STRESSES AND DISPLACEMENTS IN SEMI-INFINITE MEDIA

by

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of the requirements for the degree of
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ABSTRACT

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Submitted to the Department of Civil Engineering on
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This thesis presents a general computer oriented
method for the determination of the components of the stress
tensor and the displacement vector, the principal stresses,
and the principal directions of stress at any point of a
semi-infinite elastic medium subjected to static normal and
shearing surface loads.

This method has been programmed in FORTRAN IV for an
IBM / 360 digital computer and the program, with slight
improvements, will also provide the solution for the
homogeneous linear-viscoelastic half-space and static
loadings.

Thesis Supervisor: Fred Moavenzadeh
Title: Associate Professor
ACKNOWLEDGMENT

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. INTRODUCTION</td>
<td>5</td>
</tr>
<tr>
<td>2. LOAD AND LOADED AREA</td>
<td>11</td>
</tr>
<tr>
<td>3. COMPUTATION OF STRESSES AND DISPLACEMENT MENTS</td>
<td>16</td>
</tr>
<tr>
<td>4. IMPROVEMENT OF ACCURACY</td>
<td>20</td>
</tr>
<tr>
<td>5. COORDINATE SYSTEMS</td>
<td>43</td>
</tr>
<tr>
<td>6. PRINCIPAL STRESSES</td>
<td>49</td>
</tr>
<tr>
<td>7. PRINCIPAL DIRECTIONS</td>
<td>52</td>
</tr>
<tr>
<td>8. COMPUTER PROGRAM</td>
<td>56</td>
</tr>
<tr>
<td>9. RESULTS</td>
<td>70</td>
</tr>
<tr>
<td>10. CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK.</td>
<td>81</td>
</tr>
</tbody>
</table>

**APPENDIX**

A. DEFINITION OF SYMBOLS                                               | 82   |
B. LIST OF FIGURES                                                      | 85   |
C. LIST OF TABLES                                                       | 86   |
D. STRESS AND DISPLACEMENT COMPONENTS FOR UNIFORM NORMAL AND SHEARING CIRCULAR LOAD | 87   |
E. COMPUTER PLOTTING OF VERTICAL STRESSES                               | 125  |
F. LIST OF PROGRAMS                                                     | 129  |
CHAPTER 1

INTRODUCTION

Analysis of stresses and displacements in semi-infinite elastic media is of special interest in several problems of soil mechanics, such as the design of foundations for structures, highways and machinery. In such cases, a load is distributed over a relatively small area of a body which is only limited in one direction by a plane surface, it is generally assumed that this body is weightless, homogeneous, isotropic and its behavior is linear elastic, Figs. 1, 2, and 3.

This is a restricted case of the more general problem of analysis of stresses and displacements in layered, non-homogeneous, non-isotropic semi-infinite media with time-dependent properties, subjected to arbitrarily distributed time-varying moving normal and shearing surface loads and subjected to different interphase conditions between the layers, Fig. 4.

Most methods of analysis (see references pages 123 and 124) have attempted to present the solutions of the homogeneous half-space in the closed form, and due to the mathematical complications in obtaining this type of solution for arbitrary load distributions, most of them have been limited to axisymmetric cases whereby the load is
distributed over a circle. Closed form solutions for distributed loads over asymmetrical shapes generally involve elliptic integral expressions which have limited their application.

The solution of the basic problem of a normal point load on an elastic homogeneous half-space was obtained by Boussinesq in 1885. Terazawa in 1916 developed the solution for the stresses and displacements at any point in a semi-infinite elastic body under distributed normal loads. This solution is in the form of infinite integrals involving Bessel-Fourier expansions and is applicable to any distributed axi-symmetric loading.

Love solved the same problem through the use of potential functions in 1929. His work was summarized and extended by Fergus and Miner in 1955. Love's work was used to solve the problem of a uniform load distributed over an elliptical area by Deresiewicz in 1959.

A fairly extensive tabulation of stresses, strains, and deflections has been given, for arbitrary Poisson's ratio by Ahlvin and Ulery in 1962.

The most practical method of solution up to date is based on the influence charts developed by Newmark (10 and 11) which are general and have sufficient accuracy for most of the engineering applications but are in disadvantage of requiring excessive time for each point of
the semi-infinite medium. Design is thus somewhat limited by the lack of a flexible and accurate method that enables the design engineer to describe the characteristics of the problem (geometry of the loaded area, load distribution, properties of the materials), to obtain in real time the desired distributions of stresses and displacements at any point of the body, to modify in turn his original conception, and to introduce new data and thus follow the steps of successive approximations of the design process. These can be summarized by stating that the most desired characteristics of any method of solution for an engineering problem are:

a. Clear, accurate and simple description of the data.

b. Great flexibility of the method itself to accept and produce different types of information corresponding to different situations.

c. Known and adjustable limits of accuracy.

The purpose of this study is to present a general computer oriented method for the determination of the components of the stress tensor and the displacement vector, the principal stresses and the principal directions of stress at any point of a semi-infinite elastic medium subjected to static normal and shearing surface loads.
Shearing load Distribution

Normal load Distribution

\[ p(x,y) \]

\[ q(x,y) \]

\[ \rho(x,y,z) \]

\[ \rho(x',y',z') \]

**Unknowns**

\[
\begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z
\end{bmatrix}
\begin{bmatrix}
u \\ v \\ w
\end{bmatrix}
= \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix}
= l_1, m_1, n_1
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix}
= l_2, m_2, n_2
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix}
= l_3, m_3, n_3

**Fig. 5**
CHAPTER 2

LOAD AND LOADED AREA

To provide the method with a fair amount of generality, two basic types of load distributions are considered and superimposed:

I. Distribution of normal surface stresses $p(x,y)$

II. Distribution of shearing surface stresses $q(x,y)$

In the case of the shearing surface stresses it is assumed that they are parallel to the X axis. This assumption does not imply any limitation for the following reasons:

1. The X axis can be selected arbitrarily.
2. The expressions of the stress components can be modified if it is desired to have the shearing stresses parallel to the Y axis.
3. In case of stresses parallel to another axis in the plane $X,Y$ they can always be decomposed in $X$ and $Y$ components and their total effect can be calculated using the principle of superposition.

2a. LOADED AREA

The loaded area shall be enclosed within a grid of $M \times M$ square elements, as shown in Figure 6. Each grid element will be characterized by two numbers:

\[ i = \text{row number} \]

\[ j = \text{column number} \]
2b. **LOAD FUNCTION**

The load function is defined by two square matrices of order M, \( P(I,J) \) and \( Q(I,J) \), associated with the grid.

A. **Normal load function**

Each element \( ij \) of the first matrix will contain a number \( p_{ij} \) equal to the average intensity of the normal surface stresses applied on the element of area \( A_{ij} \) of the grid (Fig. 7).

B. **Shearing load function**

Similarly, each element \( ij \) of the second matrix will contain a number \( q_{ij} \) equal to the average intensity of the shearing surface stresses applied on the element of area \( A_{ij} \) of the grid (Fig. 8).

The elements of normal and shearing loads acting on the grid shall be treated in two different ways:

a. As elements of distributed loads of intensities \( p_{ij} \) and \( q_{ij} \) respectively.

b. As equivalent point loads applied at the center of each element \( ij \), of intensities

\[
\begin{align*}
P_{ij} &= p_{ij} \times A_{ij} \quad (1) \\
Q_{ij} &= q_{ij} \times A_{ij} \quad (2)
\end{align*}
\]
This discrimination shall be done according to the distance from the point P(x,y,z) of the semi-infinite medium to the center of the element ij of the grid. This distance is a measure of the error incurred in considering distributed loads as point loads (see article 4e).
CHAPTER 3

COMPUTATION OF STRESSES AND DISPLACEMENTS

Due to the assumption of linear elastic behavior it is possible to use the following principle:

3a. PRINCIPLE OF SUPERPOSITION

The stresses and displacements that occur at a given point $P(x,y,z)$ of the semi-infinite medium are equal to the sum of the stresses and displacements produced by each individual element of the load matrices $P(I,J)$ and $Q(I,J)$.

Let as before:

- $p_{ij}$ = average normal surface stress on element $ij$
- $q_{ij}$ = average shearing surface stress on element $ij$
- $G_{kl}$ = component of stress tensor at point $P(x,y,z)$ due to total load.

where $k = x,y,z$

$\Delta G_{kl} = $ component of stress tensor at point $P(x,y,z)$ due to load element $ij$

$\Delta u_k = $ component of displacement vector at point $P(x,y,z)$ due to load element $ij$. 

- 16 -
Then by the principle of superposition:

\[ \sigma_{kl} = \left[ \sum_{i=1}^{M} \delta_{ikl} \right]_{P} + \left[ \sum_{j=1}^{M} \delta_{jkl} \right]_{Q} \tag{1} \]

\[ u_{k} = \left[ \sum_{i=1}^{M} \delta_{ik} \right]_{P} + \left[ \sum_{j=1}^{M} \delta_{jk} \right]_{Q} \tag{2} \]

M = order of load matrices P(I,J) and Q(I,J).

3b. MATHEMATICAL PROCEDURE

For the computation of the components \( \sigma_{kl} \) of the stress tensor and \( u_{k} \) of the displacement vector, the elements of distributed load \( p_{ij} \) and \( q_{ij} \) are substituted by the equivalent point loads given by expressions (1) and (2) page 13, and the numerical values of \( \delta_{ikl} \) and \( \delta_{ik} \) are then obtained by Boussinesq's expressions for normal and shearing point loads (Appendix D, formulas (1) to (12)).

This representation of the load function by a finite number of point loads has inherent the following sources of inaccuracies:

a. Point loads produce infinite discontinuities in the stress and displacement distributions at the point of application.

b. The distributions of the stresses and displacements for the distributed loads and the equivalent point
loads vary significantly in the vicinity of the point of application of the load; this variation gradually reduces as the distance to the point of application is increased. Fig. 9 shows a comparison of the distributions of $\sigma_z$ for a point load of radius $R = 1$ at a relative depth $Z/R = 1$. The rate of decrease of this difference varies with the intensity and the radius assigned to the equivalent distributed load.

c. As a consequence of b, the degree of accuracy will vary with the size of the grid elements or equivalently, with the number $M$ of subdivisions of the loaded area and with the distance from the point of the semi-infinite medium. This point is discussed further in article 4e of the next chapter.
CHAPTER 4

IMPROVEMENT OF ACCURACY

The evaluation and the improvement of the accuracy of this method requires further analysis of points (a), (b), and (c) of article 3b.

4a. ERROR VOLUME ASSOCIATED TO A LOAD ELEMENT

It was mentioned in section 2b, page 15, that a measure of the difference in the distributions of stresses and displacements corresponding to point and distributed loads is the distance from the point of the semi-infinite medium that is being considered to the load element ij. Assuming that the magnitude if the accepted error is $\frac{\varepsilon}{\varepsilon_{\text{max}}}$, it is possible to define for each load element ij a volume of the half-space outside of which the error is less than $|\frac{\varepsilon}{\varepsilon_{\text{max}}}|$. This volume shall be designed as the "error volume" associated to the load element ij. Where the point $P(x,y,z)$ of the half space is within this error volume, the element ij is treated as a distributed load. The actual shape and dimensions of this volume will depend on items (a), (b), and (c) of the previous section. After the computation of the preliminary results of article 4e, it was found convenient and simple to adopt as an error volume a square prism of side $2\rho$ and depth ZUM.
Error Volume of Load Element $ij$
FIG. 11. Error volume of total load.
4b. ERROR VOLUME ASSOCIATED TO THE TOTAL LOAD

The error volume as corresponding to the total load is defined as the integral of the error volumes of all the load elements $ij$ as shown in Fig. 11, and its dimensions are $2\text{XLIM}$, $2\text{YLIM}$ and $\text{ZLIM}$.

4c. NUMBER OF DISTRIBUTED LOAD ELEMENTS

The procedure to determine these elements is very simple, and is used only when the point $P(x,y,z)$ of the semi-infinite medium is within the error volume associated with the total load.

Drawing the error volume of side $2\rho$ and depth $\text{ZLIM}$ with its longitudinal axis passing through $P(x,y,z)$ as shown in Fig. 12, the intersection with the grid encloses the elements to be counted (only those whose centers are within the square of side $2\rho$).
4d. **PROCEDURE TO HANDLE THE DISTRIBUTED LOAD ELEMENTS**

Assuming that the load elements shown in Fig. 12 are those to be treated as distributed loads, when considering them one by one with their actual shape, the stress and displacement components $4\sigma_{ij}$ and $4\epsilon_{ij}$ have, for arbitrary points of the semi-infinite medium, complicated expressions generally involving elliptic integrals. Even the expressions for uniform circular loads and arbitrary points of the half space involve elliptic integrals, but the latter can be reduced to rather simple forms for points on the vertical axis passing through the center of the circle. These expressions have been obtained by integration of expressions (1) to (12) in Appendix D.

This point suggests the following approximation, which has been programmed and used successfully in this study.

Considering the results obtained in article 3c, in order to maintain the error to $\varepsilon_{max} = \pm 1\%$, the length of the side of the error volume must be:

$$2 \rho = 0.4 \times R$$

$R = \text{maximum diameter of the loaded area.}$
The size of the grid divisions must be:

\[ \Delta \rho = 0.5 \times \rho \]

The number of grid elements contained in the side of a square of side length \( 2\rho \) is:

\[ N = \frac{2\rho}{\Delta \rho} = \frac{0.4 \times R}{0.5 \times \rho} = \frac{0.4 \times 2}{0.5 \times 0.4 \times R} = \frac{0.8}{0.2} = 4 \]

These results are illustrated in Fig. 13.

As it will be explained in Chapter 5, the center of coordinates will be located at the center of the grid of 20 x 20 elements and position of each grid element is specified by the coordinates of its center, it was found more convenient to assign to the side of the error volume a length equal to 5 grid elements as shown in Fig. 14.
Square ABCD = error volume associated with element ij. This square is divided into three concentric square rings as shown in Fig. 15 such that:
\[ R_1 = \frac{\Delta p}{2} = \text{radius of ring 1} \]
\[ R_2 = \frac{3 \Delta p}{2} = \text{radius of ring 2} \]
\[ R_3 = \frac{5 \Delta p}{2} = p = \text{radius of ring 3} \]

\[ K_{1\text{MAX}} = 1 = \text{number of elements in ring 1} \]
\[ K_{2\text{MAX}} = 8 = \text{number of elements in ring 2} \]
\[ K_{3\text{MAX}} = 16 = \text{number of elements in ring 3} \]

\[ K_1 = \text{number of elements enclosed by ring 1} \]
\[ \text{in a particular case} \quad (0 \leq K_1 \leq K_{1\text{MAX}}) \]

\[ K_2 = \text{number of elements enclosed by ring 2} \]
\[ \text{in a particular case} \quad (0 \leq K_2 \leq K_{2\text{MAX}}) \]

\[ K_3 = \text{number of elements enclosed by ring 3} \]
\[ \text{in a particular case} \quad (0 \leq K_3 \leq K_{3\text{MAX}}) \]

The approximation can be described as follows:

1. Check if point \( P(x,y,z) \) is within error volume corresponding to the whole load function.
   If NO use expressions for point loads (expressions (3) to (20)).
   If YES use algorithm 2.

2. Draw prism of radius \( R_1 = p \) with longitudinal axis passing through \( P(x,y,z) \).

3. Count the numbers \( K_1, K_2 \) and \( K_3 \) of load elements that fall within rings \( R_1, R_2 \) and \( R_3 \) respectively.
4. Calculate the averages

\[
\rho_{av1} = \frac{\sum_{i=1}^{k1} \rho_{ij}}{k1}
\]

\[
\rho_{av2} = \frac{\sum_{i=1}^{k2} \rho_{ij}}{k2}
\]

\[
\rho_{av3} = \frac{\sum_{i=1}^{k3} \rho_{ij}}{k3}
\]

\[
q_{av1} = \frac{\sum_{i=1}^{k1} q_{ij}}{k1}
\]

\[
q_{av2} = \frac{\sum_{i=1}^{k2} q_{ij}}{k2}
\]

\[
q_{av3} = \frac{\sum_{i=1}^{k3} q_{ij}}{k3}
\]

5. Calculate the fractions $K1/K1\text{MAX}$, $K2/K2\text{MAX}$, $K3/K3\text{MAX}$.

6. Place equivalent uniform normal and shearing circular ring loads of intensities:

\[
\rho_{av1} \quad \rho_{av2} \quad \rho_{av3}
\]

\[
q_{av1} \quad q_{av2} \quad q_{av3}
\]
on top of the square rings 1, 2, and 3 respectively.

7. Compute $\Delta G_{kl}$ and $\Delta u_k$ using formulas for points on the vertical axis of circular loads (Appendix D).

8. Multiply the results by the fractions obtained in (5) respectively.

9. Add these results to those corresponding to the other elements of the grid considered as point loads.
This method of handling the elements within the error volume involves two types of approximations:

a. It considers a number $K$ of square loads as a fraction $K/K_{\text{MAX}}$ of a circular load.

b. It reduces the actual distribution of surface stresses $p_{ij}$ and $q_{ij}$ within the three square rings to the six average intensities shown in page 29.

The determination of the errors introduced by these two approximations leads to the following considerations:

With respect to approximation (1), the diagram, Fig. 16 (see also reference 12, page 124), shows the distribution of vertical stresses $\sigma_z$ for equivalent circular and square loads (for the same loaded area and the same load intensity) as a function of the depth $Z$. For $X=0$ and $Z=0$ they are coincident and for depths $Z/R$ between 0 and 1 the percentage error is less than 0.01%.

With respect to approximation (2), the results obtained by this method have been compared to those obtained with Newmark's influence charts for the same points of the error volume of the total load, and an accuracy of the order of 99% is obtained for well behaved load functions.
In order to determine the influence of the size of the grid elements on the accuracy of the procedure and in order to determine the radius of the error volume corresponding to one grid element, the following procedure was developed:

a. Consider, for the purpose of this discussion, the loaded area as a circle of radius $R$, with a uniform circular load of intensity $\rho = 1$.

b. Calculate the stress and displacement components for points on the $Z$ axis (vertical axis passing through the center) using formulas obtained in Appendix D.

c. Draw a circle of radius $\rho$ concentric to the first, and consider the load on top of this circle as uniformly distributed.

d. Divide the annular ring of width $(R-\rho)$ into $m$ concentric rings of width $\Delta \rho$, where $\Delta \rho$ will be equivalent in this case to the length of the side of one square element of the grid in a real case.

e. Divide the annular ring $(R-\rho)$ into $n$ equal sectors, $n$ being calculated as follows:

$$2\pi \rho = n \times \Delta \rho \quad \therefore \quad n = \frac{2\pi \rho}{\Delta \rho}$$
f. Place on the centers of each of the m x n elements of area $A_{ij}$ an equivalent point load of intensity

$$P_{ij} = p_{ij} \times A_{ij} = 1 \times A_{ij}$$

g. Calculate the components of the stress tensor and the displacement vector for the same points of the semi-infinite medium as was done in (b) above.

h. Calculate the difference between results of h and b, and calculate the percentage of error based on values in part (b) above.

i. Repeat this process for different values of $\Delta p/p$ and $p/R$.

j. Define a value of the accepted error, $|\varepsilon_{\max} \%|$ and select the number m that produces this error. Take this value of m one half of the order of the load matrices or equivalently as one half of the number of divisions of the grid sides.

This process was carried out only for the stress component $\sigma_z$ and the following data was used:

$$R = 1$$

$$\frac{p}{R} = 0.05 \text{ to } 0.20 \text{ with increments of } 0.025$$

$$\Delta \frac{p}{p} = 0.1 \text{ to } 0.5 \text{ with increments of } 0.1$$
Then for any value of $\rho$:

$$\frac{\Delta \rho_i}{\rho_i} = 0.1$$
$$\frac{\Delta \rho_i}{\rho_i} = 0.2$$
$$\frac{\Delta \rho_i}{\rho_i} = 0.5$$

For each case $Z/R$ was varied from 0 to 4 with intervals $\Delta Z/R = 0.1$

This process was programmed for an IBM 360 computer, the flow chart and the program are shown in pages 36 and 37 respectively.

Table II (page 39) shows in a compact form the final selection of $M$ (order of the load matrices).

Adopting a magnitude of the error

$$\left| t \varepsilon_{\text{max}} \% \right| = 1 \%$$

the value of $M$ that produces the closest error interval is (see Table II):

$$M = 2m = 20$$

and for $M = 20$

$$\rho = 0.2 \times R \quad \text{Radius of error volume}$$
$$\Delta \rho = 0.5 \times \quad \text{Magnitude of a grid element.}$$
<table>
<thead>
<tr>
<th>DESCRIPTION</th>
<th>SYMBOL</th>
<th>SYMBOL IN PROGRAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intensity of distributed load</td>
<td>q</td>
<td>Q</td>
</tr>
<tr>
<td>Radius of loaded area</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>Radius of central element</td>
<td>( p_j )</td>
<td>( \text{RHO}(J) )</td>
</tr>
<tr>
<td>Width of the concentric rings</td>
<td>( \Delta p_{jk} )</td>
<td>( \text{DRHO}(JK) )</td>
</tr>
<tr>
<td>Coefficient of width of the rings</td>
<td>a (_j)</td>
<td>( A(J) )</td>
</tr>
<tr>
<td>Number of divisions in each ring</td>
<td>n (_j)</td>
<td>( \text{FN}(J) )</td>
</tr>
<tr>
<td>Number of rings</td>
<td>m</td>
<td>M</td>
</tr>
<tr>
<td>Depth</td>
<td>z</td>
<td>Z</td>
</tr>
<tr>
<td>Vertical component of stress for each depth, corresponding to uniform load distribution</td>
<td>( g_{zi} )</td>
<td>( \text{SIG}(I) )</td>
</tr>
<tr>
<td>Idem for mixed distribution</td>
<td>( g )</td>
<td>S</td>
</tr>
<tr>
<td>Error</td>
<td>( \epsilon )</td>
<td>( \text{EPS.} )</td>
</tr>
<tr>
<td>Depth subscript</td>
<td>i</td>
<td>I</td>
</tr>
<tr>
<td>Center element radius subscript</td>
<td>j</td>
<td>J</td>
</tr>
<tr>
<td>Ring width and width coeff. subscript</td>
<td>k</td>
<td>K</td>
</tr>
<tr>
<td>Ring subscript</td>
<td>l</td>
<td>L</td>
</tr>
<tr>
<td>Distance from point loads to center of circle</td>
<td>d</td>
<td>D</td>
</tr>
<tr>
<td>Int. radius of the rings</td>
<td>r</td>
<td>RI</td>
</tr>
</tbody>
</table>
STRESS DISTRIBUTION STUDY.

DIMENSION SIG(200), DRHO(10,10), RHO(10), A(10), FN(10)

10 READ (5,1) QsR, IMAX, Z, DZ
1 FORMAT (F2.0, F2.0, I4, F5.1, F5.1)
READ (5,2) (RHO(I), I=1,7)
2 FORMAT (7F5.3)
READ(5*16) (A(K)*K=1,5)
16 FORMAT (5F5.3)
WRITE(6,30)
30 FORMAT (11X, 1HI, 10X, 3HZ/R, 13X, 6HSIG(1),//)
DO 4 I=1,IMAX
SIG(I)=Q*(1.-(Z**3)/(SQRT((R**2.+Z**2.)**3.)))
WRITE(6,33) I, Z, SIG(I)
3 FORMAT (I12, 8X, F5.1, F15.8)
4 Z=Z+DZ
WRITE(6,20)
20 FORMAT (1H )
DO 5 J=1,7
DO 5 K=1,5
5 DRHO(JK)=A(K)*RHO(J)
DO 6 K=1,5
6 FN(K)=3.14/(2.0*A(K))
DO 9 J=1,7
WRITE(6,25)
9 FORMAT (1H )
DO 9 K=1,5
WRITE(6,26)
26 FORMAT (1H )
WRITE(6,31)
31 FORMAT (11X, 1HI, 10X, 3HZ/R, 10X, 9HDRHO(J), //)
WRITE(6,33) J, RHO(J), DRHO(JK)
15 FORMAT (10X, 1H, 12X, 12X, F6.3, 4XF15.8, //)
WRITE(6,32)
32 FORMAT (11X, 1HI, 10X, 19X, 1HS, 19X, 3HDIF, 17X, 3HEPS, //)
M=(R-RHO(J))/DRHO(JK)
AIMAX=IMAX
Z=Z-AIMAX*DZ
DO 9 I=1,IMAX
RI=RHO(J)
S=Q*(1.-(Z**3)/(SQRT((RHO(J)**2.+Z**2.)**3.)))
DO 7 L=1,M
P=Q*3.14*((RI+DRHO(J)K)**2.-RI**2.)/FN(K)
FL=L
D=RI+DRHO(J,K)/2.
S=S+P/(2.*3.14*(Z**2.)*(SQRT(1.+(D/Z)**2.))*5.)
7 RI=RI+DRHO(J,K)
DIF=S-SIG(I)
EPS=DIF*100./SIG(I)
WRITE(6,8) I, Z, S, DIF, EPS
8 FORMAT (112, 8X, F5.1, 5X, F15.8, 5X, F15.8, 5X, F15.8)
9 Z=Z+DZ
GO TO 10
END
/DATA
1.1. 40 0.1 0.1
0.05 0.075 0.10 0.125 0.15 0.175 0.20
0.1 0.2 0.3 0.4 0.5
/END OF FILE
### TABLE II

<table>
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4f. DEPTH OF THE ERROR VOLUME

In order to determine the value of ZLIM, the circle of radius $R$ is divided into concentric rings of width $\Delta \rho / \rho = 0.5$ and $n$ sectors, $n$ being as before:

$$n = \frac{2\pi R}{\Delta \rho}$$

Equivalent point loads $P_{ij} = P_{ij} \times A_{ij}$ are placed at the center of each element and the values of the vertical stress $G_z$ are calculated for points on the $Z$ axis and compared with those obtained in item (b) of the previous section.

The program corresponding to this process is shown in page 41.

The substitution of distributed loads by equivalent point loads can be started, as shown in Table III, at a depth $Z/R = 0.4$ in order to maintain an error of $\pm 1\%$. But for purposes of practical convenience it is an accepted value:

$$Z/R = 1$$

$$\therefore \quad ZLIM = R$$
C STRESS DISTRIBUTION STUDY

10 READ(5,1) Q,R,IMAX,Z,DZ,RHO,A,DRHO
   1 FORMAT(2F9.0,2F5.1,F12.8,F5.1,F5.1)
   WRITE(6,2)
   2 FORMAT(11X,1HI,10X,3HZ/R,12X,3HSIG,20X,1HS,19X,3HDIF,17X,3HEPS/)
   DO 9 I=1,IMAX
      SIG=Q*(1.-(Z**3.))/(SQRT((R**2.+Z**2.)**3.)))
      FN=3.14/(2.*A)
      M=(R-RHO)/DRHO
      RI=RHO
      S=0.
      DO 7 L=1,M
         P=Q*3.14*((RI+DRHO)**2.-RI**2.)/FN
         D=RI+DRHO/2.
         S=S+FN*3.*P/(2.*3.14*(Z**2.)*(SQRT(1.+(D/Z)**2.))**5.))
      7 RI=RI+DRHO
      DIF=S-SIG
      EPS=DIF*100./SIG
      WRITE(6,8) I,Z,SIG,S,DIF,EPS
   8 FORMAT(112S8X,F5.1,5X,F15.8,5X,F15.8,5X,F15.8,5X,F15.8)
   9 Z=Z+DZ
   GO TO 10
END

/DATA
   1. 1. 100 0.1 0.1 0.00000001 0.5 .1
/END OF FILE
TABLE III

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CHAPTER 5

COORDINATE SYSTEMS

This method basically utilizes two systems of coordinates:

1. A main cartesian system of fixed axes X,Y,Z. Its origin 0 is located at the center of the square grid of 20 x 20 elements. The X and Y axes are parallel to the sides of the grid and the Z axis is perpendicular to the grid plane, their positive directions are shown in Fig. 17.

2. An auxiliary cartesian system of movable axes X',Y',Z'. Its origin O' is, at each step ij of the process, located at the center of the corresponding ij element of the grid (Fig. 17).

These axes are parallel to the X,Y,Z axes respectively and have the same positive directions.

5a. COORDINATES OF A GRID ELEMENT

The coordinates \(X_A, Y_A, Z_A\) of a grid element \(ij\) with respect to the main system (Fig. 18) are given by the following relations:
a. Quadrant 1

\[ 1 \leq i \leq 10 \]
\[ 1 \leq j \leq 10 \]

\[ X_A = -(10-j) \cdot \Delta \rho - \frac{\Delta \rho}{2} = -(10-j+\frac{1}{2}) \cdot \Delta \rho = -(10.5-j) \cdot \Delta \rho \]
\[ Y_A = (10-i) \cdot \Delta \rho + \frac{\Delta \rho}{2} = (10+\frac{1}{2}-i) \Delta \rho = (10.5-i) \cdot \Delta \rho \]

\[ Z_A = 0 \]

b. Quadrant 2

\[ 1 \leq i \leq 10 \]
\[ 11 \leq j \leq 20 \]

\[ X_A = (j-10) \cdot \Delta \rho - \frac{\Delta \rho}{2} = (j-10-\frac{1}{2}) \cdot \Delta \rho = (j-10.5) \cdot \Delta \rho \]
\[ Y_A = (10-i) \cdot \Delta \rho + \frac{\Delta \rho}{2} = (10+\frac{1}{2}-i) \Delta \rho = (10.5-i) \cdot \Delta \rho \]

\[ Z_A = 0 \]
c. Quadrant 3

\[ 11 \leq i \leq 20 \]
\[ 1 \leq j \leq 10 \]

\[ x_A = -(10 - j) \Delta \rho - \frac{\Delta \rho}{2} = -(10 - j + \frac{1}{2}) \Delta \rho = (j - 10.5) \Delta \rho \]

\[ y_A = -(i - 10) \Delta \rho - \frac{\Delta \rho}{2} = -(i - 10 - \frac{1}{2}) \Delta \rho = (10.5 - i) \Delta \rho \]

\[ Z_A = 0 \]

d. Quadrant 4

\[ 11 \leq i \leq 20 \]
\[ 11 \leq j \leq 20 \]

\[ x_A = (j - 10) \Delta \rho - \frac{\Delta \rho}{2} = (j - 10 - \frac{1}{2}) \Delta \rho = (j - 10.5) \Delta \rho \]

\[ y_A = -(i - 10) \Delta \rho - \frac{\Delta \rho}{2} = -(i - 10 - \frac{1}{2}) \Delta \rho = (10.5 - i) \Delta \rho \]

\[ Z_A = 0 \]
This shows that for the expressions for $X_A$, $Y_A$, $Z_A$ are equal in the four quadrants:

\[ X_A = (j - 10.5) \cdot \Delta \rho \]  \hspace{1cm} (a)

\[ Y_A = (10.5 - i) \cdot \Delta \rho \]  \hspace{1cm} (b)

\[ Z_A = 0 \]  \hspace{1cm} (c)

5b. COORDINATES OF POINT P(x,y,z) WITH RESPECT TO SYSTEM X', Y', Z'.

From Fig. 17 it can be seen that the coordinates of any point P(x,y,z) of the half-space with respect to the auxiliary system X', Y', Z' are:

\[ X' = X - X_A \]
\[ Y' = Y - Y_A \]
\[ Z' = Z - 0 = Z \]
CHAPTER 6

PRINCIPAL STRESSES

The principal stresses are the characteristic values of the matrix:

\[
\begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z
\end{bmatrix}
\]

The three characteristic values of these matrices can be obtained solving the following determinantal equation:

\[
\begin{vmatrix}
\sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z - \sigma
\end{vmatrix} = 0
\]

Developing this determinant and rearranging terms, the following characteristic equation is obtained:

\[
\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0
\]

where \(I_1, I_2, \text{ and } I_3\) are the stress invariants.
\[ I_1 = \sigma_x + \sigma_y + \sigma_z \]

\[ I_2 = \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2 \]

\[ I_3 = \sigma_x \sigma_y \sigma_z + 2 \tau_{xy} \tau_{yz} \tau_{xz} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2 \]

A simple direct method of solution for this cubic equation is to reduce it to the form:

\[ \sigma^3 - I'_1 \sigma' - I'_3 = 0 \]

This is the cubic equation for the principal stresses expressed in their deviatoric form and in terms of the deviatoric stress components, where

\[ I'_1 = \frac{1}{3} \left( \sigma_x + \sigma_y + \sigma_z \right) \]

\[ I'_2 = 3 I'_1^2 - I_2 \]

\[ I'_3 = I_3 - I_2 I'_1 + 2 I'_1^3 \]

\[ \sigma'_x = \sigma_x - I'_1 \]

\[ \tau_{xy}' = \tau_{xy} \]

\[ \sigma'_y = \sigma_y - I'_1 \]

\[ \tau_{yz}' = \tau_{yz} \]

\[ \sigma'_z = \sigma_z - I'_1 \]

\[ \tau_{xz}' = \tau_{xz} \]
The principal stresses expressed in their deviatoric form are:

\[ G'_x = G_x - I'_1 \]
\[ G'_y = G_y - I'_1 \]
\[ G'_z = G_z - I'_1 \]

These three stresses are equal to:

\[ G'_x = c \cdot \cos \alpha \]
\[ G'_y = c \cdot \cos (\alpha + \frac{2\pi}{3}) \]
\[ G'_z = c \cdot \cos (\alpha - \frac{2\pi}{3}) \]

where

\[ c = 2 \sqrt[3]{\frac{I'_2}{3}} \]

\[ \cos \alpha = \frac{\sqrt{2} \cdot I'_3}{\left[ \sqrt[3]{\frac{2}{3} I'_2} \right]^3} \]

This solution has been programmed into a subroutine together with the method for the determination of the principal direction cosines explained in section 5.
CHAPTER 7

PRINCIPAL DIRECTION COSINES

The direction cosines \( l_1m_1n_1, l_2m_2n_2, \) and \( l_3m_3n_3 \) of the principal stresses \( \sigma_1, \sigma_2, \sigma_3 \) are the solutions of the following system of homogeneous linear equations:

\[
\begin{align*}
(\sigma_x - \sigma)l + \tau_{xy} m + \tau_{xz} n &= 0 \\
\tau_{yx} l + (\sigma_y - \sigma)m + \tau_{yz} n &= 0 \tag{1} \\
\tau_{zx} l + \tau_{zy} m + (\sigma_z - \sigma)n &= 0
\end{align*}
\]

When \( \sigma_1 \) is substituted the solution will be \( l_1, m_1, n_1 \)

When \( \sigma_2 \) is substituted the solution will be \( l_2, m_2, n_2 \)

When \( \sigma_3 \) is substituted the solution will be \( l_3, m_3, n_3 \)

The direction cosines of each principal stress must satisfy the following relations:

\[
\begin{align*}
 l_1^2 + m_1^2 + n_1^2 &= 1 \\
 l_2^2 + m_2^2 + n_2^2 &= 1 \\
 l_3^2 + m_3^2 + n_3^2 &= 1
\end{align*} \tag{2}
\]
System [1] can be written in the following form:

\[ a_1 x_i + b_1 y_i + c_1 z_i = 0 \]  
(1)

\[ a_2 x_i + b_2 y_i + c_2 z_i = 0 \]  
(2)

\[ a_3 x_i + b_3 y_i + c_3 z_i = 0 \]  
(3)

and relations [2] can be written as:

\[ x_i^2 + y_i^2 + z_i^2 = 1 \]  
(4)

where \( i = 1, 2, 3 \)

Assuming \( x_i = A \)

from (1)

\[ y_i = \frac{1}{b_1} (- c_1 z_i - a_1 A) \]  
(5)

from (2)

\[ y_i = \frac{1}{b_2} (- c_2 z_i - a_2 A) \]  
(6)

from (5) and (6)

\[ z_i = \frac{b_1 b_2 A (a_2 - a_1)}{b_2 c_1 - b_1 c_2} \]  
(7)

substituting in (5)

\[ x_i = A \]  
(8)

\[ y_i = \frac{A}{b_1} \left[ \frac{- c_1 b_1 b_2 (a_2 - a_1)}{b_2 c_1 - b_1 c_2} - a_1 \right] \]  
(9)

\[ z_i = \frac{b_1 b_2 A (a_2 - a_1)}{b_2 c_1 - b_1 c_2} \]  
(10)
The constant $A$ can be determined making use of relation (4):

$$A^2 \left\{ 1 + \frac{1}{b_1^2} \left[ \frac{-c_1 b_1 b_2 (\theta_2 - \theta_1)}{b_2 c_1 - b_1 c_2} - \theta_1 \right]^2 + \left[ \frac{b_1 b_2 (\theta_2 - \theta_1)}{b_2 c_1 - b_1 c_2} \right]^2 \right\} = 1$$

$$A = \left\{ 1 + \frac{1}{b_1^2} \left[ \frac{-c_1 b_1 b_2 (\theta_2 - \theta_1)}{b_2 c_1 - b_1 c_2} - \theta_1 \right]^2 + \left[ \frac{b_1 b_2 (\theta_2 - \theta_1)}{b_2 c_1 - b_1 c_2} \right]^2 \right\}^{-\frac{1}{2}}$$

Substituting $A$ into (8), (9), and (10) we obtain $X_1$, $Y_1$, and $Z_1$. In case that the denominator $b_1 c_1 - b_2 c_2 = 0$ it is necessary to prepare another combination in the program, for example using equations (2) and (3).

From (2) and (3)

$$Z_i = b_3 \frac{b_2 (\theta_3 - \theta_2)}{b_2 c_2 - b_1 c_3}$$

$$X_i = A$$

$$Y_i = A \frac{b_2}{b_2} \left[ \frac{-c_3 b_2 b_3 (\theta_3 - \theta_2)}{b_3 c_2 - b_2 c_3} - \theta_2 \right]$$
The constant $A$ again can be determined from relation (4):

$$A^2 \left\{ 1 + \frac{1}{b_2^2} \left[ \frac{-c_2 b_2 b_3 (\alpha_3 - \alpha_2)}{b_3 c_2 - b_2 c_3} - c_2 \right]^2 + \left[ \frac{b_2 b_3 (\alpha_3 - \alpha_2)}{b_3 c_2 - b_2 c_3} \right]^2 \right\} = 1$$

$$A = \left\{ 1 + \frac{1}{b_2^2} \left[ \frac{-c_2 b_2 b_3 (\alpha_3 - \alpha_2)}{b_3 c_2 - b_2 c_3} - c_2 \right]^2 + \left[ \frac{b_2 b_3 (\alpha_3 - \alpha_2)}{b_3 c_2 - b_2 c_3} \right]^2 \right\}^{-\frac{1}{2}}$$
CHAPTER 8

COMPUTER PROGRAM

The method described in the previous chapters was programmed in FORTRAN IV for an IBM/360 digital computer. The flow chart and the program are shown in pages 57 to 69. Table IV (page 78) shows the output form for each point of the half-space.

Compilation time for the program is of the order of 0.96 minutes and execution time for each point is approximately 15 seconds.
STRESSES AND DISPLACEMENTS IN SEMI-INFINITE MEDIA.

MAIN PROGRAM.

CAPABILITIES OF THE SYSTEM.

FOR ANY POINT OF THE SEMI-INFINITE MEDIUM

FOR ANY SHAPE OF THE LOADED AREA

STRESS TENSOR DUE TO NORMAL LOADS.

STRESS TENSOR DUE TO SHEARING LOADS.

STRESS TENSOR DUE TO SUPERIMPOSED NORMAL AND SHEARING LOADS.

VERTICAL COMPONENT OF DISPLACEMENT DUE TO NORMAL LOADS.

VERTICAL COMPONENT OF DISPLACEMENT DUE TO SHEARING LOADS.

VERTICAL COMPONENT OF DISPLACEMENT DUE TO SUPERIMPOSED NORMAL

AND SHEARING LOADS.

PRINCIPAL STRESSES.

PRINCIPAL DIRECTION COSINES.

DIMENSION P(20,20),Q(20,20)

DIMENSION U1(25),U2(25),U3(25),T1(25),T2(25),T3(25)

DIMENSION AP(3),AQ(3),AT(3)

DIMENSION S(3),B(3,3)

COMMON S,B,XX,SY,ZZ,SX,SY,SXZ,SYZ

READ(5,2) NMAX,L,POISS,ELAST,R

FORMAT(216,F7.3,F215.5)

READ(5,3) (P(I,J),J=1,10),I=1,20

READ(5,3) (P(I,J),J=11,20),I=1,20

READ(5,3) (Q(I,J),J=1,10),I=1,20

READ(5,3) (Q(I,J),J=11,20),I=1,20

FORMAT(10F7.2)

READ(5,30) X,Y,Z,DX,DY,DZ

FORMAT(6F10.5)

PRINT DATA.

WRITE(6,2) NMAX,L,POISS,ELAST,R

WRITE(6,988)

WRITE(6,100) (P(I,J),J=1,20),I=1,20

WRITE(6,988)

WRITE(6,100) (Q(I,J),J=1,20),I=1,20

WRITE(6,988)

FORMAT(1H)

FORMAT(2OF6.0)

WRITE(6,789) X,Y,Z,DX,DY,DZ

FORMAT(6F10.5,\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\n
ELAST=MODULUS OF ELASTICITY.

POISS=POISSON'S RATIO

FL AND GL=LAME CONSTANTS.

V=1.-2.*POISS

FL=POISS*ELAST/(POISS+1.)*V)

E628=6.28*ELAST

GL=ELAST/(2.*(POISS+1.))

RHO=0.2*R

DRH0=0.5*RHO

R1=DRH0/SQRT(3.,14)

R2=3.*R

R3=5.*R1

RHO3=3.*DRH0

DRH02=DRH0**2
XLIM=12.*DRHO
YLIM=XLIM
ZLIM=R
4 DO 40 N=1,NMAX
C INITIAL VALUES OF STRESS AND DISPLACEMENT COMPONENTS
C AT POINT (X,Y,Z) OF HALF-SPACE.
P1XX=0.
P1YY=0.
P1ZZ=0.
P1XY=0.
P1XZ=0.
P1YZ=0.
P2XX=0.
P2YY=0.
P2ZZ=0.
P2XY=0.
P2XZ=0.
P2YZ=0.
P1XX=0.
P1YY=0.
P1ZZ=0.
P1XY=0.
P1XZ=0.
P1YZ=0.
P2XX=0.
P2YY=0.
P2ZZ=0.
P2XY=0.
P2XZ=0.
P2YZ=0.
P1XX=0.
P1YY=0.
P1ZZ=0.
P1XY=0.
P1XZ=0.
P1YZ=0.
P2XX=0.
P2YY=0.
P2ZZ=0.
P2XY=0.
P2XZ=0.
P2YZ=0.
C IS POINT (XY,Z) WITHIN SENSITIVE VOLUME (2*XLIM*2YLIM*2ZLIM).
C IF YES M=1 IF NO M=0
43 IF(Z-ZLIM) 43,43,44
44 IF(ABS(X)-XLIM) 45,44,44
45 IF(ABS(Y)-YLIM) 46,44,44
46 M=1
GO TO 47
47 K1=1
K2=1
K3=1
DO 9 I=1,20
DO 9 J=1,20
C COORDINATES OF LOAD ELEMENT (I,J) WITH RESPECT TO MAIN SYSTEM.
ZA=0.
HI=I
HJ=J
XA=(HJ-10.5)*DRHO
YA=(10.5-HI)*DRHO
C COORDINATES OF POINT (X,Y,Z) WITH RESPECT TO LOAD ELEMENT (I,J).
XP=X-XA
YP = Y - YA
ZP = Z
IF (M=1) 5, 5, 6
C IS POINT (X, Y, Z) WITHIN SENSITIVE ZONE OF LOAD ELEMENT (I, J)
C IN EITHER CASE GO TO CORRESPONDING ROUTINE.
6 IF (ABS(XP) - R3) 7, 5, 5
7 IF (ABS(YP) - R3) 8, 5, 5
C POINT (X, Y, Z) OUTSIDE OF SENSITIVE ZONE OF LOAD ELEMENT (I, J)
C ROUTINE FOR CONCENTRATED LOADS.
5 D = SQRT(XP*XP + YP*YP + ZP*ZP)
DZP2 = (D + ZP)**2
R2 = XP*XP + YP*YP
A = D**2
AA = D**3
C = D**5
CCC = 6.28*C
E = XP*XP
F = YP*YP
G = ZP**2
H = ZP**3
IF (P(I, J)) 77, 78, 77
77 PP = P(I, J) * DRHO2
A1XY = 0.
TTT = 3. * PP / CCC
A1YZ = TTT * YP * G
A1XZ = TTT * XP * G
A1ZZ = TTT * H
UUU = 3. * ZP / C
VVV = R2 * D * (D + ZP)
SSS = ZP / (R2 * AA)
A1XY = PP * (F * UUU - V* ((F - E) / (VVV + E * SSS))) / 6.28
A1XY = PP * (E * UUU - V* ((E - F) / (VVV + E * SSS))) / 6.28
DP1 = PP * (1. * POISS) * (G/AA + V*2)/D)/(6.28*ELAST)
GO TO 79
78 A1XY = 0.
A1XX = 0.
A1YY = 0.
A1ZZ = 0.
A1XZ = 0.
A1YZ = 0.
DP1 = 0.
79 IF (Q(I, J)) 87, 88, 87
87 QQ = Q(I, J) * DRHO2
B1XZ = (-3. * QQ * E * ZP) / CCC
B1YZ = (-3. * QQ * XP * YP * ZP) / CCC
1/(6.28 * AA)
B1ZZ = (-3. * QQ * XP * G) / CCC
1/(6.28 * AA)
GO TO 89
88 B1XY = 0.
B1XX = 0.
B1YY = 0.
B1ZZ=0.
B1XZ=0.
B1YZ=0.
DQ1=0.

89 P1XX=P1XX+A1XX
P1YY=P1YY+A1YY
P1ZZ=P1ZZ+A1ZZ
P1XY=P1XY+A1XY
P1YZ=P1YZ+A1YZ
Q1XX=Q1XX+B1XX
Q1YY=Q1YY+B1YY
Q1ZZ=Q1ZZ+B1ZZ
Q1XY=Q1XY+B1XY
Q1XZ=Q1XZ+B1XZ
Q1YZ=Q1YZ+B1YZ
WP1=WP1+DP1
WQ1=WQ1+DQ1
GO TO 9

C POINT (X,Y,Z) INSIDE OF SENSITIVE ZONE OF LOAD ELEMENT (I,J)

8 IF(ABS(XP)-R1) 101,101,102
101 IF(ABS(YP)-R1) 103,103,102
103 U1(K1)=P(I,J)
T1(K1)=Q(I,J)
GO TO 91
102 IF(ABS(XP)-R2) 104,104,105
104 IF(ABS(YP)-R2) 106,106,105
106 U2(K2)=P(I,J)
T2(K2)=Q(I,J)
GO TO 92
105 U3(K3)=P(I,J)
T3(K3)=Q(I,J)
GO TO 93
91 K1=K1+1
GO TO 9
92 K2=K2+1
GO TO 9
93 K3=K3+1
9 CONTINUE

C ROUTINE FOR DISTRIBUTED LOADS.
K1MAX=K1-1
K2MAX=K2-1
K3MAX=K3-1
IF(K1MAX+K2MAX+K3MAX) 50,50,51
50 A2XX=0.
A2YY=0.
A2ZZ=0.
A2XY=0.
A2XZ=0.
A2YZ=0.
B2XX=0.
B2YY=0.
B2ZZ=0.
B2XY=0.
B2XZ=0.
B2YZ = 0.
DP2 = 0.
DQ2 = 0.
GO TO 52
51 SUMP1 = 0.
SUMP2 = 0.
SUMP3 = 0.
SUMQ1 = 0.
SUMQ2 = 0.
SUMQ3 = 0.
DO 600 K1 = 1, K1MAX
SUMP1 = SUMP1 + U1(K1)
600 SUMQ1 = SUMQ1 + T1(K1)
DO 601 K2 = 1, K2MAX
SUMP2 = SUMP2 + U2(K2)
601 SUMQ2 = SUMQ2 + T2(K2)
DO 602 K3 = 1, K3MAX
SUMP3 = SUMP3 + U3(K3)
602 SUMQ3 = SUMQ3 + T3(K3)
R1MAX = K1MAX
R2MAX = K2MAX
R3MAX = K3MAX
IF (R1MAX) 900, 900, 901
900 AP(1) = 0.
AQ(1) = 0.
GO TO 902
901 AP(1) = SUMP1 / R1MAX
AQ(1) = SUMQ1 / R1MAX
902 IF (R2MAX) 903, 903, 904
903 AP(2) = 0.
AQ(2) = 0.
GO TO 905
904 AP(2) = SUMP2 / R2MAX
AQ(2) = SUMQ2 / R2MAX
905 IF (R3MAX) 906, 906, 907
906 AP(3) = 0.
AQ(3) = 0.
GO TO 908
907 AP(3) = SUMP3 / R3MAX
AQ(3) = SUMQ3 / R3MAX
908 A = ZP ** 2
AA = ZP * A
C1 = R1 ** 2
C2 = R2 ** 2
C3 = R3 ** 2
D1 = C1 * R1
D2 = C2 * R2
D3 = C3 * R3
E1 = SQRT(C1 + A)
E2 = SQRT(C2 + A)
E3 = SQRT(C3 + A)
F1 = E1 ** 3
F2 = E2 ** 3
F3 = E3 ** 3
G1 = ALOG((R1 + E1) / ZP)
\[
G_2 = \text{ALOG}\left(\frac{R_2 + E_2}{ZP}\right)
\]
\[
G_3 = \text{ALOG}\left(\frac{R_3 + E_3}{ZP}\right)
\]
\[
H_1 = \text{ATAN}\left(\frac{R_1}{ZP}\right)
\]
\[
H_2 = \text{ATAN}\left(\frac{R_2}{ZP}\right)
\]
\[
H_3 = \text{ATAN}\left(\frac{R_3}{ZP}\right)
\]
\[
\text{DO } 753 \ M = 1 \times 2
\]
\[
\text{IF}(M - 2) \ 300, 301, 301
\]
\[
300 \ \text{AT}(1) = \text{AP}(1)
\]
\[
\text{AT}(2) = \text{AP}(2)
\]
\[
\text{AT}(3) = \text{AP}(3)
\]
\[
D_1XX = \left(\frac{AA}{F_1} - 2 \cdot (1 + \text{POISS}) \cdot ZP / E_1 + V + 6\right) / 2
\]
\[
D_2XX = \left(\frac{AA}{F_2} - 2 \cdot (1 + \text{POISS}) \cdot ZP / E_2 + V + 6\right) / 2
\]
\[
D_3XX = \left(\frac{AA}{F_3} - 2 \cdot (1 + \text{POISS}) \cdot ZP / E_3 + V + 6\right) / 2
\]
\[
D_1XY = 0
\]
\[
D_2XY = 0
\]
\[
D_3XY = 0
\]
\[
D_1XZ = 2 \cdot D_1 / (3 \cdot 14 \cdot F_1)
\]
\[
D_2XZ = 2 \cdot D_2 / (3 \cdot 14 \cdot F_2)
\]
\[
D_3XZ = 2 \cdot D_3 / (3 \cdot 14 \cdot F_3)
\]
\[
D_1YY = 0
\]
\[
D_2YY = 0
\]
\[
D_3YY = 0
\]
\[
D_1YZ = 0
\]
\[
D_2YZ = 0
\]
\[
D_3YZ = 0
\]
\[
D_1ZZ = 1 \cdot -AA / F_1
\]
\[
D_2ZZ = 1 \cdot -AA / F_2
\]
\[
D_3ZZ = 1 \cdot -AA / F_3
\]
\[
D_1W = (1 + \text{POISS}) \cdot (R_1 / E_1 + 2 \cdot V \cdot G_1) / \text{ELAST}
\]
\[
D_2W = (1 + \text{POISS}) \cdot (R_2 / E_2 + 2 \cdot V \cdot G_2) / \text{ELAST}
\]
\[
D_3W = (1 + \text{POISS}) \cdot (R_3 / E_3 + 2 \cdot V \cdot G_3) / \text{ELAST}
\]
\[
\text{GO TO } 700
\]
\[
301 \ \text{AT}(1) = \text{AQ}(1)
\]
\[
\text{AT}(2) = \text{AQ}(2)
\]
\[
\text{AT}(3) = \text{AQ}(3)
\]
\[
D_1XX = \left(-4 \cdot (1 + 6 \cdot \text{POISS}) \cdot R_1 / E_1 + 8 \cdot \left(3 \cdot F_1 / 3\right) - 8 \cdot G_1 + V \cdot (8 \cdot E_1 / \text{RHO3} + 1 - 32 \cdot ZP / (3 \cdot R_1) - 16 \cdot H_1 / 3)\right) / 6 \cdot 28
\]
\[
D_2XX = \left(-4 \cdot (1 + 6 \cdot \text{POISS}) \cdot R_2 / E_2 + 8 \cdot \left(3 \cdot F_2 / 3\right) - 8 \cdot G_2 + V \cdot (8 \cdot E_2 / \text{RHO3} + 1 - 32 \cdot ZP / (3 \cdot R_2) - 16 \cdot H_2 / 3)\right) / 6 \cdot 28
\]
\[
D_3XX = \left(-4 \cdot (1 + 6 \cdot \text{POISS}) \cdot R_3 / E_3 + 8 \cdot \left(3 \cdot F_3 / 3\right) - 8 \cdot G_3 + V \cdot (8 \cdot E_3 / \text{RHO3} + 1 - 32 \cdot ZP / (3 \cdot R_3) - 16 \cdot H_3 / 3)\right) / 6 \cdot 28
\]
\[
D_1XY = \left((4 \cdot 20 \cdot V / 3) \cdot R_1 / E_1 + 4 \cdot D_1 / (3 \cdot F_1) - 4 \cdot G_1 + V \cdot (-20 \cdot E_1 / \text{RHO3} + 132 \cdot ZP / \text{RHO3} + 16 \cdot H_1 / 3)\right) / 6 \cdot 28
\]
\[
D_2XY = \left((4 \cdot 20 \cdot V / 3) \cdot R_2 / E_2 + 4 \cdot D_2 / (3 \cdot F_2) - 4 \cdot G_2 + V \cdot (-20 \cdot E_2 / \text{RHO3} + 132 \cdot ZP / \text{RHO3} + 16 \cdot H_2 / 3)\right) / 6 \cdot 28
\]
\[
D_3XY = \left((4 \cdot 20 \cdot V / 3) \cdot R_3 / E_3 + 4 \cdot D_3 / (3 \cdot F_3) - 4 \cdot G_3 + V \cdot (-20 \cdot E_3 / \text{RHO3} + 132 \cdot ZP / \text{RHO3} + 16 \cdot H_3 / 3)\right) / 6 \cdot 28
\]
\[
D_1XZ = 0
\]
\[
D_2XZ = 0
\]
\[
D_3XZ = 0
\]
\[
D_1YY = (8 \cdot \text{POISS} \cdot R_1 / E_1 + 4 \cdot D_1 / (3 \cdot F_1) - 4 \cdot G_1 + V \cdot (-20 \cdot E_1 / \text{RHO3} + 144 \cdot ZP / \text{RHO3} + 4 \cdot H_1)\right) / 6 \cdot 28
\]
\[
D_2YY = (8 \cdot \text{POISS} \cdot R_2 / E_2 + 4 \cdot D_2 / (3 \cdot F_2) - 4 \cdot G_2 + V \cdot (-20 \cdot E_2 / \text{RHO3} + 144 \cdot ZP / \text{RHO3} + 4 \cdot H_2)\right) / 6 \cdot 28
\]
D3YY=(8*POISS*R3/E3+4*D3/(3*F3)-4*G3+V*(-20*E3/RHO3+144*ZP/RHO3+4*H3))/6.28
D1YZ=-3*ZP*(2*(3*ZP)+A/(3*F1)-1/E1)/3.14
D2YZ=-3*ZP*(2*(3*ZP)+A/(3*F2)-1/E2)/3.14
D3YZ=-3*ZP*(2*(3*ZP)+A/(3*F3)-1/E3)/3.14
D1ZZ=2*D1/(3.14*F1)
D2ZZ=2*D2/(3.14*F2)
D3ZZ=2*D3/(3.14*F3)
D1W=2*(1+POISS)*(V*Gl/(V+2*ELAST)-ZP*Rl/El+V*(2*ZP/Rl-2*El/Rl)/(V+2*ELAST))/3.14*ELAST
D2W=2*(1+POISS)*(V*G2/(V+2*ELAST)-ZP*R2/E2+V*(2*ZP/R2-2*E2/R2)/(V+2*ELAST))/3.14*ELAST

IF(AT(1)-AT(2)) 710 IF(AT(1)-AT(3)) 712 IF(AT(2)-AT(3)) 714

C MIN=AT(1) INT=AT(2) MAX=AT(3)

ALFA=AT(1)
BETA=AT(2)-AT(1)
GAMMA=AT(3)-AT(2)
C1XX=ALFA*D3XX
C2XX=BETA*(D3XX-D1XX)
C3XX=GAMMA*(D3XX-D2XX)
C1XY=ALFA*D3XY
C2XY=BETA*(D3XY-D1XY)
C3XY=GAMMA*(D3XY-D2XY)
C1XZ=ALFA*D3XZ
C2XZ=BETA*(D3XZ-D1XZ)
C3XZ=GAMMA*(D3XZ-D2XZ)
C1YY=ALFA*D3YY
C2YY=BETA*(D3YY-D1YY)
C3YY=GAMMA*(D3YY-D2YY)
C1YZ=ALFA*D3YZ
C2YZ=BETA*(D3YZ-D1YZ)
C3YZ=GAMMA*(D3YZ-D2YZ)
C1ZZ=ALFA*D3ZZ
C2ZZ=BETA*(D3ZZ-D1ZZ)
C3ZZ=GAMMA*(D3ZZ-D2ZZ)
C1W=ALFA*D3W
C2W=BETA*(D3W-D1W)
C3W=GAMMA*(D3W-D2W)
GO TO 750

C MIN=AT(1) INT=AT(3) MAX=AT(2)

ALFA=AT(1)
BETA=AT(3)-AT(1)
GAMMA=AT(2)-AT(3)
C1XX=ALFA*D3XX
C2XX=BETA*(D3XX-D1XX)
C3XX=GAMMA*(D2XX-D1XX)
C1XY=ALFA*D3XY
C2XY=BETA*(D3XY-D1XY)
C3XY=GAMMA*(D2XY-D1XY)
C1XZ=ALFA*D3XZ
C2XZ = BETA * (D3XZ - D1XZ)
C3XZ = GAMMA * (D2XZ - D1XZ)
C1YY = ALFA * D3YY
C2YY = BETA * (D3YY - D1YY)
C3YY = GAMMA * (D2YY - D1YY)
C1YZ = ALFA * D3YZ
C2YZ = ALFA * (D3YZ - D1YZ)
C3YZ = GAMMA * (D2YZ - D1YZ)
C1ZZ = ALFA * D3ZZ
C2ZZ = BETA * (D3ZZ - D1ZZ)
C3ZZ = GAMMA * (D2ZZ - D1ZZ)
C1W = ALFA * D1W
C2W = BETA * (D3W - D1W)
C3W = GAMMA * (D2W - D1W)
GO TO 750

C
MIN = AT(3)  INT = AT(1)  MAX = AT(2)

713  ALFA = AT(3)
BETA = AT(1)
GAMMA = AT(2) - AT(1)
C1XX = ALFA * D3XX
C2XX = BETA * D2XX
C3XX = GAMMA * (D2XX - D1XX)
C1XY = ALFA * D3XY
C2XY = BETA * D2XY
C3XY = GAMMA * (D2XY - D1XY)
C1XZ = ALFA * D3XZ
C2XZ = BETA * D2XZ
C3XZ = GAMMA * (D2XZ - D1XZ)
C1YY = ALFA * D3YY
C2YY = BETA * D2YY
C3YY = GAMMA * (D2YY - D1YY)
C1YZ = ALFA * D3YZ
C2YZ = BETA * D2YZ
C3YZ = GAMMA * (D2YZ - D1YZ)
C1ZZ = ALFA * D3ZZ
C2ZZ = BETA * D2ZZ
C3ZZ = GAMMA * (D2ZZ - D1ZZ)
C1W = ALFA * D1W
C2W = BETA * D2W
C3W = GAMMA * (D2W - D1W)
GO TO 750

714  IF (AT(1) - AT(3)) 720, 720, 721
C
MIN = AT(2)  INT = AT(1)  MAX = AT(3)

720  ALFA = AT(2)
BETA = AT(2) - AT(1)
GAMMA = AT(3) - AT(2)
C1XX = ALFA * D3XX
C2XX = BETA * D1XX
C3XX = GAMMA * (D3XX - D1XX)
C1XY = ALFA * D3XY
C2XY = BETA * D1XY
C3XY = GAMMA * (D3XY - D1XY)
C1XZ = ALFA * D3XZ
C2XZ = BETA * D1XZ
C3XZ = GAMMA * (D3XZ - D1XZ)
C1YY = ALFA*D3YY
C2YY = BETA*D1YY
C3YY = GAMMA*(D3YY-D1YY)
C1YZ = ALFA*D3YZ
C2YZ = BETA*D1YZ
C3YZ = GAMMA*(D3YZ-D1YZ)
C1ZZ = ALFA*D3ZZ
C2ZZ = BETA*D1ZZ
C3ZZ = GAMMA*(D3ZZ-D1ZZ)
C1W = ALFA*D3W
C2W = BETA*D1W
C3W = GAMMA*(D3W-D1W)
GO TO 750

C MIN = AT(2) INT = AT(3) MAX = AT(1)

721 ALFA = AT(2)
BETA = AT(1) - AT(2)
GAMMA = AT(3) - AT(2)
C1XX = ALFA*D3XX
C2XX = BETA*D1XX
C3XX = GAMMA*(D3XX-D2XX)
C1XY = ALFA*D3XY
C2XY = BETA*D1XY
C3XY = GAMMA*(D3XY-D2XY)
C1XZ = ALFA*D3XZ
C2XZ = BETA*D1XZ
C3XZ = GAMMA*(D3XZ-D2XZ)
C1YY = ALFA*D3YY
C2YY = BETA*D1YY
C3YY = GAMMA*(D3YY-D2YY)
C1YZ = ALFA*D3YZ
C2YZ = BETA*D1YZ
C3YZ = GAMMA*(D3YZ-D2YZ)
C1ZZ = ALFA*D3ZZ
C2ZZ = BETA*D1ZZ
C3ZZ = GAMMA*(D3ZZ-D2ZZ)
C1W = ALFA*D3W
C2W = BETA*D1W
C3W = GAMMA*(D3W-D1W)
GO TO 750

C MIN = AT(3) INT = AT(2) MAX = AT(1)

715 ALFA = AT(3)
BETA = AT(2) - AT(3)
GAMMA = AT(1) - AT(2)
C1XX = ALFA*D3XX
C2XX = BETA*D2XX
C3XX = GAMMA*D1XX
C1XY = ALFA*D3XY
C2XY = BETA*D2XY
C3XY = GAMMA*D1XY
C1XZ = ALFA*D3XZ
C2XZ = BETA*D2XZ
C3XZ = GAMMA*D1XZ
C1YY = ALFA*D3YY
C2YY = BETA*D2YY
C3YY = GAMMA*D1YY
C1YZ=ALFA*D3YZ
C2YZ=BETA*D2YZ
C3YZ=GAMMA*D1YZ
C1ZZ=ALFA*D3ZZ
C2ZZ=BETA*D2ZZ
C3ZZ=GAMMA*D1ZZ
C1W=ALFA*D3W
C2W=BETA*D2W
C3W=GAMMA*D1W

750 A2XX=C1XX+C2XX+C3XX
A2XY=C1XY+C2XY+C3XY
A2XZ=C1XZ+C2XZ+C3XZ
A2YY=C1YY+C2YY+C3YY
A2YZ=C1YZ+C2YZ+C3YZ
A2ZZ=C1ZZ+C2ZZ+C3ZZ
DW=C1W+C2W+C3W
IF(M-2) 751,752,752

751 A2YY=A2XX
A2YZ=A2XZ
P2XX=P2XX+A2XX
P2XY=P2XY+A2XY
P2XZ=P2XZ+A2XZ
P2YY=P2YY+A2YY
P2YZ=P2YZ+A2YZ
P2ZZ=P2ZZ+A2ZZ
WP2=WP2+DW
GO TO 753

752 A2XZ=A2YZ
Q2XX=Q2XX+A2XX
Q2XY=Q2XY+A2XY
Q2XZ=Q2XZ+A2XZ
Q2YY=Q2YY+A2YY
Q2YZ=Q2YZ+A2YZ
Q2ZZ=Q2ZZ+A2ZZ
WQ2=WQ2+DW
CONTINUE

C STRESS COMPONENTS DUE TO NORMAL LOAD.
52 PXX=P1XX+P2XX
PYY=P1YY+P2YY
PZZ=P1ZZ+P2ZZ
PXY=P1XY+P2XY
PXZ=P1XZ+P2XZ
PYZ=P1YZ+P2YZ

C VERTICAL DISPLACEMENT DUE TO NORMAL LOAD.
WP=WP1+WP2

C STRESS COMPONENTS DUE TO SHEARING LOAD.
QXX=Q1XX+Q2XX
QYY=Q1YY+Q2YY
QZZ=Q1ZZ+Q2ZZ
QXY=Q1XY+Q2XY
QXZ=Q1XZ+Q2XZ
QYZ=Q1YZ+Q2YZ

C VERTICAL DISPLACEMENT DUE TO SHEARING LOAD.
WQ=WQ1+WQ2

C STRESS COMPONENTS DUE TO SUPERIMPOSED NORMAL AND SHEARING LOADS.
SXX=PXX+QXX
SYY=PYY+QYY
SZZ=PPZZ+QZZ
SXY=PXY+QXY
SXZ=PXZ+QXZ
SYZ=PYZ+QYZ

C VERTICAL DISPLACEMENT DUE TO SUPERIMPOSED NORMAL AND SHEARING LOADS.
W=WP+WQ

C PRINT OUTPUT
WRITE (6,11) X, Y, Z
11 FORMAT (20X,2HX=F15.8,20X,2HY=F15.8,20X,2HZ=F15.8,/) WRITE (6,12)
12 FORMAT (20X,2HXX,18X,2HYY,18X,2HZZ,18X,2HXY,18X,2HXZ,18X,2HYZ,/) WRITE (6,13) PX, PY, PZZ, PX, PY, PX, PY, PYZ
13 FORMAT (5X,1HP,6F20.8,/) WRITE (6,14) QXX, QYY, QZZ, QXY, QXZ, QYZ
14 FORMAT (5X,1HP,6F20.8,/) WRITE (6,15) SXX, SYY, SZZ, SXY, SXZ, SYZ
15 FORMAT (5X,1HP,6F20.8,/) WRITE (6,16) WP, WQ, W
16 FORMAT (19X,3HP=15.8,19X,3TQ=15.8,19X,3T =15.8,/) IF (L=2) 17, 18
18 CALL MHOR

C PRINCIPAL STRESSES AND PRINCIPAL DIRECTIONS OF STRESS
WRITE (6,55)
55 FORMAT (55X,18HPRINCIPAL STRESSES)
WRITE (6,19) S(1), S(2), S(3)
19 FORMAT (17X,5HS(1)=F15.8,17X,5HS(2)=F15.8,17X,5HS(3)=F15.8,/) WRITE (6,56)
56 FORMAT (50X,27HPRINCIPAL DIRECTION COSINES)
WRITE (6,20) B(1,1), B(2,1), B(3,1)
WRITE (6,21) B(1,2), B(2,2), B(3,2)
WRITE (6,22) B(1,3), B(2,3), B(3,3)
20 FORMAT (15X,7HB(1,1)=F15.4,15X,7HB(2,1)=F15.4,15X,7HB(3,1)=F15.4,/) 21 FORMAT (15X,7HB(1,2)=F15.4,15X,7HB(2,2)=F15.4,15X,7HB(3,2)=F15.4,/) 22 FORMAT (15X,7HB(1,3)=F15.4,15X,7HB(2,3)=F15.4,15X,7HB(3,3)=F15.4,/) 1,/ / C INCREASE X, Y, Z
17 X=X+DX
Y=Y+DY
40 Z=Z+DZ
CALL EXIT
END
SUBROUTINE MHOR
C SUBROUTINE FOR PRINCIPAL STRESSES AND PRINCIPAL DIRECTIONS.
DIMENSION S(3), B(3, 3)
COMMON S, B, SXX, SYY, SZX, SYZ
C STRESS INVARIANTS.
SNV1 = (SXX + SYY + SZZ) / 3.
SNV2 = (SXX * SYY + SYY * SZZ + SZZ * SXX - SXY**2 - SXZ**2 - SYZ**2) / 3
SNV3 = -(SXX * SYY * SZZ + 2. * SXY * SYZ * SZX - SXX * SYZ * SYZ - SYY * SZX * SZX - 1 - SZZ * SXY * SXY)
C SOLUTION OF THE CUBIC EQUATION.
X = 3. * (SNV1**2) - SNV2
Y = SNV3 - SNV2 * SNV1 + 2. * (SNV1**3)
Z = SQRT(2. * X / 3.)
C3A = Y * SQRT(2.) / (Z**3)
S3A = SQRT(1. - (C3A**2))
T3A = S3A / C3A
ALFA = (ATAN(T3A)) / 3.
C = Z * SQRT(2.)
C VOLUMETRIC PRINCIPAL STRESSES.
SS1 = C * COS(ALFA)
SS2 = C * COS(ALFA + 6.28 / 3.)
SS3 = C * COS(ALFA + 12.56 / 3.)
C PRINCIPAL STRESSES.
S(1) = SS1 + SNV1
S(2) = SS2 + SNV1
S(3) = SS3 + SNV1
C PRINCIPAL DIRECTIONS.
B(1, 1), B(2, 1), B(3, 1) = DIRECTION COSINES OF PRINCIPAL STRESS S(1).
B(1, 2), B(2, 2), B(3, 2) = DIRECTION COSINES OF PRINCIPAL STRESS S(2).
B(1, 3), B(2, 3), B(3, 3) = DIRECTION COSINES OF PRINCIPAL STRESS S(3).
DO 5 J = 1, 3
U1 = (SYY - S(J)) * SXZ - SXY * SYZ
U2 = SYZ**2 - (SYY - S(J)) * (SZZ - S(J))
IF(U1) 10, 11, 10
10 V1 = SXX * SXY * (SYY - S(J)) * (SXY - SXX)
W1 = SXY**2 - (V1 / U1 - (SXX - S(J))) / SXY
A = 1. / SQRT(1. + ((V1 / U1 - (SXX - S(J))) / SXY)**2 + (W1 / U1)**2)
GO TO 12
11 V1 = -SXY * (SYY - S(J)) * SYZ * (SXZ - SXY)
W1 = (SYY - S(J)) * SYZ * (SXZ - SXY)
A = 1. / SQRT(1. + ((V1 / U2 - SXY) / (SYY - S(J)))**2 + (W1 / U1)**2)
GO TO 13
12 B(1, J) = A
B(3, J) = A * SXY * (SYY - S(J)) * (SXY - (SXX - S(J))) / U1
B(2, J) = (-SXZ * B(3, J) - (SXX - S(J)) * A) / SXY
GO TO 5
13 B(1, J) = A
B(3, J) = A * (SYY - S(J)) * SYZ * (SXY - SXZ) / U2
B(2, J) = (-SYZ * B(3, J) - SXY * A) / (SYY - S(J))
5 CONTINUE
RETURN
END
CHAPTER 9

RESULTS

The program was tested several times with different data, an example is shown in pages 72 to 76 in which the following was utilized:

Number of points to be analyzed
Use subroutine for principal stresses
Poisson's ratio
Modulus of Elasticity
Maximum diameter of loaded area
Initial Coordinates
Increments

\[
\begin{align*}
NMAX &= 10 \\
L &= 2 \\
POISS &= 0 \\
ELAST &= 1 \\
R &= 1 \\
x &= 0 \\
y &= 0 \\
z &= 0.2 \\
Dx &= 0 \\
Dy &= 0 \\
Dz &= 0
\end{align*}
\]
Load Matrices: The two load matrices were identical and had the following form:
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<thead>
<tr>
<th>X</th>
<th>0.0</th>
<th>Y</th>
<th>0.0</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>XX</td>
<td>0.51012242</td>
<td>YV</td>
<td>0.58316684</td>
<td>ZZ</td>
</tr>
<tr>
<td>ZV</td>
<td>0.97635388</td>
<td>XV</td>
<td>0.0</td>
<td>XZ</td>
</tr>
<tr>
<td>VP</td>
<td>0.30598438</td>
<td>YQ</td>
<td>0.28001078</td>
<td>ZS</td>
</tr>
<tr>
<td>VS</td>
<td>0.30598438</td>
<td>YZ</td>
<td>0.28001078</td>
<td>0.0</td>
</tr>
<tr>
<td>WP</td>
<td>1.14883327</td>
<td>WQ</td>
<td>-0.00000200</td>
<td>WZ</td>
</tr>
</tbody>
</table>

**Principal Stresses**

| S(1) | 1.41631317 | S(2) | -0.02467448 | S(3) | -0.51874450 |

**Principal Direction Cosines**

| B(1,1) | 0.0000 | B(2,1) | -1.0000 | B(3,1) | 0.0000 |
| B(2,2) | 0.0000 | B(2,2) | 1.0000 | B(3,2) | -0.0000 |
| B(3,3) | 0.0000 | B(2,3) | -1.0000 | B(3,3) | 0.0000 |

**Principal Stresses**

| S(1) | 1.02869987 | S(2) | -0.04920877 | S(3) | 1.48535995 |

**Principal Direction Cosines**

<p>| B(1,1) | 0.0000 | B(2,1) | -1.0000 | B(3,1) | 0.0000 |
| B(2,2) | 0.0000 | B(2,2) | 1.0000 | B(3,2) | -0.0000 |
| B(3,3) | 0.0000 | B(2,3) | -1.0000 | B(3,3) | 0.0000 |</p>
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<th></th>
<th>XX</th>
<th>YY</th>
<th>ZZ</th>
<th>XY</th>
<th>XZ</th>
<th>V</th>
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</thead>
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<td>0.1773719</td>
<td>0.15050429</td>
<td>0.72884732</td>
<td>0.0</td>
<td>-0.0000387</td>
<td>0.0000000</td>
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<tr>
<td>Q</td>
<td>0.0</td>
<td>-0.000000010</td>
<td>0.00000023</td>
<td>-0.00000333</td>
<td>-0.2597153</td>
<td>-0.0000000</td>
</tr>
<tr>
<td>S</td>
<td>0.1773719</td>
<td>0.15050417</td>
<td>0.72884750</td>
<td>-0.00000333</td>
<td>-0.2597157</td>
<td>0.00000386</td>
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</tbody>
</table>

\[ \omega = \text{PRINCIPAL STRESSES} \]
\[ \omega = -0.00000025 \]
\[ \omega = 0.78534941 \]

\[ \Delta = \text{PRINCIPAL DIRECTION COSINES} \]
\[ \Delta = \text{pr}\]
<p>| | | | | | | |</p>
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<td></td>
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<tr>
<td>S(11)</td>
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<tr>
<td>PRINCIPAL DIRECTION COSINES</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>θ(1,1)</td>
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<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>θ(1,2)</td>
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<td>1.0000</td>
<td>0.0000</td>
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<tr>
<td>θ(1,3)</td>
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<td>0.0000</td>
<td>1.0000</td>
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<tr>
<td>θ(2,1)</td>
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<td>0.0000</td>
<td>0.0000</td>
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<tr>
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<td>θ(3,1)</td>
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<tr>
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<td>0.03100809</td>
<td>0.37233775</td>
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**WP** = 0.59133977  
**WQ** = -0.00000012

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<td>PRINCIPAL DIRECTION COSINES</td>
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**WP** = 0.531575720  
**WQ** = -0.00000001

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<th>S(3)</th>
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<tr>
<td>b(1,2)</td>
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<td>b(2,1)</td>
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<td>b(2,2)</td>
</tr>
<tr>
<td>b(1,3)</td>
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<td>b(3,1)</td>
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<td>b(3,2)</td>
</tr>
<tr>
<td>b(2,3)</td>
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<td></td>
<td>b(3,3)</td>
<td>-1.0000</td>
<td>b(3,3)</td>
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<table>
<thead>
<tr>
<th>X = 0.0</th>
<th>Y = 0.0</th>
<th>Z = 1.594444</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.00004650</td>
<td>0.02758582</td>
</tr>
<tr>
<td>Q</td>
<td>0.0</td>
<td>-0.00000002</td>
</tr>
<tr>
<td>S</td>
<td>0.00004650</td>
<td>0.02758584</td>
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</table>

**WP** = 0.531575720  
**WQ** = -0.00000001
<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>X'</th>
<th>Y'</th>
<th>Z'</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
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<td>0.01928773</td>
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<tr>
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<td>0.01928773</td>
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<td>-0.00000002</td>
<td>-0.3361259</td>
</tr>
</tbody>
</table>

WP = 0.48352551
WQ = -0.00000001
X' = 0.48352551

| S(1) = 0.22612745 | PRINCIPAL STRESSES | S(2) = -0.0000000025 | S(3) = 0.00000000 |
|---------------------------------|---------------------------------|---------------------------------|
| B(1,1) = 0.0000 | PRINCIPAL DIRECTION COSINES | B(2,1) = 1.0000 | d(3,1) = 0.0000 |
| B(1,2) = 0.0000 | B(2,2) = -1.0000 | d(3,2) = -0.0000 |
| B(1,3) = 0.0000 | B(2,3) = 1.0000 | d(3,3) = 0.0000 |

<table>
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<tr>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>X'</th>
<th>Y'</th>
<th>Z'</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>-0.00052694</td>
<td>0.01572476</td>
<td>0.19035697</td>
<td>0.0</td>
<td>-4.0000000025</td>
</tr>
<tr>
<td>Q</td>
<td>0.0</td>
<td>-0.00000002</td>
<td>0.00000001</td>
<td>-0.00000002</td>
<td>-4.0156589</td>
</tr>
<tr>
<td>S</td>
<td>-0.00052694</td>
<td>0.01572476</td>
<td>0.19035697</td>
<td>-0.00000002</td>
<td>-4.0130157</td>
</tr>
</tbody>
</table>

WP = 0.44216603
WQ = -0.00000001
X' = 0.44216603

| S(1) = 0.19138775 | PRINCIPAL STRESSES | S(2) = 0.00206638 | S(3) = 0.00000000 |
|---------------------------------|---------------------------------|---------------------------------|
| B(1,1) = 0.0000 | PRINCIPAL DIRECTION COSINES | B(2,1) = -1.0000 | d(3,1) = 0.0000 |
| B(1,2) = 0.0000 | B(2,2) = 1.0000 | d(3,2) = 0.0000 |
| B(1,3) = 0.0000 | B(2,3) = -1.0000 | d(3,3) = 0.0000 |
PLOT OF STRESSES ON THE VERTICAL AXIS
\( x = 0 \quad y = 0 \).

FIG. 19

RELATIVE DEPTH \( Z/R \)
### Table IV

#### Coordinates

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
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</thead>
</table>

#### Components of Stress Tensor

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<th>ZZ</th>
<th>XY</th>
<th>XZ</th>
<th>YZ</th>
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</thead>
<tbody>
<tr>
<td>Due to normal load</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Due to shearing load</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Superposition</td>
<td></td>
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</tbody>
</table>

#### Vertical Component of Displacement

<table>
<thead>
<tr>
<th></th>
<th>WP</th>
<th>WQ</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>Due to normal load</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Due to shearing load</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Superposition</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Principal Stresses

<table>
<thead>
<tr>
<th></th>
<th>σ₁</th>
<th>σ₂</th>
<th>σ₃</th>
</tr>
</thead>
</table>

#### Principal Direction Cosines

<table>
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<tr>
<th></th>
<th>m₁</th>
<th>m₂</th>
<th>m₃</th>
<th>n₁</th>
<th>n₂</th>
<th>n₃</th>
</tr>
</thead>
</table>
COMPARISON OF RESULTS WITH NEWMARK'S METHOD

The results obtained in this example can be verified by Newmark's influence charts (Reference 13). The procedure, for the case of the vertical stress $G_2$ due to the normal load, is shown in Figs. 20 and 21 which corresponds to points $P_1(0,0,1)$ and $P_2(0,0,2)$ respectively.

**POINT P(0,0,1)**

\[ G_2 = \sum_{i=1}^{n} \text{influence value} \times \text{Surface stress intensity} \]

\[ = 24 \times 0.02 \times 1 = 0.48 \]

The result obtained with the computer is $G_2 = 0.48160243$
POINT P(0, 0, 2)

The result obtained with the computer is $= 0.19035697$
CHAPTER 10

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

The advantages of this method (generality, speed and accuracy) become evident after the example and verifications shown in Chapter 9. The results produced are in very good agreement with those obtained using Newmark's influence charts. To these advantages also can be added the possibility of producing the output of this program in graphical form using any type of graphical device coupled to the computer. Appendix E shows an example of computer plotting of the vertical stress $G_z$ along a vertical axis.

Considering the actual stage and future evolution of computer science and technology it seems important to extend the capabilities of this method towards the solution of the general problem mentioned in the introduction (layered half-space with time dependent properties, subjected to time-varying moving loads) and develop the necessary structure of commands to present it definitely in the form of a problem oriented language.
APPENDIX A

DEFINITION OF SYMBOLS

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>SYMBOL</th>
<th>SYMBOL IN PROGRAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Element of normal load</td>
<td>$P_{ij}$</td>
<td>$P(I,J)$</td>
</tr>
<tr>
<td>Element of shearing load</td>
<td>$q_{ij}$</td>
<td>$Q(I,J)$</td>
</tr>
<tr>
<td>Row number of load matrices</td>
<td>$i$</td>
<td>$I$</td>
</tr>
<tr>
<td>Column number of load matrices</td>
<td>$j$</td>
<td>$J$</td>
</tr>
<tr>
<td>Coordinates of point of half-space</td>
<td>$x,y,z$</td>
<td>$X,Y,Z$</td>
</tr>
<tr>
<td>Coord. of load elem $ij$ with resp. to main system</td>
<td>$x_A', y_A', z_A$</td>
<td>$X_A', Y_A', Z_A$</td>
</tr>
<tr>
<td>Coord. of points of half-space with resp. to our system</td>
<td>$x', y', z'$</td>
<td>$X_P', Y_P', Z_P$</td>
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<tr>
<td>Equiv radius of Square ring 1</td>
<td>$R_1$</td>
<td>$R_1$</td>
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<tr>
<td>Equiv radius of Square ring 2</td>
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<td>$R_2$</td>
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<tr>
<td>Equiv radius of Square ring 3</td>
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<td>$R_3$</td>
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<tr>
<td>Side of error prism</td>
<td>$\rho$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>Side of grid element</td>
<td>$\Delta \rho$</td>
<td>$\Delta \rho$</td>
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<tr>
<td>Number of grid elements in Ring 1</td>
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<td>$K_1$</td>
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<td>Number of grid elements in Ring 2</td>
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<tr>
<td>Number of grid elements in Ring 3</td>
<td>$K_3$</td>
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<td>Max. No. of grid elems in Ring 1</td>
<td>$K_1\text{MAX}$</td>
<td>$K_1\text{MAX}$</td>
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<tr>
<td>Max. No. of grid elems in Ring 2</td>
<td>$K_2\text{MAX}$</td>
<td>$K_2\text{MAX}$</td>
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<tr>
<td>Max. No. of grid elems in Ring 3</td>
<td>$K_3\text{MAX}$</td>
<td>$K_3\text{MAX}$</td>
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<tr>
<td>Average intensity of normal load in rings $1$, $2$, $3$</td>
<td>$\rho_{101}, \rho_{202}, \rho_{303}$</td>
<td>$\rho(1), \rho(2), \rho(3)$</td>
</tr>
<tr>
<td>Average intensity of shearing load in rings $1$, $2$, $3$</td>
<td>$q_{101}, q_{202}, q_{303}$</td>
<td>$q(1), q(2), q(3)$</td>
</tr>
<tr>
<td>Increments of Coordinates</td>
<td>$\Delta x$, $\Delta y$, $\Delta z$</td>
<td>$\Delta x$, $\Delta y$, $\Delta z$</td>
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<td>VARIABLE</td>
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<td>e1, e2, e3</td>
<td>XIM, YIM, ZIM</td>
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<td>Modulus of Elasticity</td>
<td>E</td>
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<td>Stress components due to normal distribution loads</td>
<td>(2)</td>
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<td>Stress components due to shearing point loads</td>
<td>(3)</td>
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</tr>
<tr>
<td>Stress components due to shearing distribution loads</td>
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<td>Sum of 1 and 2</td>
<td>(5)</td>
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<td>Sum of 3 and 4</td>
<td>(6)</td>
<td>P3 + P4</td>
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<td>Sum of 5 and 6</td>
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<td>P5 + P6</td>
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<td>Principal Stresses</td>
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<td>2 Due to normal dist. loads</td>
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<td>5 Due to shearing distrib. load</td>
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<td>6 Sum of 4 and 5</td>
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<td>WQ</td>
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<td>7 Sum of 3 and 6</td>
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<td>that program is to be repeated</td>
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APPENDIX B

LIST OF FIGURES

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APPENDIX C

LIST OF TABLES

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<td>III</td>
<td>42</td>
</tr>
<tr>
<td>IV</td>
<td>78</td>
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</table>
APPENDIX  D

UNIFORM CIRCULAR LOAD

Expressions of the Cartesian components of the stress tensor and vertical components of the displacement vector for points on the vertical axis for normal and horizontal loads.

ARISTIDES BRYAN DOMINGUEZ

1966
1. NORMAL LOAD.

For a single normal point load, the cartesian components of the stress tensor are:

\[ \sigma_x = \frac{P}{2\pi} \left( \frac{3Zx^2}{D^3} - (1-2\nu) \left[ \frac{x^2 - y^2}{\rho^2D(D+2)} + \frac{2y^2}{\rho^2D} \right] \right) \]  \hspace{1cm} (1)

\[ \sigma_y = \frac{P}{2\pi} \left( \frac{3Zy^2}{D^3} - (1-2\nu) \left[ \frac{y^2 - x^2}{\rho^2D(D+2)} + \frac{2x^2}{\rho^2D} \right] \right) \]  \hspace{1cm} (2)

\[ \sigma_z = \frac{3P}{2\pi} \cdot \frac{Z^3}{D^5} \]  \hspace{1cm} (3)

\[ \tau_{zx} = \frac{3P}{2\pi} \cdot \frac{Z^2x}{D^5} \]  \hspace{1cm} (4)

\[ \tau_{yz} = \frac{3P}{2\pi} \cdot \frac{Z^2y}{D^5} \]  \hspace{1cm} (5)

\[ \tau_{xy} = 0 \]  \hspace{1cm} (6)

The corresponding expressions for the uniform circular load (for points on the z axis) are obtained by integration of these for the whole circle.
\[ dP = \rho \cdot dA \]

\[ x^2 + y^2 + z^2 = \rho^2 + z^2 \]

\[ x = \rho \cdot \cos \varphi \]

\[ y = \rho \cdot \sin \varphi \]

\[ dP = \rho \cdot dA = \rho \cdot \rho \cdot d\rho \cdot d\varphi \]
\[ dG_z = \frac{3 \rho}{2\pi} \int_0^{\varphi_2} \int_0^{\rho_2} \frac{\rho}{(\rho^2 + z^2)^{3/2}} d\rho \, d\varphi \]

\[ G_z = \frac{3 \rho}{2\pi} \int_0^{\varphi_2} \int_0^{\rho_2} \frac{\rho}{(\rho^2 + z^2)^{3/2}} d\rho \]

\[ = \frac{3 \rho}{2\pi} \int_0^{\varphi_2} \left[ \frac{\varphi}{\varphi_1} \left[ \frac{-1}{3(\rho^2 + z^2)^{3/2}} \right] \right] d\varphi \]

Integrating over \( \frac{\pi}{4} \) of the circle and multiplying by 4, the limits are:

\[ \varphi_1 = 0 \quad \varphi_2 = \frac{\pi}{2} \quad \rho_1 = 0 \quad \rho_2 = R \]

\[ G_z = 4 \cdot \frac{3 \rho}{2\pi} \int_0^{\varphi_2} \left[ \frac{\varphi}{\varphi_1} \left[ \frac{-1}{3(\rho^2 + z^2)^{3/2}} + \frac{1}{(R^2 + z^2)^{3/2}} \right] \right] d\varphi \]

\[ = \rho \cdot \frac{z^3}{(R^2 + z^2)^{3/2}} \left[ 1 - \frac{2^3}{(R^2 + z^2)^{3/2}} \right] = \rho \left[ 1 \right] \]

\[ G_z = \rho \left[ 1 - \frac{2^3}{(R^2 + z^2)^{3/2}} \right] \]
b. \( \tau_{xz} \). Differentiating formula (4) with respect to \( \rho \) and substituting expressions on pgs. 2.

\[
d \tau_{xz} = \frac{3 \rho}{2 \pi} Z^2 \frac{\rho^2 \cos \varphi}{(\rho^2 + Z^2)^{3/2}} \cdot d\rho \cdot d\varphi
\]

\[
\tau_{xz} = \frac{3 \rho}{2 \pi} Z^2 \int_{\varphi_1}^{\varphi_2} \frac{\rho^2}{\cos \varphi} \int_{\rho_1}^{\rho_2} \frac{\rho^2}{(\rho^2 + Z^2)^{3/2}} d\rho \cdot d\varphi
\]

\[
\tau_{xz} = \frac{3 \rho}{2 \pi} Z^2 \left[ \sin \varphi \right]^{\varphi_2}_{\varphi_1} \cdot \frac{1}{3 Z^2} \left[ \frac{\rho^3}{(\rho^2 + Z^2)^{3/2}} \right]^{\rho_2}_{\rho_1}
\]

Substituting the limits as before and multiplying by 4:

\[
\tau_{xz} = \frac{4 \rho}{2 \pi} \left[ 1 \right] \left[ \frac{R^3}{(R^2 + Z^2)^{3/2}} \right]
\]

\[
\tau_{xz} = \frac{2 \rho}{\pi} \frac{R^3}{(R^2 + Z^2)^{3/2}}
\]
c. $\tau_{yz}$: Differentiating formula (5) with respect to $\rho$ and using expressions on pg. 2:

$$d\tau_{yz} = \frac{3P}{2\pi} \frac{z^2}{(\rho^2 + z^2)^{3/2}} \frac{\rho^2 \sin \varphi}{\rho} \, d\rho \, d\varphi$$

$$\tau_{yz} = \frac{3P}{2\pi} \frac{z^2}{\rho} \int_0^{\varphi_2} \sin \varphi \int_{\rho_1}^{\rho_2} \frac{\rho^2}{(\rho^2 + z^2)^{3/2}} \, d\rho \, d\varphi$$

$$\tau_{yz} = \frac{3P}{2\pi} \frac{z^2}{\rho} \left[- \cos \varphi\right] \int_{\rho_1}^{\rho_2} \frac{\rho^3}{(\rho^2 + z^2)^{3/2}} \, d\rho$$

Substituting the limits and multiplying by 4:

$$\tau_{yz} = -4 \cdot \frac{P}{2\pi} \left[0 - 1\right] \frac{R^3}{(R^2 + z^2)^{3/2}}$$

$$\tau_{yz} = \frac{2P}{\pi} \frac{R^3}{(R^2 + z^2)^{3/2}}$$
Differentiating formula (1) with respect to $\rho$ and using expressions on pag. 2:

\[
d\sigma_x = \frac{\rho}{2\pi} \left\{ 3Z \frac{\rho^3 \cos^2 \varphi}{(\rho^2 + Z^2)^{3/2}} - (1-2\nu) \left[ \frac{\rho^3 (\cos^2 \varphi - \sin^2 \varphi)}{\rho^2 (\rho^2 + Z^2)^{3/2}} \right] \right\} d\rho \, d\varphi
\]

\[
d\sigma_x = \frac{\rho}{2\pi} \left\{ 3Z \frac{\rho^3 \cos^2 \varphi}{(\rho^2 + Z^2)^{3/2}} - (1-2\nu) \left[ \frac{\rho \cos 2\varphi}{(\rho^2 + Z^2) + Z(\rho^2 + Z^2)^{3/2}} \right] \right\} d\rho \, d\varphi
\]

1st term.

\[
3Z \int_{\varphi_1}^{\varphi_2} \frac{\cos^2 \varphi}{\rho_1} \left( \int_{\rho_1}^{\rho_2} \frac{\rho^3}{(\rho^2 + Z^2)^{3/2}} \right) \, d\rho \, d\varphi =
\]

\[
= 3Z \left[ \frac{\varphi_2}{2} + \frac{\sin 2\varphi}{4} \right]_{\varphi_1}^{\varphi_2} \left[ \frac{-1}{(\rho^2 + Z^2)^{1/2}} + \frac{Z^2}{3(\rho^2 + Z^2)^{3/2}} \right]_{\rho_1}^{\rho_2}
\]

Substituting the limits and multiplying by 4:

\[
3\pi Z \left[ \frac{Z^2}{3(R^2 + Z^2)^{3/2}} - \frac{1}{(R^2 + Z^2)^{1/2}} + \frac{2}{3Z} \right]
\]
2nd. term.

\[
\cos 2\varphi \frac{\rho(\rho^2 + z^2) - 2\rho(\rho^2 + z^2)^{1/2}}{\rho^2(\rho^2 + z^2)} \, d\rho \, d\varphi =
\]

\[
= \cos 2\varphi \left[ \frac{1}{\rho} - \frac{z}{\rho(\rho^2 + z^2)^{1/2}} \right] \, d\rho \, d\varphi
\]

\[
\int_{\varphi_1}^{\varphi_2} \cos 2\varphi \int_{\rho_1}^{\rho_2} \left[ \frac{1}{\rho} - \frac{z}{\rho(\rho^2 + z^2)^{1/2}} \right] \, d\rho \, d\varphi =
\]

\[
= \frac{1}{2} \left[ \sin 2\varphi \right]_{\varphi_1}^{\varphi_2} \left[ \ln \rho - z \left( - \frac{1}{z} \ln \left| \frac{z + (\rho^2 + z^2)^{1/2}}{\rho} \right| \right) \right]_{\rho_1}^{\rho_2}
\]

Substituting the limits and multiplying by 4:

\[
4 \times \frac{1}{2} \left[ 0 - 0 \right] \left[ \ln \rho + \ln \left| \frac{z + (\rho^2 + z^2)^{1/2}}{\rho} \right| \right]_{\rho_1}^{\rho_2} = 0
\]

3rd. term.

\[
z \int_{\varphi_1}^{\varphi_2} \sin^2 \varphi \int_{\rho_1}^{\rho_2} \frac{\rho}{(\rho^2 + z^2)^{3/2}} \, d\rho \, d\varphi =
\]

\[
= z \left[ \frac{\varphi}{2} - \frac{\sin 2\varphi}{4} \right]_{\varphi_1}^{\varphi_2} \left[ \frac{-1}{(\rho^2 + z^2)^{1/2}} \right]_{\rho_1}^{\rho_2}
\]
\[
\left[ 6 + (z-1) + \frac{v(z^2+z\vartheta)}{z(z+1)} - \frac{v^2z}{z} \right] \frac{\partial}{\partial d} = z^C
\]

\[
\left\{ \left[ \frac{v(z^2+z\vartheta)}{z} \right] \frac{\partial}{\partial (z^2-1)} - \frac{vz}{z} \frac{\partial}{\partial z} \right\} \frac{\delta z}{\delta} = z^C
\]

\[
\left[ \frac{v(z^2+z\vartheta)}{z} + 1 \right] \mathcal{U} = \left[ \frac{z}{z} + \frac{v(z^2+z\vartheta)}{z} \right] \cdot \left[ \frac{4}{1} \right] z
\]

Solving the limits and modifying by \( z \).
Differentiating formula (2) with respect to \( \rho \) and substituting expressions on pg. 2:

\[
d\theta = \frac{\rho}{2\pi} \left\{ 3z \frac{\rho^3 \sin^2 \varphi}{(\rho^2 + z^2)^{3/2}} - (1 - 2\varphi) \frac{\rho^3 \cos 2\varphi}{\rho^2 (\rho^2 + z^2)^{3/2}} \right\} \, d\rho \, d\varphi
\]

**1st. term.**

\[
3z \int_{\rho_1}^{\rho_2} \sin^2 \varphi \int_{\rho_1}^{\rho_2} \frac{\rho^3}{(\rho^2 + z^2)^{3/2}} \, d\rho \, d\varphi =
\]

\[
= 3z \left[ \frac{\rho}{2} - \frac{\sin 2\varphi}{4} \right]_{\rho_1}^{\rho_2} \left[ \frac{-1}{(\rho^2 + z^2)^{1/2}} + \frac{Z^2}{3(\rho^2 + z^2)^{3/2}} \right]_{\rho_1}^{\rho_2}
\]

Substituting the limits and multiplying by 4:

\[
4 \times 3z \left[ \frac{\pi}{4} \right] \left[ \frac{-1}{(R^2 + Z^2)^{1/2}} + \frac{1}{Z} + \frac{Z^2}{3(R^2 + Z^2)^{3/2}} - \frac{Z^2}{3Z^3} \right] =
\]

\[
= 3\pi z \left[ \frac{Z^2}{3(R^2 + Z^2)^{3/2}} - \frac{1}{(R^2 + Z^2)^{1/2}} + \frac{2}{3Z} \right]
\]
2nd. term.

\[
\cos 2\varphi \cdot \frac{\rho (\rho^2 + z^2) - Z \rho (\rho^2 + z^2)^{1/2}}{\rho^2 (\rho^2 + z^2)} \, d\rho \, d\varphi
\]

\[
\int_{\varphi_1}^{\varphi_2} \int_{\rho_1}^{\rho_2} \cos 2\varphi \cdot \left[ \frac{1}{\rho} - \frac{Z}{\rho (\rho^2 + z^2)} \right] \, d\rho \, d\varphi =
\]

\[
= \frac{1}{2} \left[ \sin 2\varphi \right]_{\varphi_1}^{\varphi_2} \left[ \ln \rho - Z \left( - \frac{1}{Z} \ln \left| \frac{Z + (\rho^2 + z^2)^{1/2}}{\rho} \right| \right) \right]_{\rho_1}^{\rho_2}
\]

Substituting the limits and multiplying by 4:

\[
4 \times \frac{1}{2} \left[ 0 - 0 \right] \left[ \ln \rho - Z \left( - \frac{1}{Z} \ln \left| \frac{Z + (\rho^2 + z^2)^{1/2}}{\rho} \right| \right) \right]_{\rho_1}^{\rho_2} = 0
\]

3rd. term.

\[
2^2 \int_{\varphi_1}^{\varphi_2} \int_{\rho_1}^{\rho_2} \frac{\rho}{(\rho^2 + z^2)^{3/2}} \cdot d\rho \, d\varphi = 2 \left[ \frac{\varphi}{2} + \frac{\sin 2\varphi}{4} \right]_{\varphi_1}^{\varphi_2} \cdot
\]

\[
\cdot \left[ \frac{-1}{(\rho^2 + z^2)^{1/2}} \right]_{\rho_1}^{\rho_2}
\]

Substituting the limits and multiplying by 4:

- 10 -
\[ 4. \, z \left[ \frac{\pi}{4} \right] \left[ \frac{-1}{(R^2 + Z^2)^{1/2}} + \frac{1}{Z} \right] = \mathcal{J} \left[ 1 - \frac{Z}{(R^2 + Z^2)^{1/2}} \right] \]

\[ \sigma_y = \frac{P}{2\pi} \left\{ 3\pi Z \left[ \frac{Z^2}{3(R^2 + Z^2)^{3/2}} - \frac{1}{(R^2 + Z^2)^{1/2}} + \frac{2}{3Z} \right] - (1-2\nu) \mathcal{J} \left[ 1 - \frac{Z}{(R^2 + Z^2)^{1/2}} \right] \right\} \]

\[ \sigma_y = \frac{P}{2} \left[ \frac{Z^3}{(R^2 + Z^2)^{3/2}} - \frac{2(1+\nu)Z}{(R^2 + Z^2)^{1/2}} + (1-2\nu) + 6 \right] \]
2.- HORIZONTAL LOAD.

For a single horizontal point load applied in the X direction, the cartesian components of the stress tensor are:

\[ G_x = \frac{Q_x}{2\pi D^3} \left[ \frac{3x^2}{D^2} + \frac{1-2\nu}{(D+Z)^2} \left( D^2 - \frac{y^2}{D+Z} - \frac{2Dy^2}{D+Z} \right) \right] \]  
\[ G_y = \frac{Q_x}{2\pi D^3} \left[ \frac{3y^2}{D^2} + \frac{1-2\nu}{(D+Z)^2} \left( \frac{3D^2 - x^2}{D+Z} - \frac{2Dx^2}{D+Z} \right) \right] \]  
\[ G_z = -\frac{3Q_xZ^2}{2\pi D^5} \]  
\[ \tau_{xy} = \frac{Q_y}{2\pi D^3} \left[ \frac{3x^2}{D^2} + \frac{1-2\nu}{(D+Z)^2} \left( -D^2 + \frac{x^2}{D+Z} + \frac{2Dx^2}{D+Z} \right) \right] \]  
\[ \tau_{yz} = -\frac{3Q_yZ}{2\pi D^5} \]  
\[ \tau_{xz} = -\frac{3Q_xZ^2}{2\pi D^5} \]

The corresponding expressions for the uniform circular load (for points on the Z axis) are obtained by integration of these for the whole circle.
\[ D^2 = x^2 + y^2 + z^2 = \rho^2 + z^2 \]
\[ x = \rho \cdot \cos \varphi \]
\[ y = \rho \cdot \sin \varphi \]
\[ d\theta = q \cdot dA = q \cdot \rho \cdot d\rho \cdot d\varphi \]
9. \( G_z \). Differentiating formula (9) with respect to \( \rho \) and using expressions on pg. 13:

\[
\begin{align*}
\rho G_z &= - \frac{3 \varrho z^2}{2 \pi} \int_0^{\varphi_2} \cos \varphi \, \, \int_0^{\rho_2} \frac{\rho^2}{(\rho^2 + z^2)^{3/2}} \, d\rho \, d\varphi \\
&= - \frac{3 \varrho}{2 \pi} \, z^2 \left[ \sin \varphi \right]_0^{\varphi_2} \cdot \frac{1}{3z^2} \left[ \frac{\rho^3}{(\rho^2 + z^2)^{3/2}} \right]_0^{\rho_2}
\end{align*}
\]

Substituting the limits and multiplying by 4:

\[
G_z = - 4 \times \frac{\varrho}{2 \pi} \left[ 1 - 0 \right] \left[ \frac{R^3}{(R^2 + Z^2)^{3/2}} - 0 \right]
\]

\[
G_z = - \frac{2 \varrho}{\pi} \frac{R^3}{(R^2 + Z^2)^{3/2}}
\]
6. \( \tau_{yz} \). Differentiating formula (11) with respect to \( \theta \) and substituting expressions on peg. 13:

\[
d\tau_{yz} = -\frac{3q z}{2\pi} \frac{\rho^3 \sin \phi \cos \phi}{(\rho^2 + z^2)^{5/2}} \cdot dp \cdot d\phi
\]

\[
\tau_{yz} = -\frac{3q z}{2\pi} \int_{\phi_1}^{\phi_2} \frac{\rho^2}{\sin \phi \cos \phi} \int_{\rho_1}^{\rho_2} \frac{\rho^3}{(\rho^2 + z^2)^{5/2}} \cdot dp \cdot d\phi =
\]

\[
= -\frac{3q z}{2\pi} \left[ \frac{\sin^2 \phi}{2} \right]_{\phi_1}^{\phi_2} \left[ \frac{-1}{(\rho^2 + z^2)^{3/2}} + \frac{z^2}{3(\rho^2 + z^2)^{3/2}} \right]_{\rho_1}^{\rho_2}
\]

Substituting the limits and multiplying by 4:

\[
\tau_{yz} = -4 \frac{3q z}{2\pi} \left[ \frac{1 - 0}{2} \right] \left[ \frac{-1}{(R^2 + z^2)^{3/2}} + \frac{z^2}{3(R^2 + z^2)^{3/2}} \right] -
\]

\[
= \left[ -\frac{1}{(z^2)^{3/2}} + \frac{z^2}{3(z^2)^{3/2}} \right]
\]

\[
\tau_{yz} = -\frac{3q}{\pi} \left[ \frac{2}{3} + \frac{z^3}{3(R^2 + z^2)^{3/2}} - \frac{z}{(R^2 + z^2)^{3/2}} \right]
\]
c. \( \tau_{xz} \). Differentiating formula (12) with respect to \( \theta \) and substituting expressions on pág. 13:

\[
d\tau_{xz} = - \frac{3\theta}{2\pi} \frac{R^3 \cos^3 \theta}{(\rho^2 + Z^2)^{3/2}} \cdot d\rho \cdot d\phi
\]

\[
\tau_{xz} = - \frac{3\theta}{2\pi} \int_{\rho_1}^{\rho_2} \cos \phi \int_{\phi_1}^{\phi_2} \frac{R^3}{(\rho^2 + Z^2)^{3/2}} \cdot d\rho \cdot d\phi =
\]

\[
= - \frac{3\theta}{2\pi} \left[ \frac{\phi}{2} + \frac{\sin 2\phi}{4} \right]_{\phi_1}^{\phi_2} \cdot \left[ \frac{-1}{(\rho^2 + Z^2)^{1/2}} + \frac{Z^2}{3(\rho^2 + Z^2)^{3/2}} \right]_{\rho_1}^{\rho_2}
\]

Substituting the limits and multiplying by 4:

\[
\tau_{xz} = - 4 \cdot \frac{3\theta}{2\pi} \left[ \frac{\pi}{4} - 0 \right] \cdot \left[ \frac{-1}{(R^2 + Z^2)^{1/2}} + \frac{1}{(Z^2)^{1/2}} + \frac{Z^2}{3(R^2 + Z^2)^{3/2}} \right]
\]

\[
+ \frac{Z^2}{3(Z^2)^{3/2}}
\]

\[
\tau_{xz} = - \frac{3\theta}{2} \left[ \frac{2}{3} + \frac{Z^3}{3(R^2 + Z^2)^{3/2}} - \frac{Z}{(R^2 + Z^2)^{1/2}} \right]
\]
\[ dG_x = \frac{9}{2 \pi} \left\{ -3 \frac{\rho^4 \cos \varphi}{(\rho^2 + z^2)^{3/2}} + (1 - 2 \nu) \left[ \frac{\rho^2 \cos \varphi}{(\rho^2 + z^2)^{3/2}} \left[ \frac{1}{(\rho^2 + z^2)^{3/2}} \right]^2 - \frac{\rho^2 \cos \varphi \sin^2 \varphi}{(\rho^2 + z^2)^{3/2}} \frac{2 \rho^4 \cos \varphi \sin^2 \varphi}{(\rho^2 + z^2)^{3/2}} \right] \right\} \, dp \, d\varphi \]

**First term**

\[ -3 \int_{\varphi_1}^{\varphi_2} \frac{\rho^2}{\rho_1} \cos \varphi \left( \frac{\rho^4}{(\rho^2 + z^2)^{3/2}} \right) \, dp \, d\varphi = \]

\[ = -3 \int_{\varphi_1}^{\varphi_2} \left[ \sin \varphi - \frac{\sin^3 \varphi}{3} \right] \, dp \, d\varphi \]

\[ = -3 \left[ \frac{-\rho}{(\rho^2 + Z^2)^{3/2}} - \frac{\rho^3}{3(\rho^2 + Z^2)^{3/2}} + \ln \left[ (\rho^2 + Z^2)^{3/2} + \rho \right] \right] \]

Substituting the limit and multiplying by 4:

\[ -8 \left\{ \frac{-R}{(R^2 + Z^2)^{3/2}} - \frac{R^3}{3(R^2 + Z^2)^{3/2}} + \ln \left[ \frac{R^2 + (R^2 + Z^2)^{1/2}}{Z} \right] \right\} \]

**2nd term**

\[ \frac{\rho^2 \cos \varphi \, dp \, d\varphi}{(\rho^2 + Z^2)^{3/2} + 2Z(\rho^2 + Z^2) + Z^2(\rho^2 + Z^2)^{1/2}} \]

Making:

\[ (\rho^2 + Z^2)^{1/2} = x, \quad Z = \alpha, \quad \rho^2 = x^2 - \alpha^2 \]

- 17 -
\[
\frac{(x^2 - a^2) \cos \varphi \ d\rho \ d\varphi}{x^3 + 2ax^2 + ax} = \frac{(x+a)(x-a) \cos \varphi \ d\rho \ d\varphi}{x(x+a)(x-a)} = \\
= \cos \varphi \left[ \frac{x}{x(x+a)} - \frac{x}{x(x+a)} \right] d\rho \ d\varphi = \cos \varphi \left[ \frac{1}{x+a} \right. \\
- \frac{a}{x(x+a)} \right] d\rho \ d\varphi
\]

Then substituting back:

\[
\cos \varphi \left[ \frac{1}{(\rho^2+z^2)^{3/2} + z} - \frac{z}{(\rho^2+z^2)^{3/2}[(\rho^2+z^2)^{1/2} + z]} \right] d\rho \ d\varphi
\]

Multiplying and dividing each term by the conjugate of its denominator:

\[
\cos \varphi \left[ \frac{(\rho^2+z^2)^{3/2} - z}{\rho^2} - \frac{z(\rho^2+z^2) - z^2(\rho^2+z^2)^{1/2}}{\rho^2(\rho^2+z^2)} \right] d\rho \ d\varphi = \\
= \cos \varphi \left[ \frac{(\rho^2+z^2)^{1/2}}{\rho^2} - \frac{z}{\rho^2} - \frac{z}{\rho^2} + \frac{z}{\rho^2(\rho^2+z^2)^{1/2}} \right] d\rho \ d\varphi = \\
= \int_{\varphi_1}^{\varphi_2} \cos \varphi \int_{\rho_1}^{\rho_2} \left[ \frac{(\rho^2+z^2)^{1/2}}{\rho^2} - \frac{2z}{\rho^2} + \frac{z}{\rho^2(\rho^2+z^2)^{1/2}} \right] d\rho \ d\varphi = \\
= \left[ \sin \varphi \right]_{\varphi_1}^{\varphi_2} \left\{ -\frac{(\rho^2+z^2)^{1/2}}{\rho} + \ln \left[ \rho + (\rho^2+z^2)^{1/2} \right] + \frac{4z}{\rho} - \\
- \frac{(\rho^2+z^2)^{1/2}}{z^2 \rho} \right\} \left[ \rho_2 \right]_{\rho_1}^{\rho_2}
\]
\[ = \left[ \sin \varphi \right]_{\varphi_1}^{\varphi_2} \left\{ - \frac{2(\rho^2 + z^2)^{1/2}}{\rho} + \frac{4z}{\rho} + \ln \left[ \rho + (\rho^2 + z^2)^{1/2} \right] \right\} \, d\rho \, d\varphi. \]

Substituting the limits, the first and second terms become indeterminate. For \( \rho = 0 \), applying L'Hopital's rule:

1) \( -2 \cdot \frac{1}{2} \frac{(\rho^2 + z^2)^{-1/2}}{2\varphi} \bigg|_{\rho=0} = -2 \rho (\rho^2 + z^2)^{-1/2} \bigg|_{\rho=0} = 0 \)

2) \( \frac{0}{1} = 0 \)

\[ = 4 \left[ 1 - 0 \right] \left\{ - \frac{2(R^2 + z^2)^{1/2}}{R} + \frac{2z}{R} + \ln \left[ R + (R^2 + z^2)^{1/2} \right] - \ln z \right\} \]

\[ = 4 \left\{ - \frac{2(R^2 + z^2)^{1/2}}{R} + \frac{2z}{R} + \ln \left[ \frac{R + (R^2 + z^2)^{1/2}}{R} \right] \right\} \]

3rd term.

\[ \frac{\rho^4 \cos \varphi \sin^2 \varphi \, d\rho \, d\varphi}{(\rho^2 + z^2)^{3/2} + 2z(\rho^2 + z^2)^{1/2} + z^2(\rho^2 + z^2)^{3/2}} \]

Using the same substitution of the last case:

\[ \cos \varphi \sin^2 \varphi \frac{(x^2 - a^2)(x^2 - a^2)}{x^4 + 2ax^2 + a^2x^3} = \cos \varphi \sin^2 \varphi \frac{(x-a)^2}{x^3} \]

Substituting back:
\[ \cos \varphi \sin^2 \varphi \left[ \frac{\left(\rho^2 + z^2\right)^{\frac{3}{2}} - 2z}{\left(\rho^2 + z^2\right)^{\frac{3}{2}}} \right] \, d\rho \, d\varphi = \]

\[ = \cos \varphi \sin^2 \varphi \left[ \frac{\rho \left(\rho^2 + z^2\right)^{\frac{3}{2}} - 2z \left(\rho^2 + z^2\right)^{\frac{1}{2}} + z^2}{\left(\rho^2 + z^2\right)^{\frac{3}{2}}} \right] \, d\rho \, d\varphi \]

\[ = \int_{\Phi_{1}}^{\Phi_{2}} \cos \varphi \sin^2 \varphi \int_{\rho_{1}}^{\rho_{2}} \left[ \frac{1}{\left(\rho^2 + z^2\right)^{\frac{3}{2}}} - \frac{2z}{\left(\rho^2 + z^2\right)^{\frac{1}{2}}} + \frac{2^2}{\left(\rho^2 + z^2\right)^{\frac{3}{2}}} \right] \, d\rho \, d\varphi \]

\[ = \left[ \frac{\sin^3 \varphi}{3} \right]_{\Phi_{1}}^{\Phi_{2}} \left\{ \ln \left[ \rho \left(\rho^2 + z^2\right)^{\frac{3}{2}} \right] - \frac{2z}{z} \tan^{-1} \frac{\rho}{z} + \frac{z^2 \rho}{z^2 \left(\rho^2 + z^2\right)^{\frac{1}{2}}} \right\} \]

\[ = \left[ \frac{\sin^3 \varphi}{3} \right]_{\Phi_{1}}^{\Phi_{2}} \left\{ \ln \left[ \rho \left(\rho^2 + z^2\right)^{\frac{3}{2}} \right] - 2 \tan^{-1} \frac{\rho}{z} + \frac{\rho}{\left(\rho^2 + z^2\right)^{\frac{1}{2}}} \right\} \]

Substituting the limits and multiplying by 4:

\[ \frac{4}{3} \left\{ \ln \left[ \frac{R + \left(R^2 + Z^2\right)^{\frac{3}{2}}}{Z} \right] - \ln Z - 2 \tan^{-1} \frac{R}{Z} + \frac{R}{\left(R^2 + Z^2\right)^{\frac{3}{2}}} \right\} = \]

\[ = \frac{4}{3} \left\{ \ln \left( \frac{R + \left(R^2 + Z^2\right)^{\frac{3}{2}}}{Z} \right) - 2 \tan^{-1} \frac{R}{Z} + \frac{R}{\left(R^2 + Z^2\right)^{\frac{3}{2}}} \right\} \]
\[ \text{4th term} \]
\[ 2 \cos \varphi \sin^2 \varphi \ \frac{\rho^4 \ d\rho \ d\varphi}{(\rho^2 + Z^2)^{\frac{3}{2}} + 3Z(\rho^2 + Z^2) + 3Z^2(\rho^2 + Z^2)^{\frac{1}{2}} + Z^4(\rho^2 + Z^2)} \]

Using the same substitution of the last case:
\[ \frac{(x^2 - \vartheta^2)(x^2 - \vartheta^2)}{x^5 + 3\vartheta x^4 + 3\vartheta^2 x^3 + \vartheta^3 x} = \frac{(x+\vartheta)(x-\vartheta)(x+\vartheta)(x-\vartheta)}{x^2(x+\vartheta)(x+\vartheta)} \]
\[ = \frac{x^2}{x^2(x+\vartheta)} - \frac{2x \vartheta}{x^2(x+\vartheta)} + \frac{\vartheta^2}{x^2(x+\vartheta)} \]
\[ = \frac{1}{x+\vartheta} - \frac{2 \vartheta}{x(x+\vartheta)} + \frac{\vartheta^2}{x^2(x+\vartheta)} \]

Substituting back:
\[ 2 \cos \varphi \sin^2 \varphi \left[ \frac{1}{(\rho^2 + Z^2)^{\frac{3}{2}} + Z^2} - \frac{2Z}{(\rho^2 + Z^2)^{\frac{3}{2}}(\rho^2 + Z^2)^{\frac{1}{2}}} \right] \]
\[ + \frac{Z^2}{(\rho^2 + Z^2)^{\frac{3}{2}}(\rho^2 + Z^2)^{\frac{1}{2}}} \ d\rho \ d\varphi \]

Multiplying and dividing each term by the conjugate of its denominator:
\[ 2 \cos \varphi \sin^2 \varphi \left[ \frac{(\rho^2 + Z^2)^{\frac{3}{2}} - Z^2}{\rho^2} - \frac{2Z(\rho^2 + Z^2) - 2Z(\rho^2 + Z^2)^{\frac{1}{2}}}{\rho^2(\rho^2 + Z^2)} \right] \]
\[ + \frac{Z^2(\rho^2 + Z^2)(\rho^2 + Z^2)^{\frac{1}{2}} - Z^2}{(\rho^2 + Z^2)^2(\rho^2 + Z^2 - Z^2)} \]
\[ 2 \cos \phi \sin^2 \phi \left[ \frac{(\rho^2 + Z^2)^{3/2}}{\rho^2} - \frac{Z}{\rho^2} - \frac{2Z}{\rho^2} + \frac{Z^2}{\rho^2(\rho^2 + Z^2)^{1/2}} + \frac{Z^2}{\rho^2(\rho^2 + Z^2)^{1/2}} - \frac{Z^3}{\rho^2(\rho^2 + Z^2)} \right] d\rho d\phi \]

\[ 2 \int_{\phi_1}^{\phi_2} \cos \phi \sin^2 \phi \int_{\rho_1}^{\rho_2} \left[ \frac{(\rho^2 + Z^2)^{3/2}}{\rho^2} - \frac{Z}{\rho^2} + \frac{Z^2}{\rho^2(\rho^2 + Z^2)^{1/2}} - \frac{Z^3}{\rho^2(\rho^2 + Z^2)} \right] d\rho d\phi \]

\[ = 2 \left[ \frac{\sin^3 \phi}{3} \right]_{\phi_1}^{\phi_2} \left\{ - \frac{(\rho^2 + Z^2)^{1/2}}{\rho} + \ln \left[ \rho + (\rho^2 + Z^2)^{1/2} \right] + \frac{6Z}{\rho} - \frac{3Z^2(\rho^2 + Z^2)^{1/2}}{Z^2 \rho} - Z^3 \left[ - \frac{1}{2^3 \rho} - \frac{1}{2^3 \tan^{-1} \frac{\rho}{Z}} \right] \right\}_{\rho_1}^{\rho_2} \]

Substituting the limits, the first and the second terms become

undetermined, using L'Hopital's rule:

1) \(-4 \times \frac{1}{2} (\rho^2 + Z^2)^{-1/2} \rho \bigg|_{\rho=0} = 0\)

2) \(\frac{0}{1} = 0\)

\[ 4 \times 2 \left\{ \frac{(1 - 0)}{3} \left[ - \frac{4(\rho^2 + Z^2)^{1/2}}{R} + \frac{7Z}{R} + \tan^{-1} \frac{R}{Z} + \ln \left[ R + (\rho^2 + Z^2)^{1/2} \right] - \ln Z \right] \right\} \]

- 22 -
\[
G_x = \frac{9}{2\pi} \left\{ \frac{8}{3(R^2 + Z^2)^{3/2}} - \frac{8}{3} \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right. \\
\left. + (1-2\nu) \left[ \frac{8}{3} \frac{(R^2 + Z^2)^{1/2}}{R} - \frac{4}{3} \frac{R}{(R^2 + Z^2)^{1/2}} - \frac{32 \nu}{3} \right] - \frac{16}{3} \left( 1 + \frac{1}{3} \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right) \right\}
\]
Differentiating formula (8) with respect to \( \theta \) and using expressions on page 13:

\[
dB_y = \frac{9}{2\pi} \left[ -\frac{3p^4 \cos \varphi \sin^2 \varphi}{\left( p^2 + Z^2 \right)^{3/2}} + (1 - 2\lambda) \left\{ \frac{3p^2 \cos \varphi}{\left( p^2 + Z^2 \right)^{3/2}} \left[ \frac{1}{\left( p^2 + Z^2 \right)^{3/2} + Z} \right]^2 \right\} \right]
\]

\[
= \frac{9p^4 \cos^3 \varphi}{\left( p^2 + Z^2 \right)^{3/2}} - \frac{2p^4 \cos \varphi}{\left( p^2 + Z^2 \right)^{3/2} \left[ \left( p^2 + Z^2 \right)^{1/2} + Z \right]^2}
\]

1st term:

\[
-3 \int_0^{\frac{\pi}{2}} \cos \varphi \sin^2 \varphi \int_{p_1}^{p_2} \frac{p^4}{\left( p^2 + Z^2 \right)^{3/2}} \cdot dp \cdot d\varphi = 0
\]

\[
= -3 \left[ \frac{\sin^3 \varphi}{3} \right]_{\varphi_1}^{\varphi_2} \left[ \frac{-p}{\left( p^2 + Z^2 \right)^{1/2}} - \frac{p^3}{3\left( p^2 + Z^2 \right)^{3/2}} + \ln \left[ \frac{p + \left( p^2 + Z^2 \right)^{1/2}}{Z} \right] \right]_{p_1}^{p_2}
\]

Substituting the limits and multiplying by 4:

\[
= -4 \times 3 \left[ 1 - \frac{0}{3} \right] \left\{ -\frac{R}{(R^2 + Z^2)^{1/2}} - \frac{R^3}{3(R^2 + Z^2)^{3/2}} + \ln \left[ \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right] \right\} = 0
\]

\[
= -4 \left\{ -\frac{R}{(R^2 + Z^2)^{1/2}} - \frac{R^3}{3(R^2 + Z^2)^{3/2}} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right\}
\]
2nd term.

$$3 \int_{\varphi_1}^{\varphi_2} \cos \varphi \int_{\rho_1}^{\rho_2} \frac{\rho^2 \, d\rho \, d\varphi}{(\rho^2 + Z^2)^{3/2} + 2Z (\rho^2 + Z^2) + Z(\rho^2 + Z^2)^{1/2}} =$$

$$= 12 \left\{ - \frac{2(R^2 + Z^2)^{1/2}}{R} + \frac{2Z}{R} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right\}$$

3rd term.

$$\int_{\varphi_1}^{\varphi_2} \cos \varphi \int_{\rho_1}^{\rho_2} \frac{\rho^4 \, d\rho \, d\varphi}{(\rho^2 + Z^2)^{5/2} + 2Z (\rho^2 + Z^2)^2 + 2^2(\rho^2 + Z^2)^{3/2}} =$$

$$\left[ \sin \varphi - \frac{\sin^3 \varphi}{3} \right]_{\varphi_1}^{\varphi_2} \left[ \ln \left( \rho + (\rho^2 + Z^2)^{1/2} \right) - 2 \tan^{-1} \frac{\rho}{Z} + \frac{\rho}{(\rho^2 + Z^2)^{1/2}} \right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$4 \left[ 1 - \frac{1}{3} \right] \left[ \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} - 2 \tan^{-1} \frac{R}{Z} + \frac{R}{(R^2 + Z^2)^{1/2}} \right] =$$

$$\frac{8}{3} \left[ \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} - 2 \tan^{-1} \frac{R}{Z} + \frac{R}{(R^2 + Z^2)^{1/2}} \right]$$
\[ 4.\text{th. term.} \]
\[ 2 \int_{\varphi_1}^{\varphi_2} \frac{\rho^4 \, d\varphi \, d\rho}{(\rho^2 + z^2)^{3/2} + 3 \rho (\rho^2 + z^2) + 3 z^2 (\rho^2 + z^2)^{3/2} + z^2 (\rho^2 + z^2)} \]
\[ = 2 \left[ \sin\varphi - \frac{\sin^3\varphi}{3} \right] \frac{\rho^2}{\varphi_1} \left\{ - \frac{4 (\rho^2 + z^2)^{1/2}}{\rho} + \frac{7z}{\rho} + \tan^{-1} \frac{\rho}{z} + \ln \left[ \rho + (\rho^2 + z^2)^{1/2} \right] \right\} \frac{\rho^2}{\varphi_1} \]

Substituting the limits and multiplying by 4:

\[ 4 \times 2 \left[ 1 - \frac{1}{3} \right] \left\{ - \frac{4 (R^2 + z^2)^{1/2}}{R} + \frac{7z}{R} + \tan^{-1} \frac{R}{z} + \ln \frac{R + (R^2 + z^2)^{1/2}}{z} \right\} = \]

\[ = \frac{16}{3} \left\{ - \frac{4 (R^2 + z^2)^{1/2}}{R} + \frac{7z}{R} + \tan^{-1} \frac{R}{z} + \ln \frac{R + (R^2 + z^2)^{1/2}}{z} \right\} \]

\[ \sigma_Y = \frac{9}{2\pi} \left\{ -4 \left[ \frac{-R}{(R^2 + z^2)^{1/2}} - \frac{R^3}{3(R^2 + z^2)^{3/2}} + \ln \frac{R + (R^2 + z^2)^{1/2}}{z} \right] + \right. \]

\[ + (1-2\lambda) \left\{ 12 \left[ \frac{-2(R^2 + z^2)^{1/2}}{R} + \frac{2z}{R} + \ln \frac{R + (R^2 + z^2)^{1/2}}{z} \right] - \right. \]

\[ \left. - \frac{8}{3} \left[ \ln \frac{R + (R^2 + z^2)^{1/2}}{z} - 2 \tan^{-1} \frac{R}{z} + \frac{R}{(R^2 + z^2)^{1/2}} \right] - \frac{16}{3} \right\} = \frac{4(R^2 + z^2)^{1/2}}{R} + \]
\[ G_y = \frac{9}{2\pi} \left\{ \left[ 4 \left( 1 - 2\nu \right) \frac{\theta}{3} \right] \frac{R}{(R^2 + Z^2)^{1/2}} + \frac{4R^3}{3(R^2 + Z^2)^{3/2}} + \right. \\
+ \left. \left[ -4 + (1 - 2\nu)(12 - \frac{8}{3} - \frac{16}{3}) \right] \ln \frac{R+(R^2+Z^2)^{1/2}}{Z} + (1 - 2\nu) \left\{ -24 + \frac{64}{3} \right\} \right\} \\
\left. \cdot \frac{(R^2 + Z^2)^{1/2}}{R} + \left( 24 - \frac{112}{3} \right) \frac{Z}{R} + \left( \frac{16}{3} - \frac{16}{3} \right) \tan^{-1} \frac{R}{Z} \right\} \\
- 8\nu \ln \frac{R+(R^2+Z^2)^{1/2}}{Z} + (1 - 2\nu) \left\{ -\frac{8(R^2+Z^2)^{1/2}}{3R} \right\} \\
- 10 \frac{R}{Z} \]
Differentiating formula (10) with respect to $\theta$ and using expressions on page 18:

\[ d^2 r_{xy} = \frac{g}{2} \left( -\frac{3\rho^4 \sin \varphi \cos \varphi}{(\rho^2 + 2^2)^{3/2}} + \left( -\frac{\rho^4 \sin \varphi}{(\rho^2 + 2^2)^{1/2}} \right)^2 \right) \left( \frac{\rho^4 \cos \varphi \sin \varphi}{(\rho^2 + 2^2)^{1/2}} + \frac{2 \rho^4 \cos \varphi \sin \varphi}{(\rho^2 + 2^2)^{3/2}} \right) \, d\rho \, d\varphi \]

1st. term.

\[ -3 \int_{\varphi_0}^{\varphi_2} \cos \varphi \sin \varphi \int_{\rho_1}^{\rho_2} \frac{\rho^4}{(\rho^2 + 2^2)^{3/2}} \, d\rho \, d\varphi = \]

\[ = -3 \left[ -\frac{\cos^3 \varphi}{3} \right]_{\varphi_0}^{\varphi_2} \left[ -\frac{\rho}{(\rho^2 + 2^2)^{3/2}} - \frac{\rho^3}{3(\rho^2 + 2^2)^{1/2}} + \ln \left( (\rho^2 + 2^2)^{1/2} + \rho \right) \right]_{\rho_1}^{\rho_2} \]

Substituting the limits and multiplying by $4$:

\[ -4 \times 3 \left[ 0 + \frac{1}{3} \right] \left\{ -\frac{R}{(R^2 + 2^2)^{1/2}} - \frac{R^3}{3(R^2 + 2^2)^{1/2}} + \ln \left( \frac{R + (R^2 + 2^2)^{1/2}}{Z} \right) \right\} \]

\[ -4 \left\{ -\frac{R}{(R^2 + 2^2)^{1/2}} - \frac{R^3}{3(R^2 + 2^2)^{1/2}} + \ln \left( \frac{R + (R^2 + 2^2)^{1/2}}{Z} \right) \right\} \]

2nd. term.

\[ \int_{\varphi_0}^{\varphi_2} \sin \varphi \int_{\rho_1}^{\rho_2} \frac{\rho^2 \, d\rho \, d\varphi}{(\rho^2 + 2^2)^{3/2} + 2Z(\rho^2 + 2^2)^{1/2} + Z^2(\rho^2 + 2^2)^{1/2}} \]

(see pp. 17 and 18)
\[-\cos \varphi \sin ^2 \varphi \int_{\varphi_1}^{\varphi_2} \int_{\rho_1}^{\rho_2} \left( \frac{\rho^4 \, d\rho \, d\varphi}{(\rho^2 + 2^2)^3 + 2Z(\rho^2 + 2^2) + Z^2(\rho^2 + 2^2)^{3/2}} \right) =
\]

\[= \left[ -\cos \varphi \sin \varphi \right]_{\varphi_1}^{\varphi_2} \left\{ -\frac{2(\rho^2 + 2^2)^{3/2}}{\rho} + \frac{2Z}{\rho} + \ln \left[ \rho + (\rho^2 + 2^2)^{3/2} \right] \right\}_{\rho_1}^{\rho_2}
\]

Substituting the limits and multiplying by 4:

\[\frac{4}{3} \left\{ \ln \left[ \frac{R + (R^2 + 2^2)^{3/2}}{Z} \right] - 2\tan^{-1} \frac{R}{Z} + \frac{R}{(R^2 + 2^2)^{3/2}} \right\}
\]
6th. term.

\[ 2 \int \frac{\phi_2}{\gamma_1 \cos^2 \phi \sin \phi} \int \frac{\rho^4}{(\rho^2 + Z^2)^{3/2} + 3Z(\rho^2 + Z^2)^{3/2} + 3Z^2(\rho^2 + Z^2)^{3/2} + Z^3(\rho^2 + Z^2)} \, d\rho \, d\phi \]

\[ = 2 \left[ -\frac{\cos^2 \phi}{3} \right] \phi_1 \left\{ -\frac{4(R^2 + Z^2)^{1/2}}{R} + \frac{7Z}{R} + \tan^{-1} \frac{R}{Z} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right\} \]

Substituting the limits and multiplying by 4:

\[ \frac{8}{3} \left\{ -\frac{4(R^2 + Z^2)^{1/2}}{R} + \frac{7Z}{R} + \tan^{-1} \frac{R}{Z} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right\} \]

\[ \tau_{xy} = \frac{9}{2\pi} \left\{ -\frac{R}{(R^2 + Z^2)^{1/2}} - \frac{R^3}{3(R^2 + Z^2)^{3/2}} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right\} + \]

\[ + (1 - 2\nu) \left\{ -\frac{2(R^2 + Z^2)^{1/2}}{R} + \frac{2Z}{R} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{R} \right\} + \]

\[ + \frac{4}{3} \left[ \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} - 2\tan^{-1} \frac{R}{Z} + \frac{R}{(R^2 + Z^2)^{1/2}} \right] + \frac{8}{3} \left[ -\frac{4(R^2 + Z^2)^{1/2}}{R} + \right. \]

\[ + \frac{7Z}{R} + \tan^{-1} \frac{R}{Z} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \left\} \right\} \]

\[ \tau_{xy} = \frac{9}{2\pi} \left\{ (1 - 2\nu) \left( \frac{4}{3} + \frac{16}{3} \right) \frac{R}{(R^2 + Z^2)^{1/2}} + \frac{4R^3}{3(R^2 + Z^2)^{3/2}} + \right. \]

\[ - 30 - \]
\[ + \left[ 4 + (1-2\nu)(-4 + \frac{4}{3} + \frac{8}{3}) \right] \ln \frac{R + (R^2 + Z^2)^{1/2}}{R} + (1-2\nu) \left[ \frac{8 (R^2 + Z^2)^{1/2}}{R} \right] + \left\{ - 8 + \frac{3\nu}{3} \right\} \frac{Z}{R} + \left( \frac{2}{3} + \frac{8}{3} \right) \tan^{-1} \frac{R}{Z} \]
VERTICAL COMPONENTS OF THE DISPLACEMENT

1.- NORMAL LOAD

For a single normal point load, the vertical component of the displacement vector is:

\[ W = \frac{P}{2\pi E} \left( \frac{Z^2}{D^3} + \frac{(1-2\nu) Z}{D} \right) \]

Differentiating with respect to \( P \) and substituting expressions on pg. 2:

\[ dw = \frac{1+\nu}{2\pi E} P \left[ \frac{Z^2}{(P^2+Z^2)^{3/2}} + \frac{2}{(P^2+Z^2)^{1/2}} \right] \frac{dP}{dP} \]

1st. term:

\[ Z^2 \int_{\phi_1}^{\phi_2} d\phi \int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho_1^2 Z^2 (P^2+Z^2)^{1/2}} = Z^2 \left[ \phi \right]_{\phi_1}^{\phi_2} \left[ \frac{P}{Z^2 (P^2+Z^2)^{1/2}} \right]_{\rho_1}^{\rho_2} \]

Substituting the limits and multiplying by \( 4 \):

\[ 4 Z^2 \left[ \frac{\pi}{2} - 0 \right] \frac{R}{Z^2 (R^2+Z^2)^{1/2}} = 2\pi \frac{R}{(R^2+Z^2)^{1/2}} \]

2nd. term:

\[ \int_{\phi_1}^{\phi_2} d\phi \int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho_1^2 (P^2+Z^2)^{1/2}} = \left[ \phi \right]_{\phi_1}^{\phi_2} \ln \left[ P + (P^2+Z^2)^{1/2} \right] \]
Substituting the limits and multiplying by 4:

$$4 \left[ \frac{\pi}{2} \right] \ln \left[ R + (R^2 + Z^2)^{\frac{1}{2}} \right] - \ln Z =$$

$$= 2\pi \ln \frac{R + (R^2 + Z^2)^{\frac{1}{2}}}{Z}$$

$$W = \frac{1+\nu}{2\pi E} \rho \left[ 2\pi \frac{R}{(R^2 + Z^2)^{\frac{1}{2}}} + 2 \cdot 2\pi (1-\nu) \ln \frac{R + (R^2 + Z^2)^{\frac{1}{2}}}{Z} \right]$$
2. HORIZONTAL LOAD.

For a single horizontal load in the x direction, the vertical component of the displacement vector is:

\[
W = \frac{Q}{4\pi} \left[ \frac{1}{\kappa} \frac{x z}{D^3} + \frac{1}{\lambda + \kappa} \frac{x}{D(D + 2)} \right]
\]

where \( \lambda \) and \( \kappa \) are Lamé constants.

\[
\lambda = \frac{V E}{(1 + \nu)(1 - 2\nu)} \quad \kappa = \frac{E}{2(1 + \nu)}
\]

Differentiating with respect to \( \Phi \) and substituting expressions on page 13:

\[
dW = \frac{Q}{4\pi} \left[ \frac{1}{\kappa} \frac{\rho \rho' \cos \varphi}{(\rho^2 + z^2)^{3/2}} + \frac{1}{\lambda + \kappa} \frac{\rho \rho' \cos \varphi}{(\rho^2 + z^2)^{3/2} [(\rho^2 + z^2)^{3/2} + 2]} \right] d\rho d\varphi
\]

First term:

\[
2 \int_{\rho_1}^{\rho_2} \int_{\rho_1}^{\rho_2} \frac{\rho \rho' \cos \varphi}{(\rho^2 + z^2)^{3/2}} = 2 \left[ \sin \varphi \right]_{\rho_1}^{\rho_2} \left[ \frac{-\rho}{(\rho^2 + z^2)^{3/2}} + \ln \left[ \rho + (\rho^2 + z^2)^{3/2} \right] \right]
\]

Substituting the limits and multiplying by 4:

- 34 -


\[ 4 \int \left[ \frac{-R}{(R^2 + Z^2)^{1/2}} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right] \]

2nd term.

\[ \int \frac{\varphi_2}{\rho} \cos \varphi \int \frac{\rho^2}{(\rho^2 + Z^2)^{1/2}[(\rho^2 + Z^2)^{1/2} + Z]} \, d\rho \, d\varphi \]

(see pp. 18 and 19)

\[ \left[ \sin \varphi \right]^{\varphi_2}_{\varphi_1} \left\{ - \frac{2(\rho^2 + Z^2)^{1/2}}{\rho} + \frac{2Z}{\rho} + \ln \left[ \rho + (\rho^2 + Z^2)^{1/2} \right] \right\}^{\varphi_2}_{\varphi_1} \]

Substituting the limits and multiplying by 4:

\[ 4 \left[ - \frac{2(\rho^2 + Z^2)^{1/2}}{R} + \frac{2Z}{R} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right] \]

\[ w = \frac{\varrho}{\pi} \left\{ \frac{2}{\lambda + \mu} \left[ \frac{-R}{(R^2 + Z^2)^{1/2}} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right] \right\} + \frac{1}{\lambda + \mu} \left[ - \frac{2(\rho^2 + Z^2)^{1/2}}{R} + \frac{2Z}{R} + \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} \right] \]

Substituting Lamé's constants:

\[ w = \frac{2(1 + \nu)}{E} \frac{\varrho}{\pi} \left\{ \frac{1 - 2\nu}{1 - 2\nu + 2E} \ln \frac{R + (R^2 + Z^2)^{1/2}}{Z} - \frac{2R}{(R^2 + Z^2)^{1/2}} \right\} + \frac{1 - 2\nu}{1 - 2\nu + 2E} \left[ \frac{2Z}{R} - \frac{2(\rho^2 + Z^2)^{1/2}}{R} \right] \]
BIBLIOGRAPHY


APPENDIX E

COMPUTER PLOTTING VERTICAL STRESSES

The program shown in the following pages is a simplified form of the general method described in this paper and was prepared using an IBM 1620 digital computer and a GERBER plotter. The general method was reduced accordingly to the difference in capacity and speed between this computer and the IBM/360.
C VERTICAL STRESSES IN SEMI INFINITE MEDIA.
C PLOT OF VERTICAL STRESSES DUE TO NORMAL LOADS.
DIMENSION B(1), C(2)
CALL GPlot(2, 0, -100)
DIMENSION P(10, 10), U1(25), A(3)
READ 32, NMAX, R
32 FORMAT(I6, F15.5)
READ 3*( (P(I, J), J = 1, 10), I = 1, 10)
3 FORMAT(10F7.2)
READ 30, C, X, Y, Z, D, X, D, Y, D, Z
30 FORMAT(6F10.5)
C PLOT AND NUMBER X AXIS.
CALL GPlot(11, 0, 0)
DO 1 I = 500, 2500, 500
CALL GPlot(12, I, 0)
CALL GPlot(12, I, 20)
CALL GPlot(11, I - 15, 30)
F = I
H = F / 500.
CALL GNum(I - 15, -30, H, 3.1, 0.1, 1, 0)
1 CALL GPlot(11, I, 0)
C LABEL X AXIS.
IX = 2530
IY = 0
READ 10, B(1)
10 FORMAT(A4)
CALL GChar(IX, IY, B(1), 120, 1, 0)
C PLOT AND NUMBER Y AXIS.
CALL GPlot(11, 0, 0)
DO 2 I = 150, 1500, 150
CALL GPlot(12, 0, I)
CALL GPlot(12, 20, I)
CALL GPlot(11, -50, I - 5)
F = I
H = F / 1500.
CALL GNum(-75, I - 5, H, 3.1, 0.1, 1, 0)
2 CALL GPlot(11, 0, 1)
C LABEL Y AXIS.
IX = 0
IY = 1530
READ 11, C(1), C(2)
11 FORMAT(A4, A4)
CALL GChar(IX, IY, C(1), 220, 1, 0)
C RHO = 0.2 * R
DRHO = 0.5 * RHO
R1 = 5 * DRHO / SQRT(3 * 14)
RHO3 = 3 * DRHO
DRHO2 = DRHO * DRHO
XLIM = 5.4 * DRHO
YLIM = XLIM
ZLIM = R
4 DO 40 N = 1, NMAX
INITIAL VALUE OF VERTICAL STRESS AT POINT P(X, Y, Z)
P\[1\]_{ZZ} = 0,
P\[2\]_{ZZ} = 0.

IS POINT P(X, Y, Z) WITHIN SENSITIVE ZONE (XLIM, YLIM, ZLIM)

IF YES M = 1
IF NO M = 0

IF (CZ - ZLIM) \[43, 43, 44\]
43 IF (ABS(CX) - XLIM) \[45, 44, 44\]
45 IF (ABS(CY) - YLIM) \[46, 44, 44\]

M = 1
GO TO 47
M = 0

K1 = 1
DO 29 I = 1, 10
DO 29 J = 1, 10

COORDINATES OF LOAD ELEMENT (I, J) WITH RESPECT TO MAIN SYSTEM.
ZA = 0.
HI = I
HJ = J
XA = (HJ - 5.5) * DRHO
YA = (5.5 - HI) * DRHO

COORDINATES OF POINT (X, Y, Z) WITH RESPECT TO LOAD ELEMENT (I, J).
XP = CX - XA
YP = CY - YA
ZP = CZ

IF (M = 1) \[5, 6, 6\]

IS POINT (X, Y, Z) OUTSIDE OF SENSITIVE ZONE OF LOAD ELEMENT (I, J)
IN EITHER CASE GO TO CORRESPONDING ROUTINE.

IF (ABS(XP) - R1) \[7, 5, 5\]
7 IF (ABS(YP) - R1) \[8, 5, 5\]
C POINT (X, Y, Z) OUTSIDE OF SENSITIVE ZONE OF LOAD ELEMENT (I, J)
ROUTINE FOR CONCENTRATED LOADS.

A = XP*XP + YP*YP + ZP*ZP
D = SQRT(A)
DZP^2 = (D + ZP)^2

RHO = XP*XP
F = YP*YP
G = ZP*ZP
H = G*ZP

IF (P(I, J)) \[77, 78, 77\]
77 PP = P(I, J) * DRHO2
TTT = 3. * PP / CCC
A1Z = TTT * H
GO TO 89

A1Z = 0.
89 P1Z = P1Z + A1Z
GO TO 9

C POINT (X, Y, Z) INSIDE SENSITIVE ZONE OF LOAD ELEMENT (I, J).
8 U1(K1) = P(I, J)
91 K1 = K1 + 1
9 CONTINUE

IF (M = 1) \[52, 60, 60\]
C ROUTINE FOR DISTRIBUTED LOADS.

K1MAX = K1 - 1
IF(K1MAX) 50*50*51
50 A2ZZ=0.
GO TO 52
51 SUMPl=0.
DO 600 K1=1,K1MAX
600 SUMPl=SUMPl+U1(K1)
R1MAX=K1MAX
IF(R1MAX) 900,900,901
900 AP(1)=0.
A2ZZ=0.
GO TO 750
901 AP(1)=SUMPl/R1MAX
IF(CZ) 150,150,908
908 A=ZP*ZP
AA=ZP*A
C1=R1*R1
D1=C1*R1
E1=SQRT(C1+A)
F1=E1*E1*E1
201 A2ZZ=AP(1)*(1-AA/F1)
750 P2ZZ=P2ZZ+A2ZZ
52 PZZ=P1ZZ+P2ZZ
C PLOT OUTPUT
X=5.*CZ
Y=10.*PZZ
GO TO 151
150 X=5.*CZ
Y=10.*AP(1)
CALL GPLOT(11,X,Y)
GO TO 39
151 CALL GPLOT(12,X,Y)
39 CX=CX+DX
CY=CY+DY
40 CZ=CZ+DZ
CALL EXIT
END
* DATA
41 1.
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.
1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.
1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.
1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.
1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.
1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
0.05000 0.05000 0.00000 0.00000 0.00000 0.10000
+Z/R
SIGMAZ/P
ZZZZ END OF JOB 1620
### APPENDIX F

#### LIST OF PROGRAMS

<table>
<thead>
<tr>
<th></th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Flow chart Program</td>
<td>36</td>
</tr>
<tr>
<td>2</td>
<td>Program</td>
<td>37</td>
</tr>
<tr>
<td>3</td>
<td>Flow chart Program</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td></td>
<td>57</td>
</tr>
<tr>
<td></td>
<td></td>
<td>58</td>
</tr>
</tbody>
</table>