
by

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Abstract

The equations used to model reversible processes (in the thermodynamic sense) in a moist atmosphere differ from the equations used in most meteorological models; from these equations, one may derive the theoretical basis of an algorithm to determine the reference state for computation of the available energy in a moist atmosphere. This algorithm is significantly better than both of the other two methods of determining this reference state which are currently available.

Thesis Supervisor: Edward N. Lorenz, Sc. D.
Professor of Meteorology
To my parents, without whom I wouldn't be;

To Miss French, Mr. Kent, and Professor Sanders, without whom I would not have become interested in Meteorology;

And especially to Professor Lorenz, without whose patience and guidance this paper would not have been written.
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Chapter 1:

Introduction
In one of his recent writings (Ref. I), Professor Lorenz defined a quantity which he called moist available energy; it represents the energy stored in a moist atmosphere which can be converted to kinetic energy through reversible processes (in the thermodynamic sense). This quantity is equivalent to available potential energy in a dry atmosphere; however, it incorporates the effects of the presence of water.

In this paper, Lorenz notes that this quantity may be computed, like available potential energy in a dry atmosphere, by deducting the stored energy of a reference state, which is defined as the permutation of the atmosphere which, when achieved through reversible processes, has the least stored energy, from the stored energy of the atmosphere:

\[
\text{M. A. E.} = \text{S. E. Atm.} - \text{S. E. R. S.} \quad (1-1)
\]

As Lorenz states, the computation of the stored energy of the atmosphere, and of the reference state, is rather trivial; however, the determination of the reference state is not. Lorenz presents, in this paper, a graphical procedure through which the reference state can be rather crudely determined; in a subsequent manuscript (Ref. II), he translated it into a numerical procedure for use on a computer. However, as Lorenz notes, the numerical version of this procedure will, under certain conditions, produce an incorrect result.
In this text, I am presenting a new algorithm for determination of the reference state. While its usefulness may be limited by the current state of the art in computer technology, primarily in terms of capacity, and some minor approximations are used, this algorithm is significantly more accurate than any other algorithms that have been developed to date.

The equations used in the computations required for this algorithm are formulated in the next chapter. Several equations differ somewhat from the corresponding standard equations due to constraints imposed by the assumption of reversibility; for example, in the heat balance equation, there is no diabatic heating term, but there is a term for the specific heat of liquid water (which is normally taken to be zero with the assumption that the liquid water falls out as precipitation, which is an irreversible process; in the context of this problem, we must assume that the liquid water is retained as clouds in the atmosphere). The remaining equations are included in the interest of completeness.
Chapter II:

Derivation of

Equations
The potential energy of the atmosphere is given by:

\[ \text{P. E.} = \oint \rho g z \, dz \, dA \]  
\hspace{1cm} (2-1)

where the closed surface integral is taken over the Earth's surface. We assume (Ref. I) that the properties of the reference state vary only with altitude; thus the integrals may be separated, and, since the surface integral is a constant (specifically the surface area of the Earth), we may write:

\[ \text{P. E. R. S.} = S \int_0^\infty \rho g z \, dz \]  
\hspace{1cm} (2-2)

where:

\[ S = \oint dA \]  
\hspace{1cm} (2-3)

(The variation in \( S \) due to altitude is negligible, since the altitudes where the density is non-zero are small compared to the mean radius of the Earth.) Using the hydrostatic equation,

\[ dz = -\frac{\partial p}{\rho g} \]  
\hspace{1cm} (2-4)

equation (2-2) may be rewritten:
The integral on the right side of equation (2-5) may be integrated by parts; since the non-integral term is zero (the altitude is zero at one limit of the integral, and the pressure is zero at the other),

\[ \int_{0}^{P_0} z \, dp = \int_{0}^{\infty} p \, dz \quad (2-6) \]

Combining equations (2-4), (2-5), and (2-6),

\[ \text{P. E. R. S.} = \frac{S}{g} \int_{0}^{P_0} \frac{p}{\rho} \, dp \quad (2-7) \]

Combining the equation of state:

\[ p = \rho R_{\text{eff}} T \quad (2-8) \]

with equation (2-7),

\[ \text{P. E. R. S.} = \frac{S}{g} \int_{0}^{P_0} R_{\text{eff}} T \, dp \quad (2-9) \]

The stored heat energy of the atmosphere is given by:

\[ \text{S. H.} = \oint_{0}^{\infty} \rho (C_{\text{eff}} T + L_0 W/(1+ W_0)) \, dz \, dA \quad (2-10) \]
where \( W_o \) is the total (vapor and liquid) water mixing ratio and \( W \) is the water vapor mixing ratio. Making the reference state assumptions previously discussed and substituting by equation (2-3), equation (2-10) may be rewritten:

\[
S. H. R. S. = \int_0^\infty \rho \left( c_{v_{\text{eff}}} T + L_o \frac{W}{1+W_o} \right) dz \quad (2-11)
\]

Combining equations (2-4) and (2-11),

\[
S. H. R. S. = (S/g) \int_0^{P_o} \rho \left( c_{v_{\text{eff}}} T + L_o \frac{W}{1+W_o} \right) dz \quad (2-12)
\]

In computing the stored energy in the atmosphere and in the reference state, we must consider potential energy and stored heat; other forms of stored energy, such as mass, may be ignored, since they will not be changed by reversible permutation of the atmosphere. Thus, for computation of the available energy in a moist atmosphere,

\[
S. E. = P. E. + S. H. \quad (2-13)
\]

Combining equations (2-9), (2-12), and (2-13),

\[
S. E. R. S. = (S/g) \int_0^{P_o} (r_{\text{eff}} + c_{v_{\text{eff}}}) T + L_o \frac{W}{1+W_o} \) dp \quad (2-14)
\]
The effective constant pressure and constant volume specific heats and the effective specific gas constant are related:

\[ R_{\text{eff}} = C_{\text{p,eff}} - C_{\text{v,eff}} \] (2-15)

Combining equations (2-14) and (2-15),

\[ S. E. R. S. = (S/g) \int P_0 (C_{\text{p,eff}} T + L_0 W / (1 + W_0)) \, dp \] (2-16)

In determining the reference state, we must minimize the stored energy as given by these equations. Since any permutation of the atmosphere not satisfying the assumptions about the reference state could not be the reference state, we need only consider those states for which the reference state equations above are valid; we are seeking the state for which the quantity on the right side of equation (2-16) is minimum.

Minimizing the stored energy is equivalent to minimizing the relative stored energy, defined by:

\[ R. S. E. = S. E. \int \int_0^\infty -P_0 L_0 W / (1 + W_0) \, dz \, dA \] (2-17)

because the integral in equation (2-17) is constant. As before, the integral may, by equations (2-3) and (2-4), be rewritten:
\[ \mathcal{O} = \int_{0}^{p_{o}} -p_{o}w_{o} \, dz \, dA = (S/g) \int_{0}^{p_{o}} -w_{o} - w_{o}/(1 + w_{o}) \, dp \quad (2-18) \]

for the reference state. Combining equations (2-16), (2-17), and (2-18),

\[ \text{R.S.E.R.S.} = (S/g) \int_{0}^{p_{o}} \left( C_{p_{e}} T_{l} - L_{o} \frac{w_{o} - w}{1 + w_{o}} \right) \, dp \quad (2-19) \]

Writing the integrals as finite sums, as the atmosphere will be regarded as a collection of a finite number of parcels for computation rather than as a continuum, equations (2-15) and (2-19) become:

\[ \text{R.S.E.R.S.} = \frac{S_{p_{o}}}{g n} \sum_{i=1}^{n} \left( C_{p_{e}} T_{l} - L_{o} \frac{w_{o} - w}{1 + w_{o}} \right) \quad (2-20) \]

\[ \text{R.S.E.R.S.} = \frac{S_{p_{o}}}{g n} \sum_{i=1}^{n} \left( C_{p_{e}} T_{l} - L_{o} \frac{w_{o} - w}{1 + w_{o}} \right) \quad (2-21) \]

(All parcels are assumed to be of equal mass (Ref. I); hence their respective pressure increments are equal.) The quantity in parentheses before the summations in equations is a positive constant (the mass of a parcel); thus minimizing the stored or relative stored energy is equivalent to minimizing the total specific stored energy or specific relative stored energy of the parcels:
\[ S. E. \cdot R. S. = \sum_{i=1}^{n} (S. P.)_i \]  \hspace{1cm} (2-22)  

\[ R. S. E. \cdot R. S. = \sum_{i=1}^{n} (R. S. P.)_i \]  \hspace{1cm} (2-23)  

where:

\[ S. P. = \frac{C_{p_{eff}} T + L_0 W}{1 + W_o} \]  \hspace{1cm} (2-24)  

\[ R. S. P. = \frac{C_{p_{eff}} T - L_0 (W_o - W)}{(1 + W_o)} \]  \hspace{1cm} (2-25)  

As they appear in the above expressions, the effective specific gas constant and the effective constant volume and constant pressure specific heats are the mass-weighted average of the respective constants for dry air, water vapor, and liquid water; thus:

\[ R_{eff} = \frac{R + WR_w}{1 + W_o} \]  \hspace{1cm} (2-26)  

\[ C_{veff} = \frac{C_v + WC_w + (W_o - W) C_w}{(1 + W_o)} \]  \hspace{1cm} (2-27)  

\[ C_{peff} = \frac{C_p + WC_w + (W_o - W) C_w}{(1 + W_o)} \]  \hspace{1cm} (2-28)
In order to use the equations thus far derived to determine the reference state, we must formulate, either explicitly or implicitly, the behavior of temperature and water vapor mixing ratio as a function of pressure. We may use the standard humidity assumption (Ref. II) to obtain the water vapor mixing ratio:

\[ W = \text{Min}(W_0, W_s) \]  

(2-29)

where \( W_s \) is the saturation water vapor mixing ratio. We may obtain an expression for the saturation water vapor mixing ratio from the equation linking the water vapor mixing ratio to the partial pressures of water vapor and dry air:

\[ W = \frac{\varepsilon e}{(p-e)} \]  

(2-30)

by setting the partial pressure of water vapor equal to its saturation limit:

\[ W_s = \frac{\varepsilon e_s}{(p-e_s)} \]  

(2-31)

The saturation partial pressure of water vapor is given by the Clausius-Clapeyron Equation:

\[ e_s = e_{so} \left( \frac{T_o}{T} \right) \left( \frac{C_w - C_{pw}}{R_w} \right) \exp\left( \frac{L_o}{R_w \left( \frac{1}{T_o} - \frac{1}{T} \right)} \right) \]  

(2-32)
where $e_{so}$ is the saturation partial pressure of water vapor at temperature $T_0$. Of course, the liquid water mixing ratio is given by $(W_o - W)$.

The behavior of temperature may be obtained from the heat balance equation; since pressure is the independent variable, it may be written:

$$(1+W_o)C_{p}^{\text{eff}} \frac{dT}{dp} + L \frac{dW}{dp} = (1+W_o)R_c T / p$$ \hfill (2-33)

where:

$$L = L_0 - (C_w - C_{pw})T$$ \hfill (2-34)

In the case of an unsaturated parcel,

$$W = W_o$$ \hfill (2-35)

Since $W_0$ is constant for each parcel by the assumption of reversibility,

$$\frac{dW}{dp} = 0$$ \hfill (2-36)

Combining equation (2-33) and equation (2-36),
Combining equations (2-26), (2-28), and (2-35),

\[
\frac{R_{\text{eff}}}{C_{\text{peff}}} = \frac{(R+W_0 R_w)/(C_p+W_0 C_{pw})}{(R+C_p)}
\]  

(2-38)

Since \( W_0 \) is small in the atmosphere, and \((R_w/C_{pw})\) is close to \((R/C_p)\), we may approximate \((R_{\text{eff}}/C_{\text{peff}})\) by \( \kappa \) without appreciable loss of accuracy; with this substitution, equation (2-37) may be rewritten (in integrated form):

\[
T = T_0 (p/p_0)^\kappa
\]  

(2-39)

where \( T_0 \) is the temperature of the parcel at some pressure \( p_0 \).

If the parcel is saturated,

\[
W = W_s
\]  

(2-40)

Differentiating with respect to pressure,

\[
\frac{dW}{dp} = \frac{\partial W_s}{\partial p} + \frac{\partial W_s}{\partial T} \frac{dT}{dp}
\]  

(2-41)

Combining equations (2-33) and (2-41),
\[
\frac{d\tau}{dp} = \frac{(1+w_o)R_{eff}T/p - L(\partial w_s/\partial p)}{(1+w_o)c_{peff} + L(\partial w_s/\partial p)}
\] (2-42)

The two partial derivatives on the right side of equation (2-42) may be obtained by differentiating equations (2-31) and (2-32):

\[
\frac{\partial W_s}{\partial p} = -\frac{w_s}{(p-e_s)}
\] (2-43)

\[
\frac{\partial w_s}{\partial T} = w_s \left(\frac{p}{(p-e_s)}\right) \frac{L}{R_w T^2}
\] (2-44)

Thus, while equation (2-42) cannot be integrated through analytic techniques, it can be integrated numerically on a computer.

From the previously mentioned assumptions about the reference state and the assumption that all parcels of the atmosphere are of equal mass, it follows that the reference state is a vertical stacking of the parcels. The parameters of a parcel are generally measured at the average pressure of the parcel; for the i\textsuperscript{th} parcel from the ground in the reference state, this is given by:

\[
p_i = \left(1-(i-\frac{1}{2})/n\right)p_o
\] (2-45)

where the atmosphere is divided into n parcels.
Chapter III:

An Algorithm for

Determination of the

Reference State
The reference state has been defined as the permutation of the atmosphere which, when achieved through reversible processes, has the least stored energy. Since we regard the atmosphere as a collection of a finite number of parcels, and assume that the properties of the reference state vary only vertically, the reference state will obviously be some permutation of the parcels in a vertical column; the most obvious algorithm to determine the reference state is to simply compute the stored energy of all such permutations and choose the permutation which has the least stored energy. This algorithm would be acceptable for ten parcels; however, to obtain worthwhile results (results that would be a significant contribution in this field), the number of parcels must be at least one order of magnitude (and preferably two or three orders of magnitude) higher, and for even one hundred parcels, this algorithm is quite impractical. (One hundred parcels would involve approximately $10^{158}$ permutations; allowing a tenth of a microsecond per permutation, this would require over $10^{140}$ computer years!) Thus, we must devise an algorithm which is not totally dependent upon such brute force techniques.

In the last chapter, we defined the specific stored energy of a parcel (equation (2-24)). We may obtain the derivative of this quantity with respect to pressure by differentiating equation (2-24):
\[
\frac{d(S. P.)}{dp} = C_{\text{peff}} \frac{dT}{dp} + T \frac{dC_{\text{peff}}}{dp} + \frac{L_0}{1+W_0} \frac{dW}{dp}
\]  

(3-1)

Differentiating equation (2-28),

\[
\frac{dC_{\text{peff}}}{dp} = \frac{C_{PW} - C_W}{1+W_0} \frac{dW}{dp}
\]  

(3-2)

Combining equations (2-34), (3-1), and (3-2),

\[
\frac{d(S. P.)}{dp} = C_{\text{peff}} \frac{dT}{dp} \frac{L}{1+W_0} \frac{dW}{dp}
\]  

(3-3)

Combining equations (2-33) and (3-3),

\[
\frac{d(S. P.)}{dp} = R_{\text{eff}} \frac{T}{p}
\]  

(3-4)

We may define the virtual temperature of a parcel:

\[
T_v = (R_{\text{eff}}/R)T
\]  

(3-5)

Combining equations (3-4) and (3-5),

\[
\frac{d(S. P.)}{dp} = R_T \frac{T_v}{p}
\]  

(3-6)

Now, suppose there are two parcels, A and B, that we know will be in a region where, at any given pressure level in the
region,

\[ (T_v)_A > (T_v)_B \]  \hspace{1cm} (3-7)

If, in the reference state, parcel A is at pressure level \( p_1 \) and parcel B is at pressure level \( p_2 \), the change in stored energy if parcels A and B are interchanged without permuting the remaining parcels is given by:

\[ \Delta S. E. R. S. = \int_{P_1}^{P_2} \left( (d(S.P.)_A / dp) dp + \int_{P_1}^{P_2} (d(S.P.)_B / dp) dp \right) \]  \hspace{1cm} (3-8)

Combining equations (3-6) and (3-8) and writing the two integrals as one,

\[ \Delta S. E. R. S. = \int_{P_1}^{P_2} \frac{R}{p} ((T_v)_A - (T_v)_B) dp \]  \hspace{1cm} (3-9)

Since relation (3-7) holds over the entire region of integration, the integrand in equation (3-9) is positive between the limits of integration; thus:

\[ \text{Sgn} (\Delta S. E. R. S.) = \text{Sgn} (p_2 - p_1) \]  \hspace{1cm} (3-10)

It follows from the definition of the reference state that:
since otherwise there is another state which is achievable through reversible processes and has less stored energy than the reference state; this is contradictory to the definition of the reference state. Since we assumed that $p_1$ and $p_2$ are different pressure levels, equality is impossible; thus it follows from equation (3-10) and relation (3-11) that:

$$p_2 > p_1 \quad (3-12)$$

Thus, parcel A must be at a higher altitude in the reference state than parcel B.

A more practical expression for virtual temperature may be obtained by combining equations (2-26) and (3-5):

$$T_v = \frac{1+W/V}{1+W_0/V} T \quad (3-13)$$

Figure 3-1 shows the temperature and virtual temperature of a parcel as a function of pressure, based upon the reversible processes formulated in the last chapter. In the lower part of the atmosphere ($p < p_c$), the parcel is unsaturated; combining equations (2-26) and (3-13),
Figure 3-1: Temperature and Virtual Temperature of a Parcel as a Function of Pressure (Reversible Process).
In this region, both curves follow dry adiabats (indicated by dashed lines in the figure), and the virtual temperature is slightly greater than the temperature, since $\delta$ is less than unity. In the upper part of the atmosphere ($p < p_c$), the parcel is oversaturated; the temperature behaves according to equation (2-33). Because the water vapor mixing ratio decreases as pressure decreases, the $((1+\bar{w}/\varepsilon)/(1+\bar{w}_o))$ factor in equation (3-13) does likewise; the two curves intersect where this factor is unity. In the limit as pressure goes to zero, the curves become asymptotic to dry adiabats (indicated by dashed lines in the figure).

The four dry adiabats discussed above may be labeled by the temperature at which they intersect some arbitrary reference pressure, usually taken to be the nominal bottom of the atmosphere (1000 mb.), as illustrated in figure 3-1. By convention (Ref. 1), these values for the dry adiabats through the condensation point of and asymptotic to the temperature curve are known as the condensation and equivalent potential temperatures, respectively; it seems logical, by extension, to call the respective values for the virtual temperature curve the condensation and equivalent virtual potential temperatures.
respectively. Mathematically, from equation (2-39),

\[ q_c = T_c \left( \frac{p_{\text{ref}}}{p_c} \right)^\kappa \]  
(3-15)

\[ q_e = \lim_{p \to 0} T \left( \frac{p_{\text{ref}}}{p} \right)^\kappa \]  
(3-16)

\[ q_{v_c} = T_{v_c} \left( \frac{p_{\text{ref}}}{p_c} \right)^\kappa \]  
(3-17)

\[ q_{v_e} = \lim_{p \to 0} T_v \left( \frac{p_{\text{ref}}}{p} \right)^\kappa \]  
(3-18)

It should be noted that for small pressures, the temperature and virtual temperature behave according to equations (2-42) and (3-13), and the limits in equations (3-16) and (3-18) are convergent. In the case of a dry parcel, the above expressions are indeterminate; in the limit as the total water mixing ratio goes to zero, all four values approach the potential temperature of the parcel, and can be computed from any point on the curve.

The virtual temperature of an unsaturated parcel follows a dry adiabat; thus, from equation (2-39),

\[ T_v = q_{v_c} \left( \frac{p}{p_{\text{ref}}} \right)^\kappa \]  
(3-19)

From equation (3-19), it follows that for two parcels, A and
B, in a region where neither parcel can be saturated, relation (3-7) holds if and only if:

\[(e_{vc})_A > (e_{vc})_B\]  \hspace{1cm} (3-20)

Thus, a parcel of higher condensation virtual potential temperature must be above a parcel of lower condensation virtual potential temperature in a region where both parcels are unsaturated in the reference state; this argument applies, by extension, to any finite number of parcels in a region where all of the parcels are necessarily unsaturated. (This result is a generalization of a solution to the problem of determination of the reference state for computation of the available potential energy in a dry atmosphere.)

The result discussed above is due to the fact that any two dry adiabats intersect only at zero pressure (hence the virtual temperature curves of unsaturated parcels do not intersect); this is not true, in general, of the virtual temperature curves of two parcels in a region where one or both of the parcels may be saturated. The virtual temperature curves of two parcels may intersect in either of two ways, as illustrated in figure 5-2 — both parcels may be saturated, or one of the parcels may be saturated and the other unsaturated. In
Figure 3-2: Possible Intersections of Virtual Temperature Curves of Two Parcels: (A) Two Saturated Parcels; (B) One Saturated Parcel and One Unsaturated Parcel.
the first case (both parcels saturated), this is due to the effects of the specific heat of liquid water and different total water mixing ratios; this effect is small, and may be neglected, so that equivalent virtual potential temperature may be used to determine the relative position of two or more saturated parcels in the reference state in the same way that condensation virtual potential temperature may be used to determine the relative positions of two or more unsaturated parcels.

The crossing of the virtual temperature curves of two parcels in a region where one of the parcels is saturated and the other is not is due primarily to the release of latent heat; this effect is not negligible. Thus, since the virtual temperature curves of two curves cannot intersect at more than one point (discounting the zero pressure point at which all such curves intersect), we may determine the relative positions of two or more parcels in a region where at least one parcel may be saturated and at least one parcel may be unsaturated from the equivalent and condensation virtual potential temperatures of the parcels if and only if the same relationship exists between the respective values of these parameters (that is, one parcel must be at a higher altitude in the reference state than another if its condensation and equivalent virtual potential temperatures are greater
then the condensation and equivalent virtual potential temperatures, respectively, of the other parcel; in this case, relation (3-7) holds throughout the entire atmosphere). Where such a relationship does not exist (one parcel has higher condensation and lower equivalent virtual potential temperatures than another in a region where at least one of the parcels may be saturated and at least one of the parcels may be unsaturated), it is necessary to compute the stored energy (or one of the analogous quantities derived in Chapter II) of each possible permutation of those parcels, and choose the permutation having the least stored energy.

Using these results, we may establish an algorithm that will correctly determine the reference state; a flowchart of this algorithm is shown in figure 3-3. The necessary quantities (condensation and equivalent virtual potential temperatures and condensation pressure (the pressure at which the parcel is precisely saturated, as indicated in figure 3-1)) are computed and stored, and the parcels are put into a list in order of condensation virtual potential temperature. This list is then searched for regions where it deviates from the correct reference state by checking the equivalent virtual potential temperature; the condensation pressure is used to determine which parcels are saturated. Any discrepancies are then corrected by an appropriate method (if all the parcels
Figure 3-3: Global Flowchart of an Algorithm which Determines the Correct Reference State for Computation of the Available Energy in a Moist Atmosphere.
must be saturated in the region in question, the parcels are simply rearranged in order of equivalent virtual potential temperature; otherwise the possible permutations are determined and the correct permutation of those parcels is selected, as previously discussed.)
Chapter IV:

Conclusion
A computer program based on the algorithm presented in the last chapter is listed in Appendix II, along with its sub-programs. (The logic in the program differs slightly from that in the flowchart in figure 3-3 in that parts of two steps in the flowchart may sometimes be done simultaneously in the program more efficiently than if they are done separately; however, the theoretical aspects of both are the same.) As a practical matter, a limitation has been imposed on the extent to which the program will consider permutations; the parcels must be separable into not more than ten groups, one of which is reserved for parcels which would be saturated everywhere in the region, where the relative positions of the parcels in each group can be obtained from the parameters of the parcels, as discussed in Chapter III. This procedure enables the parcels in each group to be regarded as indistinguishable for purposes of determining the possible permutations, dividing the number of permutations which must be considered by the factorials of the number of parcels in each of the groups.

The program was run with a test data base to obtain the sample reference state which is included with the program in Appendix II. (This is an example of the case where Lorenz notes that his procedure fails (Ref. II); this result is correct. This algorithm will also yield the correct result in all cases for which Lorenz's procedure does so.) The execu-
tion time for this run (which required a total of twenty numerical integrations with pressure increments of one percent of the pressure and consideration of nearly one hundred eighty-five thousand permutations of twenty parcels) was under thirty-five seconds; the execution time which would have been required by the brute force method discussed at the beginning of Chapter III would be measured in years. While the situations of which this data base is representative are relatively infrequent in the real atmosphere, it gives a fair impression of the capability of this algorithm. In cases where Lorenz's procedure yields the correct result, this algorithm will do so in about the same amount of execution time (assuming that the programs implementing both methods are of a similar level of efficiency); in other cases, like the case in Appendix II, it is the only algorithm devised as of this writing that can be used, within the limits of practicality, to determine the reference state.

The program in Appendix II can be used directly to print the reference state of the atmosphere, as was done with the data base listed therein to obtain the sample output; however, it will probably be more desirable, for most applications, to use it as a subroutine for a program which computes the available energy in the atmosphere. In such an application, some modification of the data input and output routines may be
The algorithm to determine the reference state for computation of available energy in a moist atmosphere which I have presented herein seems to be considerably better than the brute force method discussed in Chapter III, in terms of both speed and practical capacity, and Lorenz's procedure, which cannot always determine the correct result. This algorithm will give the correct result in a reasonable amount of time.
Appendix I:

Symbols of Physical Constants
<table>
<thead>
<tr>
<th>Symbol in: Text Program</th>
<th>Written Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>g</td>
<td>Earth's Gravitational Acceleration</td>
</tr>
<tr>
<td>s</td>
<td>Earth's Surface Area</td>
</tr>
<tr>
<td>( \infty )</td>
<td>Highest Altitude of the Earth's Atmosphere</td>
</tr>
<tr>
<td>p_o</td>
<td>( \text{P}_{\text{E}} ) Pressure at Bottom of Earth's Atmosphere (Mean)</td>
</tr>
<tr>
<td>l_o</td>
<td>( \text{L}_{\text{O}} ) Heat of Vaporization of Water at 0(^{\circ})K</td>
</tr>
<tr>
<td>r</td>
<td>R Specific Gas Constant of Dry Air</td>
</tr>
<tr>
<td>r_w</td>
<td>RW Specific Gas Constant of Water Vapor</td>
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<tr>
<td>c_v</td>
<td>Constant Volume Specific Heat of Dry Air</td>
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<tr>
<td>c_vw</td>
<td>Constant Volume Specific Heat of Water Vapor</td>
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<tr>
<td>C_w</td>
<td>CW Specific Heat of Liquid Water</td>
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<tr>
<td>c_p</td>
<td>( \text{C}_{\text{P}} ) Constant Pressure Specific Heat of Dry Air</td>
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<tr>
<td>c_{pw}</td>
<td>( \text{C}_{\text{Pw}} ) Constant Pressure Specific Heat of Water Vapor</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>( \frac{R}{RW} ) Ratio of Molecular Mass of Water to Mean Molecular Mass of Dry Air</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>( \frac{R}{RW} ) Ratio of Specific Gas Constant of Dry Air to Constant Pressure Specific Heat of Dry Air</td>
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Appendix II:

A Fortran Program to Determine the Correct Reference State
References


II. Lorenz, Edward N. "Numerical Evaluation of Moist Available Energy." (No Publication Information.)
DIMENSION W0(100),NUMBER(100),PCOND(100),PLEVEL(100),TPVC(100),TP1E(100),NHOLD(100),SP(100,100),NCOUNT(10),NEWPRM(100),NSTACK(10,100)

REAL K,L,L0
LOGICAL LSTPRM
COMMON CP,CPW,CW,ESATO,K,L0,PREF,R,RW
CP=0.239
CPW=0.465
CW=1.0
ESATO=6.11*273.15**((0.535/0.11)*EXP((595./273.15+0.535)/0.11))
K=0.0683/0.239
L0=595.+(0.535*273.15
PREF=1000.
R=0.0683
RW=0.11
N=1

READ(5,50)END=9) P0,T0,RH
W0(N)=WSAT(P0,T0)*RH/100.
P=P0
T=T0
W=W0(N)
WMMP=W/(W+R/RW)
TPVCN=T*(PREF/P)**K
IF (W.GT.0) GO TO 2
PCONDN=0.
TPVEN=TPVCN
N=N-1
GO TO 7
2 IF (W.LE.WSAT(P,T)) Go TO 5
3 DTDP=DTDPM(P,T,W)
T=T+DTDP*P*0.01
P=P*1.01
IF (W.GT.WSAT(P,T)) Go TO 3
4 ESAT=ESATO*T**((CPW-CW)/RW)*EXP(-L0/(RW*T))
ERROR=WMMR-ESAT/P
DT=ERROR*P/(ESAT*(1/(P*DTDP-(L0/T+CPW-CW)/(RW*T))))
T=T+DT
P=P+DT/DTDP
IF (ABS(ERROR).GE.0.01*WMMR) GO TO 4
PCONDN=P
TPVCN=T*(PREF/P)**K*(1+W*RW/R)/(1+W)
P=P0
T=T0
GO TO 6
5 ESAT=ESATO*T**((CPW-CW)/RW)*EXP(-L0/(RW*T))
ERROR=WMMR-ESAT/(PREF*(T/TPVCN)**(1/K))
T=T-ERROR*T/((WMMR-ERROR)**(1/K-(L0/T+CPW-CW)/RW))
IF (ABS(ERROR).GE.0.01*WMMR) GO TO 5
P=PREF*(T/TPVCN)**(1/K)
PCONDN=P
TPVCN = TPVCN*(1+W*R/RW)/(1-W)

6 DTDP = DTDPMAP(W, T, W)

P = P*0.99
IF (HT.DTDP.LT.0.99*K*T/P .AND. P.GT.1.) GO TO 6

TPVEN = T/(1+W)*(PREF/P)**K

I = N-1

7 IF (I.EQ.0) GO TO 8
IF (TPVCN.GT.TPVC(NUMBER(I))) GO TO 8
IF (TPVCN.EQ.TPVC(NUMBER(I))).AND. TPVEN.GE.TPVE(NUMBER(I))) GO TO 8
J = I+1
NUMBER(J) = NUMBER(I)
I = I-1
GO TO 7

8 I = I+1
NUMBER(I) = N
PCOND(N) = PCONDN
TPVC(N) = TPVCN
TPVE(N) = TPVEN
N = N+1
GO TO 1

9 NPRCLS = N-1
N = 1

10 M = NPRCLS

11 IF (TPVE(NUMBER(N)).GT.TPVE(NUMBER(M))) GO TO 15

12 M = M-1
IF (M.GT.N) GO TO 11
PLEVEL(N) = PREF*(1+(0.5-N)/NPRCLS)

13 N = N+1
IF (N.LT.NPRCLS) GO TO 10
PLEVEL(N) = PREF*(1+(0.5-N)/NPRCLS)
NPAGE = 1
WRITE(6,51) NPAGE
DO 14 N=1,NPRCLS
I = NUMBER(N)
IF (50*INT(N/50.).NE.N) GO TO 14
NPAGE = NPAGE+1
WRITE(6,51) NPAGE

14 WRITE(6,52) IPLEVEL(N), PCOND(I), TPVE(I), TPVC(I)
STOP

15 NMAX = M+1

16 IF (TPVE(NUMBER(I)).GE.TPVE(NUMBER(N))) NMAX = NMAX-1
PLNMAX = PREF*(1+(0.5-NMAX)/NPRCLS)
IF (PLNMAX.GE.PCOND(N)) GO TO 13
NSAT = 0
NUNSAT = N-1
DO 20 I = N,M

20 I = N,M
IF (PCOND(NUMBER(I)).LT.PLEVEL(N)) GO TO 19
J=NSAT
NSAT=NSAT+1
IF (J.EQ.0) GO TO 18
17 IF (TPVE(NHOLD(J)) .LT. TPVE(NUMBER(I))) GO TO 18
NHOLD(J+1)=NHOLD(J)
J=J+1
IF (J.GT.0) GO TO 17
18 NHOLD(J+1)=NUMBER(I)
GO TO 20
19 NUNSAT=NUNSAT+1
NUMBER(NUNSAT)=NUMBER(I)
20 PLEVEL(I)=PREF*(1+(0.5-I)/NPRCLS)
DO 21 I=1,10
21 NCOUNT(I)=0
I=N
22 J=1
23 IF (NCOUNT(J).EQ.0) GO TO 24
IF (TPVE(NUMBER(I)).GE.TPVE(NSTACK(J,NCOUNT(J)))) GO TO 24
J=J+1
IF (J.LE.9) GO TO 23
WRITE(6,53)
STOP
24 NCOUNT(J)=NCOUNT(J)+1
NSTACK(J,NCOUNT(J))=NUMBER(I)
I=I+1
IF (I.LE.NUNSAT) GO TO 22
DO 25 I=1,NSAT
J=I+NUNSAT
NUMBER(J)=NHOLD(I)
25 NSTACK(10,J)=NHOLD(I)
NCOUNT(10)=NSAT
N0=N-1
SEMIN=0.
DO 30 I=1,10
J0=NCOUNT(I)
IF (J0.EQ.0) GO TO 30
DO 30 J=1,J0
LSTOP=M-NCOUNT(I)+J
N0=N0+1
NEWPRM(N0)=I
NLEVEL=N+J-1
NPRCL=NSTACK(I, J)
W=W0(NPRCL)
IF (PCOND(NPRCL).GT.PLEVEL(NLEVEL)) GO TO 27
TPVCN=TPVC(NPRCL)*(1+W)/(1+W*RW/RW)
26 SP(NPRCL,NLEVEL)=TPVCN*(PLEVEL(NLEVEL)/PREF)**K*(CP+W*CPW)/(1+W)
NLEVEL=NLEVEL+1
IF (NLEVEL.GT.LSTOP) GO TO 30
IF (PCOND(NPRCL).LT.PLEVEL(NLEVEL)) GO TO 26
27 P=PCOND(NPRCL)
T=TPVC(NPRCL)*(P/PREF)**K*(1+W)/(1+W*RW/R)
- 42 -
28 D P = P - P LEVEL(NLEVEL)
    IF (DP .LE. P * 0.01) GO TO 29
    T = T - DTDPMA(P, T, W) * P * 0.01
    P = P * 0.99
    GO TO 28
29 T = T - DTDPMA(P, T, W) * DP
    P = P - DP
    SP(NPRCL, NLEVEL) = T * CP + (T * (CPW - CW) + LO) * WSAT(P, T) + (CW * T - LO) * W
    SP(NPRCL, NLEVEL) = SP(NPRCL, NLEVEL) / (1 + W)
    NLEVEL = NLEVEL + 1
    IF (NLEVEL .LE. LSTOP) GO TO 28
30 CONTINUE
    DO 31 I = NM
31 SEMIN = SEMIN + SP(NUMBER(I), I)
32 SE = 0.
    DO 33 I = 1, 10
33 NCOUNT(I) = 0
    CALL PRMTTN(NM, NEWPRM, LSTPRM)
    IF (LSTPRM) GO TO 36
    DO 34 NLEVEL = NM
34 J = NEWPRM(NLEVEL)
    NCOUNT(J) = NCOUNT(J) + 1
    NHOLD(NLEVEL) = NSTACK(J * NCOUNT(J))
    SE = SE + SP(NHOLD(NLEVEL), NLEVEL)
    IF (SE .GE. SEMIN) GO TO 32
    DO 35 I = NM
35 NUMBER(I) = NHOLD(I)
    SEMIN = SE
    GO TO 32
36 NM = M
    GO TO 13
50 FORMAT(3F10.2)
51 FORMAT(*1 ATOMSPHERIC REFERENCE STATE -- PAGE */12//4X,*PARCEL N
    10. PRESSURE LEVEL (MB) COND. PRESSURE (MB) EQUIV. POT. VIRTU
    2AL TEMP. (DEG K) COND. POT. VIRT. TEMP. (DEG K)*)
52 FORMAT(*1 ERROR -- TOO MANY SUBSTACKS REQUIRED -- EXECUTION T
    ERMINATING*)
END
FUNCTION WSAT(P,T)
REAL KL,LO
COMMON CP,CPW,CW,ESATO,K,LO,PREF,R,RW
ESAT=ESATO*T**((CPW-CW)/RW)*EXP(-LO/(RW*T))
WSAT=R/RW*ESAT/(P-ESAT)
RETURN
END

FUNCTION DTDPMA(P,T,W)
REAL KL,LO
COMMON CP,CPW,CW,ESATO,K,LO,PREF,R,RW
ESAT=ESATO*T**((CPW-CW)/RW)*EXP(-LO/(RW*T))
WV=R/RW*ESAT/(P-ESAT)
L=LO+(CPW-CW)*T
REFF=R+RW*WV
CPEFF=CP+CPW*WV+CW*(W-WV)
DWSATP=-WV/(P-ESAT)
DWSATT=WV/P/(P-ESAT)*L/(RW*T*T)
DTDPMA=(REFF*T/P-L*DWSATP)/(CPEFF*L*DWSATT)
RETURN
END
SUBROUTINE PRMTTN(N,M,NEWPRM,LSTPRM)
DIMENSION NEWPRM(100),NHOLD(100)
LOGICAL LSTPRM
I=M-1
1 J=I+1
   NHOLD(J)=NEWPRM(J)
   IF (NEWPRM(I)*LT.NEWPRM(J)) GO TO 2
   I=I-1
   IF (I.GE.N) GO TO 1
   LSTPRM=.TRUE.
   RETURN
2 NO=NEWPRM(I)
3 IF (NEWPRM(J)*LE.NO) GO TO 4
   J=J+1
   IF (J*LE.M) GO TO 3
4 J=J-1
   NEWPRM(I)=NEWPRM(J)
   NHOLD(J)=NO
   J=M
5 I=I+1
   NEWPRM(I)=NHOLD(J)
   J=J-1
   IF (I*LT.M) GO TO 5
   LSTPRM=.FALSE.
   RETURN
END
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**Test Data Base**

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