterlies, at the northern extremity of the ITCZ, there was a region of strong cyclonic shear. Observations show that this shear in the low level wind field across the ITCZ does indeed exist, though it is usually concentrated into a small number of mesoscale bands, so that the present model represents a smoothing.

The ITCZ also causes a considerable strengthening of the upper level westerlies. This influence is exerted through the temperature field. It was seen that, in contrast to the purely radiatively driven case, the temperature is everywhere above the radiative equilibrium value. The equator-to-pole temperature difference, which in the case of the dry circulation is less than the radiative equilibrium temperature difference, becomes greater than the radiative difference with the growth of the ITCZ.

The resulting flow is baroclinically unstable in middle latitudes. An experiment was performed showing the growth of a baroclinic wave for the dry case. It grew exponentially to finite amplitude and its energy then oscillated with an irregular period. In experiments with the strong ITCZ basic state, the release of baroclinic instability was prevented by choosing stable wavelengths. Since all perturbations were assumed to be periodic in longitude, periodicities could be chosen corresponding to linear wavelengths which were long enough, both from the viewpoint of observations and of theory, for tropical perturbations, while at the same time being baroclinically stable at middle latitudes. Thus, attention
was confined to perturbations on the ITCZ.

The low level wind field in the vicinity of the ITCZ was found to be barotropically unstable. The wavelength of maximum growth rate was about 2000 km, corresponding to the observed average wavelength of tropical disturbances. With both frictional retardation and self-heating by condensation omitted in the dynamics of the perturbations, the e-folding time of the perturbation energy for the dominant wavelength was 1.2 days. Inclusion of the frictional terms slowed down the growth rate and in one case, where the ground friction parameter was large, actually prevented the wave from growing.

Inclusion of self-heating counteracted the effects of friction, as was to be expected. With all terms active, and with reasonable values of the parameters, an e-folding time for perturbation energy of 3 days was found.

Increasing the number of wave components, which allowed the perturbation quantities to assume non-sinusoidal forms and introduced the non-linear interaction between components of different wavelength, made little qualitative difference to the results. With 2000 km as the basic periodicity at the equator, nearly all the energy remained in the longest wavelength.

In the growing stage, the perturbation derived its energy mainly by Reynolds stress conversion from the mean flow kinetic energy. After reaching a mature stage, the Reynolds stress conversion was still in the same direction but the dissipation of perturbation energy was now balanced mainly by direct conversion
to kinetic energy of the condensationally produced eddy available potential energy.

The perturbation amplitude was 6 to 10 times greater at the lower level than at the upper level and was maximum around the latitude of the ITCZ. It decreased to zero beyond 30° and also at the equator. The decrease towards the equator was not too abrupt, showing that the effect of the wall is not drastic.

The disturbances show a warm core structure with an upper ridge to the east of the surface trough. The maximum temperature and vertical motion in the middle troposphere almost coincide with the position of the surface trough. This corresponds with the positive nature of the potential to kinetic energy conversion. The warm core structure is a necessary consequence, hydrostatically, of the way the amplitude decreases with height.

The field of vertical motion, which at the initialisation of the perturbation consisted of a zonal pattern with a strong band of rising motion in the ITCZ, evolved to a highly asymmetric pattern with strong rising motion surrounded by weak sinking motion. The maximum value of the rising motion was about 2.2 cm/sec.

This pattern propagated towards the west at about 6.4 m/sec, corresponding very well with the observed rate of propagation of tropical disturbances.

The perturbations caused considerable changes in the mean flow. In the case of the free perturbations (without self-heating or friction) the effect was to increase the mean temperature in the Tropics and to increase the shear in the low level wind. In
the case with all terms included, and with three separate wavelengths, the mean temperature decreased somewhat below the unperturbed case and the shear in the low level wind was lessened. With smaller values of the frictional coefficients, it is to be expected that there would be a closer resemblance to the free perturbation case.

The picture which emerges is no longer of a strictly zonally symmetric ITCZ but of a series of wave perturbations having strong centres of rising motion propagating towards the west. The mean winds at the lower level, even when the perturbations have reached finite amplitude, are barotropically unstable. This agrees with the findings of Nitta and Yanai (1969) who studied the barotropic stability of mean wind profiles taken from observations in the Marshall Islands. It had generally been believed (see, for example, Lorenz 1967, p. 142) that the zonally averaged winds in the atmosphere are nearly always baroclinically unstable but barotropically stable.

The question arises as to what mechanism would cause tropical perturbations as studied in this thesis to amplify further and become hurricanes. One possibility is that the boundary layer humidity could be a function of the wind speed, increasing with the growth of the perturbation, rather than remaining constant as assumed here. Garstang (1967), in an observational study in the low latitude western Atlantic, has found that in disturbed conditions the transfer of latent and sensible heat from the ocean to the atmosphere increases by an order of magnitude. Allowing,
the boundary layer humidity to increase with the wind speed would cause an additional tendency to amplification.

The 'ventilation' due to the vertical shear of the wind, suggested by Gray (1968) as an influence which suppresses the growth of tropical disturbances, may also be operative here. In all cases studied, there was a vertical shear, though not a large one, in the mean zonal wind at the latitude of maximum development of the perturbation. The heating due to condensation depends only on the wind at the lower level while the advection of temperature at the middle level depends on the mean of the upper and lower level winds. Thus the presence of vertical shear implies that heat is advected away from the region of condensation, preventing concentrated warming of the atmosphere. Continental and seasonal influences are important in eliminating this vertical shear.

It is to be noted that none of the perturbations studied here had a cold-core temperature structure, though observations suggest this is a common phenomenon. The condensational heating mechanism used in this thesis necessarily leads to a warm core disturbance with maximum amplitude at the lower levels. Cold core disturbances are kinetic energy consuming, though it is possible to have disturbances which are cold core at lower levels and warm core at upper levels and are in an overall sense kinetic energy producing. With better knowledge of the way in which condensational heating is distributed in the vertical, and with the cooling due to re-evaporation taken into account, it is possible that a many-level model similar to the model here could produce disturbances of this
nature. At present, observational knowledge is insufficient to differentiate clearly between the mechanisms giving rise to disturbances which are observed to be warm core and those which are observed to be cold core.

It is believed that the exclusion of vertically propagating waves from the present model is not a serious limitation for the study of tropospheric motions. Such waves lead to propagation of energy in the vertical as pointed out by Holton (1969) and other authors, but their presence is more important for motions which gain their energy from distant sources than for motions caused by energy production in situ.

A possibility for further useful studies with the model, as mentioned in §55, would be to increase the condensational heating effect in the perturbations while still taking the weak ITCZ as basic state. In this way, disturbances which grow due to condensational heating against the stabilising effect of Reynolds stresses could be examined.

The interaction between baroclinic waves at middle latitudes and the tropical circulation with an ITCZ has not been studied in this thesis. It would, however, be entirely possible to carry out such a study also within the framework of the present model. This would demand a large amount of computing time; indeed, if a sufficient number of wave components were included to study the baroclinic regime and the tropical perturbations simultaneously, the computing time required would be almost prohibitive. A great deal of experimentation would also be required to overcome the
problem of inertial instability.

Study of the tropical branch alone indicates the importance of the ITCZ and gives a rational basis for viewing tropical disturbances as necessary components of the general circulation of the atmosphere.
Appendix

Stability of the numerical schemes

In this appendix, numerical stability criteria for individual terms in the perturbation vorticity equations (2.7.5) and (2.7.6) are derived by means of a linear stability analysis.

The numerical stability properties of the equation

\[
\frac{\partial \zeta}{\partial t} = \alpha \zeta \tag{A.1}
\]

with respect to forward, centred and implicit differencing are first examined.

(a) Forward Differencing

(A.1) becomes

\[
\frac{\zeta^{n+1} - \zeta^n}{\Delta t} = \alpha \zeta^n
\]

Hence

\[
\zeta^{n+1} = (1 + \alpha \Delta t) \zeta^n
\]

\[
= (1 + \alpha \Delta t)^n \zeta'
\tag{A.2}
\]

Comparing this with the analytic solution of (A.1), viz.,

\[
\zeta = \zeta' e^{\alpha t}
\tag{A.3}
\]

it can be seen that the numerical solution approximates to the
analytic solution if

$$|\alpha \Delta t| < 1$$  \hfill (A.4)$$

If $\alpha \Delta t < -1$, the numerical solution becomes oscillatory and bears no resemblance to the analytic solution.

(b) Centred Differencing

(A.1) now becomes

$$\frac{z^{n+1} - z^{n-1}}{2\Delta t} = \alpha z^n$$

Assuming solutions of the form

$$z = z^0 e^{\sigma n \Delta t}$$  \hfill (A.5)$$

where

$$\sigma = \sigma_r + i \sigma_i$$

it is found that

$$e^{\sigma_r \Delta t} \left[ \cos \sigma_i \Delta t + i \sin \sigma_i \Delta t \right] - e^{-\sigma_r \Delta t} \left[ \cos \sigma_i \Delta t - i \sin \sigma_i \Delta t \right] = 2\alpha \Delta t$$

Separating real and imaginary parts gives

$$\left( e^{\sigma_r \Delta t} + e^{-\sigma_r \Delta t} \right) \sin \sigma_i \Delta t = 0$$

$$\left( e^{\sigma_r \Delta t} - e^{-\sigma_r \Delta t} \right) \cos \sigma_i \Delta t = 2\alpha \Delta t$$
Hence

\[ \sigma_i \Delta t = 0, \pi \]
\[ \sigma_r \Delta t = \pm \sinh^{-1}(\alpha \Delta t) \]

Thus the solution is

\[ Z^n = Z_o e^{n[\sinh^{-1}(\kappa \Delta t)]} + Z_0 e^{-n[\sinh^{-1}(\kappa \Delta t)]} (-1)^n \quad (A.6) \]

If

\[ 0 < \alpha \Delta t < 1 \quad (A.7) \]

\( \sinh^{-1}(\kappa \Delta t) \) is real and positive, the first term approximates to the analytic solution and the parasitic second term decays.

For negative \( \alpha \) however, the parasitic solution soon predominates. In this case, centred differencing cannot be used.

(c) Implicit differencing

(A.1) now becomes

\[ \frac{Z^{n+1} - Z^n}{\Delta t} = \alpha Z^{n+1} \]

Hence

\[ Z^{n+1} = \left( \frac{1}{1 - \alpha \Delta t} \right) Z^n \]
\[ = \left( \frac{1}{1 - \alpha \Delta t} \right)^n Z^1 \]

This is absolutely stable for negative \( \alpha \) and approximates to the analytic solution if
Individual terms in the vorticity equations (2.7.5), (2.7.6) will now be examined.

(i) Mean flow vorticity advection

These give

\[
\frac{\partial}{\partial t} (A + B) = (u+w) a_j C + (u-w) a_j D
\]

\[
\frac{\partial}{\partial t} (A - B) = (u+w) a_j C - (u-w) a_j D
\]

\[
\frac{\partial}{\partial t} (C + D) = -(u+w) a_j A - (u-w) a_j B
\]

\[
\frac{\partial}{\partial t} (C - D) = -(u+w) a_j A + (u-w) a_j B
\]

where

\[
\begin{pmatrix}
A \\
B \\
C \\
D
\end{pmatrix} = \begin{pmatrix}
\hat{\nabla} \times \hat{\nabla} \times \hat{\varphi}_w \\
\hat{\nabla} \times \hat{\nabla} \times \hat{\varphi}_v \\
\hat{\nabla} \times \hat{\nabla} \times \hat{\varphi}_x \\
\hat{\nabla} \times \hat{\nabla} \times \hat{\varphi}_z
\end{pmatrix}
\]

Hence

\[
\frac{\partial A}{\partial t} = (u+w) a_j C
\]

\[
\frac{\partial B}{\partial t} = (u-w) a_j D
\]

\[
\frac{\partial C}{\partial t} = -(u+w) a_j A
\]

\[
\frac{\partial D}{\partial t} = -(u-w) a_j B
\]
For the purposes of the analysis, these reduce to the two coupled equations

\[ \frac{\partial A}{\partial t} = \alpha C \]  
\[ \frac{\partial C}{\partial t} = -\alpha A \]

where

\[ \alpha = (u \pm w) a_i \]

Assuming solutions of the form

\[ \begin{bmatrix} A \\ C \end{bmatrix}^n = \begin{bmatrix} A \\ C \end{bmatrix}^0 e^{\delta n \Delta t} \]

it is found that forward differencing is unconditionally unstable.

Examining the case of centred differencing, it is found that

\[ (e^{\delta \Delta t} - e^{-\delta \Delta t}) A^0 - (2\alpha \Delta t) C^0 = 0 \]
\[ (2\alpha \Delta t) A^0 + (e^{\delta \Delta t} - e^{-\delta \Delta t}) C^0 = 0 \]

The discriminant gives

\[ e^{\delta \Delta t} - e^{-\delta \Delta t} = \pm i (2\alpha \Delta t) \]

Hence, separating real and imaginary parts,

\[ \left( e^{\delta \Delta t} - e^{-\delta \Delta t} \right) \cos \sigma_i \Delta t = 0 \]
\[ \left( e^{\delta \Delta t} + e^{-\delta \Delta t} \right) \sin \sigma_i \Delta t = \pm 2\alpha \Delta t \]
i.e.

$$\delta_i = 0$$

$$\sigma: \Delta t = \pm \delta \lambda^{-1} (\alpha \Delta t)$$

The criterion for stability is then seen to be

$$|\alpha \Delta t| < 1$$

i.e.

$$|(u \pm \omega) a_i \Delta t| < 1$$  \hspace{1cm} (A.11)

(ii) **Lateral friction**

This is given by

$$\frac{\partial}{\partial t} (\hat{\psi} \hat{\psi}) = \bar{\kappa} (\hat{\psi} \hat{\psi})$$

$$= \bar{\kappa} \left[ \left\{ (1-y^2) \frac{\partial^2}{\partial y^2} - 2y \frac{\partial}{\partial y} \right\} \hat{\psi} \hat{\psi} - \frac{\alpha^2}{1-y^2} \hat{\psi} \hat{\psi} \right]$$  \hspace{1cm} (A.12)

The first term on the right is unconditionally unstable for centred differencing (Richtmyer, 1957, p. 94).

Expressing the first two terms on the right by forward differencing over the interval $\Delta t$, it is found that

$$\frac{Z_{i+1}^n - Z_i^{n-1}}{2 \Delta t} = \bar{\kappa} \left[ (1-y^2) \frac{\partial^2}{\partial y^2} - 2y \frac{\partial}{\partial y} \right] Z_i^{n-1}$$

Assuming solutions of the form

$$Z_j = Z^0 e^{\delta \Delta t} e^{i k (\lambda y)}$$  \hspace{1cm} (A.13)
it follows that

\[
e^{-\Delta t} - e^{\Delta t} = 2\Delta t \bar{n} \left( 1 - y + \frac{\epsilon_{k^2}}{\Delta y^2} \right) - 2y \left( \epsilon_{k^2} + \frac{\epsilon}{\Delta y^2} \right) \left( e^{\Delta t} - 1 \right)
\]

i.e.

\[
e^{\Delta t} = 1 - (1-y) \left( \frac{4\bar{n}(\Delta t)}{\Delta y^2} \right) \sin^2 \frac{k_a y}{x} - i \left( \frac{4\bar{n}(\Delta t)}{\Delta y^2} \right) y \sin \frac{k_a y}{x} \cos \frac{k_a y}{x}
\]

The stability criterion for the highest order term is seen to be

\[
\varepsilon_1 \equiv \frac{4\bar{n}(\Delta t)}{\Delta y^2} < 2
\]  

(A.14)

That this also suffices to guarantee the stability of the first order term can be seen as follows:

\[
|e^{\Delta t}|^2 = \left[ 1 - (1-y)\varepsilon_1 \sin^2 \frac{k_a y}{x} \right] + \left[ \varepsilon_1 y \sin \frac{k_a y}{x} \cos \frac{k_a y}{x} \right]^2
\]

\[
\leq 1 - \varepsilon_1 \sin^2 \frac{k_a y}{x} \left[ 2(1-y) - (1-y)^2 \varepsilon_1 \sin^2 \frac{k_a y}{x} \cos^2 \frac{k_a y}{x} \right]
\]

\[
\leq 1 - \varepsilon_1 \sin^2 \frac{k_a y}{x} \left[ 2(1-y) - 2(1-y)^2 - 2y^2 \Delta y^2 \right]
\]

\[
\leq 1 - \varepsilon_1 \sin^2 \frac{k_a y}{x} \left[ 2y^2 \right] \left[ 1 - (1-y)^2 - \Delta y^2 \right]
\]

\[
\leq 1 - \varepsilon_1 \sin^2 \frac{k_a y}{x} \left[ 2y^2 \right] \left[ 1 - (1-\Delta y)^2 - \Delta y^2 \right]
\]

\[
\leq 1 - \varepsilon_1 \sin^2 \frac{k_a y}{x} \left[ 2y^2 \right] \left[ 2\Delta y \right] \left[ 1 - \Delta y \right]
\]

\[
< 1
\]
The last term on the right of (A.12) is also unconditionally unstable for centred differencing as can be seen by referring to (A.7).

Forward differencing is impractical because, in order to ensure stability near the pole, (A.4) would demand too small a time increment.

Implicit differencing is the only feasible possibility, being absolutely stable in the present case. The truncation error criterion (A.8) becomes

\[ \left( \frac{\bar{\kappa} a_i^z}{1 - y^2} \right) (2 \Delta t) < 1 \]  

(A.15)

(iii) Vertical friction

This gives

\[ \frac{d}{dt} (A + B) = 0 \]
\[ \frac{\partial}{\partial t} (A - B) = -\kappa' (A - B) \]

Applying (A.4), it is seen that forward differencing is stable provided

\[ \kappa' (2 \Delta t) < 1 \]  

(A.16)

(iv) Surface friction

This gives

\[ \frac{\partial}{\partial t} (A + B) = -\bar{\kappa} B \]
\[ \frac{\partial}{\partial t} (A - B) = \bar{\kappa} B \]
i.e.
\[
\frac{\partial A}{\partial t} = 0 \\
\frac{\partial B}{\partial t} = -\hat{\kappa} B
\]

Again applying (A.4), it is seen that forward differencing is stable provided

\[
\hat{\kappa}(2\Delta t) < 1
\]  \hspace{1cm} (A.17)

(v) Condensational heating

This gives
\[
\frac{\partial}{\partial t} (A + B)_j = 0 \\
\frac{\partial}{\partial t} (A - B)_j = \hat{\xi}_j
\]

i.e.
\[
\frac{\partial A_j}{\partial t} = \frac{\hat{\xi}_j}{2} \hspace{1cm} (A.18)
\]
\[
\frac{\partial B_j}{\partial t} = -\frac{\hat{\xi}_j}{2} \hspace{1cm} (A.19)
\]

where
\[
\hat{\xi}_j = 4\mu^* y_e \eta \hat{\omega}_e F_j
\]

For the purposes of the stability analysis, \( \eta \) is taken as a constant and the boundary layer pumping \( \hat{\omega}_e \) is linearized as in (3.1.7), giving
\( \hat{\omega}_L = \left( \frac{\kappa}{4+\kappa^2a^2} \right) \omega_L \)

\( = -\left( \frac{\kappa}{4+\kappa^2a^2} \right) \left( \rho_+ \left| \bar{u}_j \right| \frac{\theta}{\chi_c} \right) \left( \sum_{j=1}^{J} \nabla^2 \psi_j F_i \right) \)

\( = -\frac{\hat{\kappa}}{2\mu \gamma} \sum B_i F_i \)

Hence

\( \hat{\epsilon}_i = 2\eta \hat{\kappa} B_j \)

Applying criterion (A.7) to (A.19), it can then be seen that centred differencing can be used provided

\[ \eta \hat{\kappa} \Delta t < 1 \]  \hspace{1cm} (A.20)

With the differencing as expressed in (2.7.5) and (2.7.6), and with (A.11), (A.14), (A.16), (A.17) and (A.20) satisfied, the numerical integration was stable.
BIBLIOGRAPHY


BIOGRAPHICAL SKETCH

The author was born on October 24, 1940 at Kilmore Quay, County Wexford, a village on the south coast of Ireland, where he spent his childhood.

He attended secondary school at St. Peters College, Wexford, from 1953 to 1958, devoting most of his time to the study of Irish, Latin and Greek.

On obtaining a University scholarship, he decided to study physics and mathematics at University College, Dublin, which he attended from 1958 to 1962.

Summers spent fishing lobsters around the Saltee Islands off the Wexford Coast acquainted him intimately with the practical aspects of meteorology, but stormy times when the rosy-fingered dawn was not so rosy left no doubt in his mind that it was a subject better considered from a safely detached vantage point.

Graduating from college with first class honours in 1962, he shortly afterwards joined the Irish Meteorological Service and spent some time forecasting at Shannon Airport before coming to M.I.T. on a Ford Foundation Fellowship in September, 1964.