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# THE DERIVATION OF EFFICIENT SETS

by

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## Abstract

This paper presents an algorithm that solves the parametric quadratic programming problem:

$$\text{maximize} \quad -\frac{1}{2} X'AX,$$

$$\text{subject to} \quad \begin{aligned} BX &\leq D, \\ -C'X &\leq -R, \\ X &\geq 0. \end{aligned}$$

for all R between $R_{max}$ and $R_{min}$.

The algorithm thus generates the set of efficient portfolios in the portfolio allocation problem. The algorithm essentially involves the solution of one quadratic programming problem and then one addition pivot step in the linear programming sense for each corner portfolio. A numerical example is given.

The Derivation of Efficient Sets.

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## 1. Introduction

In a recent article G. F. Alexander [1] suggested an algorithm for the derivation of efficient sets. Alexander's method essentially involves repeated applications of C. E. Lemke's complementary pivot algorithm [2] for quadratic programming. Alexander gives several numerical examples to illustrate the superiority of his algorithm to H. M. Markowitz critical line method [3].

The purpose of this paper is to present a parametric version of Lemke's algorithm that determines the efficient set completely. This algorithm essentially involves the solution of one quadratic programming problem and then one additional pivot step in the linear programming sense for each corner portfolio[1]. It is obviously superior to the technique suggested by Alexander.

A numerical example, illustrating the working of the algorithm, is given in section 3.

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1 The efficient set is completely described by $\ell$ so called corner portfolios $x^1, \ldots x^\ell$. Any efficient portfolio is a convex combination of two corner portfolios $x^i$ and $x^{i+1}$.

## 2. The Portfolio Allocation Problem

An investor may compose a portfolio from n risky and one riskless asset, say cash. The riskless asset is assumed to have a return of 0. Let us introduce the following notation:

$x_i$ = the share of wealth invested in security i (i = 1, . . n),

$c_i$ = expected rate of return of security i (i = 1, . . n),

$a_{ij}$ = covariance of rates of return of asset i and j (i = 1, . . n, j = 1, . . n),

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

The expected rate of return and the variance of the rate of return of the portfolio are then given by:

$$\text{Variance of the rate of return} = X' A X$$

$$\text{Expected rate of return} = C' X .$$

Various constraints representing legal and financial considerations with respect to the composition of the portfolio are given by the m constraints

$$BX \leq D.$$

The above set of constraints include the constraint that the sum of the x's has to be less or equal to 1.

Suppose $R_{max}$ is the maximum feasible rate of return on the portfolio and that the feasible portfolio with the lowest variance has a return of $R_{min}$. Derivation of the efficient set is then equivalent to solving the quadratic programming problem

(1.1) maximize $-\frac{1}{2} X'AX$

subject to

(1.2) $BX \leq D$

(1.3) $-C'X \leq -R$

(1.4) $X \geq 0$

for **all R between** $R_{max}$ and $R_{min}$.

The method suggested by Alexander selects 10-15 values of R between $R_{max}$ and $R_{min}$ and then solves (1) by repeated application of Lemke's complementary pivot algorithm for quadratic programming. Alexander's procedure is consequently a straightforward application of a standard quadratic programming algorithm. Moreover this technique only approximates the efficient set since the 10-15 portfolios generated by the algorithm are not corner portfolios.

Let the vector $Y_1$ and the scalar $Y_2$ denote the Lagrange multipliers for (1.2) and (1.3) respectively. Moreover let U be a column vector of the same dimension as X, let $V_1$ be a column vector of the same dimension as $Y_1$ and let $V_2$ be a scalar. The solution of (1) is then equivalent to the following linear complementarity problem:

$$(2.1) \quad \begin{pmatrix} U \\ V_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} A & B' & -C \\ -B & 0 & 0 \\ C' & 0 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ D \\ 0 \end{pmatrix} - R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(2.2) \quad U'X + V_1'\,Y_1 + V_2'Y_2 = 0$$

$$(2.3) \quad U,\ X \geq 0, \quad V_1,\ Y_1 \geq 0, \quad V_2,\ Y_2 \geq 0.$$

Lemke [2] has designed an ingenious combinatorial algorithm that solves (2) and as a consequence solves (1).

We shall denote the pairs of variables $(u_j, x_j)$ , $(v_{1i}, y_{1i})$ , $(V_2, Y_2)$ complementary pairs. The solution of (2) requires that at most one member of each pair is strictly positive.

In order to construct an algorithm that generates the efficient set we shall make some stipulations on the solutions to (2). We shall later suggest how our algorithm should be modified if these assumptions are not satisfied.

<u>Assumption 1 (Uniqueness)</u>

The solution to (2) is unique for any $R_{min} < R < R_{max}$.

Assumption 2 (Nondegeneracy)

If X is a corner portfolio, different from the one with the lowest rate of return, then only one complementary pair has both members equal to 0. All non-corner portfolios have the property that all complementary pairs have one member strictly positive.

Suppose we solve (2) for $R_{max} - \varepsilon$, where $\varepsilon$ is a small positive number. By assumption each complementary pair has one member strictly positive. Let us define 2 new vectors of variables $\omega$ and $\xi$ such that a component of $\omega$ will represent the member of the complementary pair which is positive and the corresponding component of $\xi$ will represent the other member, i.e.

$\omega_j = x_j$ and $\xi_j = u_j$ if $x_j > 0$ and

$\omega_j = u_j$ and $\xi_j = x_j$ otherwise,

$\omega_{n+i} = y_{1i}$ and $\xi_{n+i} = v_{1i}$ if $y_{1i} > 0$ and

$\omega_{n+i} = v_{1i}$ and $\xi_{n+i} = y_{1i}$ otherwise,

$\omega_{n+m+1} = Y_2$ and $\xi_{n+m+1} = V_2$ if

$Y_2 > 0$ and

$\omega_{n+m+1} = V_2$ $\xi_{n+m+1} = Y_2$ otherwise.

The system of equations given by (2.1) may then be rewritten

$$\omega = M\xi + Q_1 - R\,Q_2$$

We must have

$$H = Q_1 - R_{max} Q_2 \geq 0$$

with one component of H identically 0, otherwise $R_{max}$ would not be the maximum feasible rate of return. Moreover the corresponding component of $Q_2$ has to be strictly positive since by assumption

$$Q_1 - (R_{max} - \varepsilon) Q_2 > 0.$$

H yields the first corner portfolio. The second corner portfolio is derived by reducing R until one component of

$$H = Q_1 - R \cdot Q_2 \geq 0$$

becomes 0. Say R can be reduced to $\underline{R}$. By assumption only one component of H becomes 0. Without loss of generality suppose the first component of H becomes 0. We shall argue that $m_{11} > 0$.

Let us begin by observing that we must have $q_{21} < 0$. Suppose now that $m_{11} \leq 0$. If so choose

$$\xi_1 = \Delta > 0$$

$$R = \underline{R} + \frac{m_{11}}{q_{21}} \Delta \quad (\geq \underline{R})$$

Then for $\Delta$ sufficiently small we must obviously have that

$$R < R_{max}$$

and that

$$\Delta M_1 + Q_1 - (\underline{R} + \frac{m_{11}}{q_{21}} \Delta) Q_2 \geq 0,$$

where $M_1$ is the first column of M, since by construction

$$\Delta m_{11} + q_{11} - \underline{R} q_{21} - \frac{m_{11}}{q_{21}} \Delta q_{21} = q_{11} - \underline{R} q_{21} = 0,$$

and all, but the first component of

$$H = Q_1 - \underline{R} Q_2$$

are strictly positive by assumption. But then there is more than one solution to (2) for $R = \underline{R} + \frac{m_{11}}{q_{21}} \Delta$, a contradiction. We have thus proved that $m_{11} > 0$.

Define $\omega_1^* = \xi_1$, $\omega_i^* = \omega_i$ otherwise and $\xi_1^* = \omega_1$ and $\xi_i^* = \xi_i$ otherwise The system of equations may then be rewritten after a pivot operation has been performed.

$$\omega^* = M^* \xi^* + Q_1^* - R Q_2^* ,$$

where we have

$$Q_1^* - \underline{R} Q_2^* = Q_1 - \underline{R} Q_2 ,$$

and specifically

$$q_{11}^* - \underline{R} q_{21}^* = 0 ,$$

and

$$q_{21}^* > 0 .$$

Next R is reduced from $\underline{R}$ until one component of

$$H = Q_1^* - R\, Q_2^* \geq 0$$

becomes 0, and the third corner portfolio is determined. The process goes on until $Y_2$ becomes 0, at which point $R_{min}$ has been reached. Once the quadratic programming problem has been solved for $R_{max} - \varepsilon$ only a pivot step in the linear programming sense thus has to be performed for each new corner portfolio.

It may be instructive to illustrate the redefinition and pivot step by a simple example, where we have $\omega_1 = u_1$ and $\xi_1 = x_1$.

$$\omega_1 = \xi_1 \quad \ldots\ldots\ldots\ldots \quad -8 + R\cdot 4 ,$$

$$\omega_2 = -2\xi_1 \quad \ldots\ldots\ldots\ldots \quad -4 + R\cdot 12 ,$$

$$\cdot \quad \ldots\ldots\ldots\ldots\ldots\ldots ,$$

$$\omega_{n+m+1} = 10\xi_1 \quad \ldots\ldots\ldots\ldots \quad -2 + R\cdot 10 .$$

R is reduced till it becomes 2. We redefine $\omega_1^* = x_1$ $\omega_i^* = \omega_i$ otherwise $\xi_1^* = u_1$ and $\xi_i^* = \xi_i$ otherwise. We thus get

$$-\omega_1^* = -\xi_1^* \quad \ldots\ldots\ldots \quad -8 + R\cdot 4 ,$$

$$2\omega_1^* + \omega_2 = 0\cdot\xi_1 \quad \ldots\ldots\ldots \quad -4 + R\cdot 12 ,$$

$$\cdot \quad \cdot \quad \ldots\ldots\ldots\ldots\ldots\ldots ,$$

$$-10\omega_1^* + \omega_{n+m+1}^* = 0\cdot\xi_1^* \quad \ldots\ldots\ldots \quad -2 + R\cdot 10 .$$

We next pivot on $-\omega_1^*$ and get

$$\omega_1^* = \xi_1^* \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; +8 - 4\cdot R \;,$$

$$\omega_2^* = -2\xi_1^* \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; -20 + 20\, R \;,$$

$$\cdot \qquad .\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;,$$

$$\omega_{n+m+1}^* = 10\xi_1^* \;.\;.\;.\;.\;.\;.\;.\;.\;.\; 78 - 30\, R \;.$$

R is then further reduced and the next corner portfolio determined.

Suppose Assumption 1 and 2 do not hold. The following problems may then arise:

i) $m_{11} = 0$, i.e. the pivot operation may not be performed.

ii) $m_{11} < 0$, i.e. the pivot operation may be performed, but R may not be reduced.

iii) $m_{11} > 0$, but R may not be reduced because

$$\omega_j = 0 \quad (j > 1) \quad \text{and} \quad q_{2j} < 0.$$

If either of these situations should arise we suggest that R is reduced by $\varepsilon$, (2) next is solved using Lemke's algorithm and our procedure then is applied to the resulting solution. Practical applications of our algorithm suggest that these problems would occur relatively seldom.

Finally let us conclude that $R_{max}$ is determined by solving the linear programming problem

$$\begin{aligned} &\text{maximize} \quad C'X, \\ &\text{subject to} \\ &\qquad BX \leq D, \\ &\qquad X \geq 0. \end{aligned}$$

## 3. A Numerical Example

We shall illustrate the working of the algorithm by a numerical example and shall consider a portfolio allocation problem when m = 1, i.e. the only constraint imposed upon the composition of the portfolio is that the sum of the x's is less or equal to 1. In this special case it is not necessary to use Lemke's algorithm to determine which member of each complementary pair that is positive for $R = R_{max} - \varepsilon$.

Without loss of generality let us assume that $c_1 > 0$ and that $c_1 > c_j$, j = 2, . . n. We must obviously have that $R_{max} = c_1$ and that the first corner portfolio is (1,0 . . . .0). Now suppose that the investment in asset 1 is reduced by dx, whereas the investment in asset j is increased by dx. The resulting change in the variance and expected rate of return of the portfolio are given by:

Reduction in variance $= 2a_{11}dx - 2a_{1j}dx$

Reduction in expected rate of return $= c_1 dx - c_j dx$

Let $a_{1,n+1} = c_{n+1} - 0$ and suppose that

$$\frac{a_{11} - a_{1k}}{c_1 - c_k} \geq \frac{a_{11} - a_{1j}}{c_1 - c_j} \qquad j = 2, \ldots n+1.$$

We must then have that $x_1 = 1 - dx$, $x_k = dx$, $x_j = 0$ otherwise is an efficient portfolio.

If $k = n+1$ the next corner portfolio contains only cash and the efficient set consequently contains only 2 corner portfolios.

If $k \leq n$ we define the initial $\omega$ and $\xi$ vectors as follows

$$\omega_i = u_i, \quad \xi_i = x_i \quad \text{for } i = 1, \ldots n, \quad i \neq 1, k,$$

$$\omega_1 = x_1, \quad \xi_1 = u_1,$$

$$\omega_k = x_k, \quad \xi_k = u_k,$$

$$\omega_{n+1} = y_1, \quad \xi_{n+1} = v_1,$$

$$\omega_{n+2} = y_2, \quad \xi_{n+2} = v_2.$$

The algorithm then proceeds as outlined above.

For our numerical example suppose that

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad .$$

i.e. we have

$$(3)\quad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 & 1 & -3 \\ 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - R \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} .$$

The first corner portfolio is (1,0,0,0), where the last component of the portfolio vector represents cash. We have $k = 2$ since

$$\frac{3-1}{3-2} > \begin{cases} \dfrac{3-0}{3-1} \\[2ex] \dfrac{3-0}{3-0} \end{cases} .$$

$\omega$ and $\xi$ may now be defined.

$$\omega = \begin{pmatrix} x_1 \\ x_2 \\ u_3 \\ y_1 \\ y_2 \end{pmatrix} , \qquad \xi = \begin{pmatrix} u_1 \\ u_2 \\ x_3 \\ v_1 \\ v_2 \end{pmatrix} .$$

(3) is rewritten

$$\begin{pmatrix} x_1 \\ x_2 \\ u_3 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -3 & -1 \\ -1 & 2 & 8 & 11 & 4 \\ -2 & 3 & 11 & 18 & 7 \\ -1 & 1 & 4 & 7 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ x_3 \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \\ -11 \\ -18 \\ -7 \end{pmatrix} - R \begin{pmatrix} -1 \\ 1 \\ -4 \\ -7 \\ 3 \end{pmatrix} .$$

We have

$$H = \begin{pmatrix} -2 \\ 3 \\ -11 \\ -18 \\ -7 \end{pmatrix} - R \begin{pmatrix} -1 \\ 1 \\ -4 \\ -7 \\ -3 \end{pmatrix} .$$

R may be reduced to $\frac{11}{4}$ at which point the 3rd component of H becomes 0.
The 2nd corner portfolio is thus $(\frac{3}{4}, \frac{1}{4}, 0, 0)$.

We redefine $\omega$ and $\xi$ and rewrite the system of equations.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & -11 & 1 & 0 \\ 0 & 0 & -4 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -3 & -1 \\ -1 & 2 & -1 & 11 & 4 \\ -2 & 3 & 0 & 18 & 7 \\ -1 & 1 & 0 & 7 & 3 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \\ -11 \\ -18 \\ -7 \end{pmatrix} - R \begin{pmatrix} -1 \\ 1 \\ -4 \\ -7 \\ 3 \end{pmatrix} .$$

We pivot on -8 and get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & \frac{5}{8} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & -\frac{11}{8} & -\frac{1}{2} \\ -\frac{5}{8} & \frac{1}{4} & \frac{11}{8} & \frac{23}{8} & \frac{3}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{3}{2} & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} -\frac{5}{8} \\ \frac{1}{4} \\ \frac{11}{8} \\ -\frac{23}{8} \\ -\frac{3}{2} \end{pmatrix} - R \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ -\frac{3}{2} \\ -1 \end{pmatrix} .$$

We have

$$H = \begin{pmatrix} -\frac{5}{8} \\ \frac{1}{4} \\ \frac{11}{8} \\ -\frac{23}{8} \\ -\frac{3}{2} \end{pmatrix} - R \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ -\frac{3}{2} \\ -1 \end{pmatrix} .$$

R may be reduced to $\frac{23}{12}$ at which point the 4th component of H becomes 0. The 3rd corner portfolio is thus $(\frac{1}{3}, \frac{1}{4}, \frac{5}{12}, 0)$.

We redefine $\omega$ and $\xi$ and rewrite the system of equations.

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{8} & 0 \\ 0 & 1 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{11}{8} & 0 \\ 0 & 0 & 0 & -\frac{23}{8} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ v_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & 0 & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 & 0 \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & 0 & -\frac{1}{2} \\ -\frac{5}{8} & \frac{1}{4} & \frac{11}{8} & -1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 \end{bmatrix} + \begin{bmatrix} -\frac{5}{8} \\ \frac{1}{4} \\ \frac{11}{8} \\ -\frac{23}{8} \\ -\frac{3}{2} \end{bmatrix} - R \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ -\frac{3}{2} \\ -1 \end{bmatrix} .$$

We pivot on $-\frac{23}{8}$ and get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ v_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} \frac{6}{23} & -\frac{7}{23} & -\frac{4}{23} & \frac{5}{23} & \frac{4}{23} \\ -\frac{7}{23} & \frac{12}{23} & -\frac{3}{23} & -\frac{2}{23} & \frac{3}{23} \\ -\frac{4}{23} & -\frac{3}{23} & \frac{18}{23} & -\frac{11}{23} & \frac{5}{23} \\ \frac{5}{23} & -\frac{2}{23} & -\frac{11}{23} & \frac{8}{23} & -\frac{12}{23} \\ -\frac{4}{23} & -\frac{3}{23} & -\frac{5}{23} & \frac{12}{23} & \frac{5}{23} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - R \begin{bmatrix} -\frac{4}{23} \\ -\frac{3}{23} \\ -\frac{5}{23} \\ \frac{12}{23} \\ -\frac{5}{23} \end{bmatrix} .$$

We have

$$H = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - R \begin{pmatrix} -\frac{4}{23} \\ -\frac{3}{23} \\ -\frac{5}{23} \\ \frac{12}{23} \\ -\frac{5}{23} \end{pmatrix} .$$

R may be reduced to 0, at which point $y_2$ becomes 0 and the algorithm terminates. The 4th corner portfolio thus is (0,0,0,1).

## REFERENCES


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