MOARE

A National Energy Planning System for Argentina

by

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INTRODUCTION

Argentina has always had a varied and substantial resource base of energy. With extensive oil and gas reserves, and other primary energy sources including hydroelectric resources, the Argentine government is studying programs to achieve self-sufficiency during the 1980's. New incentives for oil and gas exploration have resulted in dramatic increases in proven reserves. In particular, Argentina's natural gas reserves have tripled since 1977 and current reserves almost 50 years of present consumption. Oil reserves equal about 20 years of current consumption. Together, oil and gas cover three-quarters of total energy demand in the country.

In 1983, the World Bank funded a contract awarded to Stone and Webster Worldwide Consultants, and administered by the Argentine state petroleum company, to develop a modeling system for national energy planning. The system, called MOARE (El Modelo Operativo de Asignacion de Recursos Energeticos), has been fully implemented, tested and permanently installed in Buenos Aires. The major purpose of MOARE is to assist Argentine energy planners in the Secretariat of Energy and state energy companies in selecting a rational strategy for the long-term exploitation and use of their natural gas and oil resources.

In this paper, we provide details about the models used in MOARE, its system features, and the sorts of results it has produced. The models are intended to capture

- explicit detail about transportation and transformation processes
- investments in infrastructure for production, transportation, distribution, and transformation processes
- investments in new energy technologies
- economies of scale in investments
- explicit financial and geological detail about oil and gas production
- endogenous end-use demand
- energy imports and exports

Mathematical programming models were selected for MOARE because they were the only ones judged capable of identifying globally efficient energy strategies reconciling the complex interactions among economic, engineering and policy factors. Goreux and Manne (1973) report on earlier applications of optimization models to energy and economic planning in Mexico. See deLucia and Jacoby (1982) for further discussions of energy modeling for developing countries. The reader is also referred to the recent paper by Hogan and Weyant (1983) for an extensive review of the role of mathematical programming in energy modeling.

Mathematical programming methodologies permit a clear and coherent construction and integration of normative models for studying four major types of decisions: energy resource allocation, capital investments in infrastructure, variable energy end-use demands, and primary energy resource development.

Energy Resource Allocation: Given end-use demands and the infrastructure of the Argentine energy economy, linear programming (LP) models are ideally suited for determining an optimal allocation of primary fuels. At the same time, the models determine an optimal utilization of energy transformation and distribution facilities. The multi-period LP models in MOARE also capture inter-fuel substitutions based on the relative costs and efficiencies of different fuels and energy technologies. Finally, demand curves for primary fuels at each basin are effectively derived from optimal LP shadow (marginal) prices computed for constraints on these resources.

Capital Investments in Infrastructure: Mathematical programming models permit the explicit modeling of long run shifts in demand due to fuel substitution effects and changes in the infrastructure of the Argentine energy economy. The models incorporated in MOARE also link capital constraints to descriptions of capacity expansion options for changing the infrastructure. Shadow prices on the capital constraints measure the tradeoffs between energy efficiency and capital costs. Mixed integer programming modeling is required to capture the capacity expansion options since they are lumpy and may have associated economies of scale.

Variable Energy End-Use Demand: Econometric models were developed to describe consumers' behavior for each combination of geographical market, end-use category and time period. The behavior was incorporated into MOARE using the concept of consumers' surplus. Each end-use demand function is integrated and inverted to produce a curve relating consumers' surplus to satisfied demand. This concave, non-linear function is then approximated by a piecewise linear function and incorporated directly into the energy economy model.

Primary Energy Resource Development: We had originally proposed statistically derived supply functions for describing oil and gas production at the basins. Although this approach is appropriate for estimating production from mature fields, we decided early on that it was highly inappropriate for undeveloped or incompletely developed fields. One reason is that statistical extrapolation methods are incapable of accurately describing future production for fields with scant or non-existent histories. A related reason is that decisions regarding drilling strategies depend on the prices the producers can expect to receive in the future for the primary fuels. Since the Argentine energy economy is undergoing major changes, past prices are largely irrelevant to future actions.

With these considerations in mind, we decided at the start of the project to construct a normative model for optimizing drilling strategies in each undeveloped or incompletely developed field, given prices that the Argentine energy economy is willing to pay for oil and gas there. Details of the model are given in a separate section below.

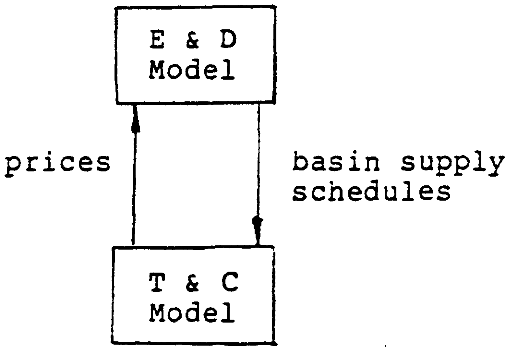
In summary, mathematical programming models were highly appropriate for the four major decision making areas discussed above. As a result, we were able to construct and validate models that were both logically consistent and understandable.

In the next section, we present a brief overview of the models imbedded in MOARE. In the two following sections, we provide details about two main submodels, namely, the Exploration and Development (E&D) model, and the Transformation and Consumption (T&C) model. We then discuss how decomposition and equilibrium methods are used to effect an integration of these two submodels. The paper concludes with three sections in which we discuss system implementation features, representative results, and areas of future modeling research and systems development.

We emphasize at the outset that the most important deliverable of the MOARE project was a system, not simply a study. Although MOARE was used in studying a number of long range questions facing Argentine energy planners before it was installed in Buenos Aires, we believe that the major benefits of the project will accrue from continuing use of the system over the years to come.

MODEL OVERVIEW

The main purpose of MOARE is to assist Argentine energy planners in determining development strategies for the country's abundant energy resources. Accordingly, MOARE was constructed by integrating an oil and gas production model, the E&D model, with an energy economy model, the T&C model. For each undeveloped or incompletely developed field, and given prices the energy economy is willing to pay for oil and gas, the E&D model uses dynamic programming to calculate a drilling strategy maximizing the net present value of the field's remaining reserves. The T&C model is a multiperiod mixed integer program that employs production strategies from the E&D model, via energy transformations and distributions, to meet variable energy end-use demands. The overall objective in MOARE is to maximize the discounted sum of net producers' surplus and energy consumers' surplus, in the various consuming regions, over the planning horizon.



Model Integration Schema
Figure 1

The E&D and T&C models are integrated via a price-directed scheme as shown in Figure 1. Given trial supply schedules for each basin that were previously calculated by the E&D model, the T&C model determines an optimal energy economy strategy for meeting variable end-use demand. As a by-product of this optimization, the T&C model also determines time prices at the basins that the energy sector is willing to pay for oil and gas, at each basin in each time period. These prices are passed to the E&D model which uses them to compute new drilling strategies for the fields within each basin. The resulting production is aggregated into a basin supply schedule and an associated total cost. The new basin supply schedules are then added to the T&C model, and it is re-optimized.

This iterative process is terminated when the prices imputed by the T&C model are consistent, in the following sense, with the supply schedules it selects. Each basin supply schedule is generated from a basin price schedule. If the T&C model selects a particular schedule, then the imputed prices should approximately equal the basin price schedule. In other words, the demand prices and the supply prices should be in approximate equilibrium. Further details describing this price-directed decomposition/integration scheme are given in a separate section below.

Exploration and Development Model

Exploration and development of primary fuels in the Argentine energy economy is modeled at the reservoir (field) level. The country is divided into six basins, each with some reservoirs capable of producing oil and/or natural gas. (The number of basins can be increased or descreased as necessary.) The reservoirs are categorized as gas or oil reservoirs depending on their prominent geological features. In each basin oil and gas reservoirs are ranked on the size of their remaining reserves. Reservoirs are then selected for individual treatment if they contain a significant fraction of the basin's proved and probable gas reserves. The remaining (small) reservoirs in each basin are grouped and treated as one reservoir. Over 100 reservoirs have been identified and their data collected.

For each reservoir the E&D model analyzes the investment decisions associated with drilling wells, operating and maintaining the wells, going to secondary recovery, gas reinjection and shutting down. Given the prices that the producer will receive at the wellhead for the oil and gas produced, and the prices for oil and gas left in the ground at the end of the planning horison, the model determines an extraction schedule for oil and gas that maximizes the net present value of the reservoir's reserves.

Supply schedules for all the reservoirs in a basin are then combined to produce basin supply schedules. The cost of a basin schedule is likewise determined by combining the costs of the reservoir schedules. Multiple supply schedules for each basin, corresponding to different sets of prices, are provided as input to the T&C model. The details of the E&D and T&C model integration are given in following sections.

For purposes of efficient modeling, oil and gas resources are treated in two primary categories:

(1) Fully developed resources which, because of low marginal costs to continue production, can be expected to be produced at the practical capacity available from each source, using presently installed primary production facilities.

(2) Resources subject to or requiring additional investment, including that portion of resources associated with category

1 that can only be produced and gathered with additional investment.

We have forecast oil and gas production for the category 1 (fully developed) resources. Gas production not currently being gathered has been deducted from this gas forecast, and is treated as a Category 2 resource. This forecast is then input to the model as a fixed volume of annual production over the planning horizon.

Category 2, resources which are subject to additional investment, include the following:

(a) Identified oil and gas reservoirs that are not fully developed at the time the model is run.

(b) Secondary recovery petroleum reserves potentially producible after future investments are made -- either for water injection or gas reinjection.

(c) Gas production now being vented but recoverable upon investment in gathering facilities.

(d) Potential new oil and gas reserves that are not presently identified but can be expected to become available subject to user-specified minimum time lags and judgements on exploratory success.

(e) Other types of oil and gas resources that may be identified in the course of using the system.

The components of Category 2 are analyzed, primarily, using a <u>dynamic programming</u> model. This modeling approach leads to an efficient algorithm for searching through a complete set of reservoir development strategies to identify the one maximizing net producers' revenues or surplus. See Huppler (1974) for a similar approach for analyzing field development.

The algorithm proceeds forward in time, enumerating successive states of reservoir development and associated decisions linking states. At the start of each year of the 20 year planning horizon, these states are defined by

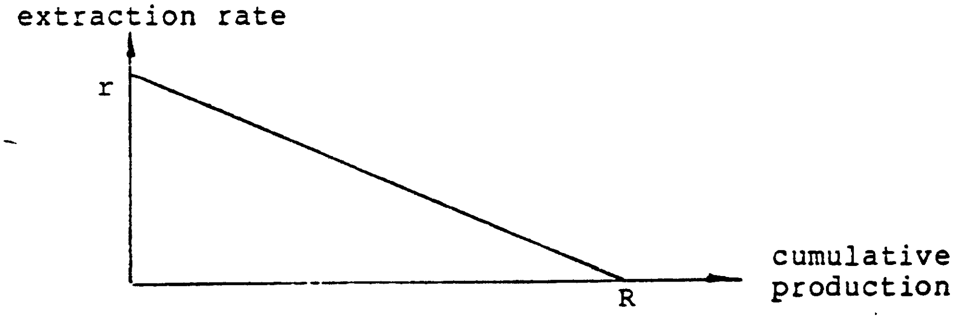
- o the number of wells drilled in previous years
- o the total remaining reserves
- o whether or not secondary recovery has been previously initiated (oil wells only).

A partial strategy produced by the dynamic programming calculations is defined as a list of decisions up to a given year. Given the initial conditions of the reservoir and a partial strategy to year t-1, the algorithm calculates the state paths through which the system may pass up to year t. The algorithm extends the best paths one year further by applying all possible decisions to the current state.

To prevent the number of states from growing exponentially, paths are eliminated from further consideration when they are dominated by other paths. A path is said to be dominated when the state that the path leads to can be reached by a different path with a higher objective function value, or when another path has such a high objective value that the difference in objective function values cannot possibly be overcome by any future decisions. By eliminating all dominated partial strategies we can keep the number of paths to a manageable size.

After the best paths have been extended to the last year the "salvage" value for the reserves left in the ground is added to the net present value of the fuels produced for each path. The highest value is then selected to identify the optimal path or strategy. This optimal path is used to generate the supply schedule, the supply costs, and the annual capital expenditures for the given reservoir.

In the dynamic programming calculations, oil and gas production is determined implicitly by the linear function shown in Figure 2.



Declining Extraction Rate Relationship
Figure 2

In this figure, R is the recoverable reserves and r is the extraction rate, which equals the number of wells times the rate of extraction per unit of time per well. The parameter r is adjusted respectively during the calculations as the algorithm deteremines the number of wells to be drilled in each year.

Given the number of wells, the function in Figure 2 is used to compute production as follows. Let

f(t) = cumulative extraction through period t when the rate of extraction is r.

Then f(t) must satisfy

$$r\left(1 - \frac{f(t)}{R}\right) = \frac{df(t)}{dt} = \text{instantaneous extraction rate at time } t$$

The solution to this differential equation is

$$f(t) = R\left(1 - e^{-rt/R}\right)$$

Hence, production between t_0 and t_1, during which time the number of wells drilled remained constant, equals

$$f(t_1) - f(t_0) = Re^{-rt_0/R}\left(1 - e^{-r(t_1-t_0)/R}\right)$$

If the algorithm decides it is optimal to drill more wells in the period starting at time t_1, the parameter r is increased to reflect this change.

For an oil reservoir one may specify a secondary recovery option. The addition of recoverable reserves obtained through the secondary recovery option is modeled through a single reserves multiplier, B. This multiplier is defined as the ratio of total recoverable reserves with secondary recovery to the total recoverable reserves without secondary recovery. Hence, if the total initial recoverable reserves are R, and the total cumulative production is S, then the redefined recoverable reserves using secondary recovery are (BR-S).

The user is able to specify the point at which the secondary recovery option becomes viable. This is done by specifying a factor between zero and one which represents the minimum fraction of reserves which must be produced by primary production techniques before the secondary recovery option may be considered. Once the decision to go to the secondary recovery has been made, the remaining reserves in the reservoir are depleted at a rate depending only on the number of wells in the field and the remaining reserves; no new wells are drilled in the reservoir.

In optimizing development of a reservoir, the E&D model takes into account the geological characteristics of the reservoir (reserves, gas-oil ratios, maximum rate of production, etc.) and their associated costs (fixed costs of drilling wells, reservoir investment costs, operating and maintenance costs, etc.). Exploration costs can be treated as a fixed cost incurred before development begins. Reinjection of associated gas is also allowed. Optimal reinjection, given prices for natural gas, is carried out by a separate procedure that is performed after the E&D model has been applied to the reservoir or group of reservoirs for which reinjection is being considered.

The discovery of unknown reserves and the results of oil and gas exploration activities are defined by the user as part of the input data. Our approach has been to forecast, over 10 years, exploration and development expenditures and reserves discovered. From this analysis a set of parameters has been developed to convert exploration and development expenditures into discovered reserves, and, after a time lag into category 2 resources. The user may specify an initial exploration and development cost, and the time lags involved to convert the undiscovered resources into Category 2 for the dynamic programming algorithm. This 10-year program does not assume that this is all the oil and gas that can be found; however, it does identify the cost of finding additional reserves, and allows the inclusion of new resources in the model.

A particular advantage of the modeling and data specification approach that has been adopted is elimination of the necessity to assign arbitrary investment and/or operating costs to gas and oil production. Essentially all costs are attached to the development and operation of identifiable physical facilities (by well, for example) that will produce both gas and oil in proportions specified in the input data. Compressor fuel and other field use are accounted for as a fractional reduction of production; otherwise no operating costs are expressed as a function of units produced.

This approach allows the energy planner to model the joint production of oil and gas. This, in turn, provides a better basis for calculating correct prices for oil and gas to be paid to the producer. This is possible because, as we shall see, the market value of both oil and gas are reflected back through the energy network of the T&C model to the wellhead. These prices are optimal (and thus correct) because they reflect an optimal allocation of energy resources. The energy planner need not depend on, for example, an arbitrary allocation of producer costs in order to calculate the producer's price of gas.

Transformation and Consumption Model

The T&C Model is a mixed integer programming model. Alternative supply schedules (e.g., from the E&D model) are one of the main inputs to the T&C model along with energy end-use demand functions, energy transformation and transportation coefficients, taxes, and other characteristics of the Argentine energy economy. The objective of the T&C model is to maximize the discounted sum of net consumers' surplus over the multiperiod planning horizon. As discussed in the following section on model integration, optimizing this objective function also maximizes net producers' surplus. In addition, the T&C model considers decisions about pipeline investment, exports, imports, and so on.

A mixed integer programming model is used to describe transformation and consumption activities because there are indivisibilities, nonlinearities and non-numeric logical constraints associated with some activities that cannot be modeled using the more familiar linear programming (LP) approach.

In particular, mixed integer programming is used to capture fixed costs associated with investment in pipeline or refinery expansions.

Strictly speaking, the T&C model is not a single model. Rather, we have designed and implemented a modeling system capable of generating a range of similar mixed integer programming models. The specific model generated and optimized during a single run of MOARE will be determined by the nature of the energy planning question to be studied.

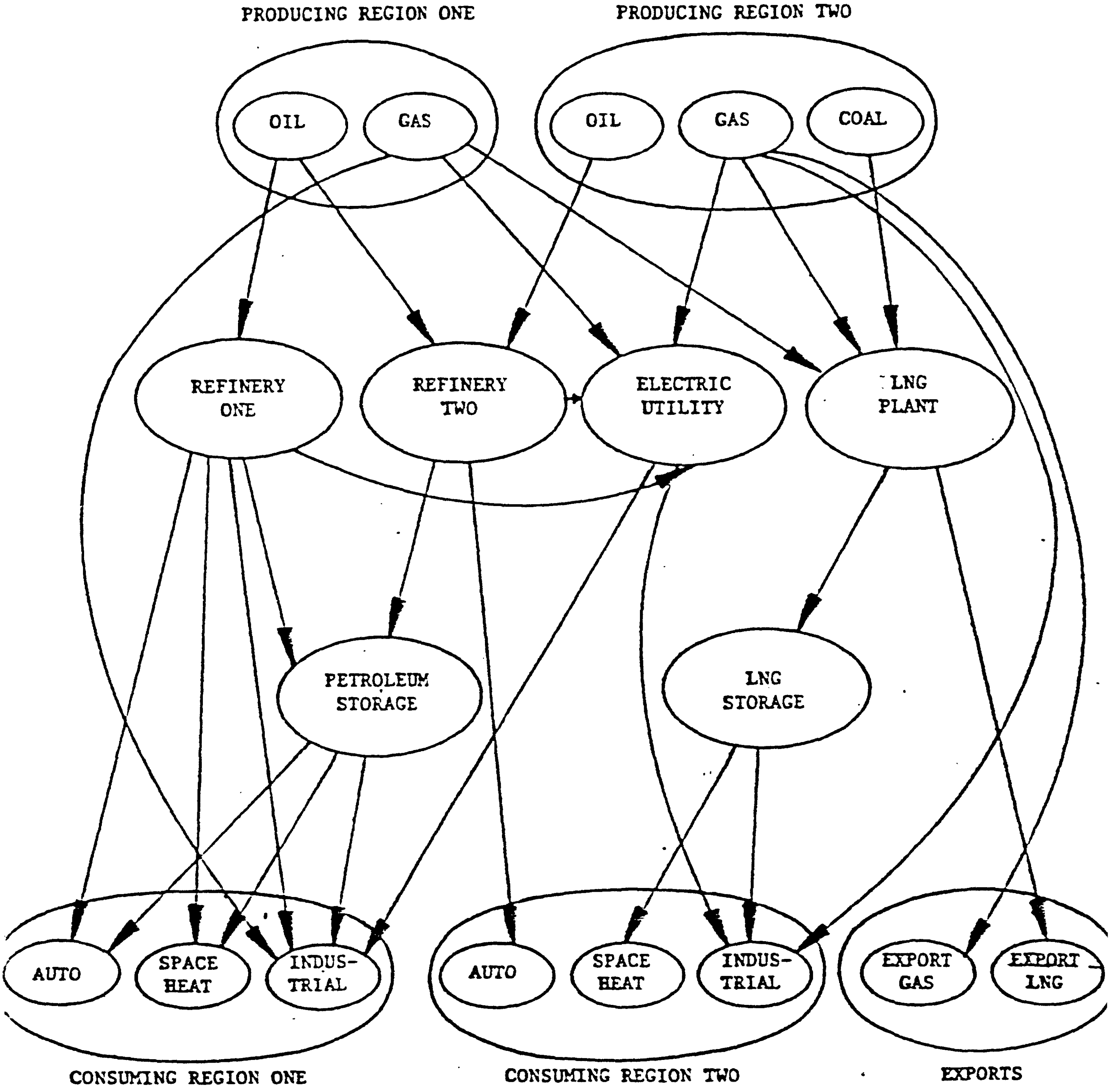
A typical T&C model is described by a descriptive modeling language that allows energy planning problems to be described in terms of nodes and arcs (see Figure 3). Nodes correspond to facilities where energy products are transformed or transshipped, and arcs to links along which energy products flow. Thus, the generation programs for the T&C model are not based on a fixed system of equations. Additional information about the descriptive modeling language, and its relationship to mixed integer programming model generation, is given in the section below devoted to implementation.

The remainder of this section is devoted to brief descriptions of the different types of energy economy constructs that can be included in a T&C model. Asterisks indicate that the information is optional.

Recipe

Characterizes any process (e.g., at gas fractionation plants, refineries, petrochemical plants, etc.) where one or more inputs are transformed into one or more outputs according to a fixed recipe.

(1)	Node identification (e.g., refinery name)	name of node where process occurs
(2)	Investment option identification*	only if recipe conditional on an option
(3)	Name of 'reference' input product	'primary' recipe input
(4)	Variable cost of recipe activity, measured on (3)*	cost per unit of reference product flow
(5)	Recipe maximum, expressed as maximum quantity of (3) per year*	
(6)	Recipe minimum, expressed as minimum quantity of (3) per year*	



T&C Model

Figure 3

(7)	Other recipe constituents (at least one)	can be inputs or outputs to recipe
	(a) name of input or output product	
	(b) Input(+)/output(-) coeffecient relative to (3)	

Investment Option

One or more processes defined by recipes may be controlled by an investment option in such a way that the recipes can only be active if the investment option is active. This feature can be used, for example, to introduce the option of starting up a recipe process or set of recipe processes in some period, at a fixed investment cost. One of 3-6 should be given.

(1)	Option name	used in recipe reference
(2)	Node identification	
(3)	Fixed option cost*	
(4)	Variable option cost*	if additional to recipe costs
(5)	Date when option can first be selected*	i.e. first possible construction date
(6)	Date when option becomes defunct*	recipes in option no longer available
(7)	Time lag*	costs can be incurred before option 'up'

End-use Conversion

Characterizes that transformation of one or more input products (e.g., fuel oil, propane) into an end-use produce (e.g., domestic heat) for the satisfaction of consumer demand. A separate conversion specification must be given for each consuming region. Not to be confused with the term 'conversion' as used to describe the alteration of plant and equipment to use a different fuel. Note, however, that the status of a conversion program, in the latter sense, can be described by an end-use conversion specification in the sense used here.

(1)	Consuming region identification	
(2)	Name of end-use	e.g., domestic heating

(3)	Period data	data can change over time
(a)	Periods in which (b)-(f) apply	
(b)	Input product name	a fuel to produce end-use
(c)	Conversion efficiency	units of fuel per unit end-use
(d)	Maximum relative amount of (b) in (2)*	proxy for current conversion technology
(e)	Minimum relative amount of (b) in (2)*	likewise
(f)	Tax rate ($/unit of (a) in (2))*	per unit of fuel used *in this end-use*

Source Node/Market Node

Characterizes any node (other than a basin) which is the ultimate source or destination of a product. If a source, then a unit purchase cost can be specified; if a destination, then a unit revenue can be specified.

(1) Node name

(2) Node type: source or market

(3) Product name

(4)	Period data	
(a)	Period(s) in which (b)-(d) apply	
(b)	Cost or revenue per unit flow*	cost at source, revenue at market
(c)	Minimum flow*	e.g., contracted minimum imports
(d)	Maximum flow*	e.g., maximum possible exports

(5) Import/Export tax*

Transport Specification

Characterizes the transport network between nodes. Transportation arcs are *not* created automatically between nodes;

product A can move from node X to node Y only if an arc is explicitly specified.

(1)	Origin node	e.g., basin, refinery, etc
(2)	Destination node	e.g., refinery, demand region
(3)	Product name	e.g., natural gas, fuel oil
(4)	Period(s) when (5)-(8) apply to arc 1-2-3	to specify more than one arc at a time
(5)	Transport cost ($/unit of (3))*	cost on unit flow
(6)	Maximum capacity*	e.g., pipeline flow maximum
(7)	Minimum required flow*	can be used to force flow

<u>End-use Demand</u>

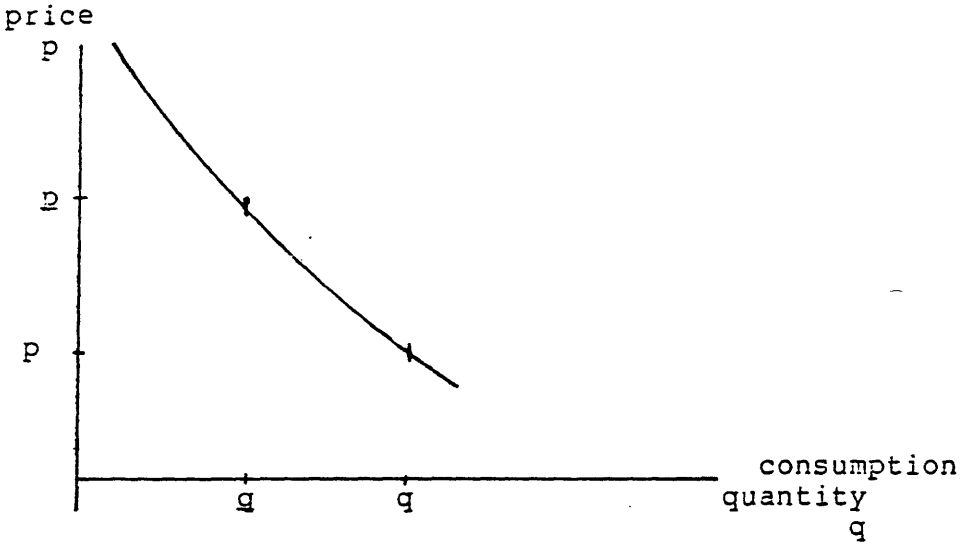
Characterizes the demand for an end-use product in a consuming region. The minimal input is a 'target' demand for the end use. If only the target demand is specified, the effect is to require MOARE to meet the target as a fixed demand. However, the input form is a fairly general one. It will allow the specification of price-elastic demand, in the form of alternative demands (above target) and a price elasticity to be applied at those demand levels to determine consumers' surplus levels.

(1)	Name of consuming region	must match with same in end-use specifications
(2)	Name of end-use or demanded product	same as in end-use specifications
(3)	Period data	
(a)	Period(s) in which (b)-(d) apply	
(b)	Target demand	minimum demand model must satisfy
(c)	Elasticity*	for use in calculating consumer surplus
(d)	Demand alternatives*	necessary if maximizing consumer surplus
(i)	Alternative demand increment	added to target, gives trial demand point

(ii) Consumer surplus	obtained from satisfying increment under stated demand elasticity calculated from other data

We conclude this section by elaborating on how MOARE treats demand endogenously in terms of consumers' surplus. For each end use energy demand category, in each consumption region, and in each time period, a supply-demand surve similar to the one in Figure 4 is used to compute this surplus. According to economic theory, the increase in consumers' surplus associated with an increase in consumption from a reference level q to the level q is given by

$$\Delta cs = \int_{p}^{p} q(p)\,dp$$



Supply-Demand Curve and Consumers' Surplus
Figure 4

where $\underline{p}$ and p are the prices associated with these consumption levels.

In MOARE, the function q(p) is assumed to have the form

$$q(p) = kp^{\varepsilon}$$

where k is a positive constant and ε is a negative constant. The parameter ε is the price elasticity of the demand function, namely,

$$\varepsilon = \frac{\frac{dq(p)}{dp}}{q/p}$$

Thus, in this case,

$$\Delta cs = \begin{cases} \frac{k}{\varepsilon+1} [\underline{p}^{\varepsilon+1} - p^{\varepsilon+1}] & \text{if } \varepsilon \neq -1 \\ k(\ln\underline{p} - \ln p) & \text{if } \varepsilon = -1 \end{cases}$$

In order to incorporate this function into MOARE, we must transform it into a function of q because quantities consumed, rather than prices, correspond directly to decision variables in the T&C model. This is easily accomplished by inverting q(p); namely,

$$p = \left(\frac{q}{k}\right)^{1/\varepsilon}$$

The result is

$$F(q) = \begin{cases} \frac{1}{(\varepsilon+1)k^{1/\varepsilon}} [\underline{q}^{\varepsilon+1/} - q^{\varepsilon+1/\varepsilon}] & \text{if } \varepsilon \neq -1 \\ k(\ln q - \ln\underline{q}) & \text{if } \varepsilon = -1 \end{cases}$$

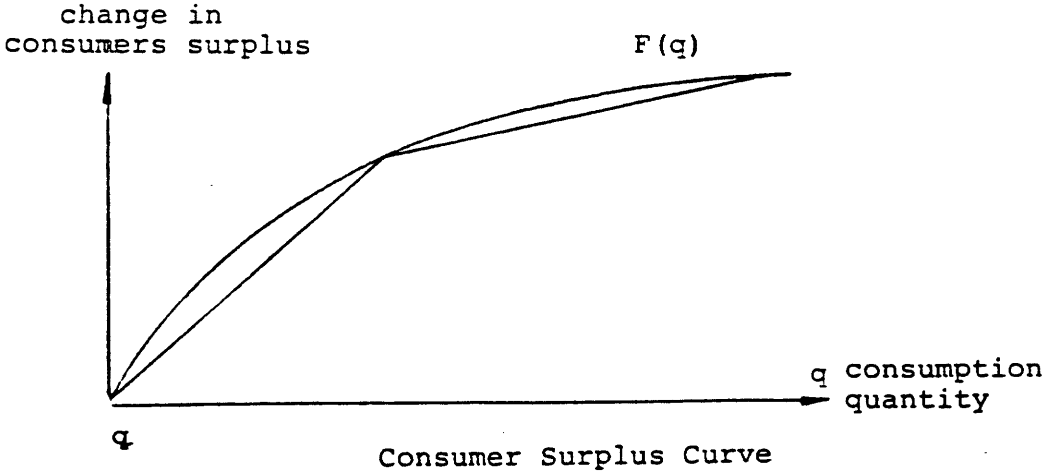
Functions with the form F(q) for each end-use category are the ones being maximized in the T&C model. In order for the maximization to be well behaved, F(q) should be concave (see Figure 4).

In fact, this is the case since for all positive consumption q,

$$\frac{d^2F(q)}{dq^2} = \begin{cases} -\frac{1}{\varepsilon^2 k^{1/\varepsilon}} \; y^{(1-\varepsilon)/\varepsilon} & \text{if } \varepsilon \neq -1 \\ -k/q^2 & \text{if } \varepsilon = -1 \end{cases}$$

is negative for all ε less than zero.

Operationally, F(q) is approximated by a piecewise linear function as shown in Figure 5. The breakpoints in the function are specified by the user as part of the demand data.



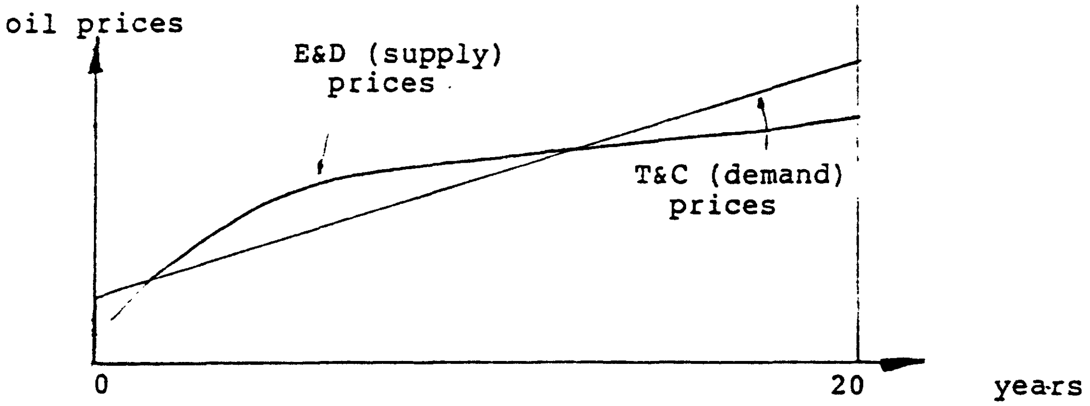
Consumer Surplus Curve
Figure 5

MODEL INTEGRATION

The theoretical concepts underlying E&D and T&C model integration are those concerned with the existence and computation of economic equilibria. See Mehring et al. (1983) for a discussion of these concepts applied to coal supply and demand modeling. Although the theory requires assumptions about supply and demand markets that do not strictly hold in a dynamic environment such as the Argentine energy economy, it does provide guidelines for a meaningful integration of the models. Moreover, we were able to extend price directed decomposition methods for <u>computing</u> economic equilibria to the more complex analyses required by MOARE. Modiano and Shapiro (1980) and Shapiro and White (1982) report on decomposition methods applied to the computation of energy equilibria.

We were obliged to generalize classical equilibrium theory because a fundamental requirement of the study was an in-depth evaluation of potential changes in the infrastructure of the Argentine energy economy. Thus, in the T&C model, we included investment decisions regarding pipeline system additions and expansions, and refinery capacity expansion. In the E&D model we included investment decisions regarding initial field development, the drilling of wells, secondary recovery and gas reinjection. As a result, the models imbedded in MOARE are lumpy and non-convex with complex inter-temporal linkages. For such models it is incorrect to expect or require that supply and demand prices and quantities equilibrate in each time period, and in each location.

Instead, in integrating the E&D and T&C models, we sought to establish the inter-temporal relationship between supply and demand prices depicted in Figure 6. The rationale for this



Intertemporal Supply and Demand Prices
Figure 5

relationship is that producers' (supply) prices must lead demand prices to stimulate production in a growing economy, especially one where we wish to reduce or eliminate imports. In addition, once production from a field has begun, it is difficult, for economic and engineering reasons, to turn it off. For this reason, the relative values of supply and demand prices may decline over time. Empirically, this pattern of prices led MOARE to yield energy strategies with several desirable properties: oil imports were largely eliminated, a smooth pattern of field development was selected for each basin, and significant and increasing fuel substitution (gas for oil) occurred.

More generally, the concept of equilibrium discussed above corresponds to an economic system in which economic agents respond to market signals such as quasi rents by investing and then competing away the rents in order to reach a new economic equilibrium. When a model such as MOARE is used for economic planning, the user becomes the overall economic agent in that he/she plans investments in order to simulate a market economy searching for an equilibrium. In simplistic terms, a divergence between prices and opportunity costs (or shadow prices) is indicative of a disequilbrium, and the user is trying to eliminate such a divergence by selecting the required investments. Thus, if there are divergences and no such investment plans are developed and included in the model, the user cannot expect to equilibrate prices and economic costs. However, given an investment plan to optimize the allocation of resources and sufficient time lags, the user should see a trend toward convergence of prices and economic costs, as illustrated in Figure 6. Indeed, this tendency was observed when the model could choose investment options that optimize the allocation of resources over time.

In the paragraphs that follow, we illustrate the price directed decomposition method by considering a simplified version of the models contained in MOARE. This section concludes with remarks about how the method is actually used to solve the more complex models that MOARE generates from its data base.

Suppose we have a national energy sector supplied by a single field producing a single depletable primary energy resource over a T period planning horizon. Let R denote the quantity of the resource available. Let Π_t for t = 1, ..., T denote a set of prices that the energy sector is offering to pay for the primary energy resource. The producer uses these prices to produce a supply schedule that maximizes the discounted net value of his reserves of the primary energy resource.

As we discussed earlier, the E&D model contains a dynamic programming routine for computing such a supply schedule. In particular, for the given price vector, the E&D model determines the supply schedule or vector.

$$\hat{s} = \begin{pmatrix} \hat{s}_1 \\ \cdot \\ \cdot \\ \hat{s}_t \\ \cdot \\ \cdot \\ \hat{s}_T \end{pmatrix}$$

and an associated scalar quantity $\hat{C}$ equal to the net present cost of the stream of costs incurred over the planning horizon in producing the supply vector $\hat{s}$. These include fixed investment costs, the cost of drilling wells, O&M, secondary recovery, and others. The cost $\hat{C}$ also includes a salvage value or credit for the quantity of resource $R - \sum_{t=1}^{T} \hat{s}_t \geq 0$ that is not developed during the planning horizon. Thus, given the prices $\hat{\Pi}_t$, the quantity

$$\sum_{t=1}^{T} \hat{\Pi}_t \hat{s}_t - \hat{C}$$

is the maximal value to the producer of the resources R. Notes that we have incorporated the discount factor α^{t-1} in the price Π_t, where α is the discount rate.

The T&C model and the E&D model are equilibrated by applying a price-directed decomposition method known in the mathematical programming literature as generalized linear programming. In this method, the T&C model is treated as the master model generating price vectors that are sent to the E&D submodels, one for each field. The submodels respond by sending supply vectors, aggregated into basin supply vectors, to the master model.

To illustrate the procedure, suppose the T&C model previously determined the price vector

$$\Pi_i = \begin{pmatrix} \Pi_{i1} \\ \cdot \\ \cdot \\ \Pi_{it} \\ \cdot \\ \cdot \\ \Pi_{iT} \end{pmatrix} \qquad \text{for } i = 1, \ldots, I$$

and used them in the E&D model to determine the supply vectors for the single field,

$$s_i = \begin{pmatrix} s_{i1} \\ \cdot \\ \cdot \\ s_{it} \\ \cdot \\ \cdot \\ s_{iT} \end{pmatrix} \qquad \text{for } i = 1, \ldots, I$$

each with their associated cost C_i. The supply vectors are put into the T&C model

$$\min \sum_{i=1}^{I} C_i \Theta_i + \sum_{t=1}^{T} \alpha^{t-1} \{c_t x_t + f_t y_t\} \qquad (1a)$$

subject to

$$\text{For } t=1,\ldots,T \quad \begin{cases} \sum_{i=1}^{I} s_{it}\Theta_i + A_t x_t = 0 & (1b) \\ -D_t x_t + Q_t y_t = d_t & (1c) \end{cases}$$

$$\sum_{i=1}^{I} \Theta_i = 1 \qquad (1d)$$

$$\Theta_i \geq 0, \; x_t \geq 0, \; y_t \geq 0$$

Problem (1) is a linear programming model in which the Θ_i are the multiple choice variables that select the particular supply vector from among the I supply vectors available. Strictly speaking, the Θ_i should also be constrained to take on values of either zero or one because the production possibility set and costs of the E&D model are lumpy and non-convex. In other words, a supply vector

$$\sum_{i=1}^{I} \begin{pmatrix} s_{i1} \\ \cdot \\ \cdot \\ s_{it} \\ \cdot \\ \cdot \\ s_{iT} \end{pmatrix} \Theta_i$$

with the cost $\sum_{i=1}^{I} C_i\Theta_i$, where more than one Θ_i is positive, may not correspond to an implementable development plan for the reservoir. However, restricting the Θ_i to zero-one values disrupts the theory underlying the existence of economic equilibria, and the convergence properties of the price-directed decomposition method for computing them. As a practical matter, unique basin strategies (only one Θ_i positive in (1d)) are often selected in the T&C model without this restriction.

The variables x_t in (1) link the supply region to the demand regions in period t, and the variables y_t transform the primary energy resource and distribute the transformed products to the energy end-use markets. For simplicity, we have characterized these markets by the exogenous demands d_t, but it is completely straightforward to let the demands be endogenous by changing the objective function in (1) to one in which we maximize net consumers' surplus. The vectors c_t and f_t reflect the costs of transportation, transformation and distribution of energy to the markets. The matrices A_t, D_t and Q_t reflect the transportation, transformation and distribution activities.

We can illustrate the nature of the decomposition method for computing equilibria by considering an optimal solution to the T&C model (1). Let Π_t denote the optimal LP shadow prices on the supply constraints (1b) and let $\tilde{\gamma}$ denote the optimal LP shadow price on the convexity row (1c). Consider any index j corresponding to a $\tilde{\Theta}_j > 0$ in this solution. By LP duality theory, we have

$$C_j - \sum_{t=1}^{T} \tilde{\Pi}_t s_{jt} - \tilde{\gamma} = 0$$

and for all i

$$C_i - \sum_{t=1}^{T} \tilde{\pi}_t s_{it} - \tilde{\gamma} \geq 0$$

Rearranging terms, we have for all i that

$$\tilde{\gamma} = \sum_{t=1}^{T} \tilde{\Pi}_t s_{jt} - C_j \geq \sum_{t=1}^{T} \tilde{\Pi}_t s_{it} - C_i \qquad (2)$$

Thus, among the I supply schedules provided to the T&C model, the supply schedule j selected by solving that model as a linear program maximizes the net present value to the producer of his reserves.

We can illustrate the nature of the decomposition method for computing equilibria by considering an optimal solution to the T&C model (1).

The decomposition method proceeds by solving the E&D model using the price vector $\tilde{\pi}_t$ to see if a better supply strategy at those prices can be obtained. The result is the new supply vector

$$s_{I+1} = \begin{pmatrix} s_{I+1,1} \\ \cdot \\ \cdot \\ \cdot \\ s_{I+1,T} \end{pmatrix}$$

with net present value

$$V_{I+1} = C_{I+1} - \sum_{t=1}^{T} \tilde{\pi}_t s_{I+1,t}$$

Since the E&D model maximizes the net present value of the reserves, given the prices, we know in advance that $V_{I+1} \geq \tilde{V}$

If $V_{I+1} = \tilde{V}$ then no improvement in the master T&C model is possible, and an equilibrium has been reached. On the other hand, if $V_{I+1} > \tilde{V}$ the new supply strategy s_{I+1} with cost C_{I+1}, can be added to the T&C model, and it can be re-optimized. The positive quantity $V_{I+1} - \tilde{V}$ is an upper bound on the objective function improvement that will result. As a practical matter, if this quantity is small, the procedure can be terminated.

Remarks:

(a) The Θ_i are not constrained to be integer in the simplified T&C model (1) for the reasons cited above. One could impose such a constraint to guarantee that a unique production strategy is selected for each basin. In this case, mixed integer programming (MIP) techniques would be used to select such a strategy. In general, the strategy chosen would not be the one maximizing producers' net revenues, but we would expect it to be close to being maximal. In particular, MIP begins with the optimal LP solution and then does a systematic search for an optimal MIP solution. We can think that the LP solution orders the I supply schedules by decreasing values of

$$\sum_{t=1}^{T} \tilde{\Pi}_t s_{it} - C_i$$

The spirit of MIP is to select a supply schedule that is feasible in the T&C model, and at the same time, that is high up on this ordered list. In other words, if supply schedule k is the one that is optimal in the MIP model, we would expect

$$\tilde{V} = \left(\sum_{t=1}^{I} \tilde{\Pi}_t s_{kt} - c_k \right)$$

to be a relatively small positive number.

(b) We have automated the "tâtonnement" procedure linking the E&D and T&C models. Our experience in running MOARE indicates, however, that human judgement in exercising these models is much more effective. Operationally, this means that the user runs the E&D model under varying price scenarios, some from earlier runs, to develop a representative set of supply schedules for each basin. If necessary, additional supply schedules can be generated based on shadow price information from newly optimized solutions to the T&C model. In most cases, no more than three optimizations of the T&C model were required to determine consistent supply and energy economy strategies.

(c) The successful implementation of MOARE has raised several issues requiring further theoretical research. For example, the economic theory of competitive markets does not extend to the situation, such as the one just described, where one or more of the economic agents is required to make lumpy investments in order to produce or consume commodities. Related to this is the question of how the "tâtonnement" dynamics characterizing theoretical equilibrium calculations can be translated into decisions taken by different economic agents in chronologically ordered time periods. The pattern depicted in Figure 6 is one example of such a translation. A third research issue is how to describe and constrain terminal conditions in a finite horizon economic planning model. We saw in MOARE that the salvage prices for oil and gas paid to the producers for reserves left at the end of the planning horizon had a strong effect on the supply strategies they selected, especially on production during the last five years of the planning horizon.

This list of interesting and challenging research questions could be extended indefinitely. The point is that one should not expect economic theory to provide perfect answers to complex, real life planning problems. Instead, the practitioner must constantly apply and re-apply the Hegelian principle of "thesis, antithesis, synthesis" in searching for meaningful solutions to these problems.

IMPLEMENTATION

The decision was made at the outset of the project to develop MOARE for an IBM mainframe computer. In addition, we intended to use existing software whenever and wherever possible. As we discuss below, this objective was realized in implementing the T&C Model. Since that model constitutes the largest building block in MOARE, and would consume the largest implementation effort, it was very important to employ existing programs as much as possible in building it. Moreover, the T&C Model would surely consume the most computing time and storage space once the system was in operation. Thus, it was doubly important to use computer codes with proven efficiencies.

On the other hand, there are no general purpose, commercially available, dynamic programming packages. Dynamic programming models must be tailored to each new application. Thus, our only choice was to construct the E&D Model, with its imbedded optimization routine, in its entirety. In so doing, we were able to streamline the model's forward dynamic programming calculations by employing efficient list processing routines, and by the incorporation of dominance tests to eliminate non-optimal partial solutions.

In building the T&C Model, we made extensive use of three software systems: ISPF, MIP/370, and LOGS. ISPF is an IBM package that greatly facilitates the implementation of full screen, menu-driven interfaces for viewing and changing data. In effect, ISPF functions were used to construct a maintenance system for the Argentine national energy data base imbedded in MOARE. The system is a general one applicable to energy data base management and planning in any developing country with oil and/or gas resources.

MIP/370 is an IBM package for optimizing mixed integer programming models. It is, in the opinion of the authors, the most powerful system available for that purpose. The primary difficulty with using MIP/370, as with all such packages, however, is generating the model to be solved.

LOGS is a proprietary system filling the gap between ISPF and MIP/370. It can be viewed as a wrapper for MIP/370 which can be integrated with ISPF to produce tailored user interfaces. For generating models, LOGS contains a Descriptive Modeling Language that corresponds closely to the analysts's intuitive understanding of his/her planning problem. Starting with a problem specification in this language, LOGS' modeling programs automatically generate a linear or mixed integer programming model in the format accepted by MIP/370. LOGS also contains programs for pre-processing the output of MIP/370 to create, in effect, pre management reports. The reader is referred to Brown, Northup and Shapiro (1984) for more details about LOGS.

RESULTS

Much of the data describing the Argentine energy economy, and the long term energy planning strategies produced by MOARE, are proprietary information. For this reason, we are unable to provide much detail in this report about results. Instead, we provide the reader with representative model outputs which we discuss in a qualitative manner.

Exploration and Development Model

Table 1 shows the output of the E&D model applied to a single field called Reservoir L. This field has combined oil and gas reserves of 83.72; this figure is the sum of gas measured in billions of cubic meters and oil measured in millions of cubic meters. The production in a given year of each primary fuel is computed by multiplying the total production by the appropriate fuel fraction.

Table 2 gives the unit prices in dollars to be paid to the producer for each cubic meter of oil, in each of the 8 periods spanned by the T&C Model's horizon. The yearly prices required by the E&D Model are then calculated by interpolation. For the run reported in Table 1, the prices of gas were taken to be zero, a typical solution. The salvage prices paid for oil and gas left in the ground at the end of the twenty year horizon are $40 per cubic meter of oil and $20 per thousand cubic meters of gas. The low salvage price for oil was used in this run to encourage drilling at the end of the planning horizon. (As we can see from Table 1, this had the desired effect, since drilling took place in periods 13 through 18.)

At the start of the twenty year planning horizon, 23 wells have just been drilled, but production has not begun. The E&D model elects to drill 25 wells in year 1, 25 wells in year 2, 5 in year 3, 5 in year 6, 5 in year 7, 2 in year 8, 5 in year 13, 2 in year 14, 5 in year 15, 2 in year 16, 5 in year 17, and 5 in year 18. (Part of the input data is a list specifying "lumps" in which new wells can be drilled at the reservoir in any year. At this reservoir 0, 2, 5, 10, or 25 wells could be drilled in any year.)

A reasonable effort went into streamlining the dynamic programming calculations. As a result, the computational effort required to develop a new set of basin production strategies was less than that required to run the T&C Model. A typical basin consisting of 30 fields can be optimized in approximately 2 minutes of CPU time on an IBM 4381, including a gas re-injection calculation for the basin as a whole that is performed after development strategies have been calculated for the individual fields. Thus, the time to compute an optimal strategy for a single field is on the order of 2 to 10 seconds of CPU time, depending on the complexity of the geological and cost data describing the field, and the economic ambiguities inherent in

```
   OPTIMAL PRODUCTION STRATEGY T7 FOR RESERVOIR
====================================================================
                    CUM         CUM        CUM
YEAR    PROD'N    PROD'N      WELLS       NPV          COST
====================================================================
  0                 0.00        23
  1       1.45      1.45        43         -68         79.2
  2       2.41      3.87        73         -90         79.3
  3       2.91      6.79        73         -47         15.5
  4       2.90      9.69                    17          3.2
  5       2.79     12.48                    69          3.2
  6       2.77     15.26        83         111         18.7
  7       2.82     18.09        88         155         18.9
  8       2.81     20.90        90         199          9.8
  9       2.72     23.63                   243          3.7
 10       2.60     26.24                   285          3.7
 11       2.49     28.74                   321          3.7
 12       2.38     31.12                   354          3.7
 13       2.34     33.47        95         382         19.2
 14       2.32     35.79        97         400         57.0
 15       2.29     38.09       102         428         21.8
 16       2.26     40.35       104         462         12.8
 17       2.21     42.57       109         494         22.2
 18       2.20     44.77       114         526         22.5
 19       2.12     46.90                   559          7.3
 20       2.01     48.91                   590          7.3
--------------------------------------------------------------------
SALVAGE            34.80                   125
====================================================================
TOTAL              83.72                   715        436.6
====================================================================
```

E & D Model Applied to a Reservoir

Table 1

Period	oil price ($/m^3)	gas price ($/m^{3}x10^3)
1	65	0
2	70	0
3	75	0
4	95	0
5	100	0
6	120	0
7	125	0
8	296	0
salvage	40	20

Oil and Gas Prices

Table 2

the oil and gas price vectors.

Transformation and Consumption Model

The full T&C Model, with all important flows and processes, and with eight-periods representing twenty years, was a large mixed-integer programming model. Depending upon the data it could typically have over 9000 variables, 300 of which were integer, and 4000 constraints. Optimization times ranged from 20 to 30 cpu minutes on an IBM 4381 depending on the complexity of the model.

Over the course of the project we solved the model under various scenarios scores of times. The model generation and optimization routines behaved quite consistently. It was sometimes possible to prove optimality; more often, the MIP search was terminated with a solution that was very close to the optimal bound.

The output from a model of this size is obviously voluminous. For the study, results were presented in five reports showing:

1) flows and material balances along all arcs
2) process activities at all nodes, and the selection of investment options
3) end-use demand levels, by demand region
4) basin strategy selection and well-head shadow prices
5) tax revenues generated by arc flows, based on user-specified tax rates and unit values for energy products

No single run was decisive for the study. Instead, we made suites of runs directed at a particular issue, such as assessing the value of export options. Within a suite, the model might be run many times. Initial runs indicated the most important options and activities to investigate in subsequent runs, and often suggested areas where the data had to be further refined or extended.

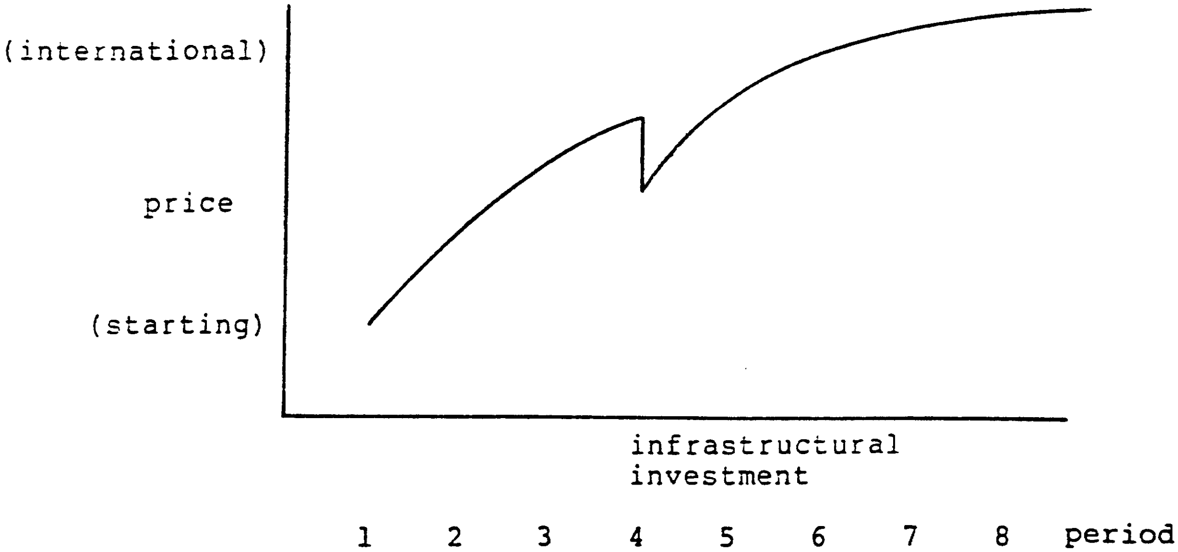
The model was first verified and calibrated by making a series of base case runs intended to represent conditions under the current infrastructure, with a reasonable rate of growth in end-use demands. No investment options were included except those, such as methanol and urea plants, which were already authorized in current development plans.

The base case matched current conditions quite well in terms of overall energy flows, balances, and the relative use of different fuels, although particular flows did not always match exactly. This is a familiar problem when trying to mimic current operations with an optimization model. It arises from the fact that current operations are seldom optimal and may be far from it, so that the model must be severely constrained if it is to display current patterns.

Following validation, suites of runs were carried out to investigate a variety of issues, some quite detailed, some at a fairly general or "strategic" level. These included, for example:

- location ad timing of major new oil and gas pipelines to support additional supplies and demands, under a variety of demand scenarios.
- amounts of gas to be exported to neighboring countries, desired selling prices, and the routes for pipelines to Argentinea's borders.
- synthetic fuel plants proved an economically attractive use of excess gas to replace oil-based fuels
- need for added gas processing capacity

Starting at a rather low value on the first model period, the equilibrium price for oil tended to rise to the international price by the last (8th) model period (i.e. over the course of 15-20 years). The upward trend was a general pattern; in some runs, the new investments coming on stream in intermediate periods could change the supply picture and temporarily interrupt the trend.



Equilibrium Oil Prices
Figure 6

Overall, MOARE indicated that several measures could be taken to use excess gas and reduce consumption of relatively scarce crude oil. However, under credible scenarios for future demand, infrastructure investment, and supply, the results still showed an excess of gas at the well-head in certain fields, in most or all periods of the model. Ultimately this reflects the irreducible effects of joint oil and gas production, as captured by the E&D Model.

CONCLUSIONS AND AREAS OF FUTURE DEVELOPMENT

We have reported on the successful design and implementation of MOARE. The system has been installed in Argentina, and extensive training of a user's group has been completed. MOARE is now in active use by members of this group.

Areas of possible future development of MOARE can be usefully divided into five categories

- data extensions
- structural model extensions
- extensions of scope
- addition of macroeconomic linkages
- implementation extensions

(i) Data Extensions.

We emphasize that the accuracy of the results produced by MOARE is limited by the quality of the data inputs. We believe these inputs were for the most part satisfactory for the project just completed. The national energy data base incorporated in MOARE will undoubtedly be updated and improved as planners use it over the coming months to study a range of energy planning questions.

In particular, the accuracy of certain cost data, such as capital and labor costs, could be improved by adjusting them to better reflect imperfections in existing financial relations between Argentina and the rest of the world. The use of spreadsheet programs integrated to MOARE's input routines would permit these costs to be built up in a flexible manner from more basic data describing costs in Argentina pesos, exchange rates, unofficial rates, and so on.

(ii) Structural Model Extensions.

This category refers to important structures in the Argentine energy economy that were omitted from the models produced thus far by MOARE, but which could be added easily if the data were available. Included are constraints and variables, by time period and type, describing more fully the effects that limitations on capital investments would have on changes in the infrastructure of the energy economy. Another example is model extensions to describe the final distribution of energy products in end-use demand markets and their costs.

(iii) Extensions of Scope.

The models generated by MOARE are intended to address strategic energy planning questions. Because they treat data in an aggregate manner and consider the Argentine energy economy in its entirety, the models are not appropriate for analyzing more detailed, shorter term planning questions. An example of such a problem is how to optimally construct an oil and gas pipeline system to service new fields in a producing basin. Another example is the problem of managing short term inventories of natural gas and LNG to meet peak demands during the winter season. Mathematical programming models would also be effective in analyzing these problems, but the model generation capabilities in MOARE are not well suited to producing such models. Moreover, the models would require data not currently included in MOARE's data bases.

(iv) Addition of Macroeconomic Linkages.

The original request for proposal that led to the development of MOARE explicitly stated that the model be independent of linkages to other sectors of the Argentine economy, and the economy as a whole. Given the volatile nature of the Argentine economy, this was a perfectly valid proscription. Nevertheless, several policy issues that have arisen as the result of MOARE analyses indicate the need to model some of these linkages. Transfer payments from the energy sector to other sectors of the economy, such as the agricultural sector, are an example of policy variables that would be useful to model and study. In this case, MOARE's model generation capabilities would easily allow an analyst to construct the necessary model extensions.

More generally, integration of MOARE with models of other economic sectors, or with a macroeconomic model of the entire Argentine economy, could be carried out by price or quantity directed decomposition methods similar to the method employed in integrating the E&D and T&C models. The major challenge of such integrations, however, would be to identify meaningful economic relationships, and to gather the data needed to numerically estimate these relationships. At the macroeconomic level, modeling even the most stable economies is a difficult and questionable task.

(v) Implementation Extensions.

The hardware and software realization of MOARE has proven very successful. Optimal strategies for the Argentine energy economy over the next twenty years can be predictably computed in 20 to 30 minutes of CPU time. Nevertheless, as the result of extensive experience with the system, and also as the result of recent changes in computer technology, we have identified several ways in which MOARE's implementation could be improved. Potential improvements in software include

- sensitivity analysis of optimal T&C strategies
- programs to save and restore optimal T&C strategies
- interactive graphics programs for reporting results
- incorporation of spreadsheet programs for preliminary data analysis and interactive solution processing

Sensitivity analysis would permit planners to determine the effect of changes in data on optimal T&C strategies. This would be useful, for example, in studying oil/gas substitution effects, or the viability of an export contract. Save and restore options would be extremely useful in those instances when the T&C model is re-run with only a few data changes.

Graphics programs could be readily integrated with MOARE's existing output reports to produce a variety of useful pictorial displays. Finally, the incorporation of spreadsheet programs would permit input and output data to be manipulated in a number of interesting ways. For instance, a spreadsheet could be used to conduct post-optimal analyses of the allocation of fixed costs (and joint production costs) to energy flows through the economy.

The recent advent of powerful minicomputers and microcomputers is a hardware development of primary importance. Using the telephone lines, the mainframe in Buenos Aires where MOARE now resides could be linked to a number of IBM AT/370 microprocessors throughout Argentina. An analyst using a micro would access MOARE to acquire data, or to make large optimization runs. Locally, he/she would be able to view and change data, view results, generate models, and in some instances, optimize the models. Thus, energy planners throughout Argentina could be analyzing problems and devloping plans based on common data and modeling methodologies. The network would also permit these planners to communicate electronically with each other regarding their energy problems and plans.

2

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