A. SPACE-CHARGE MODE THEORY OF GAP INTERACTION

We have already presented (1) a linear space-charge theory of gap interaction for a thin electron beam. We shall now take account of the space variation of the space-charge fields and present the interaction for both infinite and Brillouin magnetic-focusing fields.

![Diagram of electron beam in a drift tube with circuit fields coupled through a gap.]

Fig. VI-1. Electron beam in a drift tube with circuit fields coupled through a gap.

A system consisting of an electron stream and a gap region is shown in Fig. VI-1. We have defined (1) the gap voltage, $V_g$, and gap current, $I_g$, at the surface. Excitations in the electron stream will be represented by kinetic voltage $V$, and current $I$. In matrix form, they are given by

$$
\mathbf{B} = \begin{bmatrix} V \\ I \end{bmatrix}
$$

By superposition we can write

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1. The Undriven System: \( V_g = 0 \)

We shall assume that the gap surface is a perfect electric short. The electromagnetic fields of interest are the TM_{0n} modes; that is, \( \partial / \partial \phi = 0 \). We also assume that \( \omega_p / \omega \ll 1 \), and disregard the nonpropagating (cutoff) fields.

Case a. \( B_{0z} = \infty \)

We have found (2) an infinite set of modes

\[
E_z(z, r) = \sum_n A_{\pm n} \exp(-j\beta_{\pm n} z) J_0(p_n r)
\]

in which

\[
\beta_{\pm n} = \beta_e \mp \beta_q
\]

\[
p_n = \gamma \left[ \left( \frac{\omega_p}{\omega_q} \right)^2 - 1 \right]^{1/2}
\]

are determined from the boundary conditions, with

\[
\beta_e = \frac{\omega}{v_o}, \quad \beta_q = \frac{\omega_q}{v_o}, \quad \gamma^2 = \beta_e^2 - k^2 \quad \left( k^2 = \frac{\omega}{c} \right)
\]

Case b. Brillouin focusing (nonrelativistic)

For this case we find an infinite number of degenerate modes at \( \beta_p \), to which the external fields of the gap cannot couple, and a surface-wave mode (3):

\[
E_z(z, r) = A_{\pm B} I_0(\beta_e r) \exp(-j\beta_{\pm B} z)
\]

\[
\beta_{\pm B} = \beta_e \mp \beta_q B
\]

with \( \beta_q B \) determined from the boundary conditions.

For case a and case b the kinetic voltage and beam current for each mode satisfy a set of transmission-line equations:

\[
\frac{D}{Dz} V_n, B = jZ_{0n} B \beta_{qn}, B I_n, B
\]
For case a,
\[ V_n(z, r) = \frac{m}{e} v_o v_n(z, r) \]  
\[ I_n(z, r) = \sigma J_n(z, r) \]  
\[ Y_{0n} = \frac{1}{2} \frac{I_o}{V_o} \frac{\omega}{\omega_{qn}} \]

For case b,
\[ V_B(z) = \frac{m}{e} v_o v_z(z, b) \]  
\[ I_B(z) = 2\pi K_z(z) \]  
\[ Y_{0B} = \frac{1}{2} \frac{I_o}{V_o} \frac{\omega}{\omega_{qB}} \frac{2I_1(\beta_e b)}{\beta_e b I_o(\beta_e b)} \]

In Eqs. 8-15, \( v_o \) is the z-component of the time-average electron velocity; \( v_n(z, r) \) and \( v_z(z, b) \) are the first-order electron velocities for the \( n^{\text{th}} \) space-charge mode and surface-wave mode, respectively; \( J_n(z, r) \) is the current density of the \( n^{\text{th}} \) mode; \( K_z(z) \) is the z-component of the surface current; \( \sigma \) is the cross-section area of the beam; \( I_o \) and \( V_o \) are the dc current and voltage of the beam; and \( I_1(\beta_e b) \) and \( I_o(\beta_e b) \) are modified Bessel functions.

The solutions of Eqs. 8 and 9 give the elements of the \( \mathbf{D} \)-matrix:

\[
\frac{D}{Dz} I_n, B = j \beta_{qn}, B V_n, B
\]

\[ D_{n, B} = \begin{pmatrix} A_{n, B} & B_{n, B} \\ C_{n, B} & A_{n, B} \end{pmatrix} \]

\[ A_{n, B} = \frac{1}{2} \left[ \exp(-j \beta_{+n, B} 2\ell) + \exp(-j \beta_{-n, B} 2\ell) \right] \]

\[ B_{n, B} = \frac{1}{2} Z_{0n, B} \left[ \exp(-j \beta_{+n, B} 2\ell) - \exp(-j \beta_{-n, B} 2\ell) \right] \]

\[ C_{n, B} = \frac{1}{2} Y_{0n, B} \left[ \exp(-j \beta_{+n, B} 2\ell) - \exp(-j \beta_{-n, B} 2\ell) \right] \]

2. Circuit-to-Beam Coupling

We assume that the circuit field, \( E_c(z, r) \), can be identified in the presence of the electron stream. This is consistent with our assumption \( (\omega_p/\omega) \ll 1 \) and allows us to use a weak-coupling formalism. We have
We use the orthogonality properties of the space-charge modes (4), and neglect the electromagnetic power flow because it is small compared with the kinetic power flow. Then, the excitation of each mode is given by

\[
\frac{D}{Dz} \hat{Y}_{n,B} = jY_{0n,B} \beta_{qn,B} \hat{I}_{n,B} + C_{n,B} E_c(z)
\]

\[
\frac{D}{Dz} \hat{V}_{n,B} = jV_{0n,B} \beta_{qn,B} \hat{V}_{n,B}
\]

where the circumflex denotes that the \( r \)-dependence is omitted, and

\[
C_n = \frac{\int F_c(r) J_0(p_n r) \, da}{\int J_0^2(p_n r) \, da}
\]

\[
C_B = F_c(b)
\]

In Eqs. 24 and 25 we have written

\[
E_c(z, r) = F_c(r) E_c(z)
\]

Equations 22 and 23 can be solved by finding the impulse response \((E_c(z) = u_0(z))\) and then using the superposition integral. The elements of the \( K \)-matrix

\[
K_{n,B} = \begin{bmatrix} a_{n,B} \\ b_{n,B} \end{bmatrix}
\]

are found to be

\[
a_{n,B} = \frac{1}{2} \left[ M_{+n,B} \exp(-j\beta_{+n,B} l) + M_{-n,B} \exp(-j\beta_{-n,B} l) \right] C_{n,B}
\]

\[
b_{n,B} = \frac{1}{2} Y_{0n,B} \left[ M_{+n,B} \exp(-j\beta_{+n,B} l) - M_{-n,B} \exp(-j\beta_{-n,B} l) \right] C_{n,B}
\]

where

\[
M_{\pm n,B} = \int_{-\infty}^{\infty} \phi_{\pm n,B}(\theta_{\pm n,B}) \exp(j\theta_{\pm n,B}) \, d\theta_{\pm n,B}
\]

as defined (1) previously.
3. Beam-to-Circuit Coupling

The kinetic power theorem

\[
\text{Re} \left( \frac{V}{g} \frac{\mathscr{I}^*}{g} \right) = \text{Re} \int_{\sigma} \left[ (V^*_{\ell})_\ell - (V^*_{-\ell})_{-\ell} \right] \, da
\]  

(30)

imposes certain relationships (5) upon the matrices of Eq. 2. From Eqs. 30 and 2 we find that

\[
\Gamma_{n, B} = \begin{bmatrix} c_{n, B} & d_{n, B} \end{bmatrix}
\]

(31)

\[
c_{n, B} = \frac{1}{2} Y_{0n, B} \left[ M^*_{-\ell, B} \exp(-j\beta_{n, B} \ell) - M^*_{+\ell, B} \exp(-j\beta_{-n, B} \ell) \right] K_{n, B}
\]

(32)

\[
d_{n, B} = \frac{1}{2} \left[ M^*_{+\ell, B} \exp(-j\beta_{n, B} \ell) + M^*_{-\ell, B} \exp(-j\beta_{-n, B} \ell) \right] K_{n, B}
\]

(33)

where

\[
K_n = \frac{1}{\sigma} \int_{0}^{\pi} f_c(r) J_0(p_n r) \, da
\]

(34)

\[
K_B = \frac{C_B}{2}
\]

(35)

Equation 3 also gives the real part of the electronic loading admittance for each mode:

\[
G_{ef \ n, B} = \frac{1}{4} Y_0 \left[ |M_{+n, B}|^2 - |M_{-n, B}|^2 \right] c_{n, B} K_{n, B}
\]

(36)

The imaginary part of the electronic loading admittance is readily obtained from Eq. 2, in conjunction with the following kinetic energy theorem (5)

\[
\text{Im} \left( \frac{V}{g} \frac{\mathscr{I}^*}{g} \right) = \text{Im} \int_{\sigma} \left[ (V^*_{\ell})_\ell - (V^*_{-\ell})_{-\ell} \right] \, da
\]

\[
+ \omega J_0 \int_{\tau} \epsilon_0 |V|^2 \, d\tau
\]

\[
+ \omega \int_{\tau} \left[ \mu_0 |H|^2 - \epsilon_0 |E|^2 \right] \, d\tau
\]

(37)

Finally, Eqs. 2 and 37 yield

\[
B_{ef \ n, B} = \frac{1}{2} Y_{0n, B} \text{Im} \left[ M_{+n, B} M^*_{-n, B} \exp(j2\beta_{n, B} \ell) \right] c_{n, B} K_{n, B}
\]

(38)

\[
+ \omega \epsilon_0 \beta_p \frac{K_{n, B}}{C_{n, B}} \int_{a_n, B(z)} \, dz
\]

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(VI. MICROWAVE ELECTRONICS)

References


