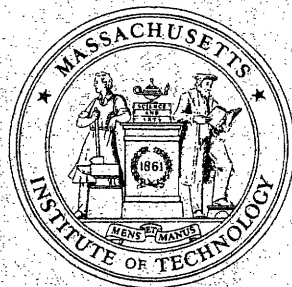# OPERATIONS RESEARCH CENTER

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Risk Independence and
Multiattributed Utility Functions*

by

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Abstract

The concepts of conditional risk aversion, the conditional risk premium, and risk independence pertaining to multiattributed utility functions are defined. The latter notion is then generalized to what is called utility independence. A number of theorems useful for simplifying the assessment of multiattributed utility functions given certain risk independence and utility independence assumptions are stated.

the conditional risk premium is introduced. The notion of risk independence is then considered in a different context which permits extensions and generalizations of the results. These are presented in the final section as representation theorems which simplify the assessment of multiattribute utility functions.

2. Definition of Risk Independence

Pratt [8] defines the local risk aversion r(x) by

$$r_i \underline{(x)} = - \frac{u''(x)}{u'(x)} , \tag{1}$$

where u(x) is a utility function[1] for the continuous scalar attribute X and u'(x) and u''(x) are respectively the first and second derivatives of u(x). By integrating (1), exponentiating, and integrating again, he showed

$$u(x) = k_1 \int e^{-\int r(x)dx} dx + k_2 , \tag{2}$$

where k_1 and k_2 are constants of integration. One can observe that the risk aversion function r(x) contains all the essential information about u(x) while eliminating the arbitrariness introduced by positive linear transformations.

For multiattributed utility functions, it seems natural to develop conditional risk aversion functions on the same basis as r(x). More specifically, consider the utility function $u(x_1, x_2, \ldots, x_n)$ for attributes X_1, $i = 1, 2, \ldots, n$ and for notational convenience, let us designate $X_1 \times X_2 \times \ldots \times X_n$ as X and

$X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n$ as $X_{\bar{i}}$. Then the <u>conditional risk aversion</u> for

X_i, which we denote by $r_i(\underline{x})$, will be defined by

$$r_i(\underline{x}) = - \frac{u_i''(\underline{x})}{u_i'(\underline{x})} ,\tag{3}$$

where $u_i'(\underline{x})$ and $u_i''(\underline{x})$ are the first and second partial derivatives[2] of $u(\underline{x})$

with respect to x_i.

We will say that X_i is <u>risk independent</u> of $X_{\bar{i}}$ if $r_i(\underline{x})$ does not depend on

$x_{\bar{i}}$. In other words, X_i is risk independent of the other attributes if the

riskiness (as measured by r_i) of lotteries involving only uncertain amounts of

X_i does not depend on the fixed amounts of the other attributes. This may be

a reasonable assumption in many situations. That is, if all of the risk is

associated with only one attribute and the other attributes are all fixed, the

decision-maker's attitudes toward risk will depend only on that attribute in-

volving the risk.

Let $x_{\bar{i}}$ represent any amount of $X_{\bar{i}}$ and let $x_{\bar{i}}^o$ be a specific amount.

Then we can define the <u>conditional utility function</u> for X_i given $x_{\bar{i}} = x_{\bar{i}}^o$ to

mean any positive linear transformation of $u(x_i, x_{\bar{i}}^o)$.

Given this notation, we can efficiently prove an important

LEMMA. If X_i is risk independent of $X_{\bar{i}}$, then

$$u(x_i, x_{\bar{i}}) = f(x_{\bar{i}})\, u(x_i, x_{\bar{i}}^o) + g(x_{\bar{i}}) ,\tag{4}$$

where $f(x_{\bar{i}}) > 0$.

Proof. Given $r_i(x_i, x_{\bar{i}}) = r_i(x_i, x_{\bar{i}}^o)$, it follows from (3) that

$$\frac{\partial}{\partial x_i} [\log u_i'(x_i, x_{\bar{i}})] = \frac{\partial}{\partial x_i} [\log u_i'(x_i, x_{\bar{i}}^o)] ,$$

so by partial integration and exponentiation,

$$u_i'(x_i, x_{\bar{i}}) \, e^{a(x_{\bar{i}})} = u_i'(x_i, x_{\bar{i}}^o) \, e^b ,$$

where $a(x_{\bar{i}})$ and b are integration constants. Integrating again,

$$u(x_i, x_{\bar{i}}) \, e^{a(x_{\bar{i}})} + c(x_{\bar{i}}) = u(x_i, x_{\bar{i}}^o) \, e^b + d$$

which becomes (4) when rearranged and $f(x_{\bar{i}}) \equiv e^{b - a(x_{\bar{i}})}$ and $g(x_{\bar{i}}) \equiv$

$[d - c(x_{\bar{i}})] \, [e^{-a(x_{\bar{i}})}]$.

The lemma becomes almost obvious when we consider that $r_i(\underline{x})$

specifies the conditional utility function for X_i uniquely up to positive trans-

formations and that $r_i(\underline{x})$ does not depend on $x_{\bar{i}}$.

This result is useful in a number of situations. For example, suppose

X_i is risk independent of $X_{\bar{i}}$ and suppose $r_i(x_i, x_{\bar{i}}^o) = c_i > 0$. That is, the de-

cision maker is constantly risk averse over X_i for some $x_{\bar{i}}$ in $X_{\bar{i}}$. Given these

conditions, it follows directly from Pratt[8] that $u(x_i, x_{\bar{i}}^o)$ must be a positive

linear transformation of $-e^{-c_i x_i}$. Then from the lemma, we know

$u(x_i, x_{\bar{i}})$ for all $x_{\bar{i}}$ in $X_{\bar{i}}$ can be expressed by

$$u(x_i, x_{\bar{i}}) = -f(x_{\bar{i}}) e^{-c_i x_i} + g(x_{\bar{i}}), \quad f(x_{\bar{i}}) > 0.$$

Stiglitz[9] considers a utility function of this form in the context of consumer

behavior under uncertainty.

3. Utility Functions and Risk Independence

In this section, we derive the functional form of a utility function with

two attributes given each attribute is risk independent of the other. As before,

if $u(x, y)$ represents the utility function for attributes X and Y, it must be

assumed u is increasing and twice continuously differentiable in each attribute.

In section 5, the concept of risk independence is viewed in a slightly different

context which allows us to eliminate these restrictions.

An important result is

THEOREM 1. Given X is risk independent of Y and Y is risk independent of

X, then $u(x, y)$ can be expressed by

$$u(x, y) = u(x, y_o) + u(x_o, y) + k\, u(x, y_o)\, u(x_o, y), \tag{5}$$

where k is an empirically evaluated constant and $u(x, y_o)$ and

$u(x_o, y)$ are consistently scaled conditional utility functions.

Proof. For reference, let us define the origin of $u(x, y)$ by

$$u(x_o, y_o) = 0. \tag{6}$$

Since X is risk independent of Y, from (4) we know

$$u(x, y) = f_1(y)\, u(x, y_o) + g_1(y) . \tag{7}$$

Similarly, Y is risk independent of X so

$$u(x, y) = f_2(x)\, u(x_o, y) + g_2(x) . \tag{8}$$

Then by evaluating (7) at $x = x_o$, we have

$$u(x_o, y) = f_1(y)\, u(x_o, y_o) + g_1(y) = g_1(y) , \tag{9}$$

and likewise, evaluating (8) at $y = y_o$ yields

$$u(x, y_o) = g_2(x) . \tag{10}$$

Now substituting (9) into (7) and (10) into (8) and equating the resulting equations,

$$f_1(y)\, u(x, y_o) + u(x_o, y) = f_2(x)\, u(x_o, y) + u(x, y_o) \tag{11}$$

which, after rearranging, is

$$\frac{f_1(y) - 1}{u(x_o, y)} = \frac{f_2(x) - 1}{u(x, y_o)} , \quad x \neq x_o, \; y \neq y_o . \tag{12}$$

In (12), a function of x is equal to a function of y, therefore, they both must equal a constant. Call this constant k, and we have

$$\frac{f_2(x) - 1}{u(x, y_o)} = k, \quad x \neq x_o , \tag{13}$$

or

$$f_2(x) = k\, u(x, y_o) + 1 . \tag{14}$$

The restriction $x \neq x_o$ of (13) is not necessary in (14) since $f_2(x_o) = 1$ as can be verified by evaluating (11) at $x = x_o$. Substituting (10) and (14) into (8) we conclude

$$u(x, y) = [k \, u(x, y_o) + 1] \, u(x_o, y) + u(x, y_o)$$

$$= u(x, y_o) + u(x_o, y) + k \, u(x, y_o) \, u(x_o, y).$$

By evaluating $u(x_1, y_1)$ for arbitrary x_1 and y_1, we find k can be evaluated from

$$k = \frac{u(x_1, y_1) - u(x_o, y_1) - u(x_1, y_o)}{u(x_o, y_1) \, u(x_1, y_o)} \, .$$

The converse of theorem 1 is also true and easily proven working directly with the definition of conditional risk aversion in (3). That is, given a utility function for two continuous scalar attributes is of the form (5), then X is risk independent of Y and Y is risk independent of X.

The usefulness of theorem 1 is that it simplifies the assessment of $u(x, y)$ provided the requisite risk independent assumptions hold. The assessment of the two-attribute utility function is reduced to assessing two one-attribute conditional utility functions and the utilities of two additional consequences. The latter are necessary to consistently scale the conditional utility functions and to evaluate k.

4. Conditional Risk Premiums

One of the important results of Pratt's work was his relating the local risk aversion to the intuitively appealing concept of a risk premium. These ideas are also valid in the current context concerning conditional utility functions.

Consider the lottery represented by $(\tilde{x}_i, x_{\bar{i}})$, where the "$\sim$" represents a random outcome, and let $p_i(x_i)$ represent the probability density function describing this outcome. Then the conditional certainty equivalent for $\tilde{x}_i$ given $x_{\bar{i}}$ is defined as the amount of X_i, call it $\hat{x}_i$, such that the decision maker is indifferent between $(\hat{x}_i, x_{\bar{i}})$ and $(\tilde{x}_i, x_{\bar{i}})$. The conditional risk premium for this lottery π_i is defined as the amount such that the decision maker is indifferent between $(\bar{x}_i - \pi_i, x_{\bar{i}})$ and $(\tilde{x}_i, x_{\bar{i}})$, where $\bar{x}_i$ is the expected value of $\tilde{x}_i$. It should be clear that $\pi_i = \bar{x}_i - \hat{x}_i$.

In general, there is no reason why the conditional certainty equivalent and conditional risk premium for $\tilde{x}_i$ would not depend on $x_{\bar{i}}$. However, it follows from (4) that when X_i is risk independent of $X_{\bar{i}}$, the conditional risk premium and conditional certainty equivalent for $\tilde{x}_i$ will not in fact depend on the condition $x_{\bar{i}}$. This is useful in that it allows us in some situations to assess the expected utility of lotteries in terms of conditional certainty equivalents.

To be specific, consider the lottery $(\tilde{x}, \tilde{y})$, where we have verified X and Y are risk independent of each other. We can now calculate the expected utility of $(\tilde{x}, \tilde{y})$ using (5) to find

$$E[u(\tilde{x}, \tilde{y})] = \int_x \int_y [u(x, y_o) + u(x_o, y) + k\ u(x, y_o)\ u(x_o, y)]\ p(x, y)\ dxdy$$

$$= E[u(\tilde{x}, y_o)] + E[u(x_o, \tilde{y})] + kE[u(\tilde{x}, y_o)\ u(x_o, \tilde{y})], \tag{15}$$

where E denotes expectation and $p(x, y)$ is the joint probability density func-

tion for $(\tilde{x}, \tilde{y})$. When X and Y are probabilistically independent, (15) becomes

$$E[u(\tilde{x}, \tilde{y})] = E[u(\tilde{x}, y_o)] + E[u(x_o, \tilde{y})] + k\ E[u(\tilde{x}, y_o)]\ E[u(x_o, \tilde{y})]. \tag{16}$$

But since X and Y are risk independent, we can reduce (16) to

$$E[u(\tilde{x}, \tilde{y})] = u(\hat{x}, y_o) + u(x_o, \hat{y}) + k\ u(\hat{x}, y_o)\ u(x_o, \hat{y}), \tag{17}$$

where $\hat{x}$ and $\hat{y}$ are respectively the conditional certainty equivalents for $\tilde{x}$

and $\tilde{y}$. It immediately follows from (17) that

$$E[u(\tilde{x}, \tilde{y})] = u(\hat{x}, \hat{y}) = u(\bar{x} - \pi_x, \bar{y} - \pi_y),$$

where π_x and π_y are the conditional risk premiums.

5. Generalization of Risk Independence

Results analogous to theorem 1 could be derived for n-attribute utility

functions provided each attribute X_i was risk independent of all the other

attributes. However, there would be some unnecessary limitations of such

results. First of all, $u(\underline{x})$ would have to be increasing and twice continuously

differentiable in each attribute for each $r_i(\underline{x})$ to be defined. Also, since the

risk aversion function is only defined for scalar attributes, there is no

corresponding risk aversion function for vector attributes.[3] Thus, an

expression like (5) would not be valid if for instance, $y = (y_1, y_2)$. Risk

independence can be considered in a slightly different context such that the

results are not limited by such restrictions.

In the notation of section 2, recall that the lemma states provided

X_i is risk independent of $X_{\bar{i}}$, the conditional utility function for X_i given

any fixed $x_{\bar{i}}$ is a positive linear transformation of the conditional utility

function for X_i given $x_{\bar{i}} = x_{\bar{i}}^o$. The converse is easily shown to be true, so

this condition is equivalent to risk independence for increasing and twice

continuously differentiable scalar attributes. However, one can see from

(4) that such a condition by itself does not require restrictions on the

attributes, and that it may hold for non-increasing, non-continuous, vector

attributes. Thus, by stating our assumptions in terms of conditional utility

functions rather than in terms of the conditional risk aversions, the results

can be extended to include many additional situations.

6. Additional Results

Rather than repeat derivations that are found elsewhere [4], we will state

only one important result which simplifies the assessment of a multidimensional

utility function provided the requisite assumptions hold.

Let $u(x_1, x_2, \ldots, x_n)$ be a utility function over consequence space

$X \equiv X_1 \times X_2 \times \cdots \times X_n$. Then define vector attributes Y and Z such that

$X = Y \times Z$, where $y = (x_1, x_2, \ldots, x_m)$ and $z = (x_{m+1}, x_{m+2}, \ldots, x_n)$ represent

specific amounts of Y and Z respectively. Given $u(y, z)$, Y is said to be

utility independent of Z if the decision-maker's relative preferences over

lotteries on Y, when Z is held fixed at z_o, are the same regardless of the

amount z_o. Given this condition, since utility functions are unique up to posi-

tive linear transformations, the conditional utility function for Y, given any

amount of Z, is a positive linear transformation of the conditional utility

function for Y given any other amount of Z. Mathematically stated, if Y is

utility independent of Z, then for any z_o.

$$u(y, z) = f(z) + g(z) u(y, z_o), \text{ all } z.$$

If Y and Z are utility independent of each other, they are said to be mutually

utility independent. Similarily, if each of the X_i's is utility independent of all

the others, they are mutually utility independent.

In the n-dimensional case, if the conditional utility function for X_i is

denoted by $u_i(x_i, x_{\bar{i}})$, where $x_{\bar{i}}$ represents a fixed amount of all the other

attributes, and if x_i^* and $^*x_i$ are arbitrarily chosen such that $u_i(x_i^*, x_{\bar{i}}) > u_i(^*x_i, x_{\bar{i}})$,

then one can prove

THEOREM 2. Given $X = X_1 \times X_2 \times \cdots \times X_n$ and the X_i are mutually utility

independent, $u(x_1, x_2, \ldots, x_n)$ is completely specified by

(a) $u_i(x_i, x_{\bar{i}})$ for arbitrary $x_{\bar{i}}$, for each X_i, and

(b) $u(x_1', x_2', \ldots, x_n')$ for all $x_i' = x_i^*$ or $^*x_i$.

When X is partitioned into Y x Z, there is the generalization of Theorem 1 which

we state as a

COROLLARY. If Y and Z are mutually utility independent, $u(y, z)$ can be

evaluated from

$$u(y, z) = u(y, z_o) + u(y_o, z) + k \, u(y, z_o) \, u(y_o, z) \, ,$$

where y_o and z_o are arbitrarily chosen and k is an empirically evaluated

constant.

Other related results using the concept of utility independence are found

in Meyer [6] and Keeney [5] .

7.   Conclusions

We have used the concept of risk aversion developed by Pratt [8] which

has proven to be important in assessing single-attribute utility functions in

defining a conditional risk aversion function relevant to multiattribute utility

functions. The functional form of the two-attribute utility function satisfying

certain reasonable assumptions concerning one's conditional risk aversion

attitudes was derived. The notion of risk independence was then generalized

and renamed utility independence. Finally, two representation theorems were

stated which simplify the assessment of a multidimensional utility function pro-

vided specified utility independence assumptions hold.

8.   Acknowledgment

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Notes

1. It is assumed u(x) is increasing and twice continuously differentiable.

2. It is assumed u(x) is increasing in each X_i and the first and second partial derivatives exist and are continuous.

3. We define a scalar attribute as a single-valued attribute and a vector attribute as a many-valued attribute. For instance, if the set of consequences $X \equiv X_1 \times X_2 \times X_3$, where the X_i are scalar attributes, then $Y \equiv X_1 \times X_2$ would be a vector attribute.