Approximating Tempered Class Rates
in Auto Insurance

by

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INTRODUCTION

In 1977, after fifty years of statemade rates, automobile insurers were authorized for the first time by the Massachusetts legislature to file individual competitive rates. Substantial changes in differentials resulted, raising questions about the justification for techniques used to price risk classes.

For example, a young male driver in urban Boston with a three year old auto and a perfect driving record might have received an annual insurance bill of $2500. An elderly driver in a rural community, driving an identical auto with identical coverage but having a poor claims record, could have paid only $160.

During the Massachusetts Division of Insurance hearings on auto insurance in October of 1977, a research paper was submitted by Dr. Joseph Ferreira, Jr., addressing this issue of equitable class rates in automobile insurance (1). Ferreira showed that all class rate alternatives result in large error distributions (undercharges and overcharges) and that in any feasible classification scheme there will be significant overlap of expected loss distributions. Therefore, the traditional actuarial rate for a class (the mean of that class' expected loss distribution) will hardly be typical of the appropriate rate for each individual in that class, and traditional techniques alone do not determine the fairest premium to charge each individual in any class.
IDENTIFYING PREFERRED RATES

Since all rating schemes result in large error distributions, choosing class rates reduces to choosing which resulting error distribution is most "equitable" (often in social or legal terms). In terms of "utility" theory, a utility function can be constructed that reflects the ratemaker's relative concern for different types and amounts of pricing errors. The "preferred" rates would then be that set of prices that maximizes expected utility while still collecting the required amount to balance expected losses in total.

Ferreira presented a method to produce exact solutions identifying preferred rates based on an exponential utility function. While the method is exact, the computations are highly complex and are based on the availability of considerable data which, at this time, is generally not available or is difficult to collect. Nevertheless, the most important features of the preferred results are easily approximated.

One method is to approximate the tempered relativities that result from Ferreira's method by mathematically raising the traditional class relativities to a power less than unity. Such an exponent would move all relativities closer to unity and provide essentially the same tempering effect as the exact method. The greater the tempering of the exact method (that is, the closer the preferred relativities are to unity), the lower the approximating exponent will be.

Table 1 shows that such an approximation can yield extremely accurate results. Using 1976 data for the 24 territorial classifications in Massachusetts, columns (3) and (4) compare traditional relativities to those implied by preferred rates obtained by Ferreira's exact method. The approximate preferred relativities of column (5) are obtained by fitting the data of
TABLE 1. Comparison of Exact and Approximated Preferred Relativities

<table>
<thead>
<tr>
<th>Territory</th>
<th>Distribution of Insureds (%)</th>
<th>Traditional Relativity</th>
<th>Exact Preferred Relativity</th>
<th>Approximate Preferred Relativities Obtained by Power Curve Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.506</td>
<td>.745</td>
<td>.789</td>
<td>.790</td>
</tr>
<tr>
<td>2</td>
<td>2.983</td>
<td>.690</td>
<td>.740</td>
<td>.742</td>
</tr>
<tr>
<td>3</td>
<td>11.581</td>
<td>.782</td>
<td>.821</td>
<td>.822</td>
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<tr>
<td>4</td>
<td>8.345</td>
<td>.806</td>
<td>.841</td>
<td>.843</td>
</tr>
<tr>
<td>5</td>
<td>13.622</td>
<td>.880</td>
<td>.904</td>
<td>.905</td>
</tr>
<tr>
<td>6</td>
<td>12.598</td>
<td>.949</td>
<td>.962</td>
<td>.963</td>
</tr>
<tr>
<td>7</td>
<td>13.198</td>
<td>.987</td>
<td>.993</td>
<td>.994</td>
</tr>
<tr>
<td>8</td>
<td>15.438</td>
<td>1.074</td>
<td>1.066</td>
<td>1.064</td>
</tr>
<tr>
<td>9</td>
<td>3.342</td>
<td>1.135</td>
<td>1.115</td>
<td>1.113</td>
</tr>
<tr>
<td>10</td>
<td>7.216</td>
<td>1.099</td>
<td>1.086</td>
<td>1.084</td>
</tr>
<tr>
<td>11</td>
<td>1.672</td>
<td>1.252</td>
<td>1.208</td>
<td>1.205</td>
</tr>
<tr>
<td>12</td>
<td>1.278</td>
<td>1.330</td>
<td>1.270</td>
<td>1.266</td>
</tr>
<tr>
<td>13</td>
<td>.840</td>
<td>1.461</td>
<td>1.372</td>
<td>1.367</td>
</tr>
<tr>
<td>14</td>
<td>.889</td>
<td>1.647</td>
<td>1.512</td>
<td>1.508</td>
</tr>
<tr>
<td>15</td>
<td>.349</td>
<td>1.401</td>
<td>1.325</td>
<td>1.321</td>
</tr>
<tr>
<td>16</td>
<td>.459</td>
<td>1.518</td>
<td>1.415</td>
<td>1.411</td>
</tr>
<tr>
<td>17</td>
<td>.458</td>
<td>1.518</td>
<td>1.415</td>
<td>1.411</td>
</tr>
<tr>
<td>18</td>
<td>.351</td>
<td>1.692</td>
<td>1.546</td>
<td>1.541</td>
</tr>
<tr>
<td>19</td>
<td>1.191</td>
<td>2.214</td>
<td>1.917</td>
<td>1.919</td>
</tr>
<tr>
<td>20</td>
<td>.265</td>
<td>2.370</td>
<td>2.022</td>
<td>2.028</td>
</tr>
<tr>
<td>21</td>
<td>.559</td>
<td>2.060</td>
<td>1.810</td>
<td>1.809</td>
</tr>
<tr>
<td>22</td>
<td>.385</td>
<td>1.335</td>
<td>1.274</td>
<td>1.270</td>
</tr>
<tr>
<td>23</td>
<td>.200</td>
<td>2.370</td>
<td>2.022</td>
<td>2.028</td>
</tr>
<tr>
<td>24</td>
<td>.282</td>
<td>2.370</td>
<td>2.022</td>
<td>2.028</td>
</tr>
</tbody>
</table>
columns (3) and (4) to a power curve of the general form

\[(\text{Preferred Relativity}) = a \cdot (\text{traditional relativity})^b\]

and then rebalancing so that the relativities average unity across all classes. Comparing columns (4) and (5) shows the exponential approximation to be a good one. In fact, the exponent of .815 implied by the above fit results in a coefficient of determination of .99995.

**ROBUSTNESS OF APPROXIMATING EXPONENT**

To determine the robustness of such an approximation, it is necessary to analyze the effect of various factors on the determination of the approximating exponent. Of primary concern are:

1. the distribution of insureds among risk classes,
2. the heterogeneity of risk classes, and
3. the ratemaker's aversion to unequal pricing errors, since it is the consideration of these factors that leads to preferred rates different from the traditional "class average" rates.

To analyze the effect of these factors on the determination of the approximating exponent, the exponent must be equated to some function of these factors. For simplicity of calculations, it will be assumed that only one "low-risk" class and one "high-risk" class exist. (This assumption is analyzed in a later section of this report.)

As an approximation of the exact solutions, preferred relativities are to be expressed as the traditional relativities raised to some power less than unity. That is,

\[(1) \quad (\text{Preferred Relativity}) = (\text{Traditional Relativity})^{\text{EXP}}\]
The traditional relativity of a class is simply the ratio of the average expected loss of a person in that class to the average expected loss of all insureds. In the traditional method, the class rate equals the class average expected loss. (In practice, of course, this average cost must be loaded for taxes and profits. Such loading can be ignored in this analysis without significantly affecting the results.) Denoting the class average as "m", the overall population average as "M", and the preferred rate by "P", equation (1) becomes:

\[(P/M) = (m/M)^{EXP}\]

Solving for EXP,

\[EXP = \left[\ln\left(\frac{P}{M}\right)\right] / \left[\ln\left(\frac{m}{M}\right)\right]\]

Ferreira's exact method uses a utility function of the form

\[U = 1 - e^{-c}\]

The parameter "c" measures the ratemaker's aversion to unequal pricing errors, with higher values of c indicating less inequality aversion. The "error" is the difference between an insured's "true" expected loss and the actual premium he is charged.

From the exact solution for the two-class case (see Appendix B, Ferreira and Peters):

\[P_1 = M + C(1-N) \left\{ \ln \left[ \frac{\text{rate}_1/m_1}{\text{rate}_1/m_1 + 1/c} \right] \right\} \]

and

\[P_2 = M + (-C)N \left\{ \ln \left[ \frac{\text{rate}_2/m_2}{\text{rate}_2/m_2 + 1/c} \right] \right\} \]
where the subscripts "1" and "2" refer to the low-rated and high-rated classes respectively. "P", "M", "C" and "n" are defined above. "N" is the fraction of insureds in the low-rated class. "k" is the square of the reciprocal of the coefficient of variation for the underlying Pearson Type III claim frequency distribution. It is a measure of relative class homogeneity, with higher values of "k" indicating higher class homogeneity.

The overall population average is simply the sum across all classes of the fraction of insureds in a class times the class average. Thus, for the two-class case:

\[ M = Nm_1 + (1-N)m_2 \]

Since \( P_1 \) will generally not equal \( P_2 \), equation (3) will produce a different exponent for each class. The single value for the exponent that will give the best power curve fit will lie somewhere between these two extreme values. For simplicity, and without loss of generality, one of the "extreme point" exponents can be analyzed as well as the "best fit" exponent. Therefore:

\[
\text{EXP} = \frac{\ln(P/M)}{\ln(D/M)} \quad \text{where} \quad \begin{cases} 
    P = Nm_1 + (1-N)m_2 + C(1-N) \left\{ \left[ \frac{\hat{k}_1/m_2}{\hat{k}_2/m_1 + 1/C} \right]^{2} \right. \\
    M = Nm_1 + (1-N)m_2 \\
    D = m_1 
\end{cases}
\]

Equation (7) equates the approximating exponent to a function of the distribution of insureds among risk classes ("N"), the heterogeneity of risk classes ("k_1" and "k_2"), and the ratemaker's aversion to unequal pricing errors ("C"). The "elasticity" of the exponent with respect to each factor can now be calculated. (4)
The elasticity of $\text{EXP}$ with respect to any variable "$X$" is defined as

$$\frac{\frac{\partial (\text{EXP})}{\partial X}}{\text{EXP}}$$ which equals \( \frac{X}{\text{EXP}} \frac{\partial (\text{EXP})}{\partial X} \)

Using equation (7), and representing a partial derivative with respect to "$X$" by "prime" notation:

$$\text{(8) elasticity of } \text{EXP} \text{ with respect to } X = \frac{\frac{\partial}{\partial X} \left[ \ln \left( \frac{\text{PM'}}{\text{PM}} \right) \right] + \frac{\partial}{\partial X} \left[ \ln \left( \frac{\text{PM'}}{\text{PM}} \right) \right] \frac{\text{PM'}}{\text{PM}}}{\ln \left( \frac{\text{PM'}}{\text{PM}} \right) + \ln \left( \frac{\text{PM'}}{\text{PM}} \right)}$$

The resulting elasticities with respect to $N$, $k_1$, $k_2$, and $C$ are summarized in Table 2 below. The derivations and numerical examples are shown in Appendices 1-3.

RESULTS FOR A BASE CASE

Figure 1 shows the expected loss distributions for a low-rated class with $k_1 = 5.88$ and $m_1 = \$166.67$, and a high-rated class with $k_2 = 4$ and $m_2 = \$300$, assuming underlying Pearson Type III distributions. Note that the low-rated class, being more homogeneous, is more "tightly" distributed about its mean. Note also that, because of imperfect classification, the distributions have some overlap.

Table 3 summarizes the resulting "point" elasticities for a base case in which 90% of the insureds are in the low-rated class, 10% are in the high-rated class, and the ratemaker's aversion to unequal pricing errors is quantized by a $C$ value of 100.
Table 2. Elasticity of Approximating Exponent-General Equations

<table>
<thead>
<tr>
<th>FACTOR</th>
<th>ELASTICITY WITH RESPECT TO FACTOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) N</td>
<td>[ N \cdot \left{ \left[ \ln \left( \frac{p_m}{p_m'} \right) \right] \left[ \frac{mp' - p'<em>{M'}}{p} \right] + M \left[ \ln \left( \frac{p_m}{p</em>{M'}} \right) \right] \right} \right} ]</td>
</tr>
<tr>
<td>(2) (k_1)</td>
<td>[ \frac{k_1 p'}{p \cdot \left[ \ln \left( \frac{p_m}{p_{M'}} \right) \right]} ]</td>
</tr>
<tr>
<td>(3) (k_2)</td>
<td>[ \frac{k_2 p'}{p \cdot \left[ \ln \left( \frac{p_m}{p_{M'}} \right) \right]} ]</td>
</tr>
<tr>
<td>(4) C</td>
<td>[ \frac{CP'}{P \cdot \left[ \ln \left( \frac{p_m}{p_{M'}} \right) \right]} ]</td>
</tr>
</tbody>
</table>
Figure 1. Pearson Type III Expected Loss Distributions

Low-Rated Class

High-Rated Class

Distribution of Expected Loss

$0 \quad $166.67 \quad $300

Expected Loss
Table 3. Elasticity of Approximating Exponent—Base Case

<table>
<thead>
<tr>
<th>FACTOR</th>
<th>ELASTICITY OF EXPONENT WITH RESPECT TO FACTOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Fraction of insureds in low-rated class (N)</td>
<td>+.142</td>
</tr>
<tr>
<td>(2) Homogeneity of low-rated class ($k_1$)</td>
<td>-.224</td>
</tr>
<tr>
<td>(3) Homogeneity of high-rated class ($k_2$)</td>
<td>+.695</td>
</tr>
<tr>
<td>(4) Aversion to unequal pricing errors (C)</td>
<td>+.471</td>
</tr>
</tbody>
</table>
A positive elasticity indicates that an increase in the specified factor produces an increase in the approximating exponent, which corresponds to a decrease in the tempering effect on the preferred relativities. The point elasticity of +.142 for N means that, from an initial base case as given, an infinitesimal increase in N will cause an increase in the approximating exponent .142 as large as the increase in N. As one would expect intuitively, increasing the fraction of insureds in the low-rated class, increasing the homogeneity of the high-rated class, and increasing the value of C (thereby decreasing aversion to unequal pricing errors) all result in less tempering.

Not so intuitive is the fact that increasing the homogeneity of the low-rated class increases tempering. As shown in Part III of Appendix 2, if the two classes are "equally homogeneous," an increase in homogeneity decreases tempering. In the base case of Table 3, however, the high-rated class is less homogeneous than the low-rated class. It can be seen that, in such a case, an increase in homogeneity of the low-rated class makes the high-rated class even less homogeneous relative to the low-rated class, and the overall effect is an increase in tempering.

Numerical examples of small changes in each specified factor are shown in the appendices. The resultant elasticities are quite close to the "point" values determined by substituting the base case data into the analytical results.

INTERPRETING THE RESULTS

While the exponent is inelastic with respect to all factors, it is particularly so with respect to the distribution of insureds among risk classes. While the difference may not appear large on the surface, it becomes significant when one considers that the distribution of insureds in actual data is
the factor that varies the least between sovereignties responsible for setting rates. For example, while it is generally true in all states that the highest-rated classes are the least populated, the use of different risk factors and classification schemes by different states will result in substantially different "k" values even if the true underlying risk distributions were identical for each state.

Some numerical examples will clarify this conclusion. Experience with real data would indicate that "N" could reasonably vary from .9 to .5, while "k" values are generally equal across classes and vary from .5 to greater than 4. Decreasing N from one end of the spectrum to the other produces only a 6.6% decrease in the approximating exponent. If k is increased from 1 to 8, however, a 159% increase occurs in the exponent. Similarly, varying C from 100 to 500 appears to span the range of reasonable amounts of inequality aversion. Increasing C through this range produces a 54.2% increase in the exponent.

Table 4 compares the changes that occur in the exponent when each factor is varied throughout its feasible range.

It appears that varying the fraction of insureds in each class has very little impact on the determination of the approximating exponent. This conclusion is reinforced when Table 5 is considered.

For the base case \( m_1 = 166.67, m_2 = 300, k_1 = 5.88, k_2 = 4, C = 100, N = .9 \), the preferred rates obtained by Ferreira's exact method are $172.29 for the low-rated class and $249.39 for the high-rated class. These preferred rates imply an approximating exponent of .56883 for the traditional low-rated class relativity.
### Table 4. Changes in Exponent Implied by Varying Factors

**BASE CASE DATA:** $m_1 = 166.67, m_2 = 300, k_1 = 5.88, k_2 = 4, C = 100, N = 0.9$

**EXponent IMPLIED BY CASE BASE:** 0.56883

<table>
<thead>
<tr>
<th>FACTOR VARIED</th>
<th>EXPONENT IMPLIED</th>
<th>% CHANGE FROM BASE CASE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 0.5$</td>
<td>0.53123</td>
<td>-6.6</td>
</tr>
<tr>
<td>$C = 250$</td>
<td>0.77784</td>
<td>+36.7</td>
</tr>
<tr>
<td>$C = 500$</td>
<td>0.87736</td>
<td>+54.2</td>
</tr>
<tr>
<td>$k_1 = k_2 = 1$</td>
<td>0.29635</td>
<td>-47.9</td>
</tr>
<tr>
<td>$k_1 = k_2 = 8$</td>
<td>0.76847</td>
<td>+35.1</td>
</tr>
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</table>
Table 5. Comparison of Exact and Approximate Preferred Rates

<table>
<thead>
<tr>
<th></th>
<th>N = .9</th>
<th></th>
<th>N = .81</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Exact</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Traditional</td>
<td>Preferred</td>
<td>Traditional</td>
<td>Preferred</td>
</tr>
<tr>
<td>Low-Rated Class</td>
<td>$166.67</td>
<td>$172.29</td>
<td>$166.67</td>
<td>$177.351</td>
</tr>
<tr>
<td></td>
<td>$180.00</td>
<td>$249.39</td>
<td>$249.39</td>
<td>$254.451</td>
</tr>
<tr>
<td>Population Average</td>
<td>180.00</td>
<td>192.00</td>
<td>192.00</td>
<td>192.000</td>
</tr>
<tr>
<td>High-Rated Class</td>
<td>300.00</td>
<td>255.299</td>
<td>300.00</td>
<td>255.299</td>
</tr>
</tbody>
</table>
If the distribution of insureds is shifted so that only 81% are in the low-rated class, the exact preferred rates become $177.351 and $254.451. (Note that the traditional premiums are the same regardless of the value of N.) Thus the preferred rates increased in both classes. The reason for this is that by decreasing N to .81, the overall population average has increased to $192.

If we use the exponent determined from the N = .9 case, .56883, to temper the traditional premiums in the N = .81 case, we get approximate preferred premiums of $177.152 and $255.299. The small difference between this approximation and the exact results (less than 85¢; i.e., less than 1/3 of one percent) would indicate that it is not necessary to do the exact calculations for every case. The exponent implied by the base case can be used to determine preferred premiums regardless of the value of N.

GENERALIZING THE RESULTS

To generalize the results of the preceding sections, the assumptions on which those results are based must be examined. There are two major assumptions that occur in this analysis: only two risk classes are assumed to exist, and expected loss distributions within each risk class are assumed to be Pearson Type III distributions. The first assumption was made to simplify the calculations of elasticities as described in the appendices, while the second assumption is made in the original work by Ferreira in which the exact method for determining preferred rates is presented.

Extending the results of this analysis to the multi-class case requires a mathematical analysis beyond the scope of this report. Nevertheless, the results can be expected to be similar to those for the two-class case; that is, the approximating exponent would remain least sensitive to changes in
the distribution of insureds among risk classes. This is due to a certain "balancing" effect that occurs in equation (7) when N is varied. Changing C, k_1, or k_2 affects only the "P" term of this equation, while changing N produces "balancing" changes in both the "P" and "M" terms, so that the overall change in the exponent is diminished.

The exact solution for the two-class case is determined by solving two equations in two unknowns (the two preferred rates). Extension to a multi-class case means simply more equations in more unknowns, but all the equations will be similar in form to Equations (4) and (5). Thus the balancing effect discussed above will continue to exist in the multi-class case, and results similar to those obtained in this analysis can be expected.

The assumption of Pearson Type III distributions within each class is well documented in the actuarial literature.(7) Nevertheless, important differences may occur in preferred rates if distributions of different shapes are assumed.

For example, consider a population with only two underlying risks: low risks with expected losses of $150, and high risks with expected losses of $450. A low-rated class containing 85 low risks and 5 high risks, and a high-rated class with 5 low and 5 high risks, produce two classes where m_1 = 166.67, m_2 = 300, k_1 = 5.88, k_2 = 4, and N = .9. (If we had access to some "perfect" risk factor, of course, there would be no misclassification and the resulting class averages of $150 and $450 would be the appropriate rates.)

The expected loss distributions for these two classes would be as shown in Figure 2. Note that while the class distributions of Figure 2 are discrete and bimodal, they have the same mean and variance as the continuous Pearson Type III distributions of Figure 1.
Figure 2. Bimodal Expected Loss Distributions

LOW-RATED CLASS

Expected Loss

HIGH-RATED CLASS

Expected Loss
The exact solution for this bimodal case is obtained by maximizing expected utility subject to the "adequacy" constraint, just as in the Pearson Type III case, except that there are now only four possible errors considered in the utility function as opposed to a continuum of possible errors in the Pearson Type III case. This solution is shown in Appendix 4.

Table 6 compares the preferred rates implied by a bimodal distribution to those implied by a Pearson Type III distribution with the same mean and variance (and thus the same "k"), for the base case shown in Figure 2. In this case, more tempering results when the bimodal assumption is made. The implied exponent is .43304, 24% lower than the exponent of .56883 implied by the Pearson Type III distribution.

A more complete analysis of the sensitivity of the results of this report to the underlying loss distributions would best be addressed in a separate report.

CONCLUSION

This report has investigated the degree to which (1) the distribution of insureds among risk classes, (2) the heterogeneity of risk classes, and (3) the ratemaker's aversion to unequal pricing errors affect the determination of an approximating exponent for tempering class relativities in order to obtain equitable class rates in automobile insurance. Using a simplified two-class case, the elasticity of the exponent was calculated with respect to each of the above factors, and it was shown that the distribution of insureds among risk classes has very little impact on the determination of the approximating exponent.

These results indicate that the power curve approximation is generally robust. The implication is that preferred rates for populations having similar homogeneity characteristics can be well approximated by the same
Table 6. Comparison of Preferred Rates Implied by Bimodal vs. Pearson Type III Distributions

<table>
<thead>
<tr>
<th></th>
<th>Traditional Premium</th>
<th>Preferred Assuming Pearson Type III</th>
<th>Preferred Assuming Bimodal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low-Rated Class</strong></td>
<td>$166.67</td>
<td>$172.29</td>
<td>$174.10</td>
</tr>
<tr>
<td><strong>High-Rated Class</strong></td>
<td>300.00</td>
<td>249.39</td>
<td>233.10</td>
</tr>
</tbody>
</table>
exponent, so that the exact solution and corresponding exponent need be computed only once.

Finally, the sensitivity of these conclusions to the assumptions of the analysis was discussed, and suggestions were made for further study.
NOTES


2. The derivation of the exact method is contained in Appendix B, "Formulae for Computing Preferred Rates," by J. Ferreira and S. Peters, Massachusetts State Rating Bureau. Briefly, expected loss distributions are assumed to be Pearson Type III distributions. The problem of maximizing expected utility, subject to the "adequacy" constraint that collected premiums equal expected losses, is then solved using a Lagrange multiplier approach.

3. Data in the first four columns of Table 1 is extracted from Table 5, p. 101, of reference in note 1 above.

4. The concept of "elasticity" arises mainly in the field of economics. The formal definition, as shown in the text, is in terms of partial derivatives; in the context of this report, it is basically the ratio of the percentage change in the exponent to the percentage change in the specified factor. A ratio resulting in an absolute value less than unity is defined as "inelastic."

5. See Ferreira's notes 16, 33, and 38 to better conceptualize "equal homogeneity."

6. For a discussion of the value judgments implied by various values of C, see pp. 91-95, and figure 8 in particular, of the reference in note 1 above.

APPENDIX 1. Elasticity of EXP with Respect to N

Since "D" in equation (7) of the text equals "m_1", D' = \( \frac{\partial D}{\partial N} = 0 \).

Therefore, equation (8) of the text is reduced to:

\[
\text{elasticity} = \frac{N \cdot \left\{ \ln \left( \frac{P M}{p} \right) \left[ \frac{M P' - P M'}{P} \right] + M \cdot \left[ \ln \left( \frac{P M}{p} \right) \right] \right\}}{M \cdot \left[ \ln \left( \frac{P M}{p} \right) \right] \cdot \left[ \ln \left( \frac{P M}{p} \right) \right]}
\]

From equation (7) of the text,

\[
P' = \frac{\partial P}{\partial N} = m_1 - m_2 - C \cdot \left\{ \ln \left( \frac{\frac{P}{M_1} + \frac{P}{M_2}}{M_1/M_2 + 1/C} \right) \right\}
\]

and

\[
M' = \frac{\partial M}{\partial N} = m_1 - m_2
\]

Assuming a base case of \( m_1 = 166.67 \), \( m_2 = 300 \), \( k_1 = 5.88 \), \( k_2 = 4 \), \( C = 100 \), and \( N = 0.9 \) implies:

\[
\begin{align*}
P &= 172.29 \\
M &= 180 \\
D &= 166.67 \\
P' &= -56.222 \\
M' &= -133.33
\end{align*}
\]

which, when substituted into the elasticity equation above, determine a point elasticity of 0.142.

EXAMPLE: Using the base case data, equation (7) of the text \( \Rightarrow \) EXP = 0.56883

Decreasing N to 0.81, equation (7) of the text \( \Rightarrow \) EXP = 0.56088

Check: \( \frac{\partial \text{EXP}}{\partial N} = -0.014 \Rightarrow -0.100 = 0.140 \), very close to the 0.142 expected for an infinitesimal change in N.
In this case, \( M' = D' = 0 \). Therefore, equation (8) in the text reduces to:

\[
(1) \quad \text{elasticity} = \frac{k_1 \cdot \left\{ \ln \left( \frac{M'}{P} \right) \right\}}{M \cdot [\ln (P/M)]} = \frac{k_1 P'}{P \cdot [\ln (P/M)]}
\]

From equation (7) in the text:

\[
P' = C(1-N) \left[ -\frac{C}{k_1/m_1} \left( \frac{k_1/m_1}{k_1/m_1 + 1/c} \right) \left( \frac{1}{k_1/m_1 + 1/c} \right) - \left( \frac{1}{k_1/m_1 + 1/c} \right) \right]
\]

Assuming the same base case as before implies:

\[
P = 172.29
\]

\[
M = 180
\]

\[
D = 166.67
\]

\[
P' = 0.287
\]

which in turn imply an elasticity of \(-0.224\).
EXAMPLE:

With base case data, $P_1 = 172.29$, $P_2 = 249.39$, and $\text{EXP} = .56883$

Increasing $k_1$ to 6.00, $P_1 = 172.32$, $P_2 = 249.09$, and $\text{EXP} = .56693$

Check: $\frac{\% \Delta \text{EXP}}{\% \Delta k_1} = -0.00334 = -0.164$
APPENDIX 2, PART II: Elasticity of EXP with respect to $k_2$

As in PART I of this Appendix, $M' = D' = 0$. Therefore, equation (8) in the text reduces to:

$$\varepsilon_{\text{elasticity}} = \frac{\frac{\partial \varepsilon}{\partial k_2} P'}{P'. [\frac{\partial \varepsilon}{\partial k_2} (\frac{P}{M})]}$$

From equation (7) in the text:

$$P' = C(1-N) \left[ \frac{k_{11}/m_1}{(k_{11}/m_1 + \sqrt{\tau})} \right] \left[ \frac{k_{12}/m_2}{(k_{12}/m_2 + \sqrt{\tau})} \right] \left[ \frac{k_{13}/m_3}{(k_{13}/m_3 + \sqrt{\tau})} \right] \left[ \frac{k_{14}/m_4}{(k_{14}/m_4 + \sqrt{\tau})} \right]$$

which parallels the same expression in PART I and reduces to:

$$P' = C(1-N) \left[ \frac{m_2}{C k_2 + m_2} + \ln \left( \frac{k_{12}/m_2}{k_{12}/m_2 + \sqrt{\tau}} \right) \right]$$

Assuming the same base case as before implies:

$$P = 172.29$$
$$M = 180$$
$$D = 166.67$$
$$P' = -1.31$$

which in turn imply an elasticity of .695.

**EXAMPLE:** With base case data, $P_1 = 172.29$, $P_2 = 249.39$, and $\text{EXP} = .56883$

Increasing $k_2$ to 4.25, $P_1 = 171.975$, $P_2 = 252.225$, and $\text{EXP} = .59261$

Check: $\frac{\% \Delta \text{EXP}}{\% \Delta k_2} = 0.0418 = 0.669$
APPENDIX 2, PART III: Elasticity of EXP with respect to $k$ when $k_1 = k_2$

Again, $M' = D' = 0$, and equation (8) in the text reduces to:

(1) \[ \text{elasticity} = \frac{\frac{k^2 P'}{P [\ln(P/M)]}} {2} \]

From equation (7) in the text:

\[ P' = C(1-N) \left[ \left( \frac{\frac{k e/m_1}{k e/m_1 + V_c}}{\frac{k e/m_1}{k e/m_1 + V_c}} \right)^{2k} \right] \left[ R - S \right] = C(1-N) \left[ \left( \frac{\frac{\beta e/m_1}{\beta e/m_1 + V_c}}{\frac{\beta e/m_1}{\beta e/m_1 + V_c}} \right)^{2k} \right] \rightarrow R - S \]

and

\[ R = \left( \frac{\beta e/m_1}{\beta e/m_1 + V_c} \right)^{2k} \left[ \left( \frac{\beta e/m_1}{\beta e/m_1 + V_c} \right)^{2k} \left( \frac{\beta e/m_1}{\beta e/m_1 + V_c} \right)^{2k} \right] \]

which reduces to:

(2) \[ P' = C(1-N) \left[ \frac{m_l}{C k e + m_l} = \frac{m_1}{C k e + m_1} + \ln(\frac{\beta e/m_1}{\beta e/m_1 + V_c}) - \ln(\frac{\beta e/m_1}{\beta e/m_1 + V_c}) \right] \]

Assuming the same base case as before, with $k = 5$, implies:

\[ P = 170.884 \]
\[ M = 180 \]
\[ D = 166.67 \]
\[ P' = -0.5732 \]

which in turn imply an elasticity of 0.323.
APPENDIX 2, PART III (continued)

EXAMPLE:

With base data, $P_1 = 170.884$, $P_2 = 262.044$, and $\text{EXP} = .67530$

Decreasing $k$ to 4, $P_1 = 171.55$, $P_2 = 256.05$, and $\text{EXP} = .62541$

Check: $rac{\Delta \text{EXP}}{\Delta k} = \frac{-.07388}{.2} = .369$
APPENDIX 3: Elasticity of EXP with respect to C

In this case, $M' = D' = 0$. Therefore equation (8) in the text reduces to:

\[(1) \text{ elasticity} = \frac{CP'}{P \cdot \Delta (P/M)}\]

From equation (7) in the text:

\[P' = (1-N) \left\{ \ln \left[ \left( \frac{P_2/m_1}{P_2/m_1 + vc} \right)^{\frac{1}{k_1}} \right] + C \left[ \frac{k_2}{k_2/m_1 + vc} \right] \right\} \left[ \frac{R - S}{(R_1/m_1)^{\frac{1}{k_1}}} \right] \]

and

\[R = \left( \frac{k_2/m_1}{k_2/m_1 + vc} \right)^k \left( \frac{k_2}{k_2/m_1 + vc} \right)^{k-1} \left[ \frac{k_2/m_1}{k_2/m_1 + vc} \right] \]

\[S = \left( \frac{k_2/m_1}{k_2/m_1 + vc} \right)^k \left( \frac{k_2}{k_2/m_1 + vc} \right)^{k-1} \left[ \frac{k_2/m_1}{k_2/m_1 + vc} \right] \]

which reduces to:

\[(2) \quad P' = (1-N) \left\{ \ln \left[ \left( \frac{P_2/m_1}{P_2/m_1 + vc} \right)^{\frac{1}{k_1}} \right] + \frac{k_2/m_1}{C k_2 + m} - \frac{k_2/m_1}{C k_2 + m} \right\} \]

Assuming the same base case as before implies:

\[P = 172.29 \]
\[M = 180 \]
\[D = 166.67 \]
\[P' = -.03554 \]

which in turn imply an elasticity of .471.

**EXAMPLE:** With base case data, $P_1 = 172.29$, $P_2 = 249.39$, and $\text{EXP} = .56883$

Increasing C to 110, $P_1 = 171.9545$, $P_2 = 252.4095$, and $\text{EXP} = .5948$

Check: $\% \Delta \text{EXP} = .0456$, $\% \Delta \text{C} = .1$, $\Delta = .456$
APPENDIX 4: Calculation of Preferred Rates in Bimodal Case (Table 6)

In this case, the utility function takes the form:

\[ U = 85(1 - e^{-\frac{P_1}{100}}) + 5(1 - e^{-\frac{P_2}{100}}) + 5(1 - e^{-\frac{P_1}{100}}) + 5(1 - e^{-\frac{P_2}{100}}) \]

We want to maximize utility subject to the constraint that premiums collected equal expected losses, which is expressed:

\[ 90P_1 + 10P_2 = 90(166.67) + 10(300) \]

which reduces to:

\[ P_2 = 1800 - 9P_1 \]

Since there is only one constraint, and it expresses \( P_2 \) in terms of \( P_1 \), the maximization problem can be solved by substituting (3) into (1) wherever \( P_2 \) occurs, differentiating (1), and setting equal to zero:

\[ \frac{dU}{dP_1} = -85 \cdot e^{-\frac{P_1}{100}} - 5 \cdot e^{-\frac{P_2}{100}} + 45 \cdot e^{-\frac{9P_1}{100}} + 45 \cdot e^{-\frac{450}{100}} = 0 \]

which reduces to:

\[ e^{\frac{P_1}{100}} (19.02) = e^{\frac{-9P_1}{100}} (692106110.9) \]

Solving for \( P_1 \) gives \( P_1 = 174.10 \), and substituting this value into equation 3 gives \( P_2 = 233.10 \)