XXII. MICROWAVE THEORY

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A. CASCADING OF BILATERAL, TWO-PORT NETWORKS BY MEANS OF THE SCHILLING FIGURE

In a previous report (1) it was shown how the Klein generalization of the well-known Pascal theorem to three dimensions could be used for analyzing a bilateral, two-port network from three arbitrary impedance or reflection-coefficient measurements. Another important problem in network theory, that of cascading bilateral, two-port networks, can be studied by using the Schilling generalization of the Hamilton theorem, a well-known theorem in spherical trigonometry (2). Hamilton's theorem states:

> "If, on a sphere, we denote the fixed diameters passing through the corners of a spherical triangle $A_1A_2A_3$ by OA_1, OA_2, and OA_3, and if we rotate the sphere consecutively around each of the diameters through angles equal to twice the (inner) angles of the triangle, then the sphere and the entire space return to their original positions."

Instead of three diameters we may assume that we have three arbitrary straight lines, L_1, L_2, and L_3, cutting the sphere. We denote the three non-Euclidean perpendiculars to the lines by L_{12}, L_{31}, and L_{23}. The non-Euclidean distance that is cut out on L_1 by L_{12} and L_{31} is λ_1'; the non-Euclidean angle between planes through L_{12} and L_1, and L_{13} and L_1, is λ_1''. Similar notations are introduced for L_2 and L_3. By using these notations, Schilling generalized the Hamilton theorem to the following form:

> "If the three straight lines L_1, L_2, and L_3 are used as axes for three consecutive helical movements, specified by the three quantities $2(\lambda_1' + j\lambda_1'')$, $2(\lambda_2' + j\lambda_2'')$, and $2(\lambda_3' + j\lambda_3'')$, then the sphere and the entire space return to their original positions."

The geometric configuration consisting of L_1, L_{12}, L_2, L_{23}, L_3, and L_{31} was called "the Schilling figure" by Klein (2).

The geometric theory of the use of the Schilling figure in cascading bilateral, two-port networks has been outlined in previous works (3, 4). Analytically, all of the constructions of the geometric theory can be performed by using the theory of invariance of quadratic forms and complex spherical trigonometry. Some preliminary examples have been worked out.

A purely geometric work on the cascading of lossy, two-port networks was published recently by de Buhr (5). He utilizes the unit sphere for tutorial purposes and performs all geometric constructions in the plane; a rather complicated method.

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B. NOISE-POWER RATIO TRANSFORMATIONS IN THREE-DIMENSIONAL SPACES

1. Geometric Connections between the Poincaré and Cayley-Klein Models of the Three-dimensional non-Euclidean Hyperbolic Space

The geometric connection between the Poincaré model of the three-dimensional hyperbolic space that has the ζ-plane, $\zeta = \xi + j\eta$, for the absolute surface (1) and the Poincaré model that has the unit sphere for the absolute surface is shown schematically in Fig. XXII-1. The first model is shown upside-down and the unit length is different for the two models. One of the models is obtained from the other by inversion in an inversion sphere.

In the geometric-analytic theory of noisy two-ports, which was presented recently (1), it was shown how a noise-power ratio transformation through a bilateral, lossless, two-port network can be interpreted as movement in the different models of the three-dimensional hyperbolic space. In the (ξ, η, θ)-space an elliptic transformation corresponds to a movement of points on circles perpendicular to a fixed semicircle which passes orthogonally through the fixed points in the (ξ, η)-plane. In the hyperbolic case a stretching is obtained along circles through the fixed points. A transformation through a bilateral, lossy, two-port network will correspond, therefore, to a loxodromic movement on a horned cyclide (2, 3). Such a curve is indicated at A in Fig. XXII-1. By inversion in the inversion sphere, the loxodromic curve is transformed so that it is situated on another horned cyclide inside the unit sphere. If we assume that the cyclide at A grows, then the loxodromic curve inside the unit sphere will fall, at a certain moment, on a parabolic cyclide (3). Further growth changes the cyclide inside the unit sphere and it becomes a spindle cyclide. An example of the spindle cyclide is shown at B in Fig. XXII-1. This example is a special case, because the fixed semicircle in the (ξ, η, θ)-space passes through (0, 0, 1), so that it is transformed into a diameter of the unit sphere.

Figure XXII-2 shows the geometric connection between the Poincaré model and the Cayley-Klein model of the three-dimensional hyperbolic space, both of which have the

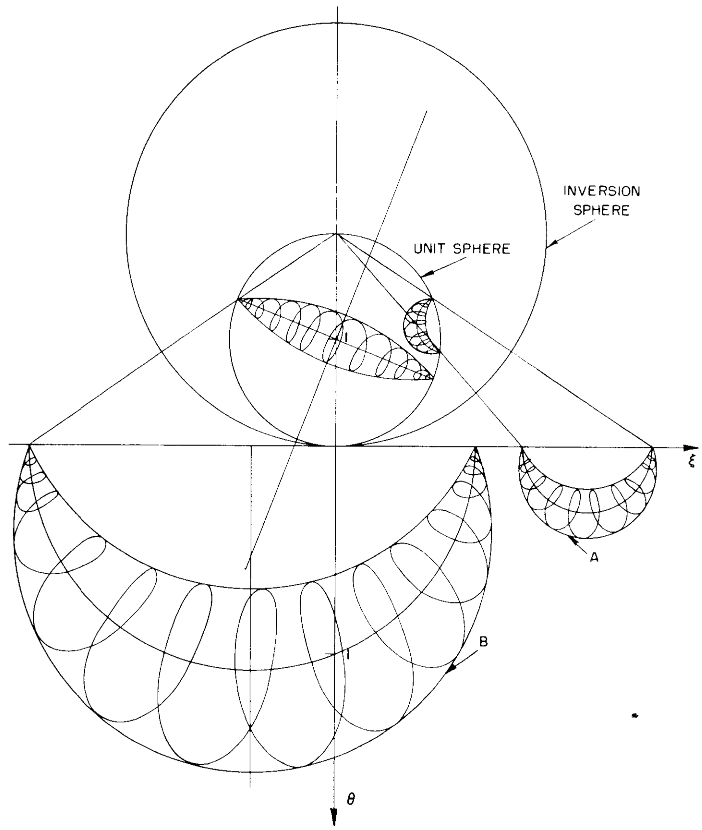

Fig. XXII-1. Geometric connection between the Poincaré models of the three-dimensional hyperbolic space.

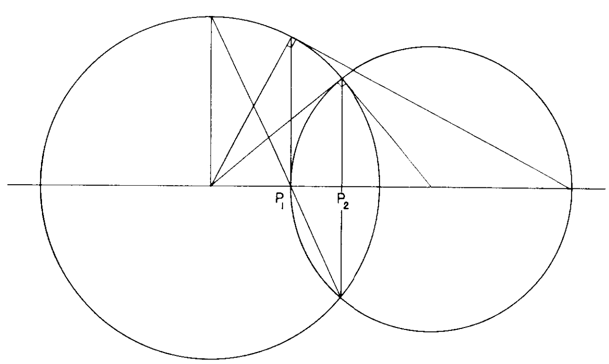

Fig. XXII-2. Darboux transformation.

unit sphere as the absolute surface. The transformation between the points P_1 and P_2 is performed by the Darboux transformation through which every pair of points which is inverse with respect to the sphere is transformed into the pole of its symmetry plane. The transformation is equivalent to the transformation B that was introduced by Deschamps (4).

2. The Isometric Sphere Method

The isometric circle method, presented in the Quarterly Progress Report, April 15, 1956, pages 123-126, can be generalized to three dimensions by extending the isometric circles to hemispheres and the symmetry line to a symmetry plane (1). Thus a noise-power ratio transformation can be performed through the following operations:

1. An inversion in the isometric sphere of the direct transformation;
2. a reflection in the symmetry plane; and
3. a rotation around a perpendicular axis through the center of the isometric sphere of the inverse transformation through an angle $-2 \arg (a+d)$.

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C. NOISE TRANSFORMATIONS IN FOUR-DIMENSIONAL SPACES

The notations that were used in the presentation of the geometric-analytic theory of noisy two-ports in earlier reports (1) will be used in this section.

1. The Coherency Matrix

In the earlier work we used the transformation

$$\psi' = T\psi \tag{1}$$

Thus

$$\psi'\psi'^{\dagger} = T\psi(T\psi)^{\dagger} = T\psi\psi^{\dagger}T^{\dagger} \tag{2}$$

where the dagger indicates a transposed conjugate quantity.

Equation 2 can also be written as

$$C' = \begin{pmatrix} Q_1'Q_2' \\ Q_3'Q_4' \end{pmatrix} = \begin{pmatrix} ab \\ cd \end{pmatrix} \begin{pmatrix} Q_1Q_2 \\ Q_3Q_4 \end{pmatrix} \begin{pmatrix} a^*c^* \\ b^*d^* \end{pmatrix} = TCT^\dagger \tag{3}$$

or as

$$C' = \begin{pmatrix} P_4' + P_3' & P_1' + jP_2' \\ P_1' - jP_2' & P_4' - P_3' \end{pmatrix} = \begin{pmatrix} ab \\ cd \end{pmatrix} \begin{pmatrix} P_4 + P_3 & P_1 + jP_2 \\ P_1 - jP_2 & P_4 - P_3 \end{pmatrix} \begin{pmatrix} a^*c^* \\ b^*d^* \end{pmatrix} \tag{4}$$

For a signal process, Eq. 4 can be considered as the transformation for the components and magnitude of the associated vector of a spinor (2). For a noise process, the Hermitian matrices C and C' in Eqs. 3 and 4 can be considered as corresponding to the "density matrix" in quantum mechanics (3) or to the "coherency matrix" in optics (4). It can be proved, by means of the Schwarz inequality, that

$$\det C \geqslant 0, \quad \det C' \geqslant 0 \tag{5}$$

We can rewrite Eq. 4 in the form

$$\sigma_1 P_1' + \sigma_2 P_2' + \sigma_3 P_3' + \sigma_4 P_4' = T(\sigma_1 P_1 + \sigma_2 P_2 + \sigma_3 P_3 + \sigma_4 P_4) T^\dagger \tag{6}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

are the Pauli matrices (refs. 5, 6).

If we set a = a' + ja", etc., we obtain:

for $P_1' = P_1$, $\sigma_1 = T\sigma_1 T^\dagger$, with $T = \begin{pmatrix} a' & jb'' \\ jc'' & d' \end{pmatrix}$;

for $P_2' = P_2$, $\sigma_2 = T\sigma_2 T^{-1}$, with $T = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ = real;

for $P_3' = P_3$, $\sigma_3 = T\sigma_3 T^{-1}$, with $T = \begin{pmatrix} e^{j\phi} & 0 \\ 0 & e^{-j\phi} \end{pmatrix}$; and

for $P_4' = P_4$, $\sigma_4 = T\sigma_4 T^\dagger$, with $T = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix}$ = unitary.

By using the Pauli matrices, Eqs. 3 and 4 can be interpreted as transformations of Q four-vectors and P four-vectors in four-dimensional spaces. Analytically, we obtain the equations Q' = LQ and P' = MP, which were used in the presentation of the geometric-analytic theory (1). The transition to wave representation can be performed easily.

In a geometric interpretation of Hermitian forms Deschamps (5) sets

$$\psi^\dagger \psi = \psi^\dagger T^\dagger T \psi \tag{8}$$

and

$$T^\dagger T = -X\sigma_1 - Y\sigma_2 - Z\sigma_3 + T\sigma_4 \tag{9}$$

thus establishing a one-to-one correspondence between the Hermitian matrices and vectors of a four-dimensional Minkowski space with real coordinates (X, Y, Z, T).

A rather extensive study has been made of Eq. 4. It has been interpreted as a four-vector transformation in a four-dimensional real space with Lorentz metric, and, by using an algebra of complex quaternions (5), it has been interpreted as a rotation in a four-dimensional (complex) Euclidean space. Such rotations are treated, for example, in works by Euler, Hamilton, Cayley, Klein, Study, Cole, Forsyth, Manning, and Ganguli.

It would seem that we can represent a bilateral, noisy, two-port theoretically by an orthogonal (complex) 5 × 5 matrix with 25 - 15 = 10 independent elements. The splitting of the noisy network into noisy and noise-free parts corresponds to replacing the 5 × 5 matrix by a noise four-vector and an orthogonal (complex) 4 × 4 matrix with 16 - 10 = 6 independent elements. The splitting of the noise-free network into lossy and lossless parts corresponds to replacing the 4 × 4 matrix by a three-vector and an orthogonal (complex) 3 × 3 matrix with 9 - 6 = 3 independent elements. The splitting of the lossless network into a symmetric lossless part and an unsymmetric lossless part corresponds

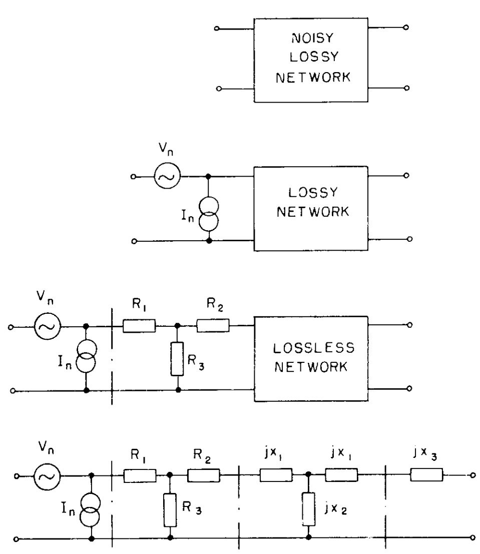

Fig. XXII-3. Splitting of a noisy two-port network.

to replacing the 3 × 3 matrix by a two-vector and an orthogonal 2 × 2 matrix with 4 - 3 = 1 independent element that can be represented by a series or shunt reactance. See Fig. XXII-3.

2. Four-dimensional Derivation of the Noise Factor

Let us assume that we have split a noisy network into noisy and noise-free parts according to Fig. XXII-3. We then attach a noisy impedance $Z_s = R_s + jX_s$ at the input of the network. The excess noise factor is

$$F_z = \frac{Q_1'}{P_s} = \frac{L_a Q}{P_s} \tag{10}$$

where L_a is a four-vector composed of the first row of the 4 × 4 matrix L in $Q' = LQ$ of the geometric-analytic theory of noisy two-ports (1), so that

$$L_a = (aa^*, ab^*, ba^*, bb^*) \tag{11}$$

and

$$P_s = 4kT\Delta f\, R_s \tag{12}$$

Thus the excess-noise factor is proportional to the scalar product of two four-vectors. For the series impedance Z_s, we have $a = 1$, $b = Z_s$, $c = 0$, and $d = 1$, so that

$$F_z = \frac{Q_1 + Z_s^* Q_2 + Z_s Q_3 + Z_s Z_s^* Q_4}{4kT\Delta f\, R_s} \tag{13}$$

which is a well known expression. The optimum excess-noise factor is obtained in the usual manner by equating the partial differentials with respect to X_s and R_s to zero. Whence we obtain

$$\left.\begin{aligned}
X_{s,\,opt} &= -\frac{Q_{2i}}{Q_4} = -\frac{P_2}{P_4 - P_3}, \quad Q_2 = Q_{2r} + jQ_{2i} \\
R_{s,\,opt} &= \frac{\sqrt{Q_1 Q_4 - Q_{2i}^2}}{Q_4} = \frac{\sqrt{P_1^2 + (\det C)^2}}{P_4 - P_3} \\
F_{z,\,opt} &= \frac{1}{2kT\Delta f}\left(\sqrt{Q_1 Q_4 - Q_{2i}^2} + Q_{2r}\right) \\
&= \frac{1}{2kT\Delta f}\left(\sqrt{P_1^2 + (\det C)^2} + P_1\right)
\end{aligned}\right\} \tag{14}$$

It is interesting to compare Eq. 10 with the expression for the transmission factor in optics or antenna theory. The transmission factor (direction of beam, from right to left) is defined as

$$T = \frac{P'_4}{P_4} = \frac{M_a P}{P_4} \tag{15}$$

where the four-vector M_a is the fourth row of the real 4×4 matrix in $P' = MP$ in the geometric-analytic theory of noisy two-ports (1), so that

$$M_a = (d_1, d_2, d_3, d_4)$$

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SUMMARY

The aim of the work that has been presented in this section for the past two years has been to investigate the means by which modern (higher) geometry can be used for solving microwave problems and simplifying solutions that are already being applied to these problems.

At the beginning of the investigation, in September 1955, the writer decided to try to follow a certain plan for performing the research work. Two rules were prescribed: (a) to start with simple problems and gradually extend the ideas and methods to more complex problems, and (b) to divide the treatment of the problems into three parts: a geometric part yielding a graphic picture of the problem, an analytic part constituting an analytic interpretation of the geometric part, and a part consisting of simple constructive examples to clarify the geometric and analytic treatments.

These rules have been strictly followed. This fact, and the fact that numerous papers in mathematics, engineering, and physics, published in six languages (German, English, French, Italian, Dutch, and Swedish), were studied led, naturally, to rather slow progress in the research work. But the thoroughness of the study has resulted in the construction of a firm foundation on which future research can be built.

One of the basic invariant properties that characterizes a geometric problem is the number of dimensions used. By using an elementary inversion method, the isometric circle method, as a starting point, problems utilizing one and two dimensions, and two-dimensional configurations imbedded in three dimensions were studied during 1956. The study of the last kind of problem, by using the surface of the unit sphere, was continued during 1957. The main geometric reason for using the surface of the unit sphere is the possibility of visualizing a complex angle (complex distance). Another advantage is that the infinity of a plane is transformed into an ordinary point on the sphere

by stereographic transformation. Analytically, we take the same step by introducing homogeneous coordinates. The analysis and synthesis of bilateral, two-port networks from three arbitrary impedance or reflection-coefficient measurements was investigated by means of Klein's three-dimensional generalization of the Pascal theorem (1). The cascading of similar networks was studied by using Schilling's three-dimensional generalization of the Hamilton theorem (see Section XXII-A). Also, different non-Euclidean geometry models that are of use in engineering and physics were investigated (2), and elementary network theory has been treated from an advanced geometric standpoint (3).

A preliminary investigation has been made on the use of three- and four-dimensional spaces in solving microwave problems. The isometric circle method has been generalized to a three-dimensional "isometric sphere method" (see Section XXII-B). A geometric-analytic theory of noisy two-port networks was presented (4). Special emphasis has been given to the cascading of noisy two-port networks (5). An analogous geometric-analytic theory of partly polarized electromagnetic waves was also presented (6). The geometric connections between the different non-Euclidean models of three-dimensional hyperbolic space have been shown (see Section XXII-B). Finally, an investigation of the use of four-dimensional spaces for noise-power transformations, showing, among other things, a four-dimensional derivation of the noise factor, has been performed (see Section XXII-C).

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