

## IX. STATISTICAL COMMUNICATION THEORY

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### A. APPROXIMATIONS TO NONLINEAR SYSTEMS

Our problem is to determine the parameters in a number of models of given form in order to approximate, as well as possible, a given realizable nonlinear system. The approximation error is the mean-square error between the output of the given nonlinear system and the output of the model when both the given system and the model have the same white Gaussian noise input. An outline of the optimization equations and experimental methods for optimizing the model are presented. Expressions for the minimum mean-square error and some examples are given.

#### 1. Model A

In a previous report (1) it was shown that optimization of model A of Fig. IX-1 requires determination of a set of  $N$  gains  $\{a_n\}$  and a set of  $N$  normalized linear filters  $\{h_n(t)\}$  that satisfy the equation

$$a_n h_n(\tau_2) = \int_0^\infty K_2(\tau_1, \tau_2) h_n(\tau_1) d\tau_1 \quad (1)$$

The magnitude of the members of the set  $\{a_n\}$  are as large as possible, and hence the set of  $N$  gains  $\{a_n\}$  and the set of  $N$  normalized filters  $\{h_n(t)\}$  are the  $N$  eigenvalues

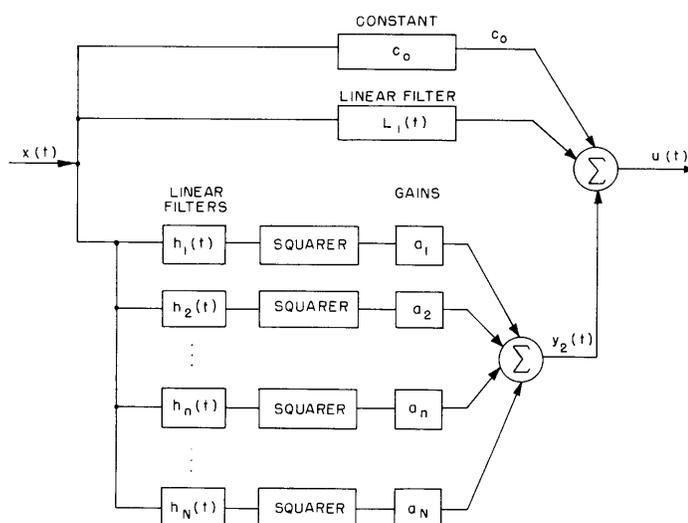


Fig. IX-1. Model A.

of largest magnitude and the corresponding  $N$  normalized eigenfunctions of Eq. 1, respectively.

First, the experimental procedure for solving Eq. 1 will be given, and then it will be proved that the procedure is correct.

The experimental procedure for determining the linear filter  $h_1(t)$  is an iterative one in which we start off with an arbitrary linear filter  $f_1(t)$ , and then by a series of measurements and a simple computation we determine a new normalized linear filter  $f_2(t)$ . The new filter  $f_2(t)$  is then substituted for  $f_1(t)$  and the procedure is repeated. The linear filter  $h_1(t)$  is then given by

$$h_1(t) = \lim_{n \rightarrow \infty} f_n(t)$$

The experimental circuit is shown in Fig. IX-2. The output of the circuit,  $f'_{n+1}(\tau)$  is the average value of the product of the output of a delay line, the output of the linear filter  $f_n(t)$  and the output  $z(t)$  of the given nonlinear system minus its average value  $\overline{z(t)}$ . The procedure is to measure the output of the circuit,  $f'_{n+1}(\tau)$  as a function of the tap position,  $\tau$ , of the delay line. The next linear filter  $f_{n+1}(t)$  is related to  $f'_{n+1}(t)$  by

$$f_{n+1}(t) = \frac{f'_{n+1}(t)}{\left( \int_0^\infty [f'_{n+1}(\tau)]^2 d\tau \right)^{1/2}} \quad (2)$$

The gain  $a_1$  is given by

$$a_1 = \lim_{n \rightarrow \infty} \frac{f'_{n+1}(t)}{2f_n(t)} \quad (3)$$

With the determination of  $h_1(t)$  and  $a_1$ , part of the output of the model is known. The experimental procedure for determining each of the remaining linear filters  $\{h_n(t)\}$  and the gains  $\{a_n\}$  is the same as that for determining  $h_1(t)$  and  $a_1$ , with the difference that the output of the part of the model that has already been determined is subtracted from the output of the given nonlinear system  $z(t)$ , and enough dc voltage is also subtracted to make the sum a zero-mean process. The experimental circuit for determining  $h_2(t)$  is shown in Fig. IX-3.

We now prove a lemma that will be useful for proving that the procedure outlined above is correct. We shall prove that the output  $r$  of Fig. IX-4 is

$$r = 2 \int_0^\infty \int_0^\infty \ell_1(\tau_1) \ell_2(\tau_2) K_2(\tau_1, \tau_2) d\tau_1 d\tau_2 \quad (4)$$

where  $\ell_1(t)$  and  $\ell_2(t)$  are arbitrary, realizable, linear filters, and  $K_2(\tau_1, \tau_2)$  is the

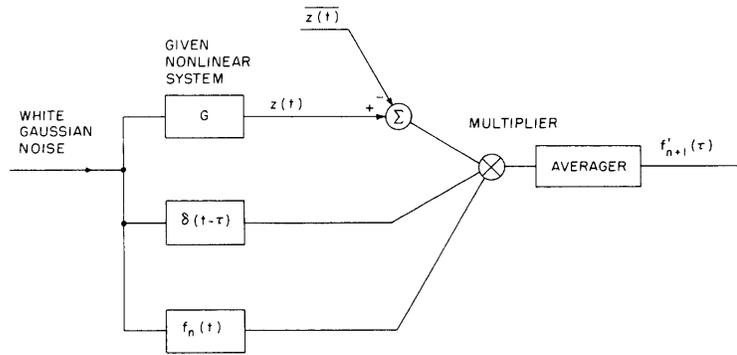


Fig. IX-2. Experimental circuit no. 1 for model A.

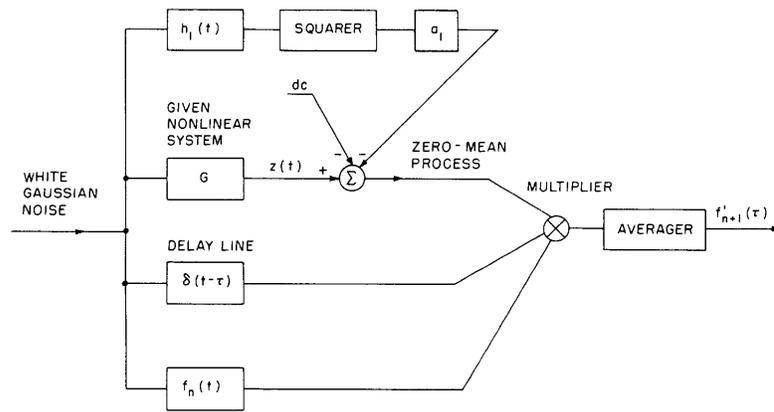


Fig. IX-3. Experimental circuit no. 2 for model A.

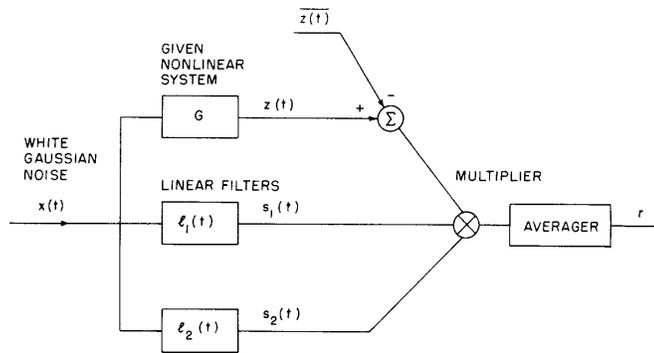


Fig. IX-4. Circuit of lemma.

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symmetric kernel associated with the second-order term in the Wiener hierarchic expansion (2) of the given nonlinear system. The input  $x(t)$  is white Gaussian noise of unity power per cycle per second.

Now,  $r$  can be written

$$r = \overline{s_1(t) s_2(t) [z(t) - \overline{z(t)}]} \quad (5)$$

where  $s_1(t)$  and  $s_2(t)$  are the outputs of the linear filters  $\ell_1(t)$  and  $\ell_2(t)$ , respectively,  $z(t)$  is the output of the given nonlinear system, and the bar indicates a time average.

We can expand the terms on the right-hand side of Eq. 5 in terms of Wiener's hierarchic expansion; that is,  $z(t)$  and  $\overline{z(t)}$  will be expanded in the hierarchic set  $G_n(t)$  and  $s_1(t) s_2(t)$  in terms of the hierarchic set  $\{P_n(t)\}$ . Thus

$$z(t) = \sum_{n=0}^{\infty} G_n(t) \quad (6)$$

$$\overline{z(t)} = G_0(t) \quad (7)$$

$$s_1(t) s_2(t) = P_0(t) + P_2(t) \quad (8)$$

The hierarchic sets have the following property:

$$\overline{G_n(t) P_m(t)} = 0, \quad \text{for } n \neq m \quad (9)$$

Substituting Eqs. 6, 7, and 8 in Eq. 5, we obtain

$$r = \overline{[P_0(t) + P_2(t)] \left[ \sum_{n=0}^{\infty} G_n(t) - G_0(t) \right]} \quad (10)$$

Using the linear independence condition given by Eq. 9, we obtain

$$r = \overline{P_2(t) G_2(t)} \quad (11)$$

Now,  $G_2(t)$  and  $P_2(t)$  are given by

$$G_2(t) = \int_0^{\infty} \int_0^{\infty} K_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2 - \int_0^{\infty} K_2(T, \tau) d\tau \quad (12)$$

$$P_2(t) = \int_0^{\infty} \int_0^{\infty} \ell_1(\tau_1) \ell_2(\tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2 - \int_0^{\infty} \ell_1(\tau) \ell_2(\tau) d\tau \quad (13)$$

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By substituting Eqs. 12 and 13 in Eq. 10 and performing the averaging, we obtain

$$r = 2 \int_0^{\infty} \int_0^{\infty} \ell_1(\tau) \ell_2(\tau) K_2(\tau_1, \tau_2) d\tau_1 d\tau_2 \quad (14)$$

which was to be proved.

We shall now prove that the experimental procedure does indeed solve Eq. 1. In order to prove that the experimental procedure for solving for  $h_1(t)$  and  $a_1$  is equivalent to the standard iterative procedure (3) for solving for the largest eigenvalue and corresponding eigenfunction, it is only necessary to show that  $f'_{n+1}(\tau)$  is given by

$$f'_{n+1}(\tau) = 2 \int_0^{\infty} K_2(\tau_1, \tau_2) f_n(\tau_1) d\tau_1 \quad (15)$$

because the experimental procedure is then identical to the standard iteration procedure for the largest eigenvalue and corresponding eigenfunction. If we equate the following parameters of Fig. IX-1 and Fig. IX-3,

$$\ell_1(t) = f_n(t) \quad (16)$$

$$\ell_2(t) = \delta(t-\tau) \quad (17)$$

$$r = f'_{n+1}(\tau) \quad (18)$$

then from Eq. 4 we obtain

$$f'_{n+1}(\tau) = 2 \int_0^{\infty} \int_0^{\infty} f_n(\tau_1) \delta(\tau_2 - \tau) K(\tau_1, \tau_2) d\tau_1 d\tau_2 \quad (19)$$

Integrating on  $\tau_1$ , we obtain the desired result

$$f'_{n+1}(\tau) = 2 \int_0^{\infty} f_n(\tau_1) K_2(\tau_1, \tau_2) d\tau_1 \quad (20)$$

The proof that the experimental procedures for obtaining the remaining linear filters and gains is correct follows from the fact that the subtraction from  $z(t)$  of the part of the model that has already been determined removes those previously determined eigenvalues and eigenfunctions from the kernel  $K_2(\tau_1, \tau_2)$ , and hence the iterative procedure converges to the next largest eigenvalue and eigenfunction. The experimental process is therefore correct.

From the error expression in the previous report (4) and an expansion of the

kernel  $K_2(\tau_1, \tau_2)$  in terms of its eigenvalues and eigenfunction, the minimum mean-square error  $\mathcal{E}_{\min}$  between the output of model A (see Fig. IX-1) and the given nonlinear system is

$$\mathcal{E}_{\min} = 2 \sum_{n=N+1}^{\infty} \lambda_n^2 + \sum_{n=3}^{\infty} \overline{G_n^2(t)} \quad (21)$$

where  $\{G_n(t)\}$  are the terms in the Wiener hierarchic expansion for the output of the given nonlinear system, and  $\{\lambda_n\}$  are the eigenvalues of the second-order kernel  $K_2(\tau_1, \tau_2)$  associated with  $G_2(t)$ . The eigenvalues are so ordered that

$$|\lambda_n| \geq \lambda_{n+1}, \quad \text{for all } n \quad (22)$$

## 2. Model B

A model for a nonlinear system is given in Fig. IX-5. It consists of an undetermined dc voltage  $c_0$ , a given set of  $N$  orthonormal linear filters  $\{S_n(t)\}$ , and an undetermined set of  $N$  linear filters  $\{h_n(t)\}$ . The input probe is white Gaussian noise.

While the quadratic part of this model will not, in general, produce as good an approximation as the quadratic part of model A, the form of model B has the advantage that its unknown parameters can be determined without iteration.

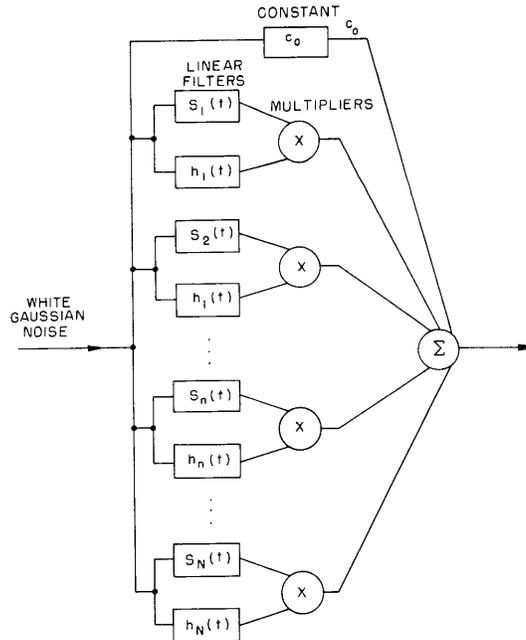


Fig. IX-5. Model B.

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An outline of an analytical method for choosing the set of filters  $\{h_n(t)\}$  will now be given. When an expression for the mean-square error between the output of the given nonlinear system and the output of model B is derived in terms of the Wiener hierarchic expansions of both systems (1), it is found that the set  $\{h_n(t)\}$  appears only in the following quadratic error term:

$$2 \int_0^\infty \int_0^\infty \left[ \frac{1}{2} \sum_{n=1}^N h_n(\tau_1) S_n(\tau_2) + \frac{1}{2} \sum_{n=1}^N S_n(\tau_1) h_n(\tau_2) - K_2(\tau_1, \tau_2) \right]^2 d\tau_1 d\tau_2 \quad (23)$$

where  $K_2(\tau_1, \tau_2)$  is the kernel corresponding to  $G_2(t)$ , the second-order term in the Wiener expansion for the output of the given nonlinear system. A variational procedure is then used to find the set  $\{h_n(t)\}$  that minimizes the error contribution of expression 23. The results of this procedure are

$$h_n(t) = h'_n(t) - \sum_{m=1}^M a_{nm} S_m(t) \quad (24)$$

where

$$a_{nm} = \int_0^\infty \int_0^\infty S_n(\tau_1) S_m(\tau_2) K_2(\tau_1, \tau_2) d\tau_1 d\tau_2 \quad (25)$$

and

$$h'_n(\tau) = 2 \int_0^\infty S_n(\tau_1) K_n(\tau_1, \tau) d\tau_1 \quad (26)$$

Using Eqs. 24-26 and the lemma of expression 4, it follows directly that the experimental procedure for determining the linear filters  $\{h_n(t)\}$ , which will now be outlined, is correct. If  $h_n(t)$  is expanded as in Eq. 24, then the output of the circuit of Fig. IX-6 is  $2a_{nm}$ . The output of the circuit of Fig. IX-7 is  $h'_n(\tau)$ , where  $\tau$  is the position of the tap of the delay line.

The expression for the minimum error with model B is developed as follows. The dc voltage  $c_0$  can duplicate exactly the  $G_0(t)$  dc term of the Wiener expansion for the output of the given nonlinear system. The quadratic part of model B can only approximate the  $G_2(t)$  term. Model B cannot approximate any of the other terms in the Wiener expansion of the output of the given nonlinear system. We now examine the quadratic error term given by expression 23.

If we now expand  $K_2(\tau_1, \tau_2)$  in the complete orthonormal set (the first N terms of which are given in Fig. IX-4), expression 23 becomes

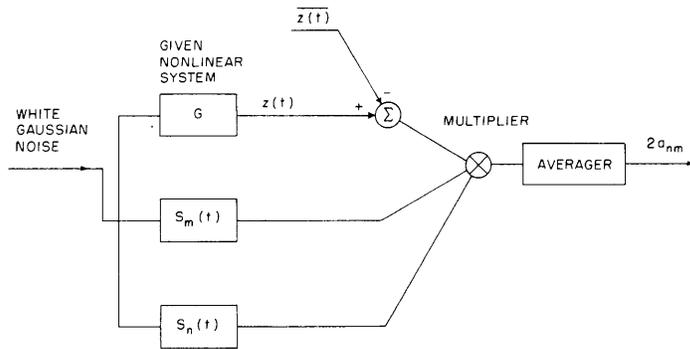


Fig. IX-6. Experimental circuit no. 1 for model B.

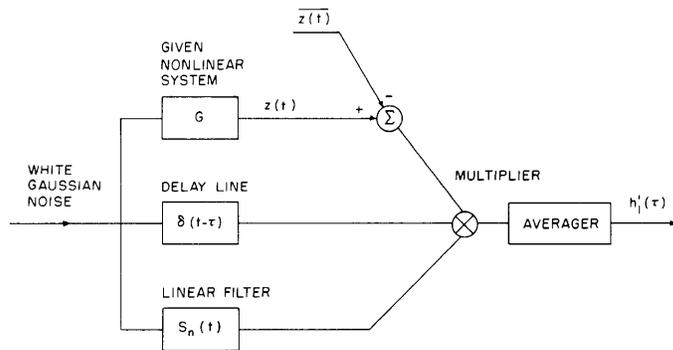


Fig. IX-7. Experimental circuit no. 2 for model B.

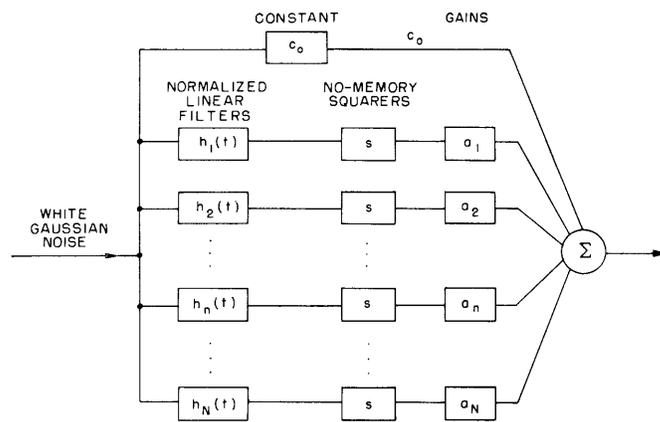


Fig. IX-8. Model C.

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$$2 \int_0^\infty \int_0^\infty \left[ \frac{1}{2} \sum_{n=1}^N h_n(\tau_1) S_n(\tau_2) + \frac{1}{2} \sum_{n=1}^N S_n(\tau_1) h_n(\tau_2) - \sum_{i=1}^\infty \sum_{j=1}^\infty b_{ij} S_i(\tau_1) S_j(\tau_2) \right]^2 d\tau_1 d\tau_2 \quad (27)$$

It can be shown that by proper choice of  $\{h_n(t)\}$ , all terms containing at least one  $S_n(t)$  ( $n \leq N$ ) in the expansion of  $K_2(\tau_1, \tau_2)$  can be removed. Any other choice of  $\{h_n(t)\}$  will increase the quadratic error in expression 27. After choosing  $\{h_n(t)\}$  in this manner, and substituting it in expression 27, the minimum quadratic error becomes

$$2 \int_0^\infty \int_0^\infty \sum_{i=N+1}^\infty \sum_{j=N+1}^\infty [b_{ij} S_i(\tau_1) S_j(\tau_2)]^2 d\tau_1 d\tau_2 \quad (28)$$

If we integrate and use the orthonormality of the set  $\{S_n(t)\}$ , we find that expression 28 reduces to

$$\sum_{i=N+1}^\infty \sum_{j=N+1}^\infty b_{ij}^2 \quad (29)$$

The total minimum mean-square error  $\mathcal{E}_{\min}$  is then given by

$$\mathcal{E}_{\min} = 2 \sum_{i=N+1}^\infty \sum_{j=N+1}^\infty b_{ij}^2 + G_1^2(t) + \sum_{n=3}^\infty G_n^2(t) \quad (30)$$

where the set  $\{G_n(t)\}$  is the Wiener expansion for the output of the given nonlinear system.

### 3. Model C

A model for a nonlinear system is given in Fig. IX-8. The dc voltage  $c_0$ , the set of  $N$  normalized linear filters  $\{h_n(t)\}$ , and a set of  $N$  gains  $\{a_n\}$  are to be determined. Each member of the set  $\{h_n(t)\}$  is restricted to being composed of linear combinations of a given set of  $M$  orthonormal linear filters  $\{S_m(t)\}$ , where  $M$  is greater than  $N$ . Each member of the set  $\{h_n(t)\}$  can be written as

$$h_n(t) = \sum_{m=1}^M c_{nm} S_m(t) \quad (31)$$

The set of linear filters  $\{h_n(t)\}$  is determined by the set of constants  $\{c_{nm}\}$ .

Without this restriction on the set  $\{h_n(t)\}$ , the quadratic part of model C would be identical with the optimum quadratic form of model A. While, in general, it increases the approximation error, the restriction allows the determination of the set  $\{h_n(t)\}$  by

measurements of constants instead of measurements of complete impulse responses.

An outline of the derivation of the optimum values for the undetermined parameters,  $c_o$ ,  $\{h_n(t)\}$ , and  $\{a_n\}$  will now be given. If the output of the given nonlinear system is expanded in terms of the Wiener hierarchy  $\{G_n(t)\}$ , and if the output of model C is similarly expanded, then the mean-square error  $\mathcal{E}$  between the output of the given nonlinear system and the output of model C is

$$\begin{aligned} \mathcal{E} = & \left[ \overline{G_o(t)} - c_o - \sum_{n=1}^N a_n \right]^2 + \overline{G_1^2(t)} + \sum_{n=3}^{\infty} \overline{G_n^2(t)} \\ & + 2 \int_0^{\infty} \int_0^{\infty} \left[ \sum_{n=1}^N a_n h_n(\tau_1) h_n(\tau_2) - K_2(\tau_1, \tau_2) \right]^2 d\tau_1 d\tau_2 \end{aligned} \quad (32)$$

where  $K_2(\tau_1, \tau_2)$  is the quadratic kernel associated with  $G_2(t)$ .

It can be seen that the mean-square error  $\mathcal{E}$  of Eq. 32 is minimized by choosing the dc voltage  $c_o$  so that

$$c_o = \overline{G_o(t)} - \sum_{n=1}^N a_n \quad (33)$$

If the symmetric kernel  $K_2(\tau_1, \tau_2)$  is expanded in terms of the complete orthonormal set  $\{S_m(t)\}$ ,

$$K_2(\tau_1, \tau_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{ij} S_i(\tau_1) S_j(\tau_2) \quad (34)$$

(where the first  $M$  terms of the set  $\{S_m(t)\}$  represent the filters that the set  $\{h_n(t)\}$  is restricted to being composed of), then the minimum value of the mean-square error  $\mathcal{E}$  is obtained when the sets  $\{a_n\}$  and  $\{h_n(t)\}$  are equal, respectively, to the largest  $N$  eigenvalues  $\{\lambda_n\}$  and normalized eigenfunctions  $\{\phi_n(t)\}$  of the expression

$$\sum_{i=1}^M \sum_{j=1}^M b_{ij} S_i(\tau_1) S_j(\tau_2) \quad (35)$$

That is, if

$$\lambda_n \phi_n(\tau_2) = \int_0^{\infty} \phi_n(\tau_1) \sum_{i=1}^M \sum_{j=1}^M b_{ij} S_i(\tau_1) S_j(\tau_2) d\tau_1 d\tau_2 \quad (36)$$

and if

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$$|\lambda_n| \geq |\lambda_{n+1}| \quad n = 1, \dots, M \quad (37)$$

then we should choose

$$a_n = \lambda_n \quad n = 1, \dots, N \quad (38)$$

$$h_n(t) = \phi_n(t) \quad n = 1, \dots, N \quad (39)$$

If the optimum values for the undetermined parameters given by Eqs. 33, 38, and 39, and the expansion for the kernel  $K_2(\tau_1, \tau_2)$  given by Eq. 34, are substituted in the error expression 32, it follows that the minimum mean-square error  $\mathcal{E}_{\min}$  is

$$\mathcal{E}_{\min} = 2 \sum_{n=N+1}^M \lambda_n^2 + 4 \sum_{i=1}^M \sum_{j=M+1}^{\infty} b_{ij}^2 + 2 \sum_{i=M+1}^{\infty} \sum_{j=M+1}^{\infty} b_{ij}^2 + \overline{G_1^2(t)} + \sum_{n=3}^{\infty} \overline{G_n^2(t)} \quad (40)$$

The experimental procedure for determining the linear filters  $\{h_n(t)\}$  and the set of gains  $\{a_n\}$  will now be presented. From the lemma of Eq. 4 it can be shown that the experimental procedure is equivalent to the standard iterative procedure for solving the eigenvalue problem of Eq. 36.

The experimental procedure for determining the set of  $M$  constants  $\{c_{1m}\}$  that determine the linear filter  $h_1(t)$  is an iterative procedure in which only constants are measured. We start off with an arbitrary set of  $M$  constants  $\{d_{1m}\}$  which determines a linear filter  $f_1(t)$  by

$$f_1(t) = \sum_{m=1}^M d_{1m} S_m(t) \quad (41)$$

By using  $f_1(t)$  in the circuit of Fig. IX-9, we can measure as the output of the circuit a set of  $M$  constants  $d'_{2,m}$  associated with the set of filters  $\{S_m(t)\}$ . We now define a new set of  $M$  constants by

$$d_{2,m} = d'_{2,m} \left( \sum_{m=1}^M [d'_{2,m}]^2 \right)^{-1/2} \quad (42)$$

A new linear filter  $f_2(t)$  is then defined as

$$f_2(t) = \sum_{m=1}^M d_{2,m} S_m(t) \quad (43)$$

The filter  $f_2(t)$  is then used in place of  $f_1(t)$  and the experimental procedure is repeated.

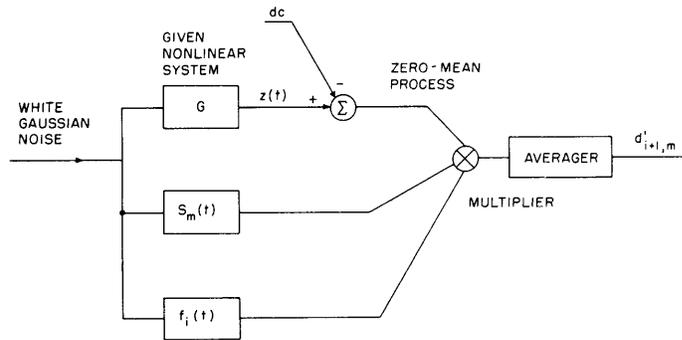


Fig. IX-9. Experimental circuit no. 1 for model C.

It follows from the standard iterative procedure for solving eigenvalue problems that

$$c_{1m} = \lim_{i \rightarrow \infty} d_{i,m} \quad \text{and that} \quad a_1 = \lim_{i \rightarrow \infty} \frac{d_{i+1,m}^1}{2d_{i,m}^1} \quad (44) \quad (45)$$

Hence the linear filter  $h_1(t)$  and the gain  $\{a_n\}$  are determined.

The procedure for determining the remaining linear filters  $\{h_n(t)\}$  and gains  $\{a_n\}$  is similar to that for determining  $h_1(t)$  and  $a_1$  with the difference that the output from the part of the model that has already been determined is subtracted from the output of the given nonlinear system.

#### 4. Examples

As examples of the effectiveness of these approximation techniques, we now examine the approximation error when the given nonlinear system of Fig. IX-10 is approximated by each of three models. This system consists of the cascade of a normalized linear filter  $g_1(t)$  and an ideal no-memory full-wave rectifier. The normalization of the linear filter gives the result

$$\int_0^{\infty} g_1^2(t) dt = 1 \quad (46)$$

The input  $x(t)$  is white Gaussian noise of unity power per cycle per second. The output of the given system  $z(t)$  is given by

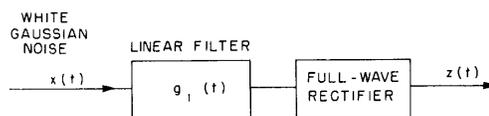


Fig. IX-10. Given nonlinear system.

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$$z(t) = \left| \int_0^{\infty} g_1(\tau) x(t-\tau) d\tau \right| \quad (47)$$

Example A<sub>1</sub>: The first model that we shall consider is that of Fig. IX-11. The dc voltage c<sub>o</sub>, the normalized linear filter h<sub>1</sub>(t), and the gain a<sub>1</sub> are all to be determined

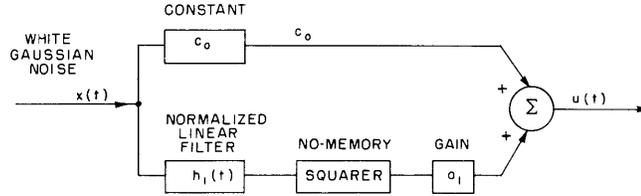


Fig. IX-11. Model for examples A<sub>1</sub> and C<sub>1</sub>.

in such a way that the normalized mean-square error,  $\mathcal{E}^*$ , between the given system output  $z(t)$  and the model output  $u(t)$  is minimized.  $\mathcal{E}^*$  is defined as

$$\mathcal{E}^* = \frac{\overline{[z(t) - u(t)]^2}}{\overline{z^2(t)}} \quad (48)$$

The result is that  $\mathcal{E}^*$  is 0.045.

Example B<sub>1</sub>: The next model that we shall consider is that of Fig. IX-12. The dc voltage c<sub>o</sub> and the linear filter h<sub>1</sub>(t) are to be determined. The given normalized linear filter S<sub>1</sub>(t) of the model is

$$S_1(t) = \frac{1}{\sqrt{2}} [g_1(t) + g_2(t)] \quad (49)$$

where

$$\int_0^{\infty} g_1^2(t) dt = 1 \quad (50)$$

$$\int_0^{\infty} g_2^2(t) dt = 1 \quad (51)$$

$$\int_0^{\infty} g_1(t) g_2(t) dt = 0 \quad (52)$$

Note that g<sub>1</sub>(t) is the linear filter of the given nonlinear system. In this case,

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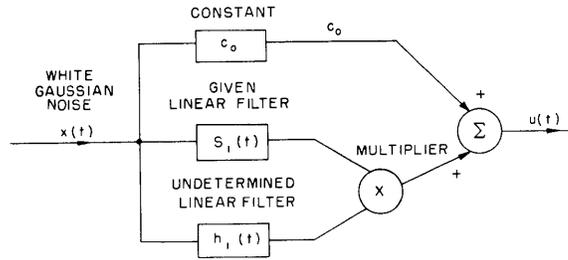


Fig. IX-12. Model for example B<sub>1</sub>.

the minimum normalized mean-square error  $\mathcal{E}^*$  is 0.05.

Example C<sub>1</sub>: The next model that will be considered has the form of Fig. IX-11. The dc voltage  $c_o$ , the linear filter  $h_1(t)$ , and the gain  $a_1$  are to be determined in such a way that the normalized mean-square error is minimized. However, the linear filter  $h_1(t)$  is restricted to be of the form

$$h_1(t) = c_{11}S_1(t) + c_{12}S_2(t) \quad (53)$$

where the orthonormal linear filters  $S_1(t)$  and  $S_2(t)$  are given as

$$S_1(t) = \frac{1}{\sqrt{2}} [g_1(t) + g_2(t)] \quad (54)$$

$$S_2(t) = \frac{1}{\sqrt{3}} [g_1(t) - g_2(t) + g_3(t)] \quad (55)$$

The functions  $g_1(t)$ ,  $g_2(t)$ , and  $g_3(t)$  are orthonormal. Note that  $g_1(t)$  is the linear filter of the given nonlinear system.

Therefore the quantities to be determined are the dc voltage  $c_o$ , the constants  $c_{11}$  and  $c_{12}$ , and the gain  $a_1$ . The result is that the minimum normalized mean-square error  $\mathcal{E}^*$  is 0.142.

D. A. Chesler

References

1. D. A. Chesler, Second-order nonlinear filters, Quarterly Progress Report No. 53, Research Laboratory of Electronics, M.I.T., April 15, 1959, pp. 68-71.
2. N. Wiener, Nonlinear Problems in Random Theory (The Technology Press, Cambridge, Mass., and John Wiley and Sons, Inc., New York, 1958).
3. F. Hildebrand, Methods of Applied Mathematics (Prentice-Hall, Inc., New York, 1952), p. 442.
4. D. A. Chesler, op. cit., p. 70, cf. Eqs. 11, 19 and 21; p. 71, Eq. 22.

(IX. STATISTICAL COMMUNICATION THEORY)

B. APPLICATION OF TRANSFORM THEORY TO CONTINUOUS NONLINEAR SYSTEMS

In previous reports (1, 2) an algebra of continuous systems was introduced. This algebra is related to the functional representation for nonlinear systems; that is, to

$$f(x) = \int h_1(\tau) x(t-\tau) d\tau + \iint h_2(\tau_1, \tau_2) x(t - \tau_1) x(t - \tau_2) d\tau_1 d\tau_2 + \dots$$

$$+ \int \dots \int h_n(\tau_1, \dots, \tau_n) x(t - \tau_1) \dots x(t - \tau_n) d\tau_1 \dots d\tau_n + \dots \quad (1)$$

By means of this algebra, systems composed of linear subsystems (with memory) and nonlinear no-memory subsystems can be expanded in terms of these component subsystems. From this expansion the system kernels or "impulse responses,"  $h_n(t_1, \dots, t_n)$ , can be determined. Just as for linear systems, analysis is greatly facilitated by the use of transforms. In this report the relationship between the algebra and the system transforms is shown. The use of nonlinear transform theory and some properties of these transforms are also developed.

1. Algebra of Systems

In the algebraic notation (1) the system shown in Fig. IX-13 is represented by

$$f = \underline{H}[x] \quad (2)$$

If

$$\underline{H}[\epsilon x] = \epsilon^n \underline{H}[x] \quad (3)$$

where  $\epsilon$  is a constant, then a subscript  $n$  is added to denote an  $n^{\text{th}}$ -order system. That is,

$$\underline{H}_n[\epsilon x] = \epsilon^n \underline{H}_n[x] \quad (4)$$

The three basic system combinations are addition, multiplication, and cascade, which

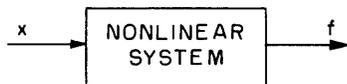


Fig. IX-13.

are represented by  $+$ ,  $\circ$ , and  $*$ , respectively. The rules for the manipulation of this algebra have been given in other reports (1, 2).

The objective is to express, or approximate, a system  $\underline{H}$  by

$$\underline{H} = \underline{H}_1 + \underline{H}_2 + \dots + \underline{H}_n \quad (5)$$

where the  $\underline{H}_n$  are expressed in terms of the component linear and nonlinear no-memory subsystems. These expansions have already been considered (1, 2). From the expansion the system impulse responses  $h_n(t_1, \dots, t_n)$  can be determined. Generally, the form of the impulse responses is cumbersome. Furthermore, calculation of the output  $f(t)$  requires that the integrals of Eq. 1 be evaluated.

These difficulties are overcome by the use of transform theory. The  $n^{\text{th}}$ -order transform pair is defined by

$$\underline{H}_n(s_1, \dots, s_n) = \int \dots \int h_n(t_1, \dots, t_n) e^{-s_1 t_1} \dots e^{-s_n t_n} dt_1 \dots dt_n \quad (6)$$

and

$$h_n(t_1, \dots, t_n) = \left(\frac{1}{2\pi j}\right)^n \int \dots \int \underline{H}_n(s_1, \dots, s_n) e^{s_1 t_1} \dots e^{s_n t_n} ds_1 \dots ds_n \quad (7)$$

The contours of integration and the values of  $s$  can be chosen to give Fourier or Laplace transformations.

It should be noted that, in this algebra, nonlinear no-memory systems are replaced by multiplication operations. Thus

$$\underline{N} * \underline{A} = n_1 \underline{A} + n_2 \underline{A}^2 + \dots + n_m \underline{A}^m \quad (8)$$

where

$$\underline{N}[x] = n_1 x + n_2 x^2 + \dots + n_m x^m \quad (9)$$

The basic relations between the algebra and the transforms are:

<u>Algebra</u>	<u>Transforms</u>	
$\underline{A}_n + \underline{B}_n$	$A_n(s_1, \dots, s_n) + B_n(s_1, \dots, s_n)$	(10)

$\underline{A}_n \cdot \underline{B}_n$	$A_n(s_1, \dots, s_n) B_m(s_{n+1}, \dots, s_{n+m})$	(11)
---	---	------

$\underline{A}_1 * \underline{B}_n$	$A_1(s_1 + s_2 + \dots + s_n) B_n(s_1, \dots, s_n)$	(12)
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A more general cascade situation occurs when, for example, we have

$$\underline{L} = \underline{A}_2 * (\underline{B}_3 + \underline{C}_1) \quad (13)$$

By use of the operation "o" (2), Eq. 13 becomes

$$\underline{L} = \underline{A}_2 \circ (\underline{B}_3 + \underline{C}_1)^2 \quad (14)$$

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or

$$\underline{L} = \underline{A}_2 \circ (\underline{B}_3^2) + 2\underline{A}_2 \circ (\underline{B}_3 \cdot \underline{C}_1) + \underline{A}_2 \circ (\underline{C}_1^2) \quad (15)$$

The transform of the term  $2\underline{A}_2 \circ (\underline{B}_3 \cdot \underline{C}_1)$  is

$$2A_2(s_1 + s_2 + s_3, s_4) B_3(s_1, s_2, s_3) C_1(s_4) \quad (16)$$

and this generalizes in a straightforward manner.

Once a system has been expanded in terms of the algebra and the associated transforms found, we are in a position to obtain the system output for a given input.

## 2. Transform Theory

We shall consider the second-order system in which

$$f_2(t) = \iint h_2(\tau_1, \tau_2) x(t - \tau_1) x(t - \tau_2) d\tau_1 d\tau_2 \quad (17)$$

First, however, consider

$$f_{(2)}(t_1, t_2) = \iint h_2(\tau_1, \tau_2) x(t_1 - \tau_1) x(t_2 - \tau_2) d\tau_1 d\tau_2 \quad (18)$$

Then

$$F_{(2)}(s_1, s_2) = H_2(s_1, s_2) X(s_1) X(s_2) \quad (19)$$

where  $X(s)$  is the transform of the input  $x(t)$ . Formally, this can be inverted to yield

$$f_{(2)}(t_1, t_2) = \left(\frac{1}{2\pi j}\right)^2 \iint F_{(2)}(s_1, s_2) e^{-s_1 t_1} e^{-s_2 t_2} ds_1 ds_2 \quad (20)$$

and the output of the second-order system  $f_2(t)$  is given by

$$f_2(t) = f_{(2)}(t, t) \quad (21)$$

Alternatively, this "association" of  $t_1$  and  $t_2$  can be made in the frequency domain. If  $F_2(s)$  is the transform of  $f_2(t)$ , then

$$F_2(s) = \frac{1}{2\pi j} \int F_{(2)}(s - u, u) du \quad (22)$$

and so the transform of  $f_2(t)$  is formed by a frequency-domain computation. These methods generalize to the higher-order transforms and can be used to

determine the system output.

Now, if the component linear subsystems are in lumped form, then the over-all system transforms are in a generalized "factorizable" form. In this case, when the input transform  $X(s)$  can be factorized, the second-order transform has the general form

$$F_{(2)}(s_1, s_2) = \sum_i \frac{A_i}{s_1 + s_2 + a_i} \frac{B_i}{s_1 + \beta_i} \frac{C_i}{s_2 + \gamma_i} \quad (23)$$

Similar forms exist for the higher-order transforms. A typical term of the second-order case is then

$$\frac{A}{s_1 + s_2 + a} \frac{B}{s_1 + \beta} \frac{C}{s_2 + \gamma} \quad (24)$$

and the use of Eq. 22 gives

$$\frac{A}{s + a} \frac{BC}{s + (\beta + \gamma)} \quad (25)$$

for the typical term of  $F_2(s)$ . Hence

$$F_2(s) = \sum_i \frac{A_i}{s + a_i} \frac{B_i C_i}{s + (\beta_i + \gamma_i)} \quad (26)$$

Examination of Eqs. 24 and 25 shows that the transformation from  $F_{(2)}(s_1, s_2)$  into  $F_2(s)$  can be made by inspection. This inspection technique can be used successively to reduce higher-order transforms  $F_{(n)}(s_1, \dots, s_n)$  to  $F_n(s)$ .

Thus it has been shown that, in the important case of factorizable transforms, frequency-domain inspection techniques can be used to reduce the higher-order transforms to the first-order transforms of the system output.

### 3. Example

For the feedback system shown in Fig. IX-14,

$$\underline{N} = \underline{N}_1 + \underline{N}_3 \quad (27)$$

and  $\underline{H}_1$  has a transform  $\frac{A}{s + a}$ . Use of the algebra gives

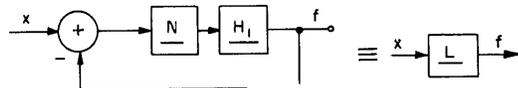


Fig. IX-14.

(IX. STATISTICAL COMMUNICATION THEORY)

$$\underline{L}_1 = \underline{H}_1 * (\underline{I} - \underline{H}_1)^{-1} \quad (28)$$

$$\underline{L}_3 = \underline{L}_1 * \underline{N}_3 \circ (\underline{I} - \underline{L}_1)^3 \quad (29)$$

$$\underline{L}_5 = 3\underline{L}_1 * \underline{N}_3 \circ \left\{ (\underline{I} - \underline{L}_1)^2 - \underline{L}_3 \right\} \quad (30)$$

for the first three terms in the expansion for  $\underline{L}$ . The corresponding transforms for  $A \gg a$  are

$$L_1(s) = \frac{A}{s + A} \quad (31)$$

$$L_3(s_1, s_2, s_3) = \frac{n_3 A}{s_1 + s_2 + s_3 + A} \frac{s_1}{s_1 + A} \frac{s_2}{s_2 + A} \frac{s_3}{s_3 + A} \quad (32)$$

$$L_5(s_1, \dots, s_5) = \frac{3n_3^2 A^2}{(s_1 + \dots + s_5 + A)(s_3 + s_4 + s_5 + A)} \left( \frac{s_1}{s_1 + A} \right) \dots \left( \frac{s_5}{s_5 + A} \right) \quad (33)$$

For the input

$$x(t) = Xu(t) \quad (34)$$

or

$$X(s) = \frac{X}{s} \quad (35)$$

use of the inspection techniques gives

$$F(s) = \frac{X}{s} + \frac{X}{s + A} \left( 1 - \frac{1}{2} n_3 X^2 - \frac{3}{4} n_3^2 X^4 \right) - \frac{X}{s + 3A} \left( \frac{1}{2} n_3 X^2 + \frac{3}{2} n_3^2 X^4 \right) + \frac{X}{s + 5A} \left( \frac{3}{4} n_3^2 X^4 \right) \quad (36)$$

or

$$f(t) \approx X - X \left( 1 - \frac{1}{2} n_3 X^2 - \frac{3}{4} n_3^2 X^4 \right) e^{-At} - X \left( \frac{1}{2} n_3 X^2 + \frac{3}{2} n_3^2 X^4 \right) e^{-3At} + X \left( \frac{3}{4} n_3^2 X^4 \right) e^{-5At}, \quad \text{for } t \geq 0 \quad (37)$$

which is a good estimate of  $f(t)$  for  $|n_3 X^2| < 0.5$ .

#### 4. Properties of Transforms

The higher-order transforms can be used directly to obtain sinusoidal steady-state output and initial and final values of the output. If

$$f_1 = \underline{H}_1[x] \quad (38)$$

and

$$x(t) = \text{Re}\{Xe^{j\omega t}\} \quad (39)$$

then

$$f_1(t) = \text{Re}\{XH_1(j\omega) e^{j\omega t}\} \quad (40)$$

in the steady state, as is well known. For the situation in which

$$f_2 = \underline{H}_2[x] \quad (41)$$

the steady-state output is given by

$$f_2(t) = \frac{1}{2} \text{Re}\{X^2 e^{j2\omega t} + |X|^2\} \quad (42)$$

For

$$f_3 = \underline{H}_3[x] \quad (43)$$

we have

$$f_3(t) = \frac{1}{4} \text{Re}\{X^3 e^{j3\omega t} + 3|X|^2 X e^{j\omega t}\} \quad (44)$$

Similar forms hold for higher-order operations.

The initial and final value theorems are similar to the linear theorems. If  $f_{(n)}(t_1, \dots, t_n)$  has a transform  $F_{(n)}(s_1, \dots, s_n)$  and

$$f_n(t) = f_{(n)}(t, t, \dots, t) \quad (45)$$

then

$$\lim_{t \rightarrow 0} f_n(t) = \lim_{\substack{s_1 \rightarrow \infty \\ \vdots \\ s_n \rightarrow \infty}} F_{(n)}(s_1, \dots, s_n) s_1 \dots s_n \quad (46)$$

and

$$\lim_{t \rightarrow \infty} f_n(t) = \lim_{\substack{s_1 \rightarrow 0 \\ \vdots \\ s_n \rightarrow 0}} F_{(n)}(s_1, \dots, s_n) s_1 \dots s_n \quad (47)$$

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The validity of these relations depends upon the existence of the limits in a manner that is similar to that of the corresponding theorems of linear transform theory.

D. A. George

### References

1. D. A. George, An algebra of continuous systems, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., July 15, 1958, p. 104.
2. D. A. George, Continuous feedback systems, Quarterly Progress Report No. 53, Research Laboratory of Electronics, M.I.T., April 15, 1959, p. 72.

### C. CONTINUOUS NONLINEAR SYSTEMS WITH GAUSSIAN INPUTS

This report is concerned with continuous nonlinear systems with random inputs — particularly with Gaussian inputs. We are interested in obtaining output averages, correlation functions, and spectra. Using the notation of the algebra of continuous systems (1), we have

$$f(t) = \underline{H}[x(t)] \quad (1)$$

and we define

$$f(t+T) = \underline{H}[x(t+T)] = \underline{H}^T[x(t)] \quad (2)$$

In general, a system is described, or approximated, by

$$\underline{H} = \underline{H}_1 + \underline{H}_2 + \dots + \underline{H}_n \quad (3)$$

in which the impulse response  $h_m(t_1, \dots, t_m)$ , or the transform  $H_m(s_1, \dots, s_m)$ , is associated with  $\underline{H}_m$ . Then

$$f(t) = \underline{H}[x(t)] = f_1(t) + f_2(t) + \dots + f_n(t) \quad (4)$$

where

$$f_m(t) = \underline{H}_m[x(t)] \quad (5)$$

Hence the average value of  $f(t)$  is given by

$$\overline{f(t)} = \overline{f_1(t)} + \overline{f_2(t)} + \dots + \overline{f_n(t)} \quad (6)$$

and the correlation function of  $f(t)$  is

$$\overline{f(t) f(t+T)} = \overline{[f_1(t) + \dots + f_n(t)][f_1(t+T) + \dots + f_n(t+T)]} \quad (7)$$

or

$$\overline{f(t) f(t+T)} = \sum_i^n \sum_j^n \overline{f_i(t) f_j(t+T)} \quad (8)$$

This report is concerned with obtaining

$$\overline{f_i(t)} \quad (9)$$

and

$$\overline{f_i(t) f_j(t+T)} \quad (10)$$

so that  $\overline{f(t)}$  and  $\overline{f(t) f(t+T)}$  can be calculated.

Now, to illustrate the method of computing  $\overline{f(t)}$ , consider, for example,

$$f_4(t) = \underline{H}_4[x(t)] \quad (11)$$

or

$$\overline{f_4(t)} = \overline{\iiint\iiint h_4(\tau_1, \tau_2, \tau_3, \tau_4) x(t - \tau_1) x(t - \tau_2) x(t - \tau_3) x(t - \tau_4) d\tau_1 d\tau_2 d\tau_3 d\tau_4} \quad (12)$$

Then, by interchanging orders of averaging and integration, we have

$$\overline{f_4(t)} = \iiint\iiint h_4(\tau_1, \tau_2, \tau_3, \tau_4) \overline{x(t - \tau_1) x(t - \tau_2) x(t - \tau_3) x(t - \tau_4)} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \quad (13)$$

In our algebraic notation this is represented by

$$\overline{f_4} = \underline{H}_4(\overline{x_1 x_2 x_3 x_4}) \quad (14)$$

Similar forms hold for higher-order situations and also for correlation calculations. If we consider

$$f_1 = \underline{H}_1[x] \quad (15)$$

and

$$f_3 = \underline{H}_3[x] \quad (16)$$

then

$$\overline{f_1(t) f_3(t+T)} = \underline{H}_1 \cdot \underline{H}_3^T(\overline{x_1 x_2 x_3 x_4}) \quad (17)$$

or

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$$\overline{f_1(t) f_3(t+T)} = \iiint\limits_{\tau_1, \tau_2, \tau_3, \tau_4} h_1(\tau_1) h_3(\tau_2, \tau_3, \tau_4) \times \overline{x(t-\tau_1) x(t-\tau_2) x(t+T-\tau_3) x(t+T-\tau_4)} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \quad (18)$$

In general, if the system impulse responses  $h_n(t_1, \dots, t_n)$  and the higher-order input correlation functions are known, then the output averages can be obtained by performing the necessary integrations. In this report we are mainly interested in the situation in which the input signal has a Gaussian distribution. In particular, if  $x(t)$  is white Gaussian, then

$$\overline{x(t)} = 0 \quad (19)$$

$$\overline{x(t_1) x(t_2)} = \delta(t_2 - t_1) \quad (20)$$

$$\overline{x(t_1) x(t_2) x(t_3)} = 0 \quad (21)$$

$$\overline{x(t_1) x(t_2) x(t_3) x(t_4)} = \delta(t_2 - t_1) \delta(t_4 - t_3) + \delta(t_3 - t_1) \delta(t_4 - t_2) + \delta(t_4 - t_1) \delta(t_3 - t_2) \quad (22)$$

and so on, for the higher-order correlation functions. In this case Eq. 10, for example, becomes

$$f_1(t) f_3(t+T) = 3 \iint h_1(\tau_1) h_3(\tau_1+T, \tau_2, \tau_2) d\tau_1 d\tau_2 \quad (23)$$

and similar results are obtained for the other terms of Eqs. 6 and 8. This white Gaussian situation has been developed by Wiener (2) and can be easily extended to the non-white Gaussian case. In the general Gaussian case, the input signal  $x(t)$  can be considered as being formed from a white Gaussian  $y(t)$  by a shaping filter  $\underline{F}_1$ . Then the shaping filter can be absorbed in the nonlinear system  $\underline{H}$  to produce a system  $\underline{K}$  with

$$\underline{K} = \underline{H} * \underline{F}_1 \quad (24)$$

Computations can then proceed with this equivalent situation in which  $\underline{K}$  operates on a white Gaussian  $y(t)$ .

The remainder of this report will be devoted to developing the use of transform-domain techniques for obtaining system output averages when the input is white Gaussian. Working in the transform domain avoids the necessity of performing the integrations illustrated in the previous equations, and also the necessity for obtaining the  $h_n(t_1, \dots, t_n)$ , which often have very clumsy functional forms. These transform

methods will be illustrated by considering a typical situation that arises in computing output averages.

Let us consider the term

$$\begin{aligned} \overline{f_2(t) f_2(t+T)} &= \iiint\limits_{\tau_1, \tau_2, \tau_3, \tau_4} h_2(\tau_1, \tau_2) h_2(\tau_3, \tau_4) \\ &\quad \times \overline{x(t - \tau_1) x(t - \tau_2) x(t + T - \tau_3) x(t + T - \tau_4)} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \end{aligned} \quad (25)$$

where  $x(t)$  is white Gaussian. Then, by use of Eq. 22,

$$\begin{aligned} \overline{f_2(t) f_2(t+T)} &= \iint h_2(\tau_1, \tau_1) h_2(\tau_2, \tau_2) d\tau_1 d\tau_2 \\ &\quad + 2 \iint h_2(\tau_1, \tau_2) h_2(\tau_1 + T, \tau_2 + T) d\tau_1 d\tau_2 \end{aligned} \quad (26)$$

If  $h_2(t_1, t_2)$  has a transform  $H_2(s_1, s_2)$  that is in factorizable form, then the term

$$\int h_2(\tau_1, \tau_1) d\tau_1 \quad (27)$$

can be obtained by

- (a) Reducing  $H_2(s_1, s_2)$  to  $K_1(s)$  by the procedure described in Section IX-B.
- (b) Taking  $\lim_{s \rightarrow 0} K_1(s) = K_1(0)$ .

Then

$$\iint h_2(\tau_1, \tau_2) h_2(\tau_2, \tau_2) d\tau_1 d\tau_2 = K_1^2(0) \quad (28)$$

This procedure is based on (a) finding the transform  $K_1(s)$  of  $h_2(t, t)$ ; (b) realizing that Eq. 27 is equivalent to the steady-state step response of a linear system with transform  $K_1(s)$ . The second term in Eq. 26 is determined by observing that

$$\iint h_2(\tau_1, \tau_2) h_2(\tau_1 + T, \tau_2 + T) d\tau_1 d\tau_2 \quad (29)$$

has a transform

$$H_2(-s_1, -s_2) H_2(s_1, s_2) \quad (30)$$

These expressions are Fourier transforms, and so  $s_1 = j\omega_1$ , and  $s_2 = j\omega_2$ . Again, if we assume factorizable transforms, Eq. 3 can be reduced to  $G(s)$  by inspection methods (that is, setting  $T_1 = T_2 = T$  by means of transform-domain techniques). This  $G(s)$  is,

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then, the Fourier transform of

$$\iint h_2(\tau_1, \tau_2) h_2(\tau_1 + T, \tau_2 + T) d\tau_1 d\tau_2 \quad (31)$$

We now have

$$\Phi(s) = K_1^2(0) \delta(s) + 2G(s) \quad (32)$$

where  $s = j\omega$  and  $\Phi(s)$  is the transform of  $\overline{f_2(t) f_2(t+T)}$ . It should be noted that, in this case, a number of terms occur in Eq. 30 with the form

$$L(s_1 + s_2) \frac{A}{s_1 + a} \frac{B}{-s_2 + a} \quad (33)$$

The contribution of such terms, when we are transforming to a function of a single variable  $s$ , is zero.

This case illustrates the situation that arises in computing output averages or correlations. Similar procedures can be applied to the higher-order terms in Eqs. 6 and 8.

For linear systems with random inputs it is usually more convenient to work in the transform (frequency) domain, rather than the time domain. This is also true for nonlinear systems, and this report has illustrated how the higher-order transforms associated with nonlinear systems can be used.

D. A. George

### References

1. D. A. George, An algebra of continuous systems, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., July 15, 1958, p. 104.
2. N. Wiener, Nonlinear Problems in Random Theory (The Technology Press, Cambridge, Mass., and John Wiley and Sons, Inc., New York, 1958).

## D. MAXIMUM SIGNAL-TO-NOISE RATIO ESTIMATION

### 1. Introduction

Consider the situation shown schematically in Fig. IX-15. The independent source, S, generates a random variable,  $x$ , which is fed to the control system, C. System C generates a set of  $N$  random variables,  $(y_1, y_2, \dots, y_N)$ , and system D produces an output,  $z$ , where  $z = f(y_1, y_2, \dots, y_N)$ . We shall denote  $(y_1, y_2, \dots, y_N)$  by  $\bar{y}$ , a convenient vector notation. We shall then assume that S is completely characterized (for our purposes) by the probability density function  $p(x)$ , C is described by the conditional probability density function  $p(\bar{y}|x)$ , and D is specified by the function  $f(\bar{y})$ .

From a physical point of view Fig. IX-15 might represent a complete communication

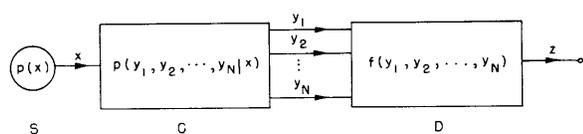


Fig. IX-15. Communication system.

system. For example, S could be an ergodic, bandlimited signal source, with output consisting of a discrete sequence of numbers representing Nyquist samples of the signal, and C could represent a communication channel that would include the usual transmitter or encoder, transmission medium, and part of the receiver with all storage elements;  $\bar{y}$  could stand for Nyquist samples of a physical, bandlimited signal actually sent through the channel. C could certainly contain random-noise sources that would interfere with the transmitted signal in some manner. D could represent the non-linear no-storage part of the detector, or decoder, in the receiver. For questions involving time averages of the variables in the system, the ergodic hypothesis would permit answers in terms of statistical averages. Then, under the assumption that the output samples of the signal source are independent, the problem can be solved by statistical methods, by following the abstract formulation of the system shown in Fig. IX-15.

We state the problem: Given  $p(x)$  and  $p(\bar{y}|x)$ : How must  $f$  be chosen so that  $z (=f(\bar{y}))$  is a "best" estimate of  $x$ , on the average? Of course, the answer generally depends upon the definition of "best." Frequently, we are interested in obtaining the least-mean-square error; that is, in minimizing  $E[(x-z)^2]$ . Some essential facts about this kind of estimate are briefly reviewed in the next section. However, there are some practical situations in which a signal-to-noise ratio criterion is more appropriate. We shall define this quantity in two ways, both of which follow common usage of the term. Then we shall find that the estimate  $z$  (and hence the function  $f$ ) that maximizes the signal-to-noise ratio (according to either definition) is the same as the least-mean-square estimate, except for an arbitrary multiplicative constant. In the discussion that follows, we shall assume that the system and signals do not have dc components; that is, that

$$E[x] = 0 \quad (1)$$

and

$$E[z] = 0 \quad (2)$$

## 2. Least-Mean-Square Estimation

We recall from the work of Bose (1) that for the system shown in Fig. IX-15,  $E[(x-z)^2]$  can be minimized by setting

$$z = f(\bar{y}) = \int xp(x|\bar{y}) dx \quad (3)$$

where

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$$p(x|\bar{y}) = \frac{p(\bar{y}|x) p(x)}{p(\bar{y})} = \frac{p(x, \bar{y})}{p(\bar{y})} \quad (4a)$$

and

$$p(\bar{y}) = \int p(x, \bar{y}) dx \quad (4b)$$

Hereafter, we shall designate the least-mean-square estimate from Eq. 3 by  $z_o$ , and the estimator, or function, by  $f_o$ , to distinguish them from other estimates and estimators.

One interesting fact, which will be useful later, follows immediately: If  $z$  is any estimate (i. e., determined by any integrable function,  $f$ , of  $\bar{y}$ ), then

$$\begin{aligned} E[xz] &= \iint xf(\bar{y}) p(x, \bar{y}) dx d\bar{y} \\ &= \int \left[ \int xp(x|\bar{y}) dx \right] f(\bar{y}) p(\bar{y}) d\bar{y} \\ &= \int f_o(\bar{y}) f(\bar{y}) p(\bar{y}) d\bar{y} \\ &= E[z_o z] \end{aligned} \quad (5)$$

We next offer a method for splitting any  $z$  into additive signal and noise components. Since we are trying to reproduce  $x$ , it appears reasonable to assume that there exists a scalar  $\alpha$ , that is so chosen that  $\alpha x$  is the signal component of  $z$ . Then, the difference between  $z$  and  $\alpha x$  can be defined as the additive noise,  $n$ ,

$$n = z - \alpha x \quad (6)$$

How is  $\alpha$  to be chosen in a unique manner? From the definition of  $n$  the signal  $\alpha x$  and noise  $n$  cannot, in general, be made statistically independent (except in the trivial case,  $\alpha = 0$ ). However, linear independence can always be achieved because

$$\begin{aligned} E[(\alpha x)n] &= E[\alpha x(z - \alpha x)] \\ &= \alpha \{E[xz] - \alpha E[x^2]\} \\ &= 0 \end{aligned} \quad (7)$$

always has the (nontrivial) solution

$$\alpha = \frac{E[xz]}{E[x^2]} \quad (8a)$$

$$= \frac{E[z_o z]}{E[x^2]} \quad (8b)$$

except in the case  $E[x^2] = 0$ , which would be rather meaningless. Equation 8b follows from Eq. 8a by using Eq. 5. If  $z = z_o$ , for the least-mean-square error, then Eq. 8b becomes

$$a_o = \frac{E[z_o^2]}{E[x^2]} \quad (8c)$$

Calculating the mean-square error for  $z = z_o$ , we obtain

$$\begin{aligned} E[(x-z_o)^2] &= E[x^2] - 2E[xz_o] + E[z_o^2] \\ &= E[x^2] - E[z_o^2] \\ &\geq 0 \end{aligned} \quad (9)$$

so that

$$0 \leq E[z_o^2] \leq E[x^2] \quad (10)$$

Hence, we have

$$0 \leq a_o \leq 1 \quad (11)$$

which seems quite reasonable; that is, the additive signal component in the least-mean-square estimate never exceeds the actual signal that we are trying to estimate.

### 3. Estimation of Maximum Signal-to-Noise Ratio

We have just seen how to split an estimate,  $z$ , into uncorrelated, additive signal and noise components. This brings us to a consideration of signal-to-noise ratio. Signal-to-noise ratio is often defined as the ratio of mean-square signal to mean-square noise, expressed as a pure number, or as ten multiplied by the logarithm to the base ten of this quantity, expressed in decibels. Since the logarithm is a strictly increasing function of its (positive real) argument, and we are primarily interested in maximizing signal-to-noise ratio, both definitions appear satisfactory. Therefore we shall accept the ambiguity associated with calling both  $\left(\frac{S}{N}\right)$  and  $\gamma$  by the name signal-to-noise ratio, and define

$$\gamma = \frac{E[(ax)^2]}{E[n^2]} \quad (12)$$

and

$$\left(\frac{S}{N}\right) = 10 \log_{10} \gamma \quad (13)$$

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In Eq. 12,  $a$  is defined by Eq. 8a or Eq. 8b;  $n$  is defined by Eq. 6.

We define the correlation coefficient,  $\rho$ , in the usual manner, using subscripts for clarity,

$$\rho_{rs} = \frac{E[rs]}{(E[r^2] E[s^2])^{1/2}} \quad (14)$$

By using Eqs. 6, 8a, and 14, Eq. 12 can be put into a more convenient form

$$\gamma = \frac{a^2 E[x^2]}{E[(z - ax)^2]} = \frac{E^2[xz]}{E[x^2] E[z^2] - E^2[xz]} = \frac{1}{\frac{1}{2} - \rho_{xz}^2} \quad (15)$$

This form makes it obvious that multiplying  $z$  by some constant will not change the signal-to-noise ratio, and this must also be true for the  $z$  that maximizes this ratio. In other words, signal-to-noise ratio is independent of any noiseless amplification inserted in the system output.

As before, let the subscript "o" be used with quantities defined when  $z = z_o$ , the least-mean-square estimate. From Eq. 14 and Eq. 5, we can verify that

$$\rho_{xz} = \rho_{z_o} z \rho_{xz_o} \quad (16)$$

But  $\rho_{xz_o}^2$  can be found in terms of  $\gamma_o$  with the aid of Eq. 15. Then

$$\rho_{xz_o}^2 = \frac{\gamma_o}{\gamma_o + 1} \quad (17)$$

Substitution from Eqs. 16 and 17 in Eq. 15 yields

$$\gamma = \frac{\gamma_o}{\frac{1}{2} (\gamma_o + 1) - \gamma_o} \rho_{z_o}^2 \quad (18)$$

It is apparent that  $\gamma$  is maximum when  $\rho_{z_o}^2$  is maximum, and the latter quantity attains its maximum value of unity when  $z$  is a scalar multiple of  $z_o$ . In other words, the signal-to-noise ratio at the output is maximized by using any arbitrary constant multiple of the least-mean-square estimate, which was to be proved.

The same result could have been obtained from Eq. 15 with a simple variational technique. However, it was worth while to derive Eq. 18 in the way in which we did for the insight that it gives into the following problem. Suppose that we have to use a single, fixed estimator (function  $f$  in box D) in the system of Fig. IX-15, but that the

statistics of source  $S$  and/or channel  $C$  are imperfectly known. Specifically, let us assume that the complete situation is described by the set of joint probability density functions,  $p_i(x, \bar{y})$ , and the associated probabilities  $P_i$ , with  $i$  ranging over a finite index set. That is, we may use any one of a finite set of joint statistics, with a certain probability of use attached to each member of the set. An interesting question would then be to ask how to choose the estimator  $f$  so that the logarithmic signal-to-noise ratio  $\left(\frac{S}{N}\right)$  of the estimate  $z$  would be, on the average, highest. Then we would have to vary  $f$  in

$$\left(\frac{S}{N}\right)_{av} = \sum_{i=1}^K P_i 10 \log_{10} \gamma_i \quad (19)$$

which, indeed, looks rather messy! However, the complication can be reduced somewhat by letting  $\gamma_{oi}$  be the best  $\gamma$  for the  $i^{\text{th}}$  statistics alone, and  $z_{oi}$  the corresponding estimate for this case. Then the problem becomes one of minimizing the product

$$\prod_{i=1}^K \left[ \left( \frac{1}{E[z_{oi}z]} \right)^2 (\gamma_{oi} + 1) - \gamma_{oi} \right]^{P_i} \quad (20)$$

subject to a constraint on  $E[z^2]$ . This problem may well be intractable in the general case, but there appears to be hope for certain simple cases.

A. D. Hause

#### References

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#### E. AN UPPER BOUND ON TREE PROBABILITY IN A PROBABILISTIC GRAPH

The probabilistic graph has been described previously (1,2). Briefly, it consists of an ensemble of linear graphs generated from a particular graph, the base graph, by randomly erasing links in such a manner that in the ensemble every link of the base graph appears independently with a probability,  $p$ . The tree probability,  $P_T$ , of a probabilistic graph is the total measure over the ensemble of the graphs that contain at least one tree.

In general, we do not know how to arrange a base graph containing  $n$  nodes and  $\ell$  (undirected) links to yield, for a given link reliability,  $p$ , the maximum tree probability. Such an optimum base graph does exist, since the number of possible  $n$ -node  $\ell$ -link graphs is finite. It is possible, however, to obtain upper bounds on the tree probability of all  $n$ -node  $\ell$ -link graphs, including the optimum graph. In this report, a rather

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interesting upper bound is derived by a direct application of the factoring theorem (2).

The factoring theorem can be stated as follows:

$$P_T = qP_T^I + pP_T^{II} \quad (1)$$

where  $q = 1 - p$  is the probability of link failure, and  $P_T^I$  and  $P_T^{II}$  are the tree probabilities of probabilistic graphs which differ from the original graph only in having one link omitted from the base graph when calculating  $P_T^I$  and by having that same link "shorted" (superimposing the nodes at either end of the link) in the base graph when calculating  $P_T^{II}$ .

Now, let  $P_1(n, \ell)$  be the tree probability in the optimum  $n$ -node  $\ell$ -link graph. Let  $P_2(n, \ell)$  be the desired upper bound on  $P_1(n, \ell)$ . From Eq. 1,

$$P_1(n, \ell) \leq qP_1(n, \ell - 1) + pP_1(n - 1, \ell - 1) \quad (2)$$

The right side of Eq. 2 is larger than the left for two reasons. First, when we remove a link from the optimum  $n$ -node  $\ell$ -link graph in order to calculate  $P_T^I$ , an  $n$ -node  $(\ell-1)$ -link graph is obtained that may not be optimum. Assuming that this  $n$ -node  $(\ell-1)$ -link graph is optimum yields an upper bound. Likewise, when we short out the same link in order to calculate  $P_T^{II}$ , we obtain an  $(n-1)$ -node structure that has, at most,  $(\ell-1)$  links. (It contains  $\ell - 1$  links if and only if no other links in the original base graph join the same two nodes as the shorted link.) Since  $P_1(n, \ell)$  is a monotonically increasing function of  $\ell$ ,  $P_T^{II}$  is bounded above by  $P_1(n - 1, \ell - 1)$ .

Now, the bound in Eq. 2 is further increased if we substitute in the right side the upper-bound probabilities  $P_2(n, \ell - 1)$  and  $P_2(n - 1, \ell - 1)$  for the probabilities  $P_1(n, \ell - 1)$  and  $P_1(n - 1, \ell - 1)$ , respectively.

$$P_1(n, \ell) \leq qP_2(n, \ell - 1) + pP_2(n - 1, \ell - 1) \quad (3)$$

The upper bound,  $P_2(n, \ell)$ , can now be defined. We shall do so by means of a difference equation and appropriate boundary conditions. Let, for  $n > 2$  and  $\ell > n - 1$ ,

$$P_2(n, \ell) = qP_2(n, \ell - 1) + pP_2(n - 1, \ell - 1) \quad (4)$$

The boundary conditions are:

$$P_2(2, \ell) = P_1(2, \ell) \quad (5a)$$

$$P_2(n, n - 1) = P_1(n, n - 1) \quad (5b)$$

It is clear from Eqs. 3 and 4 that  $P_2(n, \ell)$  is an upper bound on  $P_1(n, \ell)$ .

Let us obtain the boundary conditions explicitly. In a 2-node probabilistic graph, the probability of at least one tree is equal to the probability that at least one link exists (is not erased). Thus since  $q$  is the probability that a link is erased, and erasures are independent,

$$P_2(2, \ell) = P_1(2, \ell) = (1 - q^\ell) = p[1 + q + q^2 + \dots + q^{\ell-1}] \quad (6)$$

Likewise, if there are only  $(n-1)$  links in an  $n$ -node graph, all links must be present to obtain a tree. Therefore

$$P_2(n, n-1) = P_1(n, n-1) = p^{n-1} \quad (7)$$

It is now possible to solve Eq. 4 with the boundary conditions of Eqs. 6 and 7. The solution is

$$P_2(n, \ell) = p^{n-1} \sum_{k=0}^{\ell-n+1} \binom{n+k-2}{k} q^k \quad (8)$$

By inspection, it is evident that  $P_2(n, \ell)$  as defined in Eq. 8 satisfies the boundary conditions, Eqs. 6 and 7. To see that it satisfies Eq. 4, we substitute Eq. 8 in Eq. 4, and obtain

$$p^{n-1} \sum_{k=0}^{\ell-n+1} \binom{n+k-2}{k} q^k = q \left[ p^{n-1} \sum_{k=0}^{\ell-n} \binom{n+k-2}{k} q^k \right] + p \left[ p^{n-2} \sum_{k=0}^{\ell-n+1} \binom{n+k-3}{k} q^k \right] \quad (9)$$

Collecting terms on the right, we find that Eq. 9 becomes

$$p^{n-1} \sum_{k=0}^{\ell-n+1} \binom{n+k-2}{k} q^k = p^{n-1} \sum_{k=0}^{\ell-n+1} \left[ \binom{n+k-3}{k-1} + \binom{n+k-3}{k} \right] q^k \quad (10)$$

where, by convention,  $\binom{n}{-1} = 0$ .

Thus, in order for Eq. 10 to be an identity, it is only necessary for

$$\binom{n+k-2}{k} = \binom{n+k-3}{k-1} + \binom{n+k-3}{k} \quad (11)$$

However, Eq. 11 is a well-known recursion for binomial coefficients. (See, for example, Riordan's (3) Eq. 7.)

Several properties of  $P_2(n, \ell)$  as defined by Eq. 8 are of interest. First, let us note that as  $\ell$  tends to infinity, for any fixed  $n$ ,  $P_2(n, \ell)$  tends to 1 from below. This limiting result is obtained by noting that the sum on the right side of Eq. 8 approaches  $(1-q)^{-(n-1)}$  as  $\ell$  tends to infinity.

We next wish to determine the conditions under which  $P_2(n, \ell)$  is an attainable upper bound; that is, the conditions under which  $P_2(n, \ell) = P_1(n, \ell)$ . First, note that, as  $p$  tends to zero, the tree probability of any graph behaves as

$$P_T = N_T p^{n-1}, \quad \text{for } p \ll 1 \quad (12)$$

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where  $N_T$  is the number of trees in the graph. Equation 12 is quickly obtained by considering the tree probability as the probability of a union of nondisjoint events, the events being the occurrence of each of the possible trees in the graph. The probability of a union of nondisjoint events can be calculated by adding the probabilities of the individual events, as if they were disjoint, then subtracting the probabilities of the events occurring in pairs, adding on the probabilities of the events occurring three at a time, and so on. However, all terms in this alternating series will have a higher power of  $p$  than the first term, the term in which the events were treated as disjoint. Thus, this first term will dominate all others, as  $p$  tends to zero, and result in Eq. 12.

Let us examine the behavior of  $P_2(n, \ell)$  for small  $p$ . From Eq. 8, it is evident that

$$P_2(n, \ell) = p^{n-1} \sum_{k=0}^{\ell-n+1} \binom{n+k-2}{k}, \quad \text{for } p \ll 1 \quad (13)$$

Equation 13 can be reduced by another binomial coefficient identity (actually an iteration of Eq. 11; see Riordan's (3) Eq. 8). Using this identity, we have

$$P_2(n, \ell) = \binom{\ell}{n-1} p^{n-1}, \quad \text{for } p \ll 1 \quad (14)$$

Comparing Eq. 14 and Eq. 12, we see that if  $P_2(n, \ell)$  is the tree probability of a realizable graph, this graph must contain  $\binom{\ell}{n-1}$  trees. Thus

$$N_T = \binom{\ell}{n-1} \quad (15)$$

However, Eq. 15 states that every set of  $(n-1)$  links in the graph is a tree. In general, this is impossible. In particular, Eq. 15 can be satisfied by a realizable graph if and only if  $\ell = n - 1$  or  $\ell = n$ . If the number of links is larger than the number of nodes,  $P_2(n, \ell)$  is an unattainable upper bound.

Finally, by use of the recursion of Eq. 11, the coefficients of the powers of  $q$  in Eq. 8 can be simply obtained. See Table IX-1. Each entry in the table is the sum of the entry directly above and the entry directly to the left. The  $a_i$  are defined by writing Eq. 8 in the following way:

$$P_2(n, \ell) = p^{n-1} [a_0 + a_1 q + a_2 q^2 + \dots + a_{\ell-n+1} q^{\ell-n+1}] \quad (16)$$

Thus, for  $n = 5$  and  $\ell = 8$ ,

$$P_2(5, 8) = p^4 [1 + 4q + 10q^2 + 20q^3 + 35q^4] \quad (17)$$

Table IX-1. Coefficients of Eq. 16.

n	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	...
2	1	1	1	1	1	1	1	1	1
3	1	2	3	4	5	6	...		
4	1	3	6	10	15	21	...		
5	1	4	10	20	35	56	...		
6	1	5	15	35	70	126	...		

I. M. Jacobs

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## F. SOME CRITERIA IN PULSE TRANSMISSION PROBLEMS

This report concerns the pulse transmission system shown in Fig. IX-16, in which N is a linear network and D is a threshold detector. Pulses of the same shape are applied to the system without noise. The detector samples the network output and indicates in which interval of a finite number of possible voltage (or current) intervals the sampled output lies.

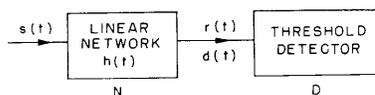


Fig. IX-16. Model for a pulse transmission system.

We wish to explore the implications of different criteria on the pulse shape. Since we assume that no random noise perturbs the pulses, the input and output levels can be quantized so that, if we send only one pulse through the system (now noisy) and we know when to sample the output, the probability that the output sample lies in an incorrect interval is negligibly small. The practicability of pulse code modulation tells us that this is a reasonable assumption. We are focusing our attention on interpulse interference.

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### 1. Definitions

The symbol  $\longleftrightarrow$  will stand for "has the Fourier transform." The following definitions will be used:  $s(t)$ , the input pulse;  $r(t)$ , the output pulse;  $h(t)$ , the output pulse resulting from a unit impulse at the input;  $g(t)$ , the input current (voltage) caused by a unit voltage (current) impulse at the input; and  $d(t)$  is an ideal output pulse that will be defined differently in different parts of our discussion.

These functions have the following Fourier transforms:

$$\begin{aligned} s(t) &\longleftrightarrow S(f) e^{j\gamma(f)} & h(t) &\longleftrightarrow H(f) e^{j\beta(f)} & g(t) &\longleftrightarrow G(f) e^{j\alpha(f)} \\ r(t) &\longleftrightarrow R(f) e^{j\delta(f)} & & & d(t) &\longleftrightarrow D(f) e^{j\eta(f)} \end{aligned}$$

The modulus functions are real, nonnegative, and even. The phase functions are real and odd, and they take on values in the interval  $-\pi < \theta \leq \pi$ . Note that  $R(f) = H(f) S(f)$  and  $\delta(f) = \beta(f) + \gamma(f)$ .

### 2. Matched-Pulse Criterion

We make the following assumptions: (a) We know the exact instants at which the detector will sample the output. (b) The sampled output at a given measuring instant,  $t_m$ , will be the real number  $K$ . (c) We are interested in sending one pulse through the network (or, equivalently, we are willing to take whatever pulse shape is dictated by the other constraints and then adjust the pulse repetition rate so that the interpulse interference will be negligible). (d) The energy of the one input pulse will be minimum, subject to the preceding constraints. (e) The input impedance of the filter is resistive (i. e.,  $g(t)$  is an impulse).

This problem can be stated mathematically by saying that we want to minimize  $\int_{-\infty}^{\infty} [s(t)]^2 dt$  by varying  $s(t)$  subject to the constraint

$$r(t_m) = \int_{-\infty}^{\infty} s(\tau) h(t_m - \tau) d\tau = K$$

The solution to this problem is well known (1,2). It is given by the formula  $s(t) = \lambda h(t_m - t)$ , where  $\lambda$  is a Lagrangian multiplier that is determined from the constraint equation

$$K = \int_{-\infty}^{\infty} \lambda h(t_m - \tau) h(t_m - \tau) d\tau$$

or

$$\lambda = \frac{K}{\int_{-\infty}^{\infty} [h(t)]^2 dt}$$

It is not difficult to solve the problem under the added constraints that  $s(t)$  and  $h(t)$  be zero for  $t$  less than zero. The solution is then

$$s(t) = \begin{cases} \lambda^* h(t_m - t) & t \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\lambda^* = \frac{K}{\int_0^{t_m} [h(t_m - \tau)]^2 d\tau} = \frac{K}{\int_0^{t_m} [h(t)]^2 dt}$$

Thus, the pulse energy required to satisfy these additional constraints is, in general, greater than before.

Other extensions of this "matched-filter" (or, more accurately, matched-pulse) argument have been made to situations in which (a) noise of an arbitrary power density spectrum enters the network with the pulse (3); and (b) the input impedance of the network is not assumed to be pure resistance (4). However, even these extended criteria for matching implicitly assume that only one pulse is to be sent through the network and that the sampling instant is known exactly. In the following discussion we shall attempt to shape the input pulses so that they can be sent through the network as rapidly as possible.

### 3. Thin-Pulse Criteria

The problems in this section are related, because here we wish to transmit pulses at a rapid rate by making each pulse as thin as possible. Such pulses might be most useful in a nonsynchronous system whose detector can recognize each pulse as it arrives, if there is enough space between pulses. However, for consistency, we shall continue to use our original model as a background.

#### a. Output-energy criterion

We shall use assumptions (a), (b), and (c) of section 2. The input pulse energy will be constrained to a fixed value  $K_1$ . Our measure of pulsewidth will be

$$\frac{\int_{-\infty}^{\infty} [r(t)]^2 dt}{r(t_m)}$$

Thus, we wish to minimize  $\int_{-\infty}^{\infty} [r(t)]^2 dt$ , subject to the constraints

$$\int_{-\infty}^{\infty} s(t) \left[ \int_{-\infty}^{\infty} s(\tau) g(t-\tau) d\tau \right] dt = K_1$$

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and  $r(t_m) = K_2$ .

Let us define  $r(t) = r_1(t - t_m)$  and  $r_1(t) \longleftrightarrow R(f) \exp [j \delta_1(f)]$ . Since, by definition, we have  $r(t) \longleftrightarrow R(f) \exp [j \delta(f)]$ , then  $\delta(f)$  and  $\delta_1(f)$  have the relation  $\delta(f) = \delta_1(f) - j2\pi f t_m$ .

Our restated problem is, then, that we must minimize  $\int_{-\infty}^{\infty} [R(f)]^2 df$  subject to the constraints  $\int_{-\infty}^{\infty} [S(f)]^2 G(f) e^{j\alpha(f)} df = K_1$  and  $\int_{-\infty}^{\infty} R(f) e^{j\delta_1(f)} df = K_2$ .

This is equivalent to minimizing the integral

$$\int_{-\infty}^{\infty} \left\{ [R(f)]^2 - \lambda R(f) e^{j\delta_1(f)} + \mu [S(f)]^2 G(f) e^{j\alpha(f)} \right\} df$$

Since the interval of integration is  $(-\infty, \infty)$ , we may discard the odd part of the integrand to obtain

$$\int_{-\infty}^{\infty} \left\{ [S(f) H(f)]^2 - \lambda S(f) H(f) \cos \delta_1(f) + \mu [S(f)]^2 G(f) \cos \alpha(f) \right\} df$$

We want the second term of this integral to be negative and large. With  $S(f)$  and  $H(f)$  given this can be accomplished by letting  $\lambda$  be positive and  $\delta_1(f)$  be zero.

Now we must select  $S(f)$  in the best possible way. Our new problem is to minimize

$$\int_{-\infty}^{\infty} \left\{ [S(f) H(f)]^2 - \lambda S(f) H(f) + \mu [S(f)]^2 G(f) \cos \alpha(f) \right\} df$$

A necessary condition for the minimization is that

$$2S(f) [H(f)]^2 - \lambda H(f) + 2\mu S(f) G(f) \cos \alpha(f) = 0$$

or

$$S(f) = \frac{\frac{\lambda}{2} H(f)}{[H(f)]^2 + \mu G(f) \cos \alpha(f)} \quad (1)$$

We determine  $\lambda$  and  $\mu$  from the two constraint equations:

$$K_1 = \int_{-\infty}^{\infty} [S(f)]^2 G(f) \cos \alpha(f) df = \int_{-\infty}^{\infty} \frac{\frac{\lambda^2}{4} [H(f)]^2 G(f) \cos \alpha(f)}{[H(f)]^2 + \mu G(f) \cos \alpha(f)} df$$

and

$$K_2 = \int_{-\infty}^{\infty} R(f) e^{j\delta_1(f)} df = \int_{-\infty}^{\infty} \frac{\frac{\lambda}{2} [H(f)]^2}{[H(f)]^2 + \mu G(f) \cos \alpha(f)} df$$

Combining Eq. 1 with our choice of  $\delta_1(f)$  yields

$$S(f) e^{j\gamma(f)} = \frac{\frac{\lambda}{2} H(f) e^{j(-\beta(f)-2\pi f t_m)}}{[H(f)]^2 + \mu G(f) \cos \alpha(f)} \quad (2)$$

and

$$R(f) e^{j\delta(f)} = \frac{\frac{\lambda}{2} [H(f)]^2 e^{-j2\pi f t_m}}{[H(f)]^2 + \mu G(f) \cos \alpha(f)}$$

If we select the values  $K_1$  and  $K_2$  arbitrarily, we have no guarantee that we can solve the constraint equations for  $\mu$  and  $\lambda$ . But, given any pair of positive real numbers  $(\mu, \lambda)$ , we can compute  $K_1$  and  $K_2$ . We may then vary  $\mu$  and  $\lambda$  to bring us closer to the desired values of  $K_1$  and  $K_2$ . The possible range of values for the pair  $(\mu, \lambda)$  is determined by the fact that  $S(f)$  must remain real and nonnegative.

Since Eq. 2 will appear as a special case of Eq. 7, this problem will not be discussed further here.

#### b. Weighted-energy criterion

Using assumptions (a), (b), (c), and (e) of section 2, we shall now take for our measure of pulsewidth

$$\frac{\int_{-\infty}^{\infty} t^2 [r_1(t)]^2 dt}{r(t_m)}$$

where  $r(t)$  and  $r_1(t)$  are related as in the previous problem. This measure puts greater emphasis on portions of the pulse that are far away from the measuring instant  $t = t_m$ . It is analogous to the variance of a probability density function.

Then  $\int_{-\infty}^{\infty} t^2 [r_1(t)]^2 dt$  must be minimized, subject to the constraints  $K_1 = \int_{-\infty}^{\infty} [s(t)]^2 dt$  and  $K_2 = r(t_m)$ .

Expressed in the frequency domain, we want to minimize

$$\int_{-\infty}^{\infty} \left| \frac{d}{df} \left( R(f) e^{j\delta_1(f)} \right) \right|^2 df$$

subject to the constraints

$$K_2 = \int_{-\infty}^{\infty} R(f) e^{j\delta_1(f)} df \quad \text{and} \quad K_1 = \int_{-\infty}^{\infty} [S(f)]^2 df$$

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If we use primes to denote differentiation with respect to  $f$ , this is equivalent to minimizing

$$\int_{-\infty}^{\infty} \left\{ [R'(f)]^2 + [\delta_1'(f) R(f)]^2 - \lambda R(f) e^{j\delta_1(f)} + \mu [S(f)]^2 \right\} df$$

subject to no constraint, and determining the values of the Lagrangian multipliers from the constraint equations.

Again, we select  $\lambda$  as positive, and  $\delta_1(f) = 0$ . This drops out the second, positively contributing term of the integrand and makes the third term real and negative.

With  $R(f) = S(f) H(f)$ , we proceed to minimize the integral

$$\int_{-\infty}^{\infty} \left\{ [S(f) H'(f) + S'(f) H(f)]^2 - \lambda S(f) H(f) - \mu [S(f)]^2 \right\} df$$

A necessary condition (5) for the minimization of this integral as we vary  $S(f)$  is that  $S(f)$  must satisfy the differential equation

$$H(f) \frac{d^2}{df^2} [S(f) H(f)] = \frac{\lambda}{2} H(f) + \mu S(f) \quad (3)$$

Note that  $S(f)$  must be zero wherever  $H(f)$  is zero and that  $\gamma(f) = -\beta(f) - 2\pi t_m f$  because  $\delta_1(f) = 0$ .

Let us remove our constraint on the input-pulse energy by letting  $\mu = 0$ . (Discarding this constraint allows  $S(f)$  to be nonzero when  $H(f)$  is zero.) Let us also assume that  $H(f) = 0$  for  $|f| > W$ . Thus

$$\frac{d^2}{df^2} [R(f)] = \frac{\lambda}{2} \quad |f| > W$$

or

$$R(f) = \frac{\lambda}{2} f^2 + C \quad |f| > W$$

where  $C$  is a constant. There is no first-degree term in  $f$ , because  $R(f)$  must be an even function.  $S(f)$  and  $R(f)$  are arbitrary outside the interval  $[-W, W]$ . We shall let  $R(f)$  and  $S(f)$  be zero outside  $[-W, W]$ . The output pulse is given by

$$R(f) e^{j\delta(f)} = \begin{cases} \left( \frac{\lambda}{2} f^2 + C \right) e^{-j2\pi t_m f} & -W \leq f \leq W \\ 0 & \text{elsewhere} \end{cases}$$

and

$$r(t) = [\lambda(W^2 - 2W + 2) + 2C] W \frac{\sin 2\pi W(t - t_m)}{2\pi W(t - t_m)}$$

Here,  $\lambda$  and  $C$  must satisfy the constraints that  $r(t_m) = K_2$  and that  $R(f)$  is nonnegative.

Gabor's (6) "signal shape which can be transmitted in the shortest effective time" is a particular solution of Eq. 3 when there is no constraint on  $r(t_m)$  (i. e.,  $\lambda = 0$ ), and  $H(f)$  is given by the formula

$$H(f) = \begin{cases} 1 & |f| \leq W \\ 0 & |f| > W \end{cases}$$

In this case

$$S(f) e^{j\gamma(f)} = \begin{cases} C \cos\left(\frac{\pi f}{2W}\right) e^{-j\beta(f)} & |f| \leq W \\ 0 & \text{elsewhere} \end{cases}$$

where  $C$  is determined by  $\int_{-\infty}^{\infty} [S(f)]^2 df = K_1$ .

The general solution of Eq. 3 for  $S(f)$  in terms of  $H(f)$  appears to be difficult. We note that if the input impedance of the network is no longer assumed to be resistive, Eq. 3 can be replaced by

$$H(f) \frac{d^2}{df^2} [S(f) H(f)] = \frac{\lambda}{2} H(f) + \mu S(f) G(f) \cos \alpha(f) \quad (4)$$

#### 4. Least-Square Criterion

Here we shall assume that there exists some ideal individual output pulse shape that, for prescribed rates of transmission, allows all possible message sequences to be recognized by the detector. The input impedance of the network is not assumed to be resistive.

Our problem is to minimize  $\int_{-\infty}^{\infty} [d(t) - r(t)]^2 dt$  by varying  $s(t)$ , subject to the constraint that the input energy per pulse,  $\int_{-\infty}^{\infty} s(t) \left[ \int_{-\infty}^{\infty} s(\tau) g(t-\tau) d\tau \right] dt$ , is some given value  $K$ . The solution of this simplified pulse-transmission problem will be applied to a particular system (8).

We shall now rephrase our problem in the frequency domain. We want to minimize the following integral:

$$\int_{-\infty}^{\infty} \left[ D^2(f) - 2D(f) R(f) e^{j[\eta(f) - \delta(f)]} + R^2(f) + \lambda S(f) e^{-j\gamma(f)} S(f) e^{j\gamma(f)} G(f) e^{j\alpha(f)} \right] df$$

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where  $\lambda$  is a Lagrangian multiplier. Since the integral is symmetric and all time functions are assumed to be real, we rewrite the integral as

$$\int_{-\infty}^{\infty} [D^2(f) - 2D(f) R(f) \cos [\eta(f) - \delta(f)] + R^2(f) + \lambda S^2(f) G(f) \cos \alpha(f)] df \quad (5)$$

Now we want to minimize this expression by choosing  $S(f) e^{j\gamma(f)}$ . We select  $\gamma(f)$  so that the integrand will be smallest when  $\eta(f) - \delta(f) = 0$ ; therefore we choose

$$\gamma(f) = \eta(f) - \beta(f) \quad (6)$$

Substituting Eq. 6 in expression 5, we obtain

$$\int_{-\infty}^{\infty} [D^2(f) - 2D(f) S(f) H(f) + S^2(f) H^2(f) + \lambda S^2(f) G(f) \cos \alpha(f)] df$$

A necessary condition for this integral to be a minimum is that

$$2S(f) H^2(f) - 2D(f) H(f) + 2\lambda S(f) G(f) \cos \alpha(f) = 0$$

or

$$S(f) = \frac{D(f) H(f)}{H^2(f) + \lambda G(f) \cos \alpha(f)} \quad (7)$$

Combining Eqs. 5 and 6 yields

$$S(f) e^{j\gamma(f)} = \frac{D(f) e^{j\eta(f)} H(f) e^{-j\beta(f)}}{H^2(f) + \lambda G(f) \cos \alpha(f)} \quad (8)$$

where  $\lambda$  must be chosen to satisfy our energy constraint. That is,

$$K = \int_{-\infty}^{\infty} \frac{D^2(f) H^2(f) G(f) \cos \alpha(f)}{[H^2(f) + \lambda G(f) \cos \alpha(f)]^2} df$$

The mean-squared error,  $E$ , is given by

$$E = \int_{-\infty}^{\infty} \left[ D(f) - \frac{D(f) H^2(f)}{H^2(f) + \lambda G(f) \cos \alpha(f)} \right]^2 df$$

We shall now consider two examples that differ only in that we shall be signaling through different networks.

In both examples, the desired pulse shape is rectangular, of 1 voltage unit in amplitude, and 1 time unit in duration. Thus  $D(f) e^{j\eta(f)} = \sin \pi f / \pi f$ . The input impedance of

each network is assumed to be 1 ohm.

$$\left. \begin{array}{l} \text{For case A: } H(f) = |f|^{1/2} \exp [-(10|f|)^{1/2}] \\ \text{For case B: } H(f) = \exp [-(10|f|)^{1/2}] \end{array} \right\} -W \leq f \leq W$$

and  $H(f)$  is zero elsewhere.

Since we shall only calculate  $K$ ,  $E$ , and  $r(t)$ , we do not need to specify the phase functions of the networks. The system has been constrained to be bandlimited. Thus the mean-square error is considered as being made up of two components – one from bandlimiting, and one that results from inaccuracy of approximation within the band. These will be called  $E_B$  and  $E_I$ .

For both cases

$$E_B(W) = 2 \int_W^\infty \frac{\sin^2 \pi f}{(\pi f)^2} df = \frac{2}{\pi} \int_{\pi W}^\infty \frac{\sin^2 x}{x^2} dx$$

This expression can be rewritten as

$$E_B(W) = \frac{2}{\pi} \left[ \frac{\sin^2 \pi W}{\pi W} + \int_{2\pi W}^\infty \frac{\sin u}{u} du \right]$$

A graph of  $E_B(W)$  is shown in Fig. IX-17. For  $W > 1$ ,  $E_B(W) \approx \frac{0.099}{W}$ . This calculation reveals the somewhat surprising fact that the power in the error signal,  $r(t) - d(t)$ , is less than 1/10 of the power in the desired signal,  $d(t)$ , when  $W$  is greater than 1.

The question then arises, How well does  $r(t)$  represent  $d(t)$  when  $E_B \approx 1/10$  (for example, when  $W = 1$ )? We now assume that  $E_I = 0$  or  $E = E_B$ . This is equivalent to throwing away our input-power constraint by letting  $\lambda$  be zero. Thus the desired frequency function is attained exactly within the band, and no attempt is made to match

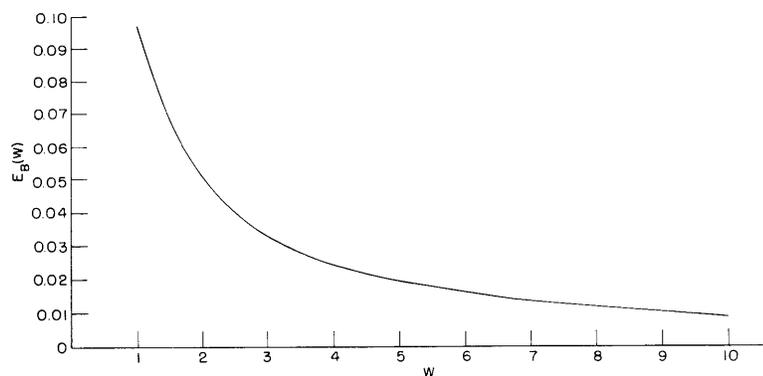


Fig. IX-17. Graph of  $E_B(W)$  for  $W > 1$ . ( $E_B(W) \approx \frac{0.099}{W}$ ;  $E_B = 1$  at  $W = 0$ ;  $E_B \rightarrow 0$  as  $W \rightarrow \infty$ .)

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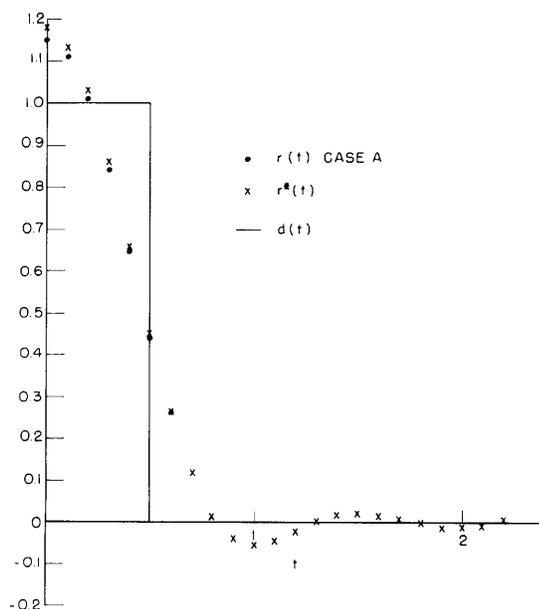


Fig. IX-18. Graphs of  $r(t)$  for case A,  $d(t)$ , and  $r^*(t)$ .

outside the band. Let  $r(t)$  under these conditions be  $r^*(t)$ . We obtain  $r^*(t)$  as follows.

$$r^*(t) = \int_{-W}^W \frac{\sin(\pi f)}{\pi f} e^{j2\pi ft} df$$

$$\frac{d}{dt} r^*(t) = 2j \int_{-W}^W \sin(\pi f) e^{j2\pi ft} df = -2 \int_{-W}^W \sin(\pi f) \sin(2\pi ft) df$$

$$= \frac{-2 \sin 2\pi W \left(t - \frac{1}{2}\right)}{2\pi \left(t - \frac{1}{2}\right)} + \frac{2 \sin 2\pi W \left(t + \frac{1}{2}\right)}{2\pi \left(t + \frac{1}{2}\right)}$$

$$r^*(t) = \int_{-\infty}^t \frac{d}{dt} r^*(t) dt = \frac{1}{\pi} \int_{2\pi W \left(t - \frac{1}{2}\right)}^{2\pi W \left(t + \frac{1}{2}\right)} \frac{\sin x}{x} dx$$

In Fig. IX-18  $r^*(t)$  is depicted with  $d(t)$  for  $W = 1$ . One property of  $r^*(t)$  is that for  $t > 1$  the envelope of  $r^*(t)$  drops off faster than  $0.056/t^2$ . Thus, if we receive pulses shaped like  $r^*(t)$  once each second, if their separate peak values are either +1 or -1, and if the detector measures the output where each peak should lie, the maximum possible interpulse interference is less than

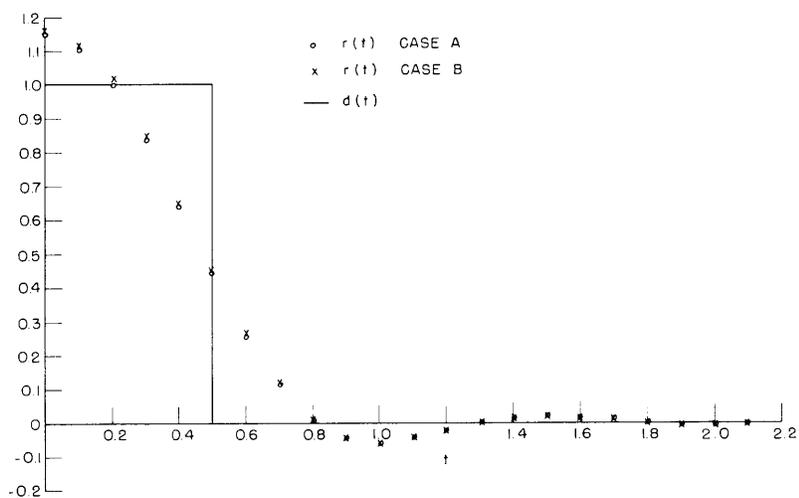


Fig. IX-19. Graphs of  $d(t)$ ,  $r(t)$  for case A, and  $r(t)$  for case B.

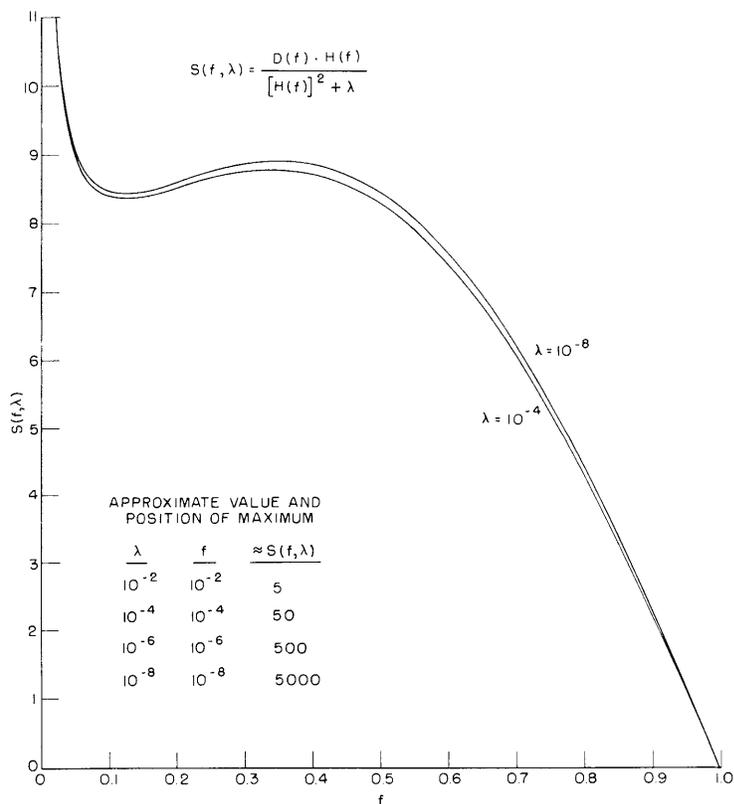


Fig. IX-20. Graphs of  $S(f)$  for case A with two different values of  $\lambda$ .

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$$\sum_{n=1}^{\infty} \frac{0.06}{n^2} = (0.06) \frac{\pi^2}{6} \approx 0.1$$

If the detector-measuring instants drift slightly from the prescribed positions, then the upper bound to the interpulse interference can only decrease. However, the signal voltage will also decrease.

We now reintroduce our input-power constraint and consider signaling through the networks of cases A and B. Again, we let  $W = 1$ . With the input pulses given by Eq. 8 and with  $\lambda = 10^{-4}$ , the corresponding output pulses for each case are plotted in Fig. IX-19. In the scale of this graph,  $r(t)$  for case B is not distinguishable from  $r^*(t)$  in Fig. IX-18.

Table IX-2.

$\lambda$	$K(\lambda)$	$E_I(\lambda)$
$10^{-8}$	260	not computed
$10^{-6}$	130	$3.1 \times 10^{-6}$
$10^{-4}$	110	$1.0 \times 10^{-3}$
$10^{-2}$	21	$2.6 \times 10^{-1}$

The output pulses for these two cases were calculated on the IBM 704 computer in the M.I.T. Computation Center.

Figure IX-20 shows  $S(f)$  of case A for two different values of the parameter  $\lambda$ . Table IX-2 indicates how the input energy,  $K$ , and the mean-square error within the band,  $E_I$ , vary with  $\lambda$  for case A.

The writer wishes to acknowledge the work of Mrs. Mabel Greber, of the Computation Center, M.I.T., in computing the data for the curves of Figs. IX-19 and IX-20 and the values of  $K$  and  $E_I$  in Table IX-2.

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8. The reader may wish to compare the results of this section with results of an investigation of a more sophisticated model. Cf. D. W. Tufts, *Some optimization problems of digital pulse transmission*, S.M. Thesis, Department of Electrical Engineering, M.I.T., June 1958.