Qubit Protection in Nuclear-Spin Quantum Dot Memories

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We present a mechanism to protect quantum information stored in an ensemble of nuclear spins in a semiconductor quantum dot. When the dot is charged the nuclei interact with the spin of the excess electron through the hyperfine coupling. If this coupling is made off-resonant, it leads to an energy gap between the collective storage states and all other states. We show that the energy gap protects the quantum memory from local spin-flip and spin-dephasing noise. Effects of nonperfect initial spin polarization and inhomogeneous hyperfine coupling are discussed.

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An essential ingredient for quantum computation and long-distance quantum communication is a reliable quantum memory. Nuclear spins in semiconductor nanostructures are excellent candidates for this task. With a magneton 3 orders of magnitude weaker than electron spins, they are largely decoupled from their environment, and the hyperfine interaction with electron spins allows one to access ensembles of nuclear spins in a controlled way [1–10]. In particular, the quantum state of an electron spin can be mapped onto the nuclear spins, giving rise to a long-term memory [3–7]. Nevertheless, memory lifetimes are limited, e.g., by dipole-dipole interactions among the nuclei. In this Letter, we demonstrate that the presence of the electron spin in the quantum dot substantially reduces the decoherence of this collective memory associated with the storage state.

Consider a quantum dot charged with a single excess electron as indicated in Fig. 1. The electron spin \( \hat{S} \) is coupled to the ensemble of underlying nuclear spins \( \hat{I} \) by the Fermi contact interaction,

\[
\hat{H}_{\text{hf}} = \mathcal{A} \sum_j g_j [I_{jz} \hat{S}_z + \frac{1}{2} (I_{j+} \hat{S}_- + I_{j-} \hat{S}_+)].
\]

where \( \mathcal{A} \) is the average hyperfine interaction constant, \( \mathcal{A} = 90 \mu eV \) for GaAs, and \( g_j \) is the electron density at the position of the \( j \)th nucleus, \( \sum_j g_j = 1 \). For convenience, we introduce the collective operators \( \hat{A} = \sum_j g_j \hat{I} \). The first term in Eq. (1) yields the Overhauser field, an effective magnetic field for the electron, and also the Knight shift for each nuclei. The flip-flop terms in Eq. (1), \( \hat{H}_{\text{IC}} = \frac{\mathcal{A}}{2} (\hat{A}_+ \hat{S}_- + \hat{A}_- \hat{S}_+) \), can be used to polarize the nuclear spins [1,2], and to map the electron’s spin state into a collective spin mode of the nuclei [3–5]. As will be shown here, the same can be used to provide a protective energy gap.

\[
\hat{H}_{\text{IC}} \propto \sum_j g_j [I_{j+} \hat{S}_- + I_{j-} \hat{S}_+] \Rightarrow \hat{H}_{\text{IC}} = \frac{\mathcal{A}}{2} (\hat{A}_+ \hat{S}_- + \hat{A}_- \hat{S}_+) \]

respectively. \( \hat{H}_{\text{IC}} \) couples the state \( |0\rangle \langle 1| \) to \( |1\rangle \langle 0| \) with an angular frequency \( \Omega = \mathcal{A} \sum_j g_j (2I_j)^{1/2} \). The detuning between these two states, \( \delta = \delta_{\text{OH}} + \delta_{\text{el}} \), comes from the Overhauser field, \( \delta_{\text{OH}} = -\hat{A} I \), and from the electron’s intrinsic energy splitting \( \delta_{\text{el}} \) due to, e.g., an external magnetic field or a spin-state dependent Stark laser pulse [11]. Coherent flip-flops between the electron and nuclear spins can be brought into resonance (\( \delta \ll \Omega \)) through \( \delta_{\text{el}} \). Then \( |0\rangle (\alpha |\uparrow\rangle + \beta |\downarrow\rangle) \) is rotated to \( (\alpha |\downarrow\rangle + \beta |\uparrow\rangle) |\uparrow\rangle \), and the

![FIG. 1 (color online). Left: Charged quantum dot with a single, polarized excess electron. Right: Spectrum of the effective nuclear Hamiltonian in the presence of a polarized electron. Off-resonant hyperfine coupling results in a gap \( \Delta_{\text{gap}} \) between the storage state \( |1\rangle \) and the nonstorage states \( |1\rangle \). \( \Delta_K \) denotes the Zeeman shift due to the effective magnetic field associated with the electron spin (Knight shift).](image-url)
quantum information is transferred from the electron to the nuclear-spin ensemble and back [3,4].

Assume that, after the qubit has been written into the nuclei, the polarized electron is not removed from the dot but the hyperfine flipflops are tuned off-resonant (δ ≫ Ω). Now, real transitions can no longer take place between [1]|i⟩ and (0)|c⟩. However, the residual virtual transitions repel the two states from each other, in analogy to the dynamic Stark effect. As a result, after adiabatic elimination of the electronic states, the energy of state |1⟩ gets shifted by Δ_gap = −Ω^2/4δ. The other, orthogonal states also having exactly one spin flipped (denoted by |1⟩_q) in Fig. 1 are “subradiant,” i.e., are not coupled via HJC to the electron. Therefore, they are unaffected by the shift. This is the origin of the energy gap.

To understand the protection scheme, let us introduce nuclear-spin waves. For highly polarized nuclei, one can introduce bosonic operators through the Holstein-Primakoff transformation: \( \hat{a}_j = \hat{p}_j/\sqrt{2\tilde{\epsilon}} \) and \( \hat{\hat{a}}_j^\dagger \hat{a}_j = \hat{p}_j^2/2 + \hat{I} \). This allows us to define the bosonic spin waves

\[
\hat{\Phi}_q = \sum_j n_{qj} \hat{a}_j, \quad \hat{\Phi}_q^\dagger = \sum_j n_{qj}^\dagger \hat{a}_j^\dagger,
\]

where the unitary matrix \( n_{qj} \) describes the mode functions. We identify the storage mode \( q = 0 \) as the one given by \( n_{0j} = \sqrt{\tilde{\epsilon}/2} g_j \) and write |1⟩ = \( \hat{\Phi}_0^\dagger |0⟩ \). This is the mode which is directly coupled to the electron spin. In fact, \( \hat{H}_{JC} = \Omega/2 (\hat{\Phi}_0^\dagger \hat{\hat{a}}_+ + \hat{\hat{a}}_- \hat{\Phi}_0) \) is a Jaynes-Cummings coupling in the bosonic approximation. After adiabatically eliminating the electronic states, \( \hat{H}_{JC} \) reduces to \( \hat{H}_{gap} = \Delta_{gap} \hat{\Phi}_0^\dagger \hat{\Phi}_0 \). As shown in Fig. 1, \( \hat{H}_{gap} \) lifts the degeneracy between states of different number of storage-mode excitations. This is the key feature of our protection scheme: any decoherence process that is associated with a transition from the storage mode \( \hat{\Phi}_0 \) to any other mode \( \hat{\Phi}_q \) now has to bridge an energy difference. If this gap is larger than the spectral width of the noise, the effect of the noise is substantially reduced.

A more detailed analysis shows that the off-resonant interaction with the electron spin—which itself is coupled, e.g., to phonons—leads in general also to an additional decoherence mechanism for the nuclear spins. If the corresponding electron spin dephasing rate \( \gamma \) is small compared to the electron’s precession frequency \( \delta \), the decay rate for the storage mode is reduced by the low probability of exciting the electron spin state: \( \gamma \Omega^2/\delta^2 \ll \gamma \).

In addition to the gap, the electron is also responsible for the Knight shift \( \hat{H}_K = \mathcal{A} \hat{\hat{A}}_z (\hat{S}_z) \). The relative shift between the |0⟩ and |1⟩ states, \( \Delta_K = -\frac{1}{2} \sum_j g_j^2/\sum_j g_j^2 \), is typically much less than \( \Delta_{gap} \). When the hyperfine coupling is inhomogeneous, however, |1⟩ fails to be eigenstate of \( \hat{H}_K \). Instead, \( \hat{H}_K |1⟩ = (-\frac{1}{2} \delta^{OH} + \Delta_K) |1⟩ + \zeta |1^\perp⟩ \), where |1^\perp⟩ is orthonormal to |1⟩ and \( \zeta^2 = 1 - \sum_j g_j^2/\sum_j g_j^2 \) characterizes the inhomogeneities. Thus, the storage mode is only an approximate eigenmode, and it gradually mixes with nonstorage modes as time passes. This causes loss of the stored qubit. \( |1^\perp⟩ \) is, however, off-resonant due to the energy gap, and the corresponding probability of finding the system in state \( |1^\perp⟩ \) is bounded by \( 4\zeta^2/\Delta_{gap}^2 \), so the detrimental effect of the inhomogeneous Knight shift is suppressed by the energy gap. In addition, since the admixture of \( |1^\perp⟩ \) is a coherent process, it can be cancelled by refocusing (echo) methods.

A larger gap can be achieved by bringing the hyperfine interaction closer to resonance. A nonzero external magnetic field or laser induced ac Stark shifts [11] can partially cancel the Overhauser field, such that \( \delta \ll \delta^{el} = -\delta^{OH} = A\mathcal{I} \). (Of course, \( \delta \) should be kept sufficiently large so that the hyperfine coupling remains off-resonant.) The requirement of separation of time scales implies \( \zeta \ll |\Delta_{gap}| \ll \Omega \ll |\delta| \). To estimate the orders of magnitude of the different energies, we take an oblate Gaussian electron density of ratio (1, 1, 1/3) and spin-\( \frac{1}{2} \) nuclei. Then we find that \( \Delta_K \) and \( \zeta \) are inversely proportional to the number of nuclei \( N \), whereas \( \Omega, \Delta_{gap} \propto N^{-1/2} \) only [Fig. 2(a)].

To analyze the decoherence suppression, we first consider a simplistic noise model where the nuclear spins are coupled to fluctuating, classical fields. The corresponding interaction Hamiltonian is \( \hat{\mathcal{V}} = \sum_j B_j \cdot \hat{\hat{V}} \). We assume isotropic Gaussian noise with zero mean and \( B_{\mu}(t)B_{\nu}^*(t') = \delta_{\mu\nu} \xi_{jk} C^{-1/2} \left| \mathbf{r} - \mathbf{r}' \right| \) for \( \mu, \nu = x, y, z \), where \( \xi_{jk} \) specifies the spatial correlations of the noise acting on different nuclei. For simplicity, the noise spectrum is assumed to be Lorentzian with a width \( \Gamma \), although similar results hold for other spectra with high-frequency cutoff.

Let us first discuss the dephasing part of the noise,

\[
\hat{\mathcal{V}}_d = \sum_j B_j^2 \hat{a}_j^\dagger \hat{a}_j = \sum_{pq} (\sum_j B_j^2 \eta_{pj} \eta_{qj}) \hat{\Phi}_p^\dagger \hat{\Phi}_q,
\]

FIG. 2. Hyperfine Rabi frequency (\( \Omega \)), protective energy gap (\( \Delta_{gap} \)), Knight shift difference between the logical states (\( \Delta_K \)), symmetry breaking couplings due to inhomogeneities (\( \zeta \) and \( \omega \)), and qubit decoherence rates due to dipolar spin diffusion without (\( \Gamma_D \)) and with (\( \Gamma_D^\prime \)) protection. (a) The fully polarized (zero temperature) case shown versus the number of spin-\( \frac{1}{2} \) nuclei (\( N \)) taking part in the storage, i.e., located within \( 3\sigma \) of the oblate Gaussian electron distribution with in-plane variance \( \sigma \). (b) Estimated energies in dark states \( |D_{\mu,\nu}⟩ \) with \( n \) spins flipped from the fully polarized state for \( N = 10^3 \). Energy units are chosen to match GaAs. \( \Delta_{gap} \) is obtained by taking \( \delta = 10 \Omega \).
expressed by the bosonic spin-wave operators (3). Dephasing of individual nuclear spins means transfer of excitations between different spin-wave modes. Especially, it leads to both real and virtual transitions from |1⟩ to a nonstorage state |1⟩. As the latter state is “subradiant” and, thus, equivalent to |0⟩ when the memory is read out, this process essentially results in damping (for real transitions) and dephasing (for virtual transitions) of the stored logical qubit [12]. Assuming the zero temperature limit with all nonstorage modes ˆΦq≡0 in the vacuum state and formally eliminating them in the Markov approximation together with the classical fields, we derive a master equation for the storage mode: ½ dρ/dt = i[ˆρ, ˆE] + ˆL±(ˆρ), with energy shift ˆE = (1 − Ξ)CΔgap/(Γ2 + Δ2 gap) and

\[ L±(ˆρ) = γ_1(2 ˆΦ^†_0 ˆΦ_0 + 2 ˆΦ^†_0 ˆΦ_0 - 2 ˆΦ^†_0 ˆΦ_0) + γ_2(2 ˆΦ^†_0 ˆΦ_0 - 2 ˆΦ^†_0 ˆΦ_0), \]

Here, γ_1 is the damping rate of the stored qubit while γ_2 describes its dephasing. The two rates are given by

\[ γ_1 = \frac{CT}{Γ^2 + Δ^2_{gap}}(1 - Ξ), \quad γ_2 = \frac{C}{Γ} Ξ, \]

where we have introduced the dimensionless parameter Ξ ≡ Σj,k ξjk ε^2 / (Σj ε^2) containing the spatial part of the noise correlator.

When the correlation length of the classical noise is smaller than the internuclear distance (local uncorrelated noise, ξjk ∼ δjk), Ξ scales inversely with the number of nuclei (Fig. 3). In this case, the dephasing rate γ_2 vanishes as 1/N, which is an effect of the collective nature of the storage states [12]. The storage is based on encoding the logical qubit states in a large, delocalized ensemble of physical spins. As the decoherence has strongly local character, it has only a very small effect on the dephasing of the qubit. Secondly, the loss of the stored qubit is associated with a change in the number of excitations in the storage mode. Such transitions are strongly suppressed, and the damping rate γ_1 is decreased if Δ_{gap} is large compared to the width of the noise spectrum Γ (or the corresponding cutoff frequency). Finally, the opposite limit of infinite spatial correlation length (ξjk = 1) corresponds to a homogeneous random field resulting, e.g., from a global external source. In that case, Ξ = 1 (see Fig. 3), and there is no protection against dephasing.

Following a similar but slightly more involved procedure, we can discuss the spin-flip part ˆVxy = 1/2 Σj (B_j^x ˆI_j^− + B_j^y ˆI_j^+) of the noise. When deriving a master equation for this case, we need to keep higher order terms in the Holstein-Primakoff approximation: in the next order, ˆI_j^+ = √2I(1 − λ ˆa_j^† ˆa_j) ˆa_j with λ = 1 − (1/2I)^1/2. Here, we have neglected the probability of double or more excitations on the same site j, which is reasonable in high polarization (T = 0) limit and exact for spin-1/2 nuclei. The Lindbladian describing decoherences due to spin flips reads, in leading order of 1/N:

\[ L_{xy}(ˆρ) = (γ_3 + γ_4)(2 ˆΦ^†_0 ˆΦ_0 + 2 ˆΦ^†_0 ˆΦ_0 - 2 ˆΦ^†_0 ˆΦ_0) + γ_5(2 ˆΦ^†_0 ˆΦ_0 - 2 ˆΦ^†_0 ˆΦ_0), \]

which describes decay with rate γ_4, dephasing with rate γ_5, and additionally thermalization (relaxation to the identity matrix) with rate γ_5. The rates read

\[ γ_3 = \frac{CTN}{Γ^2 + Δ_{gap}^2} \quad γ_4 = \frac{2CTΛ^2}{Γ^2 + Δ_{gap} + Δ_{K}^2}, \quad γ_5 = \frac{4CTΛ^2}{Γ^2 + Δ_{gap} + Δ_{K}^2} \quad ξ_j^2 \]

In the limit of vanishing spatial correlations of the spin-flip noise, Ξ^2 ≡ Σj,k ξjk ε^2 / (Σj ε^2) tends to 1 (Fig. 3), and we have protection against thermalization (γ_5) as the separation of |0⟩ and |1⟩ by Δ_{gap} + Δ_{K}. The decay corresponding to γ_4 is due to spin-flip induced transitions between |1⟩ and 1_{p}, 1_{q}⟩ (the latter containing a total of two excitations but none in the storage mode), and the energy to bridge is in the order of Δ_{gap} − Δ_{K} (see Fig. 1). Finally, the last factor in the dephasing rate γ_5 scales as 1/N, indicating that it is the collective nature of the storage that leads to protection. Note that the nonlinearity of the Holstein-Primakoff representation is responsible for this dephasing: the virtual nonstorage excitations are interacting with the storage mode.

Another potential source of decoherence is nuclear-spin diffusion due to dipole-dipole interaction between nuclear spins [13]. The energy gap gives protection against this effect, too. The dipolar interaction between the pairs of spins is described in the secular approximation by

\[ ˆH_D = Σ_{j≠k} B_{jk}( ˆI_j^+ ˆI_k^− - 2 ˆI_j^+ ˆI_k^+) = 2I Σ_{j≠k} B_{jk} ˆa_j^† ˆa_k. \]

FIG. 3. The parameters Ξ and Ξ' describing the effects of spatial correlations in the classical noise (ξjk = e^−r/λ) for different number of nuclei. The same family of Gaussian electron densities was used as in Fig. 2. The bullets on the curves denote the linear size of the dot given by the variance σ.
where $B_{jk} \propto (3 \cos^2 \theta_{jk} - 1)/r_{jk}^3$, $r_{jk} = r_j - r_k$ is the distance between two nuclei, $\theta_{jk}$ is the zenith angle of the vector $r_{jk}$, and we used the first order Holstein-Primakoff approximation. At full polarization, we can rewrite Eq. (9) using the bosonic spin-wave mode operators (3) as $\hat{H}_D = \sum_{pq} B_{pq} \hat{B}_p \hat{B}_q$. As if it were a central spin coupled to a mesoscopic spin bath [14,15], the storage mode is coupled to a bath of nonstorage modes that present a fluctuating, effective transversal magnetic field. If the electron were not present, these fluctuations would lead, in the mean-field approximation, to a dephasing rate $\Gamma_D \sim (2\sum_{q=0}^1 |\hat{B}_{0q}|^2)^{1/2}$, which is numerically found to be in the order of 100 Hz for GaAs [Fig. 2(a)]. With the protective gap, however, the storage-mode operator $\hat{\Phi}_0$ rotates rapidly with respect to the other ones, and the above coupling averages out. The strength of the remaining coupling between the storage mode and mode $q$ is proportional to $\Delta_{gap}^{-1} \sum_{r \neq 0} B_{or} \hat{B}_{qr}$, and the corresponding fluctuations yield a reduced dephasing rate of $\Gamma_D^{\prime} \sim \Delta_{gap}^{-1} (2\sum_{q=0}^1 |\hat{B}_{0q}|^2)^{1/2}$ as indicated in Fig. 2(a). Depending on the dot size, the effects of spin diffusion can be suppressed by several orders of magnitude.

Nonperfect nuclear-spin polarization.—It has been shown that partially polarized nuclei (at finite temperature) can also be used for storing a qubit state [4]. Instead of the fully polarized state $|0\rangle$, the initial preparation drives the nuclear ensemble into a statistical mixture of dark states $|D_{n,\beta}\rangle$ defined by $\hat{A}_{\pm} |D_{n,\beta}\rangle = 0$ and characterized by the total number of spins flipped $n$ and the permutation group quantum number $\beta$. As the detuning $\delta$ is adiabatically swept from far negative to far positive, a superposition of the $|1\rangle_c$ and $|1\rangle_e$ electron spin states is mapped into the mixture of superpositions of the nuclear-spin states $|D_{n,\beta}\rangle$ and $|\xi_{n,\beta}\rangle = \frac{1}{\sqrt{2}} (\hat{A}_+ |D_{n,\beta}\rangle + \hat{A}_- |D_{n,\beta}\rangle)$, and the qubit state is efficiently written into the memory [4].

When the electron is left in the quantum dot, it feels different Overhauser fields for different dark states; hence, the detuning should be adjusted such that $\delta_{nOH}^c + \delta_{nOH}^e \gg \text{Var}(\delta_{nOH})$. Moreover, the hyperfine Rabi frequency $\Omega_n$ also varies with $n$ and the energy gap $\Delta_{gap,n}$ is not the same for all dark states. This inhomogeneous broadening would result in dephasing of the qubit, but can be avoided by a symmetric spin echo sequence [4].

Because of the inhomogeneous nature of the hyperfine coupling, the $\hat{A}_{\pm}$ operators do not follow the angular momentum commutation relation. Therefore, $|D_{n,\beta}\rangle$ is not an eigenstate of the Knight shift operator, but it is partially mapped into an orthogonal state: $\hat{H}_K |D_{n,\beta}\rangle = -\frac{i}{2} \delta_{nOH}^c |D_{n,\beta}\rangle + \omega_n |D_{n,\beta}\rangle$. Furthermore, $|\xi_{n,\beta}\rangle$ is neither an eigenstate of $\hat{H}_{gap}$ nor of $\hat{H}_K$: $\hat{H}_K |\xi_{n,\beta}\rangle = (-\frac{i}{2} \delta_{nOH}^c + \Delta_{K,n} + \Delta_{gap,n}) |\xi_{n,\beta}\rangle + \xi_n |\xi_{n,\beta}\rangle$. The parameters can be expressed as expectation values in $|D_{n,\beta}\rangle$: