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by

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ON THE NUMBER OF LATENT SUBSETS OF INTERSECTING COLLECTIONS

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Abstract

Given two collections F_1 and F_2 of sets each member of one intersecting each member of the other, let the collections of latent sets F_i^L i=1,2 consist of the sets that are contained in members of F_i but that are not themselves members of F_i. If lower case letters indicate the size of the collections we then have

$$f_1^L f_2^L \geq f_1 f_2 \ .$$

This result is used to prove that a self-intersecting subfamily F of a simplicial complex G having the property that any element of F contains s_1 or s_2 can be no larger than the lesser of the number of elements of G containing s_1 and the number containing s_2. Certain extensions and a related conjecture of Chvátal are described.

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Introduction

Two collections F_1 and F_2 of subsets of a given finite set S are called _intersecting_ if each set in F_1 has a non-empty intersection with every set in F_2. _Latent subsets_ F_j^L of F_j are defined to be those subsets of S which are a subset of a set in F_j but which are not themselves in F_j, i.e.,

$$F_j^L = \{A \subseteq S: \quad A \subseteq A_1 \in F_j \text{ and } A \notin F_j\}.$$

We let f_j and f_j^L denote respectively the number of sets in F_j and F_j^L. In this note, we prove that if F_1 and F_2 are intersecting then the following equality is satisfied:

$$f_1^L f_2^L \geq f_1 f_2 \ . \tag{1}$$

We begin by developing a canonical form for F_1 and F_2 which preserves both their size and intersection property and does not increase the number of latent subsets of either collection. Then we prove the asserted inequality for the intersecting collections in canonical form. We conclude by giving several extensions of the inequality and an application.

Canonical Form

In [1], one of the authors introduced a canonical form for intersecting collections which he used to obtain bounds on the number of sets in certain collections. We show that the same technique may be used for our latent subset problem.

Let us order the elements of S as $s_1, s_2, \ldots, s_n$. We define the following set of mappings m_j, for $1 \leq j \leq n-1$, acting on subsets of S:

$$m_j(A) = \begin{cases} A + s_j - s_{j+1} & \text{if } s_j \notin A,\ s_{j+1} \in A \\ A & \text{otherwise} \end{cases}$$

where $A + s_j - s_{j+1}$ is used in place of $A \cup \{s_j\} - \{s_{j+1}\}$. The same convention will be applied for one element sets throughout this paper. The mappings m_j "push" elements of A toward the lower ordered elements.

For any collection F of subsets of S, we define $m_j(F)$ acting on F by:

$$m_j(F)(A) = \begin{cases} m_j(A) & \text{if } m_j(A) \notin F \\ A & \text{if } m_j(A) \in F. \end{cases}$$

Beginning with two intersecting collections F_1 and F_2 of subsets of S, it is shown in [1] that $m_j(F_1)$ and $m_j(F_2)$ are intersecting and that after a finite number of repeated applications of $m_1, m_2, \ldots, m_{n-1}$ the resulting collections, called the canonical form for F_1 and F_2, will be invariant under every m_j transformation.

We now note that $m_j(F)$, which has the same number of members as F, has no more latent subsets than F. Suppose A is a latent subset of $m_j(F)$ and not of F. Then, by the nature of the m_j transformation, that it affects only sets containing one and not both of s_j and s_{j+1}, we can conclude that A must contain one and not both of these elements. We shall show that under the given circumstances the set A' obtained from A by removing the one of s_j, s_{j+1} contained in A and inserting the other, must have been a latent subset of F and is not one of $m_j(F)$. This is all we need prove here.

Two cases can be distinguished. If A contains s_j, it cannot be in F nor can A' be. It is latent in $m_j(F)$ because some B satisfying $B \supseteq A$ lies in that family, and not in F. But then the member of F whose image under m_j was B contains A', which is therefore latent in F. And A' can be latent in $m_j(F)$, only if A is latent in F. For, if C in $m_j(F)$ contains A', then the set C' obtained from C by interchanging s_j with s_{j+1} (which may be C itself if both are present in it) must be in F, and will contain A.

If A contains s_{j+1} rather than s_j, then A must be latent in $m_j(F)$ and not in F through the fact that it is in F and not in $m_j(F)$ and is contained in some B that lies in $m_j(F)$. But then A' must be in $m_j(F)$ and not in F, and must be contained in B' (defined as C' above) which must be in F. Thus A' is latent in F but not in $m_j(F)$ which was to be proven.

Finally, given two intersecting collections F_1 and F_2 in canonical form it will be convenient to partition the families as:

$$F_j = G_j \cup H_j \cup I_j \qquad (j=1,2) \qquad (2)$$

where

(i) $s_n \notin A$ for any $A \in G_1$.

(ii) if $A \in H_1$, then $s_n \in A$ and $A - s_n$ intersects every set in F_2;

(iii) if $A \in I_1$, then $s_n \in A$ and $A - s_n$ is disjoint from a set in F_2

and similarly for F_2.

For j=1,2 let $\overline{H}_j=\{A-s_n: A \in H_j\}$, $\overline{I}_j=\{A-s_n: A \in I_j\}$. We observe that the sets in $\overline{I}_1$ and $\overline{I}_2$ may be paired in the sense that if $A \in \overline{I}_1$ then $S-A-s_n \in \overline{I}_2$ and conversely. Furthermore, if $A \in \overline{I}_1(\overline{I}_2)$, then A intersects every set in $F_2(F_1)$ except for S-A. To prove both assertions simply note that if $B \in F_2$, $A \cap B = \emptyset$ and $B \subseteq S-A-s_k$ then by adding s_k to A one obtains a member of F_1 (since F_1 is in canonical form) that is disjoint from B.

Main Result

We now prove our main result. If F is any collection of sets we use notation from the introduction letting F^L denote its latent subsets and f its cardinality. We assume throughout this section that S is a given finite set.

Theorem 1:

Let F_1 and F_2 be two intersecting collections of subsets of S. Then

$$f_1^L f_2^L \geq f_1 f_2 \quad . \qquad (3)$$

Proof 1:

By the results of the previous section, the theorem is true if it holds for the canonical form for F_1 and F_2. Thus we assume that F_1 and F_2 are in canonical form and that each collection has been partitioned as in (2). If either F_1 or F_2 is empty the result is obvious; thus we assume that f_1, $f_2 \geq 1$.

We use induction on n the number of elements in S. For a given value of n we use induction on $i_1 = i_2$. If n = 1, the left and right hand sides of (3) are both equal to one (the empty set is latent in each collection). Suppose that n=k. If $i_1=i_2=0$, then G_1 and $\bar{H}_1$ both intersect each of G_2 and $\bar{H}_2$.

Thus by induction on n,

$$g_1^L g_2^L \geq g_1 g_2 \quad , \qquad \bar{h}_1^L \bar{h}_2^L \geq h_1 h_2$$
$$g_i^L \bar{h}_j^L \geq g_i \bar{h}_j \qquad i \neq j \in \{1,2\}. \qquad (4)$$

Note that if T is a latent subset of $\bar{H}_j$, then T + s_n is a latent subset of H_j so that $f_j^L \geq g_j^L + \bar{h}_j^L$. Thus the result follows in this case by adding the four inequalities in (4) and using $f_j = g_j + \bar{h}_j$.

Next assume that $i_1=i_2=p$ and suppose without loss of generality that $(f_1 + f_1^L) \leq (f_2 + f_2^L)$. Let $T \varepsilon \bar{I}_1$. By our observations concerning the canonical form of F_1 and F_2, $S - T \varepsilon I_2$ and T intersects every set in F_2 other than $S - T$.

Define $\hat{F}_1 = F_1 \cup \{T\}$, $\hat{F}_2 = F_2 - \{S - T\}$. Then $\hat{F}_1$ and $\hat{F}_2$ are intersecting and $T + s_n$ is in $\hat{H}_2$ not $\hat{I}_2$. Also $f_1^L = \hat{f}_1^L+1$ (since T is latent in F_1 and not $\hat{F}_1$) and $f_2^L \geq \hat{f}_2^L - 1$ (possibly $(S-T) \varepsilon \hat{F}_2^L$ whereas $(S-T) \not\varepsilon F_2^L$). But $\hat{i}_1=p-1$, so by induction:

$$\hat{f}_1^L \hat{f}_2^L \geq \hat{f}_1 \hat{f}_2$$

thus,

$$(f_1^L - 1)\ (f_2^L + 1) \geq (f_1 + 1)\ (f_2 - 1)$$

or

$$f_1^L f_2^L \geq f_1 f_2 + (f_2+f_2^L) - (f_1+f_1^L) \geq f_1 f_2\ .$$

This completes both inductions and proves the theorem.

We note that with the hypothesis of Theorem 1 the related conjecture that $f_1^L + f_2^L \geq f_1 + f_2$ is not valid. As a counter-example, let F_1 consist of the set $\{s_1, s_2\}$ and F_2 the sets $\{s_1\}$, $\{s_2\}$, $\{s_1, s_2\}$, $\{s_1, s_3\}$, $\{s_2, s_3\}$, $\{s_1, s_2, s_3\}$.

The following results are easy consequences of Theorem 1. For any collection F of subsets of a given set S we let $F^{L'}=\{A\subseteq S: A\supseteq A_1 \in F \text{ and } A \notin F\}$ denote the latent supersets of F.

<u>Corollary</u>:

(a) Let $F_1,\ldots, F_r$ be pairwise intersecting collections of subsets of S. Let ϕ be a real valued function defined on $R^{r(r-1)/2}$ that is non-decreasing in each argument. Then $\phi(f_1^L f_2^L,\ldots, f_i^L f_j^L,\ldots, f_{r-1}^L f_r^L)\geq\phi(f_1f_2,\ldots,f_if_j,\ldots, f_{r-1}f_r)$ $i\neq j\in\{1,\ldots,r\}$. In particular,

$$\sum_{\substack{j\neq i\\ i=1}}^{r} f_i^L f_j^L \geq \sum_{\substack{j\neq i\\ i=1}}^{r} f_i f_j \quad\text{and}\quad \prod_{i=1}^{r} f_i^L \geq \prod_{i=1}^{r} f_j .$$

(b) Let F be a collection of subsets of S with the property that no two sets in F are disjoint. Then $f^L \geq f$.

(c) Let F_1 and F_2 be two collections of subsets of S with the properties:

(i) no set in F_1 intersects any set in F_2, and

(ii) $A \in F_1$ implies $S-A \notin F_2$. Then $f_1^{L'} f_2^{L'} \geq f_1 f_2$.

<u>Proof</u>:

(a) and (b) are clear. (c) follows by considering the intersecting collections $\tilde{F}_j$ j=1,2 whose elements are complements (with respect to S) of the sets in F_j.

The following result is an application of Theorem 1. If F is interpreted as a collection of committees each chaired by one of two men, it shows that an equal number of sub-committees may be chosen with a common chairman.

Theorem 2:

Let F be an intersecting collection (i.e. no two sets in F are disjoint) of subsets of S with the property that each set in F contains at least one of the elements s_1, $s_2 \in S$. Then there is an intersecting collection F' subordinate to F (i.e., $A \in F'$ implies that for some A' $A \subseteq A_1 \in F$) that satisfies:

(i) $f' \geq f$

(ii) either s_1 is contained in every set in F' or s_2 is.

Proof:

If either $\{s_1\}$ or $\{s_2\}$ is a member of F there is nothing to prove; we therefore assume otherwise; that $\{s_j\} \notin F$, $j = 1,2$.

Let $G_j = \{A \in F: \ s_j \in A, \ s_{3-j} \notin A\}$,

and let $\bar{G}_j = \{A - s_j: \ A \in G_j\} j=1,2$.

By the hypothesis on F, $\bar{G}_1$ and $\bar{G}_2$ are intersecting, thus by Theorem 1, $\bar{g}_1^L \bar{g}_2^L \geq \bar{g}_1 \bar{g}_2$ so that $\bar{g}_1^L \geq \bar{g}_2$ say. The set F',

$F' = G_1 \cup G_3 \cup \{A + s_1: \ A \in \bar{G}_1^L\}$ with $G_3 = F - G_1 - G_2$, then satisfies the conditions above.

The analog of Theorem 2 when each member of F contains one of three elements s_1, s_2, s_3 is not valid. As a counterexample, let F consist of the sets (s_j, s_4, s_5), (s_j, s_4, s_6), (s_j, s_5, s_6) $j = 1,2,3$. Note that in this example we can select F', $f' \geq f$, with the property that each of its members contains s_4.

A conjecture to this effect has been proposed by V. Chvátal.

Conjecture:

Let F be a collection of subsets of a finite set S such that $X \in F$, $Y \subseteq X \Rightarrow Y \in F$. Then there exists on $s \in S$ and an intersecting collection F_s each member of which contains s such that for any intersecting subfamily G of F, $f_s \geq g$.

The result above settles this conjecture whenever any maximal cardinality sub family of F contains at least one of two elements of S; and for any F, for all G which contain at least one of two elements of S.

Note added: A recent result of Chvátal (preprint) settles the above conjecture when F is in the canonical form defined above.

References

1. D. J. Kleitman, "On a Conjecture of Milner on k-Graphs with Non-Disjoint Edges", J. Comb. Th., 5 (1968), 153-156.

2. V. Chvátal "Intersecting Subfamilies of Hereditary Families" (To be published)